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THE COMMUNICATION THEORY AND THE EQUATION OF HEAT MOTION IN FLUID DYNAMICS BY FOURIER

- A COMMUNICATION POINT FROM CLASSICAL MECHANICS TO QUANTUM MECHANICS

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ABSTRACT.
We discuss Fourier's the heat communication theory and the heat equations of motion in fluid, explaining the theoretical background produced in the rivalry with Lagrange and Poisson. We pick up Poisson's direct method for definite integral in regarding to the problems between real and imaginary, that is the life-work theme Euler and Laplace also struggled to solve. We point out this problem based on the then continuum concept, which is the bridge point over classical mechanics into classical quantum mechanics like Boltzmann, and moreover into new quantum mechanics like Schrödinger. Through this wide range as possible, we like to attention to mathematical aspect of Fourier and his surroundings.

1. INTRODICTION

[1,2,3,4] Fourier's works are summerized by Dirichlet, a disciple of Fourier, as follows :

- a sort of *singularity of passage* from the finite to the infinite
- to offer a new example of the *prolificity* of the analytic process

The first is our topics which Fourier and Poisson point this problem in life and the other is, in other words, the sowing seeds to be solved from then on. Dirichlet says in the following contents, Fourier (1768-1830) couldn't solve in life the question in relation to the mathematical theory of heat, in *Solution d'une question relative a le thëorie mathèmatiques de la chaleur* (The solution of a question relative to the mathematical theory of heat) [5].

1.1. The outline of the situations surrounding Fourier.

About the situations around Fourier, we can summarize as follows :

1. Fourier's manuscript 1807, which had been unknown for us until 1972, I. Grattan-Guinness [15] discovered it. Fourier's paper 1812 based on the manuscript was prized by the academy of France. We consider that Fourier, in his life work of the heat theory, begins with the communication theory, and he devoted in establishing this theme as the priority.

2. Owing to the arrival of continuum theory, many mathematical physical works are introduced, such as that Fourier and Poisson struggle to deduce the trigonometric series in the heat theory and heat diffusion equations. In the curent of formularizing process of the fluid dynamics, Navier, Poisson, Cauchy and Stokes struggle to deduce the wave equations and the Navier-Stokes

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[1]Basically, we treat the exponential / trigonometric / logarithmic / π / et al. / functions as the transcendental functions.

[2]Translation from Latin/French/German into English mine, except for Boltzmann.

[3]To establish a time line of these contributor, we list for easy reference the year of their birth and death: Euler(1707-83), d'Alembert(1717-83), Lagrange(1736-1813), Laplace(1749-1827), Fourier(1768-1830), Poisson(1781-1840), Cauchy(1789-1857), Dirichlet(1805-59), Riemann(1826-66), Boltzmann(1844-1906), Schrödinger (1887-1961).

[4]The symbol ($\Downarrow$) means our remark not original, when we want to avoid the confusions between our opinion and sic.

equations. Of cource, there are many proceding researches before these topics, however, for lack of space, we must pick up at least, the essentials such as following contents :

3. We introduce the heat theory and heat diffusion equations based on the oscillating equations of cords, namely wave equations. We treate the theoretical contrarieties between Fourier and Lagrange, and next, between Fourier and Poisson, and then, the microscopically descriptive fluid equations, however, we omit the theoretical contrariety between Navier and Poisson, and the collaboration on the proof of describability of the trigonometric series of an arbitrary function up to the 20th Centuries.

4. Fourier [13] combines heat theory with the Euler's equation of incompressible fluid dynamics and proposes the equation of heat motion in fluid in 1820, however, this paper was published in 1833 after 13 years, it was after 3 years since Fourier passed away. Fourier seems to have been doutful to publish it in life.

5. After Fourier's commnunication theory, the gas theorists like Maxwell, Kirchhoff, Boltzmann [1] study the transport equations with the concept of collision and transport of the molecules in mass. In both principles, we see almost same relation between the Fourier's communication and transport of heat molecules and the Boltzmann's collision and transport of gas molecules.

6. Since 1811, Poisson issued many papers on the definite integral, containing transcendental, and remarked on the necessity of careful handling to the diversion from real to imaginary, especially, to Fourier explicitly. To Euler and Laplace, Poisson owes many knowledge, and builds up his principle of integral, consulting Lagrange, Lacroix, Legendre, etc. On the other hand, Poisson feels incompatibility with Laplace's 'passage', on which Laplace had issued a paper in 1809, entitled : On the 'reciprocal' passage of results between real and imaginary, after presenting the sequential papers on the occurring of 'one-way' passage in 1782-3.

7. To these passages, Poisson proposed the direct, double integral in 1811,13,15,20 and 23. The one analytic method of Poisson 1811 is using the round braket, contrary to the Euler's integral 1781. The multipl integral itself was discussed and practical by Laplace in 1782, about 20 years before, when Poisson applied it to his analysis in 1806.

8. As a contemporary, Fourier is made a victim by Poisson. To Fourier's main work : *The analytical theory of heat* in 1822, and to the relating papers, Poisson points the diversion applying the what-Poisson-called-it 'algebraic' theorem of De Gua or the method of cascades by Roll, to transcendental equation. Moreover, about their contrarieties, Darboux, the editor of *Œuvres de Fourier*, evaluates on the correctness of Poisson's reasonings in 1888. Drichlet also mentions about Fourier's method as a sort of *singularity of passage* from the finite to the infinite.

9. About the describability of the trigonometric series of an arbitrary function, nobody succeeds in it including Fourier, himself. According to Dirichlet, Cauchy is the only person challenges it in vain. Poisson tries it from another angle. Dirichlet and Riemann step into the kernel of the question. Up to the middle of or after the 20th Centuries, these collaborations are continued, finally in 1966, by Carleson proved in L^2, and in 1968, by Hunt in L^p.

1.2. **The preliminary discources on Fourier from the Nota to I.Grattan-Guinness.** To see the Fourier's motivation for works, now, we pay attention to the historical changes : from

(1) the Nota of Prize paper, Part 2 1826

to the each narratives in the preliminary discourses on Fourier's works

(2) English translated edition of Fourier 1822 by A. Freeman 1878, [14, pp.1-12]
(3) edition of Fourier's Oeuvres by G. Darboux 1888, pp. XV-XXVIII
- Foreword (Avant-propos) by Darboux (V.1) 1887,
- Preliminary discourse by Fourier (V.1) 1822,
- Afterword (Avertissement) by Darboux (V.2) 1890, p.VII

(4) the book by I. Grattan-Guinness 1972 [15]

1.2.1. *The Nota of Prize paper 1826 (Part 2).* The first analytic studies by the author were aimed at the communication between the disjoint masses : the paper is the first part. The problems on the continuum were solved by the author several years ago. He submitted at first this theory with the manuscript belongs to the Institute de France at the last part of 1807, and published a extract on the BSP 1808, page 112.[5] He added afterward to the first version (manuscript) (1) conversion of series, (2) the heat diffusion in the infinite prism, (3) its emission in the space of vacuum, (4) the constructive methods useful to work the principal theories ; and finally, (5) the notes on the then epoch-making solution of a question, (6) periodic motion of heat on the surface on earth.

The second paper (namely, the prize paper 1812) : sur la propagation de la chaleur was submitted to the archive of the Institute de France on 28, Sept., in 1811 : it was composed of the preceding papers and the then collected notes. The author deleted only the geometric structure and the detail of analysis unrelated to the physical problem, and added the general equation which explains the state of the surface. It is this work which he rewarded in the early part of 1812, and the paper was jointed in the collection of the Memoires. It was permitted by Mr. Delambre to print the paper in 1821. Namely, the first part was issued from MAS in 1819, the second in the following issue. (trans. mine.) [6]

1.2.2. *The Fourier's Oeuvres edited by G. Darboux.* The preliminary discource by Fourier, edited by G. Barboux, says in 1820 : Our first analytic studies of the communication of heat were aimed at the distribution between the disjoint masses ; we have kept the paper in the Section 2 of the chapter 4. The problems on the continuum were solved several years ago ; his theory have been submitted at the first time with the manuscript belongs to the Institute de France at the last part of 1807, and published a extract on the BSP (in year 1808, pp. 112-116.)[7] We have added afterward to the first version (manuscript) and succeeded the Notes by the full version in relation to (1) conversion of series, (2) the diffusion of the heat in the infinite prism, (3) its emission in the space of vacuum, (4) the constructive methods useful to work the principal theories, (5) the analysis of the periodic motion (of heat) on the surface of the earth. (trans. mine.), where, item (5) (the then-epoch-making solution of a question) is deleted from the manuscript by Fourier. G. Darboux says in his first edition in 1888 : The works relating to the heat theory by Fourier appear in the late 18C. It has been submitted to the Academy of Science, in Dec. 21, 1807. his first publication is unknown for us : we don't know except for an extract of 4 pages of BSP in 1808 ; It was read by the Committee, however, may be withdrawn by Fourier during 1810. The Committee of Academy, held in 1811, decided the following judgment : " Make clear the mathematical theory on the propagation of heat, and compare this theory with the exact result of experiments." (trans. mine.) [8]

After two years of editing work, G. Darboux, however, says in his Avertissment of second edition in 1890 as follows :

As Navier has been charged, after Fourier's death, to publish the uncompleted works entitled

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[5](⇓) BSP : Bulletin des Sciences par la Société philomatique. There are some expressions : Bulletin de la Société philomatique, Bulletin des Sciences, Société Philomatique. etc.. The extract 1808 was put not by Fourier but by Poisson. However, Grattan-Ginness mentions another existent Fourier's extract of 10 pages. [15, p.26, p.497], [19, p.25]. We don't know about it except for Poisson 1808.

[6](⇓) This Nota uses the third personal style with 'Fourier' or 'he', however, although it is almost same contents with Nota, Fourier's Preliminary discourse 1822 uses 'we'. The Nota 1826 was put earlier than Fourier's book 1822. We don't know the name of the then secretary of the Academy, the writer of Nota 1826, who is Delambre (?) who was the predecesor to Fourier. Fourier succeeded the permanent secretary to him after his death in 1822. The first part was published in 1824 as the MAS issue 1819-20 and the second in 1826 as the MAS issue 1821-22. In 1822, Fourier published by himself (not by MAS) his changed paper from the prized paper 1811. This publication was scheduled later than MAS issue, however, performed earlier.

[7](⇓) The writer's problem is same as above footnote.

[8](⇓) About the extract, same as above footnote. Lagrange was a member of the Committee of judgement and poses against Fourier's paper 1807. cf [25]. G.Darboux lists as follows : Lagrange, Laplace, Malus, Haüe and Legendre. [3, p.vii].

" Analysis of the determined equations, " we had thought that the manuscript of Fourier, must be charged him and could be consigned to the library of National School of Civil Engineering after this eminent engineer's death. (trans. mine.) [9]

1.2.3. *The Fourier 1822 by A. Freeman and The Fourier 1807 edited by I. Grattan-Guinness.* In 1878, A. Freeman published the first English translated Fourier's second version, of which the preliminary is completely the same as G. Darboux 1888, ten years later than A. Freeman. In 1972, I. Grattan-Guinness discovered the manuscript 1807. He pays attentions to the Avertissment in the second edition by G. Darboux as above we mention.

2. The theoretical contrarieties to Fourier

2.1. Lagrange and Fourier on the trigonometric series. Riemann studies the history of research on Fourier series up to then (*Geschichte der Frage über die Darstellbarkeit einer willkührlich gegebenen Function durch eine trigonometrische Reihe*, [25, pp.4-17].) We cite one paragraph of his interesting description from the view of mathematical history as follows :

> Als Fourier in einer seiner ersten Arbeiten über die Wärme, welche er der französischen Akademie vorlegtet [10], (21. Dec. 1807) zuerst den Satz aussprach, daß eine ganz willkührlich (graphisch) gegebene Function sich durch eine trigonometrische Reihe ausdrücken laße, war diese Behauptung dem greisen Lagrange's unerwartet, daß er ihr auf das Entschiedenste entgegentrat. Es soll [11] sich hierüber noch ein Schriftstrück in Archiv der Pariser Akademie befinden. Dessenungeachtet verweist [12] Poisson überall, wo er sich der trigonometrischen Reihen zur Darstellung willkürlicher Functionen bedient, auf eine Stelle in Lagrange's Arbeiten über die schwingenden Saiten, wo sich diese Darstellungensweise finden soll. Um diese Bahauptung, die sich nur aus der bekannten Rivalität zwischen Fourier und Poisson erklären laßt [13], zuwiderlegen, sehen wir uns genöthigt, noch einmal auf die Abhandlung Lagrange's zurüchzukommen ; denn über jeden über jenen Vorgang in der Akademie findet sich nichts veröffentlicht. [25, p.10]
>
> Man findet inder That an der von Poisson citirten Stelle die Formel:
>
> $$y = 2\int Y\sin X\pi dX \ \ \sin x\pi + 2\int Y\sin 2X\pi dX \ \ \sin 2x\pi + \cdots + 2\int Y\sin nX\pi dX \ \ \sin nx\pi, \qquad (1)$$
>
> de sort que, lorsque $x = X$, on aura $y = Y$, Y étant l'ordonné qui répond à l'abscisse X. Diese Formel sieht nun allerdinga ganz so aus wie die Fourier'sche Reihe ; so daßbei flüchtigerAnsicht eine Verwerwechselung leicht möglich ist ; aber dieser Schein rührt bloss daher, weil Lagrange das Zeichen $\int dX$ anwendte, wo er heute das Zeichen $\sum \Delta X$ angewandt haben würde. $\cdots$ Wenn man aber seine Abhandlung durchliest, so sieht man, daßer weit davon entfernt ist zu glauben, eine ganz willkührliche Function laße sich wirklich durch eine unendliche Sinusreihe darstellen. [25, pp.10-11]

Lagrange had stated (1) in his paper of the motion of sound in 1762-65. [17, p.553]

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[9]($\Downarrow$) Navier edited and published the first part of this Fourier's algebraic book 1831 [12] after Fourier's death in 1830. We see G. Darboux had discover Fourier's manuscript 1807, and regarded it had been passed to Navier, for the above process, when G. Doubroux edited the second volume of Fourier's Oeuvres in 1890.

[10]sic. Bulletin des sciences p. la soc. philomatique Tome I. p.112

[11]sic. Nach einer mündlichen Mittheilung des Herr Professor Dirichlet.

[12]sic. Unter Andern in den verbreiteten Traité de mécanique Nro. 323. p. 638.

[13]sic. Der Bericht in bulletin des sciences über die von Fourier der Akademie vorgelegte Abhandlung ist von Poisson.

2.2. Fourier and Poisson on the heat theory. Poisson [22] traces Fourier's work of heat theory, from the another point of view. Poisson emphasizes, in the head paragraph of his paper, that although he totally takes the different approaches to formulate the heat differential equations or to solove the various problems or to deduce the solutions from them, the results by Poisson are coincident with Fourier's. Poisson [22] considers the proving on the convergence of series of periodic quantities by Lagrange and Fourier as the manner lacking the exactitude and vigorousness, and wants to make up to it.

> Dans le mémoire cité dans ce n.^{o}, j'ai considéré directment les formules de cette espèce qui ont pour objet d'exprimer des portions de fonctions, *en séries de quantités périodiques*, dont tous les termes satisfont à des conditions données, relatives aux limites de ces fonctions. Lagrange, dans les anciens Mémoires de Turin, et M. Fourier, dans ses Recherches sur la théorie de la chaleur, avaient déjà fait usage de sembles expressions ; mais *il m'a semblé qu'elles n'avaient point encore été démonstrées d'une manière précise et rigoureuse ; et c'est à quoi j'ai tâché de suppléer dans ce Mémoire, par rapport à celles de ces formules qui se présentent le plus souvent dans les applications.* [22, §2, ¶28, p.46] (Italics mine.)

Poisson proposes the different and complex type of heat equation with Fourier's $(a)_P$. For example, we assume that interior ray extends to sensible distance, which forces of heat may affect the phenomina, the terms of series between before and after should be differente.

3. Poisson's paradigm of universal truth on the definite integral

Poisson mentions the universality of the method to solve the differential equations as follows:

> A défaut de méthodes générales, dont nous manquerons peut-être encore longtemps, il m'a semblé que ce qu'il y avait de mieux à faire, c'état de chercher à intégrer isolément les équations aux différences partiellles les plus importantes par la nature des questions de mécanique et de physique qui y conduisent. C'est la l'objet que je me suis proposé dans ce nouveau mémoire. [21, p.123]

Poisson attacks the definite integral by Euler and Laplace, and Fourier's analytical theory of heat, and manages to construct universal truth in the paradigms.

One of the paradigms is made by Euler and Laplace. The formulae deduced by Euler, are the target of criticism by Poisson. Laplace succeeds to Euler and states the passage from real to imaginary or reciprocal passage between two, which we mention in below.

The other is Fourier's application of De Gua. The diversion is Fourier's essential tool for the analytical theory of heat.

Dirichlet calls these passages a sort of *singularity of passage* from the finite to the infinite. cf. Chapter 1. We think that Poisson's strategy is to destruct both paradigms and make his own paradigm to establish the univarsal truth between mathematics and physics. We would like to show it from this point of view in our paper.

4. Argument between Fourier and Poisson on applying the theorem of De Gua

[14] There were the strifes between Poisson and Fourier to struggle for the truth on mathematics or mathematical physics for the 23 years since 1807, when Fourier submitted his manuscript paper. Poisson [24, p.367] asserts that :

- It is not able to apply the rules served the algebra to assure that an equation hasn't imaginary, to the transcendental equation.
- Algebraic theorems are unsuitable to apply to transcendental equations.
- Generally speaking, it is not allowed to divert the theorems or methods from real to transcendental, without careful and strict handling.

[14] We have submitted [20], in which we cites more bibliographies about this topics.

On the other hand, Fourier [10, p.617] refutes Poisson :

- Algebraic equations place no restriction on analytic theorems of determinant ; It is applicable to all transcendental, what we are considering, in above all, heat theory.
- It is sufficient to consider the convergence of the series, or the figure of curve, which the limits of these series represent them in order.
- Generally speaking, it is able to apply the algebraic theorems or methods to the transcendental or all the determined equations.

4.1. **Is the method of De Gua avairable for applying real or/and imaginary ?** Poisson explains the *méthode des cascades*, which means the method of De Gua, as follows :

> Soit $X = 0$ un équation quelconque dont l'inconnue est x ; désignons, pour abréger, par X', X'', $\cdots$, les coefficients différentiels successifs de X, parrapport à x : si le produit $X \cdot X''$ est négatif en même temps que $X' = 0$, que le produit $X' \cdot X'''$ soit négatif en même temps que $X'' = 0$, que $X'' \cdot X^{(4)}$ soit négatif en même temps que $X''' = 0$, et ainsi de suite jusqu'à ce qu'on parvienne à une équation $X^{(i)} = 0$, dont on soit assuré que toutes les racines sont réelles, et qui soit telle que la condition $X^{(i-1)} \cdot X^{(i+1)}$ négatif pour toutes ses racines soit aussi remplie, il sera certain que l'équation proposée $X = 0$ n'a de même que des racines réelles ; et réciproquement, si l'on parvenient à une équation $X^{(i)} = 0$, qui ait des racines imaginaires, ou pour laquelle le produit $X^{(i-1)} \cdot X^{(i+1)}$ soit positif, l'équation $X = 0$ aura aussi des racines imaginaires. [23, pp.382-3]

Here, Poisson puts a very simple example of transcendental equation and iterates the differential :

$$X = e^x + be^{ax} = 0 \tag{2}$$

where, we assume $a > 0$ and b : an arbitrary, given quantities. The equation of an arbitrary degree with respect to i is also $X^{(i)} = e^x + be^{ax} = 0$, $X^{(i-1)} = ba^{i-1} \cdot e^{ax}(1-a) = 0$, $X^{(i+1)} = ba^i \cdot e^{ax}(a-1) = 0$, then $X^{(i-1)} \cdot X^{(i+1)} = -b^2a^{2i-1} \cdot e^{2ax}(1-a)^2 = 0$. Finally, Poisson concludes : the transcendental equation of example (2) has numberless imaginaries : if $b < 0$, (2) has only real root, and if $b > 0$ no root. [23, p.383]. G.Darboux comments if $b \leq 0$, (2) has only real root, it is true, however, Poisson doesn't put the case of $b = 0$. cf. Chapter ??.

5. Fourier's heat equation of motion in fluid

Fourier esteems Euler's fluid dynamic equations, saying in the preface of "The analysis of the heat motion in the fluid." We cite Fourier's English translated paper as follows :

We have become to explain the conditions of fluid motion, by the general, partial differential equations. The discoveries of one of the most beautiful works by the modern mathematicians are due to d'Alembert and Euler. The former is proposed in titled : " The Essay on fluid resistance. " Euler, in 1755, Memoires et l'Academie de Berlin, proposes it under the same theme. He gives this equations under the simple and clear formation including the all possible cases, and he proves it by the praiseworthy clearance, which is the principle characteristic through all his works.

The general equations include four expressions, in which the top three explain the motion of accelerators and the last, mass conservation law. To see the motion of fluid, at the each instants, we must determine in each time the actual velocity of an arbitrary molecule and the pressure acting on the point of fluid mass. Therefore, in this analysis, as the unknown quality, of the direction of three orthogonal axes, we observe the three quantities of the partial velocity of the molecule itself of only one of their directions, and the pressure measuring forth quality. α, β and γ are the orthogonal velocities of a molecule on the each coordinates : x, y, z., and ε : the density variable of this molecule, θ : temperature, t : elapsed time.

In the first part of our explanation, we stated the equations of motion of heat expressing inside and on the surface of the solid as follows : If we examine these questions with the careful attention, as we mentioned it, we will be able to understand the following : the mathematical principles become clear, even in respect to the strictness of proof, it is not inferior to that of the dynamical theory ;

To solve this, we must consider, a given space interior of mass, for example, by the volume of a rectangular prism composed of six sides, of which the position is given. We investigate all the successive alterations which the quality of heat contained in the space of prism obeys. This quantity alternates instantly and constantly, and becomes very different by the two things. One is the property, the molecules of fluid have, to communicate their heat with sufficiently near molecules, when the temperatures are not equal.

The question is reduced into to calculate separately : the heat receiving from the space of prism due to the communication and the heat receiving from the space due to the motion of molecules.

We know the analytic expression of communicated heat, and the first point of the question is plainly cleared. The rest is the calculation of transported heat : it depend on only the velocity of molecules and the direction which they take in their motion.

We calculate, at first, how much heat enters through one of the faces of prism by the communication, or by the reason of fluid flow ; next, how much heat goes out through the opposite face. [13, pp.507-514.].

Fourier combines heat theory with the Euler's equation of incompressible fluid dynamics and proposes the equation of heat motion in fluid in 1820, however, this paper was published in 1833 after 13 years, it was after 3 years since Fourier passed away. Fourier seems to have been doutful to publish it in life. Here, ε is the variable density and θ is the variable temperature of the molecule respectively. K : proper conductance of mass, C : the constant of specific heat, h : the constant determining dilatation, e : density at $\theta = 0$.

$$\begin{cases} \frac{1}{\varepsilon}\frac{dp}{dx} + \frac{d\alpha}{dt} + \alpha\frac{d\alpha}{dx} + \beta\frac{d\alpha}{dy} + \gamma\frac{d\alpha}{dz} - X = 0, \\ \frac{1}{\varepsilon}\frac{dp}{dy} + \frac{d\beta}{dt} + \alpha\frac{d\beta}{dx} + \beta\frac{d\beta}{dy} + \gamma\frac{d\beta}{dz} - Y = 0, \\ \frac{1}{\varepsilon}\frac{dp}{dz} + \frac{d\gamma}{dt} + \alpha\frac{d\gamma}{dx} + \beta\frac{d\gamma}{dy} + \gamma\frac{d\gamma}{dz} - Z = 0. \\ \frac{d\varepsilon}{dt} + \frac{d}{dx}(\varepsilon\alpha) + \frac{d}{dy}(\varepsilon\beta) + \frac{d}{dz}(\varepsilon\gamma) = 0, \quad \varepsilon = e(1 + h\theta). \\ \frac{d\theta}{dt} = \frac{K}{C}\left(\frac{d^2\theta}{dx^2} + \frac{d^2\theta}{dy^2} + \frac{d^2\theta}{dz^2}\right) - \left[\frac{d}{dx}(\alpha\theta) + \frac{d}{dy}(\beta\theta) + \frac{d}{dz}(\gamma\theta)\right]. \end{cases}$$

where, α, β, γ, p, ε, θ are the function of x, y, z, t. X, Y, Z are the outer forces.

We think, Fourier seems to feel an inferiority complex to the fluid dynamics by Euler and he divers the Euler equation as the transport equation from Euler 1755 [7, p.65].

6. **From Fourier to Boltzmann**

In 1878, ten years earlier than G. Darboux, A. Freeman [14] published the first English translated Fourier's second version 1822. To this work, Lord Kelvin (William Thomson) contributes to import the Fourier's theory into the England academic society.[15] The microscopically-description of hydromechanics equations are followed by the description of equations of gas theory by Maxwell, Kirchhoff and Boltzmann. Above all, in 1872, Boltzmann formulated the Boltzmann equations, expressed by the following today's formulation :

After Stokes' linear equations, the equations of gas theories were deduced by Maxwell in 1865, Kirchhoff in 1868 and Boltzmann in 1872. They contributed to formulate the fluid equations and to fix the Navier-Stokes equations, when Prandtl stated the today's formulation in using the nomenclature as the "so-called Navier-Stokes equations" in 1934, in which Prandtl included the three terms of nonlinear and two linear terms with the ratio of two coefficients as 3 : 1, which arose from Poisson in 1831, Saint-Venant in 1843, and Stokes in 1845. From Fourier's equation of heat, Boltzmann's gas transport equation is deduced. We summarize the

[15] A.Freeman puts the name of W. Thomson in his acknoledgement. cf. [14, errata].

geist of the equations.

In general, according to Ukai [28], we can state the Boltzmann equations as follows: [16]

$$\partial_t f + \mathbf{v}\cdot\nabla_{\mathbf{x}} f = Q(f,g), \quad t>0, \quad \mathbf{x},\mathbf{v}\in\mathbb{R}^n (n\geq 3), \quad \mathbf{x}=(x,y,z), \quad \mathbf{v}=(\xi,\eta,\zeta), \tag{3}$$

$$Q(f,g)(t,x,v)=\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B(v-v_*,\sigma)\{g(v'_*)f(v')-g(v_*)f(v)\}d\sigma dv_*, \quad g(v'_*)=g(t,x,v'_*), \tag{4}$$

$$v'=\frac{v+v_*}{2}+\frac{|v+v_*|}{2}\sigma, \quad v'=\frac{v+v_*}{2}+\frac{|v-v_*|}{2}\sigma, \quad \sigma\in\mathbb{S}^{n-1} \tag{5}$$

where, $f = f(t,\ x,\ v)$ is interpretable as several meanings such as density distribution of a molecule, / number density of a molecule, / probability density of a molecule, at time : t, place : $\mathbf{x}$ and velocity : $\mathbf{v}$. $f(v)$ means $f(t,x,v)$ as abbreviating t and x in the same time and place with $f(v')$. $Q(f,g)$ of the right-hand-side of (3) is the Boltzmann bilinear *collision operator.* $\mathbf{v}\cdot\nabla_{\mathbf{x}} f$ is the *transport operator.* $B(z,\sigma)$ of the right-hand-side in (4) is the non-negative function of *collision cross-section.* $Q(f,g)(t,x,v)$ is expressed in brief as $Q(f)$. $(v,\ v_*)$ and $(v',\ v'_*)$ are the velocities of a molecule before and after collision. According to Ukai [29], the *transport operators* are expressed with two sort of terms like Boltzmann's descriptions : including the collision term $\nabla_{\mathbf{v}}\cdot(\mathbf{F}f)$ by exterior force $\mathbf{F}$. Boltzmann defines the model of the *collision* between the molecule m_1 calling the point of it and the molecule m wich we call the point m. The instant when the molecule m passes vertically throught the disc of m_1 molecule, is defined as *collision.* According to Boltzmann[2, pp.110-115], [17] his equations (so-called *transport equations*) are the following :

> Since now $V_1+V_2+V_3+V_4$ is equal to the increment $dn'-dn$ of dn : number of molecules during time dt, and this according to Equation $(101)_B$ must be equal to $\frac{\partial f}{\partial t}dod\omega dt$, one obtains on substituting all the appropriate value and deviding by $dod\omega dt$ the following partial differential equation for the function f.

(Here, Equation $(101)_B$: $dn'-dn=\frac{\partial f}{\partial t}do\ d\omega\ dt$.) Boltzmann explains an increase of dn as a result of the following *four different causes* of V_1 : increment by *transport* through do, V_2 : increment by *transport* of external force, V_3 : increment as a result of *collisions* of m-molecules with m_1-molecules, and V_4 : increment by *collision* of molecules with each other. The top two correspond to Fourier's transport of heat, which are owing to Euler, and the last two correspond to Fourier's comunication of heat.

7. From Kepler to the quantum mechanics

Kepler (1571-1630) 1634 [16] proposes laws on the motions of planets in reserving many analytical open problems. Huygens (1625-95) 1678 observes the wave propagation and Fresnel (1788-1827) corrects its wave principles. Euler (1707-1783) 1748 proposes the wave motion of string. Navier (1785-1836) and Poisson (1781-1840) propose wave equations in elasticity respectively. Fourier (1768-1830) 1820 [8] combines his communication theory with the Euler equation 1755 and puts the heat equation of motion in fluid, in which he expresses the molecular motion with communication and transportation of molecules before Boltzmann's modeling with collision and transportation. Navier, Poisson, Cauchy, Stokes, et al. struggle to configurate the microscopically-descriptive fluid equations with mathematical and practical adaptation, to which Plandtl 1934 uses the nomenclature as the Navier-Stokes equations. Sturm (1803-55) and Liouville (1809-82) propose the differential equation of Sturm-Liouville 1836-7 [18, 27], solving the boundary value problem. Boltzmann (1844-1904) 1895 proposes the gas theory, ending the microscopically descriptive equations such as the original Navier-Stokes equations. However, Boltzmann's motion theories aren't satisfied with the law of Newton (1643-1727) and are 'thrown into oblibion.'

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[16]($\Downarrow$) We refer the *Lecture Note* by S.Ukai: *Boltzmann equations: New evolution of theory, Lecture Note of the Winter School in Kyushu of Non-linear Partial Differential Equations*, Kyushu University, 6-7, November, 2009.

[17]($\Downarrow$) Boltzmann(1844-1906) had put the date in the foreword to part I as September in 1895, part II as August in 1898.

7.1. The modeling of Schrödinger equation. Schrödinger (1887-1961) [26] bases his original quantum theory on the classic mechanics of Kepler motion, showing some examples to apply the eigenvalue problem on the differential equations of Sturm-Liouvill type : [18]

$$(1)_S \quad L[y] = py'' + p'y' - qy, \qquad (2)_S \quad L[y] + E\rho y = 0$$

where, L is the differential operator, E is an eigenvalue of constant to find, $y = y(x)$. p, p', q are unrelated functions with the variable x. $\rho = \rho(x)$ is a wide-ranging-continuous function. The solutions $y(x)$ relate to the equation $(2)_S$, namely, the eigen function. Here, all the eigenvalues are real and positive. [26, (2), pp.514-5].
Schrödinger is necessary the new quantum mechanics based on the analogical ground from classical mechanics or the mathematics such as :

- the motion theory of planets by Kepler in classic principle for modeling the modern theory of atomic structure,
- collision of electron with nucleus like Fourier's or gas-theorists' molecular collision,
- entropy concept like energy conversion in gas theory unsatisfied with Newton theory since Clausius 1865,
- light wave theory unsatisfied with Newton theory since Huygens' wave principle,
- application of the Sturm-Liouville theory and its differential equation to the boundary value problem in atomic mechanics, etc.

8. Conclusions

1. Fourier's theoretical works in life are : theorem on the discriminant of number and range of real root, heat and diffusion theory and equations, practical use of transcendental series, theoretical reasons to the wave and fluid equations and many seeds to be done in the future as like Dirichlet's expression : to offer a new example of the *prolificity* of the analytic process.
2. Poisson's objections are very useful for Fourier to prove the series theory, however, in vain for Fourier's passing away. It is toword a sort of *singularity of passage* from the finite to the infinitem like Dirichlet's expression.
3. Poisson's method of definite integral is a mere one, widely, univarsally applicable to the integral problems. Euler's and Laplace's are some deductive reasonings to discover it, and these are also important.
4. Boltzmann's concept of collision and transport with entropy and probability are treated as the classical quantum mechanics. In this sense, Fourier's communication theory and the equation of motion in the fluid stand on the communication point between the classical mechanics and new quantum mechanics by Schrödinger.

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[18](⇓) Schrödinger gets this problem from Courant-Hilbert. V.§5, 1, p.238 f. [26, (3), p.440.], not from French Sturm-Liouvill's bibliographies, or like Euler's French papers on the wave equation.

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