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Parabolic Reduction, Stability and the Mass

I. Special Linear Groups

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Abstract: Basic formulas exposing intrinsic relations on volumes of fundamental domains, that of their stable portions and various zetas are obtained for special linear groups, via theory of Eienstein series and parabolic reductions. Parallel theory for bundles over curves on finite fields is reviewed as well. Based on all this, conjectures for general reductive groups are formulated.

1 Number Fields

1.1 Siegel's Volume Formula

For special linear group SL_n defined over $\mathbb{Q}$, there are 3 naturally associated groups, namely, the real Lie group $SL_n(\mathbb{R})$, its maximal compact subgroup $SO_n(\mathbb{R})$ and the full modular group $SL_n(\mathbb{Z})$. It is well known that the double quotient space $SL_n(\mathbb{Z})\backslash SL_n(\mathbb{R})/O_n(\mathbb{R})$ may be interpreted as the space of isometric classes of rank n lattices of volume one in the Euclidean space $\mathbb{R}^n$. Indeed, the metrics on $\mathbb{R}^n$ are parametrized by matrices $A\cdot A^t$ with $A\in GL_n(\mathbb{R})$, and up to $O_n(\mathbb{R})$-equivalence, the metric is uniquely determined by A. As such, then the lattice structures are finally determined modulo the automorphism group $SL_n(\mathbb{Z})$ of $\mathbb{Z}^n$.

Denote by $\mathbb{M}_{\mathbb{Q},n}[1]$ the moduli space of all full rank lattices in $\mathbb{R}^n$ of volume one. The above discussion exposes the following

Fact 1. (**Arithmetic versus Geometry**) *There is a natural one-to-one correspondence*

$$SL_n(\mathbb{Z})\backslash SL_n(\mathbb{R})/SO_n(\mathbb{R})\simeq \mathbb{M}_{\mathbb{Q},n}[1].$$

Associated the natural measure on $SL_n(\mathbb{R})$, we may ask what is the corresponding volume of the above space. Surprisingly, while the space $SL_n(\mathbb{Z})\backslash SL_n(\mathbb{R})/SO_n(\mathbb{R})$, or the same, $\mathbb{M}_{\mathbb{Q},n}[1]$, is highly non-abelian, or the same, non-commutative, according to Siegel, its volume can be expressed in terms of the special values of Riemann zeta function, which is abelian in nature.

Fact 2. *(Siegel) (***Volume of Fundamental Domain***)*

$$m_{\mathbb{Q},n} := \mathrm{Vol}\Big(\mathbb{M}_{\mathbb{Q},n}[1]\Big) = \widehat{\zeta}_{\mathbb{Q}}(1)\widehat{\zeta}_{\mathbb{Q}}(2)\cdots\widehat{\zeta}_{\mathbb{Q}}(n)$$

where $\widehat{\zeta}_{\mathbb{Q}}(s)$ *denotes the complete Riemann zeta function and*

$$\widehat{\zeta}_{\mathbb{Q}}(1) := \mathrm{Res}_{s=1}\widehat{\zeta}_{\mathbb{Q}}(s).$$

1.2 Stability

Among all lattices, motivated by Mumford's fundamental work in algebraic geometry, we independently introduced the semi-stable lattices in our studies of non-abelian zeta functions. By definition, a lattice Λ is called *semi-stable* if for all sub-lattices Λ_1 of Λ

$$\mathrm{Vol}(\Lambda_1)^{\mathrm{rank}\Lambda} \geq \mathrm{Vol}(\Lambda)^{\mathrm{rank}\Lambda_1}.$$

Denote by $\mathbb{M}_{\mathbb{Q},n}^{\mathrm{ss}}[1]$ the moduli space of rank n semi-stable lattices of volume 1. One checks that $\mathbb{M}_{\mathbb{Q},n}^{\mathrm{ss}}[1]$ is a closed compact subset of $\mathbb{M}_{\mathbb{Q},n}[1]$. With the induced metric, define

$$m_{\mathbb{Q},n}^{\mathrm{ss}} := \mathrm{Vol}\Big(\mathbb{M}_{\mathbb{Q},n}^{\mathrm{ss}}[1]\Big).$$

A natural question is what is the volume $u_{\mathbb{Q},n}$.

1.3 High rank non-abelian zeta functions

Similarly, denote by $\mathbb{M}_{\mathbb{Q},n}^{\mathrm{ss}}$ the moduli space of rank n semi-stable lattices and by $\mathbb{M}_{\mathbb{Q},n}^{\mathrm{ss}}[T]$ its volume T part. Then we have a natural decomposition

$$\mathbb{M}_{\mathbb{Q},n}^{\mathrm{ss}} = \cup_{T\in\mathbb{R}_{>0}}\mathbb{M}_{\mathbb{Q},n}^{\mathrm{ss}}[T].$$

Easily one checks that there is a natural isomorphism

$$\mathbb{M}_{\mathbb{Q},n}^{\mathrm{ss}}[T] \simeq \mathbb{M}_{\mathbb{Q},n}^{\mathrm{ss}}[T'] \qquad \forall\, T,\, T' \in \mathbb{R}_{>0}. \qquad (*)$$

Using the above measure on $\mathbb{M}_{\mathbb{Q},n}^{\mathrm{ss}}[T]$ and the invariant Haar measure $\frac{dT}{T}$ on $\mathbb{R}_{>0}$, we obtain a natural measure $d\mu$ on $\mathbb{M}_{\mathbb{Q},n}^{\mathrm{ss}}$.

Moreover, there is a genuine cohomology theory $h^i(F,\Lambda), i = 0,1$ for lattices Λ over number fields F for which the arithmetic analogue of the duality, the Riemann-Roch theorem, the vanishing theorem holds. For details, please refer to [W]. In the case of $F = \mathbb{Q}$,

$$h^0(\mathbb{Q},\Lambda) = \log\Big(\sum_{\mathbf{x}\in\Lambda} e^{-\pi\|\mathbf{x}\|^2}\Big)$$

which was introduced earlier in [GS]. Denote by $d(\Lambda)$ the Arakelov degree of Λ, which over $\mathbb{Q}$ is simply $-\log \mathrm{Vol}(\Lambda)$. Following [W], define the associated *rank n non-abelian zeta function* $\widehat{\zeta}_{\mathbb{Q},n}(s)$ by

$$\widehat{\zeta}_{\mathbb{Q},n}(s) := \int_{\mathrm{M}_{\mathbb{Q},n}^{\mathrm{ss}}} \Big(e^{h^0(\mathbb{Q},\Lambda)} - 1\Big)\cdot (e^{-s})^{d(\Lambda)}\, d\mu, \qquad \mathrm{Re}(s) > 1.$$

Then using the basic property of the above cohomology theory for h^i's, namely the duality, the RR and the vanishing theorem, tautologically, we have the following

Fact 3. *(Weng) (0) (***Relation with Abelian Zeta***)*

$$\widehat{\zeta}_{\mathbb{Q},1}(s) = \widehat{\zeta}_{\mathbb{Q}}(s);$$

*(i) (***Meromorphic Extension***) $\widehat{\zeta}_{\mathbb{Q},n}(s)$ is a well-defined holomorphic function in* $\mathrm{Re}(s) > 1$*, and admits a unique meromorphic extension to the whole s-plane;*
*(ii) (***Functional Equation***)*

$$\widehat{\zeta}_{\mathbb{Q},n}(1-s) = \widehat{\zeta}_{\mathbb{Q},n}(s);$$

*(iii) (***Singularities***) $\widehat{\zeta}_{\mathbb{Q},n}(s)$ has only two singularities, all simple poles, at $s = 0, 1$. Moreover*

$$\boxed{\mathrm{Res}_{s=1}\widehat{\zeta}_{\mathbb{Q},n}(s) = m_{\mathbb{Q},n}^{\mathrm{ss}} := \mathrm{Vol}\Big(\mathrm{M}_{\mathbb{Q},n}^{\mathrm{ss}}[1]\Big)}$$

In particular we see that $m_{\mathbb{Q},n}^{\mathrm{ss}}$ is naturally related to the special value of the non-abelian zeta function $\widehat{\zeta}_{\mathbb{Q},n}(s)$.

1.4 Parabolic Reduction: Analytic Theory

The high rank zeta functions are closely related with Eisenstein series. In fact, we have

Fact 4. *(Weng) (i) (***High Rank Zeta and Eisenstein Series***)*

$$\widehat{\zeta}_{\mathbb{Q},n}(s)=\int_{\mathcal{M}_{\mathbb{Q},n}^{\mathrm{ss}}[1]}\widehat{E}(\Lambda,s)\,d\mu=\int_{\left(SL_n(\mathbb{Z})\backslash SL_n(\mathbb{R})/O_n(\mathbb{R})\right)^{\mathrm{ss}}}\widehat{E}^{SL_n/P_{n-1,1}}(\mathbf{1},g;s).$$

Here $\widehat{E}(\Lambda,s)$ denotes the complete Eisenstein series associated to the lattice Λ, $\widehat{E}^{SL_n/P}(\mathbf{1},g;*)$ denote the relative (complete) Eisenstein series on $SL_n(\mathbb{R})$ induced from the constant function $\mathbf{1}$ on the Levi factor of the maximal parabolic subgroup P and $P_{n-1,1}$ denotes the standard parabolic subgroup of SL_n corresponding to the partition $n=(n-1)+1$, and $\left(SL_n(\mathbb{Z})\backslash SL_n(\mathbb{R})/O_n(\mathbb{R})\right)^{\mathrm{ss}}$ the part corresponding to the semi-stable lattices via Fact 1, which for our convenience will also be viewed as a subset of $SL_n(\mathbb{R})$;
*(ii) (***Analytic Truncation versus Arithmetic Truncation***)*

$$\Lambda^0\mathbf{1}=\chi_{\left(SL_n(\mathbb{Z})\backslash SL_n(\mathbb{R})/O_n(\mathbb{R})\right)^{\mathrm{ss}}}$$

Namely, Arthur's truncation of the constant function $\mathbf{1}$ is simply the characteristic function of the subset $\left(SL_n(\mathbb{Z})\backslash SL_n(\mathbb{R})/O_n(\mathbb{R})\right)^{\mathrm{ss}}$ consisting of semi-stable points.

This is a number theoretic analogue of a result of Laffourge on the relation between analytic truncation and arithmetic truncation for function fields.

Consequently,

$$\widehat{\zeta}_{\mathbb{Q},n}(s)=\int_{SL_n(\mathbb{Z})\backslash SL_n(\mathbb{R})/O_n(\mathbb{R})}\Lambda^{\mathbf{0}}\widehat{E}^{SL_n/P_{n-1,1}}(\mathbf{1},g;s).$$

Here $\Lambda^{\mathbf{0}}\widehat{E}^{SL_n/P_{n-1,1}}(\mathbf{1},g;s)$ denotes the Arthur's truncation of the Eisenstein series $\widehat{E}^{SL_n/P_{n-1,1}}(\mathbf{1},g;s)$. On the other hand, by Langlands' theory of Eisenstein series, we know that

$$\widehat{E}^{SL_n/P_{n-1,1}}(\mathbf{1},g;s)=\mathrm{Res}_{\langle\lambda-\rho,\alpha_i^\vee\rangle=0,i=1,2,\dots,n-2}\widehat{E}^{SL_n/P_{1,\dots,1}}(\mathbf{1},g;\lambda)$$

where $\alpha_i=\alpha_i-\alpha_{i+1}$ denotes the simple roots of the root system A_{n-1} associated to SL_n, and $\rho=\frac{1}{2}\sum_{\alpha>0}\alpha$ the Weyl vector.

With this, now notice that the moduli space $\mathcal{M}_{\mathbb{Q},n}[1]$ is compact, and that on the Levi of the Borel subgroup, $\mathbf{1}$ is cuspidal. So we can evaluate the *Eisenstein period*

$$\int_{SL_n(\mathbb{Z})\backslash SL_n(\mathbb{R})/O_n(\mathbb{R})}\Lambda^{\mathbf{0}}\widehat{E}^{SL_n/P_{1,\dots,1}}(\mathbf{1},g;\lambda).$$

This then gives a very precise expression of non-abelian zeta function $\widehat{\zeta}_{\mathbb{Q},n}(s)$ as a combination of terms consisting of products of rational functions coming from the symmetry depending only on the root system, and abelian zeta functions. For details, please see [W]. As a direct consequence, we have the following

Fact 5. *(Weng) (***Parabolic Reduction, Stability & the Volumes***)*

$$m_{\mathbb{Q},n}^{\mathrm{ss}}=\sum_{k\geq 1}(-1)^{k-1}\sum_{n_1+\cdots+n_k=n,n_i>0}\frac{1}{\prod_{j=1}^{k-1}(n_j+n_{j+1})}\cdot\prod_{j=1}^{k}m_{\mathbb{Q},n_j}.$$

Geometrically, this means that the semi-stable part can be obtained from the fundamental domain associated to SL_n by deleting the tubular neighborhoods of cusps corresponding to parabolic subgroups which parametrize the same type of canonical flags of unstable lattices, and whose volumes, up to the lattice extensions, are completely determined by that associated to the simple factors of related Levi factors. Undoubtedly, this parabolic reduction is also the 'heart' of the theory of the truncations, both, analytic and arithmetic.

1.5 Parabolic Reduction: Geometric Theory

During our Sept, 2012's stay at IHES, Kontsevich introduced us their beautiful formula relating $m_{\mathbb{Q},n}$'s and $m_{\mathbb{Q},n}^{\mathrm{ss}}$'s. This basic relation is obtained within their lecture notes on the wall-crossing.

Fact 6. *(Kontsevich-Soibelman) (***Parabolic Reduction, Stability & the Volume***)*

$$\frac{1}{n}\cdot m_{\mathbb{Q},n}=\sum_{k\geq 1}\sum_{n_1+\cdots+n_k=n,n_i>0}c_{n_1,n_2,\ldots,n_k}\cdot\prod_{j=1}^{k}m_{\mathbb{Q},n_j}^{\mathrm{ss}}.$$

Here $c_{n_1,n_2,\ldots,n_k}:=\dfrac{1}{n_1(n_1+n_2)\cdots(n_1+n_2+\cdots+n_k)\cdots(n_{k-1}+n_k)n_k}.$

Indeed, the essence of this is the existence of the so-called canonical filtration, namely, the Harder-Narasimhan filtration of a lattice: *For a rank n lattice Λ, there exists a unique filtration of sub-lattices*

$$0=\Lambda_0\subset\Lambda_1\subset\Lambda_2\subset\cdots\subset\Lambda_k=\lambda$$

such that

(i) $G_i(\Lambda):=\Lambda_i/\Lambda_{i+1}$ *is semi-stable*; and

(ii) $\mathrm{Vol}\Big(G_i(\Lambda)\Big)^{\mathrm{rank}\big(G_{i+1}(\Lambda)\big)}\geq\mathrm{Vol}\Big(G_{i+1}(\Lambda)\Big)^{\mathrm{rank}\big(G_i(\Lambda)\big)}.$

2 Function Fields/$\mathbb{F}_q$

The interactions between studies of number fields and that of function fields (over finite fields) have been proven to be very fruitful, based on formal analogues between these two types of fields, despite the fact that many working mathematicians, not without their own reasons, believe otherwise. The works presented here are yet another group of beautiful examples.

2.1 Weil's Formula: Tamagawa Numbers

Motivated by Siegel's volume formula above, which originally was done in the theory of quadratic forms, Weil reinterpreted it in terms his famous Tamagawa number one conjecture. For SL_n, this goes as follows.

Let X be an irreducible, reduced, regular projective curve defined over V. Denotes its function field by F and its ring of adeles by $\mathbb{A}$. Fix a vector bundle E_0 of rank n on X with determinant λ.

Consider the group $SL_n(\mathbb{A})$ with $\mathbb{K}(E_0)$ the maximal compact subgroup associated to E. Then there is a natural morphism π from the quotient space $SL_n(F)\backslash SL_n(\mathbb{A})/\mathbb{K}(E_0)$ to the stack $\mathbb{M}_{X,n}(\lambda)$ of rank n bundles with fixed determinant λ.

Fact 7. *The natural morphism*

$$\pi : SL_n(F)\backslash SL_n(\mathbb{A})/\mathbb{K}(E_0) \to \mathbb{M}_{X,n}(\lambda)$$

is surjective with the fiber $\pi^{-1}(E_0^g)$ at the vector bundle E_0^g associated to $g \in SL_n(\mathbb{A})$ consisting of $\#\Big(\mathbb{F}_q^*/\det \mathrm{Aut}(E_0^g)\Big)$. Here $\det \mathrm{Aut}(E_0^g)$ denotes the image of $\det \mathrm{Aut}(E_0^g)$ in $\mathbb{F}_q^*$ under the determinant mapping.

Denote by $\mathbb{M}_{X,n}(d)$ the moduli stack of rank n bundle of degree d on X, and introduce the total mass for rank n and degree d bundles on X by

$$m_{X,n}(d) := \sum_{E\in\mathbb{M}_{X,n}(d)} \frac{1}{\#\mathrm{Aut}(E)}.$$

Denote by

$$\widehat{\zeta}_X(s) := \zeta_X(s) \cdot (q^s)^{g-1}$$

the complete Artin zeta function associated to X, and

$$\widehat{\zeta}_X(1) := \mathrm{Res}_{s=1}\widehat{\zeta}_X(s) \cdot \frac{1}{\log q}.$$

Then Weil's result that the Tamagara number of the quotient space $SL_n(F)\backslash SL_n(\mathbb{A})$ equals one is equivalent to the following

Fact 8. *(Weil) (***Tamagawa Number***)*

$$m_{X,n}(d) = m_{X,n} := \widehat{\zeta}_X(1)\widehat{\zeta}_X(2)\cdots\widehat{\zeta}_X(n).$$

2.2 Non-Abelian Zeta Functions for X

Denote by $\mathbb{M}_{X,n}^{ss}(d)$ the moduli stack of rank n semi-stable bundle of degree d on X. Then define the *pure non-abelian zeta function of rank n for X* by

$$\widehat{\zeta}_{X,n}(s) := \sum_{k\in\mathbb{Z}}\sum_{E\in\mathbb{M}_{X,n}^{ss}(kd)} \frac{q^{h^0(X,E)}-1}{\#\mathrm{Aut}(E)}\cdot (q^{-s})^{\chi(X,E)}.$$

Write

$$\widehat{\zeta}_{X,n}(s) = \zeta_{X,n}(s)\cdot (q^s)^{n(g-1)}, \qquad Z_{X,n}(t) := \zeta_{X,n}(s) \text{ with } t = q^{-s}$$

Introduce the partial mass of semi-stable bundles by

$$\alpha_{X,n}(d) := \sum_{E\in\mathbb{M}_{X,n}^{ss}(d)} \frac{q^{h^0(X,E)}-1}{\#\mathrm{Aut}(E)}, \qquad \beta_{X,n}(d) := \sum_{E\in\mathbb{M}_{X,n}^{ss}(d)} \frac{1}{\#\mathrm{Aut}(E)}.$$

Then tautologically,

$$Z_{X,n}(t) = \sum_{m=0}^{(g-1)-1} \alpha_{X,n}(mn)\cdot\Big(T^m + Q^{(g-1)-m}\cdot T^{2(g-1)-m}\Big)$$
$$+\,\alpha_{X,n}\big(n(g-1)\big)\cdot T^{g-1} + (Q-1)\beta_{X,n}(0)\cdot\frac{T^g}{(1-T)(1-QT)}.$$

where $T := t^n$ and $Q := q^n$. This exposes the following

Fact 9. *(Weng) (i) (***Relation with Artin Zetas***)* $\zeta_{X,1}(s) = \zeta_X(s)$, *the Artin zeta function for $X/\mathbb{F}_q$;*

*(ii) (***Rationality***) There exists a degree $2g$ polynomial $P_{X,r}(T)\in\mathbb{Q}[T]$ of T such that*

$$Z_{X,r}(t) = \frac{P_{X,r}(T)}{(1-T)(1-QT)} \quad \text{with} \quad T = t^r,\ Q = q^r;$$

*(iii) (***Functional Equation***)*

$$\widehat{\zeta}_{X,n}(1-s) = \widehat{\zeta}_{X,n}(s);$$

*(iv) (***Residues***)*

$$\widehat{\zeta}_{X,n}(1) := \mathrm{Res}_{s=1}\widehat{\zeta}_{X,n}(s)\cdot\frac{1}{\log Q} = \beta_{X,n}(0)\Big(= m_{X,n}^{ss}(0)\Big).$$

2.3 Parabolic Reduction, Stability and the Mass: Geometric Theory

To go further, make a normalization by introducing

$$\widetilde{m}_{X,n}^{\mathrm{ss}}(d) = \frac{1}{q^{\frac{n(n-1)}{2}(g-1)}} \cdot m_{X,n}^{\mathrm{ss}}(d).$$

Then the parabolic reduction via the Harder-Narasimhan filtration leads to the following relation involving infinite sums:

Fact 10. *(Harder-Narasimhan, Desale-Ramanan) (***Parabolic Reduction***)*

$$m_{X,n}(d) = \sum_{k\geq 1} \sum_{n_1+\cdots+n_k=n, n_i>0} \sum_{\substack{\frac{d_1}{n_1}>\cdots>\frac{d_k}{n_k} \\ d_1+\cdots+d_k=d}} q^{-\sum_{i<j}(d_i n_j - d_j n_i)} \prod_{j=1}^{k} \widetilde{m}_{X,n_j}^{\mathrm{ss}}(d_j).$$

2.4 Parabolic Reduction, Stability and the Mass: Combinatorial Aspect

With the above result, Zagier proved the following fundamental result, hidden in his paper on Verlinder formula:

Fact 11. *(Zagier) (***Parabolic Reduction, Stability & the Mass***)*

$$\widetilde{m}_{X,n}^{\mathrm{ss}}(d) = \sum_{k\geq 1}(-1)^{k-1} \sum_{\substack{n_1+\cdots+n_k=n \\ n_i>0, i=1,\ldots,k}} \prod_{j=1}^{k-1} \frac{q^{(n_j+n_{j+1})\cdot\{(n_1+\cdots+n_j)\cdot\frac{d}{n}\}}}{q^{(n_j+n_{j+1})}-1} \cdot \prod_{j=1}^{k} m_{X,n_j}.$$

This formula should be compared with with our Fact 5 for number fields. The structure are very much similar: in fact if we let $q \to 1$, then we would get the number theoretic identity there. Indeed, the original formula of Zagier is a bit different: original coefficients depends on g. Motivated by our number fields analogue, we reorganize it with $\widetilde{m}$ for m and $\widehat{\zeta}_X$ for ζ_X. Consequently, our coefficients are environmentally free. That is, independent of the curve X and the genus.

2.5 Parabolic Reduction, Stability and the Mass: New Formula

The above relation of Harder-Narasimhan, Ramanan-Desale and Zagier for function fields correspond to our own formula listed as Fact 5.

So naturally, what should be the one appeared in the theory of parabolic reduction, stability and the volumes obtained by Kontsevich-Soibelman. This was recently obtained by Zagier during his visit to Fukuoka in May, 2012.

Fact 12. *(Zagier) For an ordered partition $n = n_1 + n_2 + \cdots + n_k$, fix $\delta_i \in \{0, , \ldots, n_1 - 1\}$, then fix $v_i \in [0,1) \cap \mathbb{Q}$ satisfying*

$$v_i \equiv \frac{\delta_i}{n_i} - \frac{\delta_{i+1}}{n_{i+1}} \pmod 1.$$

Also set $N_i = n_1 + n_2 + \cdots + n_i$ and $N_i' = n - N_i$ for $i = 1, 2, \ldots n$.
*(i) (**On Average**)*

$$n \cdot m_{X,n} = \sum_{k \geq 1} \sum_{\substack{n_1+n_2+\cdots+n_k=n \\ n_i>0,\, i=1,2,\ldots,k}} \sum_{\substack{\delta_i \in \{0,1,\ldots,n_i-1\} \\ j=1,2,\ldots,k}} \prod_{i=1}^{k-1} \frac{q^{v_i N_i N_i'}}{q^{N_i N_i'} - 1} \cdot \prod_{j=1}^{k} \widetilde{m}_{X,n_j}^{\mathrm{ss}}(\delta_j).$$

*(ii) (**Individuality**) For all $d = 0, 1, \ldots, n-1$,*

$$m_{X,n} = \sum_{k \geq 1} \sum_{\substack{n_1+n_2+\cdots+n_k=n \\ n_i>0,\, i=1,2,\ldots,k}} \frac{1}{n}$$
$$\times \sum_{\substack{\delta_i \in \{0,1,\ldots,n_i-1\} \\ j=1,2,\ldots,k}} \left(\sum_{\zeta_n :\, \zeta_n^n = 1} \zeta_n^{n-d} \cdot \prod_{h=1}^{k-1} \frac{\zeta_n^{v_h N_h} q^{v_h N_h N_h'}}{\zeta_n^{N_h} q^{N_h N_h'} - 1} \right) \prod_{j=1}^{k} \widetilde{m}_{X,n_j}^{\mathrm{ss}}(\delta_j).$$

The first is obtained by taking average on d from the relation of Fact 11, while the second is obtained directly from that of Fact 11. With geometric picture in mind, formula (ii) should be further polished, so as to get everything done according to the real world structure. This may prove to be a bit complicated due to the fact that usually

$$\mathbb{M}_{X,n}(d) \not\simeq \mathbb{M}_{X,n}(d'), \qquad \forall d,\, d' \in \{0, 1, \ldots, n-1\},\ d \neq d'.$$

This is very different from the cases for number fields, where we always have the isomorphism $(*)$ between different levels.

We remind the reader that while all relations in function field case are obtained using geometric methods, our basic relation for number fields are obtained analytically using Eisenstein series.

3 Parabolic Reduction, Stability and the Mass: General Reductive Groups

Motivated by the above discussion, more generally, for a split reductive group G defined over a number field F, B a fixed Borel, ..., denote by $G(\mathbb{A})^{\mathrm{ss}}$ the adelic elements of G corresponding to semi-stable principle G-lattices ([G]). Write $\mathbb{K}_G$ for the associated maximal compact subgroup. Also for a standard parabolic subgroup P, write its Levi decomposition as $P = UM$ with U the unipotent radical and M its Levi factor. Denote the corresponding simple decomposition of M as $\prod_i M_i$ with M_i's the simple factors of M. Introduce invariants

$$m_{F;P} := \prod_i \mathrm{Vol}\Big(\mathbb{K}_{M_i}\backslash M_i^1(\mathbb{A})/M_i(F)Z_{M_i^1(\mathbb{A})}\Big)$$

and

$$m_{F;P}^{\mathrm{ss}} := \prod_i \mathrm{Vol}\Big(\mathbb{K}_{M_i}\backslash M_i^1(\mathbb{A})^{\mathrm{ss}}/M_i(F)Z_{M_i^1(\mathbb{A})}\Big).$$

In parallel, we have similar constructions for function fields $F = \mathbb{F}_q(X)$.

Denote by

$$n_i := \#\{\alpha > 0 : \langle \rho, \alpha^\vee \rangle = i\} - \#\{\alpha > 0 : \langle \rho, \alpha^\vee \rangle = i+1\}$$

and by v_G the volume of $\{\sum_{\alpha\in\Delta} : a_\alpha \alpha^\vee : a_\alpha \in [0,1)\}$.

Fact 13. *(Langlands) (***Volume of Fundamental Domain***) For the field of rationals,*

$$\mathrm{Vol}\Big(\mathbb{K}_G\backslash G^1(\mathbb{A})/G(\mathbb{Q})Z_{G^1(\mathbb{A})}\Big) = v_G \cdot \prod_{i\geq 1} \widehat{\zeta}(i)^{-n_i}.$$

Based on all this, then we have the following

Conjecture 1. *(Weng) (***Parabolic Reduction***) Let G/F be a split reductive group with B/F a fixed Borel. Then, for each standard parabolic subgroup P of G, there exist constants $c_P \in \mathbb{Q}$, $e_P \in \mathbb{Q}_{>0}$, independent of F, such that*

$$m_{F;G} = \sum_P c_P \cdot m_{F;P}^{\mathrm{ss}}, \qquad m_{F;G}^{\mathrm{ss}} = \sum_P \mathrm{sgn}(P) \cdot e_P \cdot m_{F;P},$$

where P runs over all standard parabolic subgroups of G, and $\mathrm{sgn}(P)$ *denotes the sign of P.*

The exact values of e_P's can be written out in terms of the associated root system. Indeed, if

$$W_0 := \Big\{w \in W : \{\alpha \in \Delta : w\alpha \in \Delta \cup \Phi^-\} = \Delta\Big\},$$

then there is a natural one-to-one correspondence between W_0 and the set of subsets of Δ, and hence to the set of standard parabolic subgroups of G. Thus we will write

$$W_0 := \Big\{w_P : \ P \text{ standard parabolic subgroup}\Big\},$$

and, for $w = w_P \in W_0$, write $J_P \subset \Delta$ the corresponding subset.

Conjecture 2. *(Weng) Let G be a split type reductive group with P its maximal parabolic subgroup.*
(1) For a number field F,
*(i) (***Relation to Zetas with Symmetries***)*

$$m_{F;G}^{\mathrm{ss}} = \mathrm{Res}_{s=-c_P}\widehat{\zeta}_F^{(G,P)}(s) = \mathrm{Res}_{\lambda=\rho}\omega_F^G(\lambda);$$

*(ii) (***Parabolic Reduction, Stability & the Mass***)*

$$m_{F;G}^{\mathrm{ss}} = \sum_P \frac{(-1)^{\mathrm{rank}(P)}}{\prod_{\alpha\in\Delta\setminus w_J J_P}(1-\langle w_J\rho,\alpha^\vee\rangle)} \cdot m_{F;P};$$

(2) For an irreducible reduced regular projective curve X,
*(i) (***Relation to Zetas with Symmetries***)*

$$\log q \cdot m_{F;G}^{\mathrm{ss}} = \mathrm{Res}_{s=-c_P}\widehat{\zeta}_X^{(G,P)}(s) = \mathrm{Res}_{\lambda=\rho}\omega_X^G(\lambda);$$

*(ii) (***Parabolic Reduction, Stability & the Mass***)*

$$m_{F;G}^{\mathrm{ss}} = \sum_P \frac{(-1)^{\mathrm{rank}(P)}}{\prod_{\alpha\in\Delta\setminus w_J J_P}(1-q^{\langle w_J\rho,\alpha^\vee\rangle-1})} \cdot m_{F;P}.$$

Remark. Calculations in [Ad] for lower rank groups indicate that, for number fields, $\frac{1}{c_P} \in \mathbb{Z}_{>0}$. It would be very interesting to find a close formula for them.

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