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Novikov方程式の多重ソリトン解とピーコン極限

Multisoliton solutions of the Novikov equation and their peakon limit

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Abstract

The Novikov equation is an integrable generalization of the Degasperis-Procesi equation. We develop a systematic method for solving the Novikov equation. In particular, we present a parametric representation for the smooth bright multisoliton solutions on a constant background and investigate their property. We show that the tau-functions associated with the soliton solutions are related to those of a model equation for shallow-water waves introduced by Hirota and Satsuma. We also construct a new type of singular solution by specifying a complex phase parameter. We demonstrate that both smooth and singular solitons recover the peaked waves (or peakons) when the background field tends to zero.

1. Introduction

We consider the Novikov equation [1]

$$m_t + u^2 m_x + 3uu_x m = 0, \qquad m = u - u_{xx}, \qquad u = u(x,t), \tag{1.1}$$

subjected to the boundary condition $u \to u_0$ as $x \to \pm\infty$. The Novikov equation is an integrable generalization of the following Degasperis-Procesi (DP) equation

$$m_t + um_x + 3u_x m = 0. \tag{1.2}$$

There exists another type of integrable equation with cubic nonlinearity known as the modified Camassa-Holm (CH) equation [2]

$$m_t + [m(u^2 - u_x^2)]_x = 0, \tag{1.3}$$

which is an integrable generalization of the CH equation [3]

$$m_t + um_x + 2u_x m = 0. \tag{1.4}$$

The purpose of this paper is:

- to develop a systematic method for obtaining the soliton solutions of the Novikov equation
- to investigate the properties of the smooth and singular soliton solutions.

The details have been published in [4] and hence we summarize the main results.

2. Reciprocal transformation and SWW equation

2.1. Reciprocal transformation

We introduce the the coordinate transformation $(x, t) \to (y, \tau)$

$$dy = m^{2/3}\,dx - m^{2/3}u^2\,dt, \qquad d\tau = dt. \tag{2.1}$$

It follows from (2.1) that the variable $x = x(y, \tau)$ satisfies a system of linear PDEs

$$x_y = m^{-2/3}, \qquad x_\tau = u^2. \tag{2.2}$$

We apply the transformation (2.1) to the Novikov equation and find that it can be recast into the form

$$m_\tau + 3m^{5/3}uu_y = 0. \tag{2.3}$$

On the other hand, u from (1.1) can be rewritten in terms of m as

$$u = m + m^{4/3}u_{yy} + \frac{2}{3}m^{1/3}m_y u_y. \tag{2.4}$$

If we define the new variables V and W by $V = m^{2/3}$ and $W = um^{1/3}$, respectively, then equations (2.3) and (2.4) can be put into the form

$$\left(\frac{1}{V}\right)_\tau = \left(\frac{W^2}{V}\right)_y, \tag{2.5}$$

$$W_{yy} + UW + 1 = 0, \tag{2.6a}$$

where

$$U = -\frac{V_{yy}}{2V} + \frac{V_y^2}{4V^2} - \frac{1}{V^2}. \tag{2.6b}$$

The integrability of the Novikov equation is evidencede by the existence of the Lax representation. Actually, it can be written in terms of the variables y and τ as

$$\psi_{yyy} + U\psi_y = \lambda^2\psi, \qquad \psi_\tau = \frac{1}{\lambda^2}\left(W\psi_{yy} - W_y\psi_y\right) - \frac{2}{3\lambda^2}\,\psi. \tag{2.7}$$

Proposition 2.1. *The variables U and W satisfies a linear partial differential equation (PDE)*

$$U_\tau + 3W_y = 0. \tag{2.8}$$

If we eliminate the variable W from (2.6a) and (2.8), we obtain a single equation for U:

$$UU_{\tau yy} - U_y U_{\tau y} + U^2 U_\tau + 3U_y = 0. \tag{2.9}$$

2.2. SWW equation

We first seek the N-soliton solution of equation (2.9) of the form

$$U = U_0 + 6\,(\ln f)_{yy}, \qquad f = f(y, \tau). \tag{2.10}$$

The above dependent variable transformation enables us to recast (2.9) to the bilinear equation for f

$$(D_\tau D_y^3 - 3W_0 D_y^2 + U_0 D_\tau D_y) f \cdot f = 0, \quad U_0 = -u_0^{-4/3}, \quad W_0 = u_0^{4/3}. \tag{2.11}$$

Here, the bilinear operators D_y and D_τ are defined by

$$D_y^m D_\tau^n f \cdot g = \left(\frac{\partial}{\partial y} - \frac{\partial}{\partial y'}\right)^m \left(\frac{\partial}{\partial \tau} - \frac{\partial}{\partial \tau'}\right)^n f(y, \tau) g(y', \tau')\Big|_{y'=y, \tau'=\tau}.$$

Recall that the bilinear equation (2.11) can be transformed to a model equation for shallow-water waves (SWW)

$$q_\tau + 3\kappa^4 q_y - 3\kappa^2 q q_\tau + 3\kappa^2 q_y \int_y^\infty q_\tau dy - \kappa^2 q_{\tau yy} = 0, \qquad q = q(y, \tau), \tag{2.12}$$

through the dependent variable transformation $q = 2(\ln f)_{yy}$, where the positive parameter κ has been introduced for later convenience by the relation $\kappa = u_0^{2/3}$ so that $U_0 = -\kappa^{-2}$ and $W_0 = \kappa^2$. Substituting (2.10) into equation (2.8) and integrating once with respect to y under the boundary condition $W \to \kappa^2$, $|y| \to \infty$, we obtain the expression of W in terms of the tau-function f

$$W = \kappa^2 - 2(\ln f)_{\tau y}. \tag{2.13}$$

Finally, it follows from (2.2) and the definition of W that the variable $x = x(y, \tau)$ obeys the linear PDE

$$x_\tau = W^2 x_y. \tag{2.14}$$

Thus, the problem under consideration is to solve (2.14) with the known function W from (2.13).

2.3. Bilinear identities for the tau-functions

The tau-function f for the N-solton solution of the SWW equation is given compactly by

$$f = \sum_{\mu=0,1} \exp\left[\sum_{i=1}^{N} \mu_i \xi_i + \sum_{1\le i<j\le N} \mu_i \mu_j \gamma_{ij}\right], \tag{2.15a}$$

with

$$\xi_i = k_i\left[y - \frac{3\kappa^4}{1-(\kappa k_i)^2}\tau - y_{i0}\right], \qquad (i = 1, 2, ..., N), \tag{2.15b}$$

$$\mathrm{e}^{\gamma_{ij}} = \frac{(k_i - k_j)^2[(k_i^2 - k_i k_j + k_j^2)\kappa^2 - 3]}{(k_i + k_j)^2[(k_i^2 + k_i k_j + k_j^2)\kappa^2 - 3]}, \qquad (i, j = 1, 2, ..., N; i \neq j). \tag{2.15c}$$

Here, k_i and y_{i0} are the amplitude and phase parameters of the ith soliton, respectively, and the notation $\sum_{\mu=0,1}$ implies the summation over all possible combinations of $\mu_1 = 0, 1, \mu_2 = 0, 1, ..., \mu_N = 0, 1$.

To proceed, let us introduce some notations. The N-soliton solution from (2.15) is parametrized by the N phase variables ξ_i $(i = 1, 2, ..., N)$ and hence we use a vector notation $f = f(\xi)$ with an N-component row vector $\xi = (\xi_1, \xi_2, ..., \xi_N)$. Let $\phi = (\phi_1, \phi_2, ..., \phi_N)$ be an N-component row vector with the elements

$$\mathrm{e}^{-\phi_i} = \sqrt{\frac{(1 - \frac{\kappa k_i}{2})(1 - \kappa k_i)}{(1 + \frac{\kappa k_i}{2})(1 + \kappa k_i)}}, \qquad (i = 1, 2, ..., N). \tag{2.16}$$

Define the tau-functions f_1, f_1', f_2 and f_2' by making use of the above notation

$$f_1 = f(\xi - \phi), \quad f_1' = f(\xi - 2\phi), \quad f_2 = f(\xi + \phi), \quad f_2' = f(\xi + 2\phi). \tag{2.17}$$

Proposition 2.2. *The tau-functions f, f_1' and f_2' satisfy the bilinear identities*

$$D_y f_1' \cdot f_2' + \frac{2}{\kappa} f_1' f_2' = \frac{2}{\kappa^3}(\kappa^2 f^2 - D_\tau D_y f \cdot f), \tag{2.18}$$

$$D_\tau f_1' \cdot f_2' + 2\kappa^3 f_1' f_2' = \frac{2}{\kappa^3}(\kappa^6 f^2 + D_\tau^2 f \cdot f), \tag{2.19}$$

$$D_y^3 f_1' \cdot f_2' + \frac{6}{\kappa} D_y^2 f_1' \cdot f_2' + \frac{11}{\kappa^2} D_y f_1' \cdot f_2' + \frac{6}{\kappa^3}(f_1' f_2' - f^2) = 0, \tag{2.20}$$

$$D_\tau f_1' \cdot f_2' + \kappa D_\tau D_y f_1' \cdot f_2' + \frac{\kappa^2}{4} D_\tau D_y^2 f_1' \cdot f_2' + \kappa^3(f_1' f_2' - f^2) + \frac{\kappa^4}{2} D_y f_1' \cdot f_2' + \frac{\kappa^5}{2} D_y^2 f_1' \cdot f_2'$$
$$= \frac{1}{2\kappa}(D_\tau^2 D_y^2 f \cdot f + \kappa^6 D_y^2 f \cdot f). \tag{2.21}$$

3. The N-soliton solution

Let us introduce the tau-function $g = g(\xi)$

$$g = \sum_{\mu,\nu=0,1} \exp\left[\sum_{i=1}^{N}(\mu_i+\nu_i)\xi_i + \sum_{i=1}^{N}(2\mu_i\nu_i - \mu_i - \nu_i)\ln a_i \right.$$

$$\left. +\frac{1}{2}\sum_{\substack{i,j=1\\(i\neq j)}}^{N}(\mu_i\mu_j+\nu_i\nu_j)A_{2i-1,2j-1} + \frac{1}{2}\sum_{\substack{i,j=1\\(i\neq j)}}^{N}(\mu_i\nu_j+\mu_j\nu_i)A_{2i-1,2j}\right]. \qquad (3.1a)$$

Here

$$a_i = \sqrt{\frac{1-\frac{\kappa^2k_i^2}{4}}{1-\kappa^2k_i^2}}, \qquad (i=1,2,...,N), \qquad (3.1b)$$

$$\exp\left[A_{2i-1,2j-1}\right] = \frac{(p_i-p_j)(q_i-q_j)}{(p_i+q_j)(q_i+p_j)}, \qquad (i,j=1,2,...,N; i\neq j), \qquad (3.1c)$$

$$\exp\left[A_{2i-1,2j}\right] = \frac{(p_i-q_j)(q_i-p_j)}{(p_i+q_j)(q_i+q_j)}, \qquad (i,j=1,2,...,N; i\neq j), \qquad (3.1d)$$

$$p_i = \frac{k_i}{2}\left[1+\frac{2}{\kappa k_i}\sqrt{\frac{1}{3}\left(1-\frac{1}{4}\kappa^2k_i^2\right)}\right], \qquad (i=1,2,...,N), \qquad (3.1e)$$

$$q_i = \frac{k_i}{2}\left[1-\frac{2}{\kappa k_i}\sqrt{\frac{1}{3}\left(1-\frac{1}{4}\kappa^2k_i^2\right)}\right], \qquad (i=1,2,...,N), \qquad (3.1f)$$

and ξ_i $(i=1,2,...,N)$ are already given by (2.15b).

The tau-functions g_1 and g_2 are defined by

$$g_1 = g(\xi-\phi), \qquad g_2 = g(\xi+\phi), \qquad (3.2)$$

where ϕ is the N-component row vector introduced by (2.16).

Theorem 3.1. *The Novikov equation (1.1) admits the parametric representation for the N-soliton solution*

$$u^2 = u^2(y,\tau) = \kappa^3 + \frac{1}{2}\frac{\partial}{\partial\tau}\ln\frac{g_1}{g_2}, \qquad (3.3a)$$

$$x = x(y,\tau) = \frac{y}{\kappa} + \kappa^3\tau + \frac{1}{2}\ln\frac{g_1}{g_2} + d, \qquad (3.3b)$$

where the tau-functions g_1 and g_2 are given by (3.1) and (3.2) and d is an arbitrary constant.

Remark 3.1. The tau-function g has already appeared in constructing the N-soliton solution of the DP equation.

Proposition 3.1. *The following relation holds among the tau-functions g, f_1 and f_2*

$$g = f_1 f_2 + \kappa D_y f_1 \cdot f_2, \tag{3.4}$$

where f_1 and f_2 are defined by (2.17).

Proposition 3.2. *The tau-functions f, g_1 and g_2 satisfy the relations*

$$\left(D_y + \frac{2}{\kappa}\right) g_1 \cdot g_2 = \frac{2}{\kappa} f^4, \tag{3.5a}$$

$$\left(D_\tau + 2\kappa^3\right) g_1 \cdot g_2 = \frac{2}{\kappa}(\kappa^2 f^2 - D_\tau D_y f \cdot f)^2. \tag{3.5b}$$

4. Properties of soliton solutions

4.1. One-soliton solution

4.1.1. Smooth soliton

The tau-functions corresponding to the one-soliton solution are given by

$$g_1 = 1 + \frac{4(1-\alpha)}{2+\alpha}\mathrm{e}^{\xi} + \frac{2-\alpha}{2+\alpha}\frac{1-\alpha}{1+\alpha}\mathrm{e}^{2\xi}, \tag{4.1a}$$

$$g_2 = 1 + \frac{4(1+\alpha)}{2-\alpha}\mathrm{e}^{\xi} + \frac{2+\alpha}{2-\alpha}\frac{1+\alpha}{1-\alpha}\mathrm{e}^{2\xi}, \tag{4.1b}$$

with

$$\xi = k(y - \tilde{c}\tau - y_0), \qquad \tilde{c} = \frac{3\kappa^4}{1-\alpha^2}, \tag{4.1c}$$

where we have put $\xi = \xi_1, k = k_1, \alpha = \kappa k_1$ and $y_0 = y_{10}$ for simplicity. We assume $k > 0$ hereafter and the condition $0 < \alpha < 1$ is imposed to assure the smoothness of the solution.

The parametric representation of the smooth one-soliton solution follows from (3.3) and (4.1). It can be written in the form

$$\begin{aligned} u^2 &= \kappa^3 + \frac{12k\alpha\tilde{c}}{4-\alpha^2}\frac{\cosh\xi + \frac{1}{2}\frac{2+\alpha^2}{1-\alpha^2}}{\cosh 2\xi + \frac{8(2+\alpha^2)}{4-\alpha^2}\cosh\xi + \frac{3(4-\alpha^2+3\alpha^4)}{(1-\alpha^2)(4-\alpha^2)}} \\ &= \frac{2\kappa^3\left(\cosh\xi + \frac{1+2\alpha^2}{1-\alpha^2}\right)^2}{\cosh 2\xi + \frac{8(2+\alpha^2)}{4-\alpha^2}\cosh\xi + \frac{3(4-\alpha^2+3\alpha^4)}{(1-\alpha^2)(4-\alpha^2)}}, \end{aligned} \tag{4.2a}$$

$$X \equiv x - ct - x_0 = \frac{\xi}{\alpha} + \frac{1}{2}\ln\left(\frac{\tanh^2\frac{\xi}{2} - \frac{2}{\alpha}\tanh\frac{\xi}{2} + \frac{4-\alpha^2}{3\alpha^2}}{\tanh^2\frac{\xi}{2} + \frac{2}{\alpha}\tanh\frac{\xi}{2} + \frac{4-\alpha^2}{3\alpha^2}}\right), \tag{4.2b}$$

where

$$c = \frac{\tilde{c}}{\kappa} + \kappa^3 = \frac{\kappa^3(4-\alpha^2)}{1-\alpha^2}, \tag{4.2c}$$

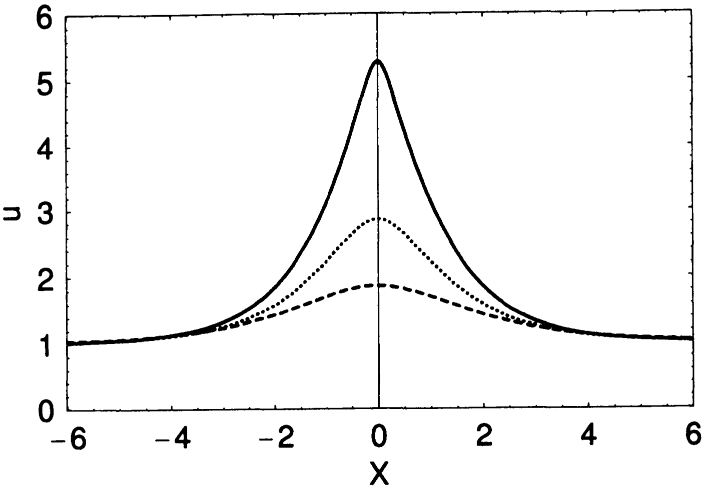

Figure 1. The profile of smooth solitons with $\kappa = 1$. $\alpha = 0.7$ (dashed curve), $\alpha = 0.85$ (dotted curve), $\alpha = 0.95$ (solid curve).

is the velocity of the soliton in the (x, t) coordinate system and $x_0 = y_0/\kappa$.

Figure 1 depicts the profile of smooth solitons against the stationary coordinate X for three distinct values of α with $\kappa = 1$. The one-soliton solution represents a bright soliton on a constant background $u = \kappa^{3/2}$ whose center position x_c is located at $x_c = ct+x_0$. The amplitude of the soliton with respect to the background field, which we denote by A, is found to be as

$$A = \kappa^{3/2}\left(\frac{2+\alpha^2}{\sqrt{(1-\alpha^2)(4-\alpha^2)}} - 1\right). \tag{4.3}$$

Eliminating the parameter α from (4.2c) and (4.3), we obtain the amplitude-velocity relation

$$c = \frac{1}{2}\left[(A+\kappa^{3/2})^2 + 4\kappa^3 + (A+\kappa^{3/2})\sqrt{(A+\kappa^{3/2})^2 + 8\kappa^3}\right]. \tag{4.4}$$

4.1.2. Singular soliton

The singular soliton is obtained from the smooth soliton (4.2) if one replaces the phase variable x_0 and y_0 by $x_0 + \pi i/\alpha$ and $y_0 + \pi i/k$, respectively. In this setting, $\cosh\xi \to -\cosh\xi$ and $\tanh(\xi/2) \to \coth(\xi/2)$, giving rise to the parametric representation of u^2

$$u^2 = \frac{2\kappa^3\left(-\cosh\xi + \frac{1+2\alpha^2}{1-\alpha^2}\right)^2}{\cosh 2\xi - \frac{8(2+\alpha^2)}{4-\alpha^2}\cosh\xi + \frac{3(4-\alpha^2+3\alpha^4)}{(1-\alpha^2)(4-\alpha^2)}}, \tag{4.5a}$$

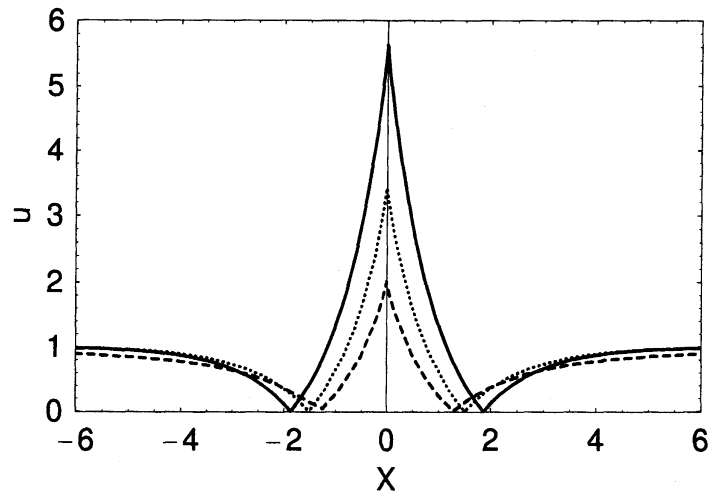

Figure 2. The profile of singular solitons with $\kappa = 1$. $\alpha = 0.1$ (dashed curve), $\alpha = 0.85$ (dotted curve), $\alpha = 0.95$ (solid curve).

$$X \equiv x - ct - x_0 = \frac{\xi}{\alpha} + \frac{1}{2}\ln\left(\frac{\coth^2\frac{\xi}{2} - \frac{2}{\alpha}\coth\frac{\xi}{2} + \frac{4-\alpha^2}{3\alpha^2}}{\coth^2\frac{\xi}{2} + \frac{2}{\alpha}\coth\frac{\xi}{2} + \frac{4-\alpha^2}{3\alpha^2}}\right). \tag{4.5b}$$

Figure 2 shows the typical profile of singular solitons for three distinct values of α with $\kappa = 1$. We can observe that the singularities appear both at the crest $X = 0$ and at $X = \pm X_0$, where X_0 is a positive constant.

4.1.3. Peakon

It has been shown that the Novikov equation admits no smooth solutions which vanish at infinity. Under the same boundary condition, however it exhibits a peaked wave (or peakon) solution of the form

$$u = \sqrt{c}\,\mathrm{e}^{-|x-ct-x_0|}. \tag{4.6}$$

We can show analytically that the smooth soliton recovers the peakon in the limit of $\kappa \to 0$ with the velocity c of the soliton being fixed, which we term the peakon limit. Here, we provide a numerical evidence for the validity of the limiting procedure. The passage to the peakon solution is illustrated in figure 3 for four distinct values of κ. We can observe that the profile drawn by the thin solid curve fits very well with the peakon solution (4.6) with $c = 1$. Figure 4 show the limiting process of the singular soliton as well. Obviously, the singular soliton recovers the peakon in the peakon limit.

4.2. Two-soliton solution

The tau-functions g_1 and g_2 for the two-soliton solution are given by

$$g_1 = 1 + 2b_1\mathrm{e}^{\xi_1} + 2b_2\mathrm{e}^{\xi_2} + (a_1b_1)^2\mathrm{e}^{2\xi_1} + (a_2b_2)^2\mathrm{e}^{2\xi_2} + 2\nu b_1b_2\mathrm{e}^{\xi_1+\xi_2} + 2\delta b_2(a_1b_1)^2b_2\mathrm{e}^{2\xi_1+\xi_2}$$

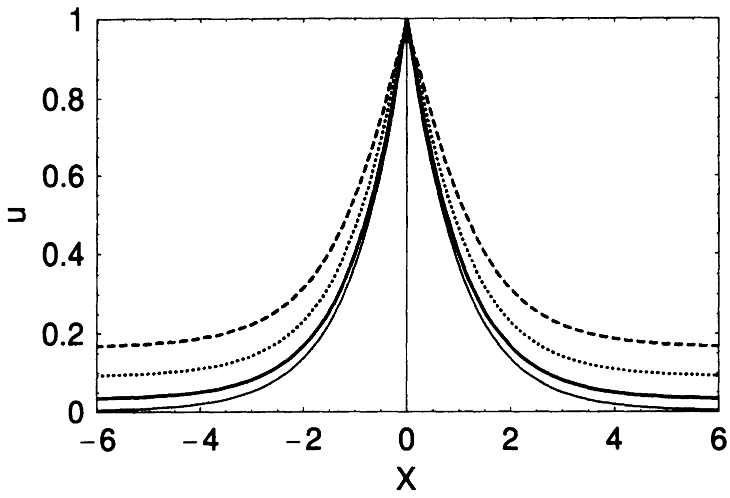

Figure 3. The peakon limit of the smooth soliton with $c = 1$. $\kappa = 0.3$ (dashed curve), $\kappa = 0.2$ (dotted curve), $\kappa = 0.1$ (bold solid curve), $\kappa = 0.01$ (thin solid curve).

$$+2\delta b_1(a_2b_2)^2 e^{\xi_1+2\xi_2} + \delta^2(a_1a_2b_1b_2)^2 e^{2\xi_1+2\xi_2}, \tag{4.7a}$$

$$g_2 = 1 + \frac{2}{a_1^2b_1}e^{\xi_1} + \frac{2}{a_2^2b_2}e^{\xi_2} + \frac{1}{(a_1b_1)^2}e^{2\xi_1} + \frac{1}{(a_2b_2)^2}e^{2\xi_2} + \frac{2\nu}{(a_1a_2)^2b_1b_2}e^{\xi_1+\xi_2}$$

$$+\frac{2\delta}{(a_1a_2)^2b_1^2b_2}e^{2\xi_1+\xi_2} + \frac{2\delta}{(a_1a_2)^2b_1b_2^2}e^{\xi_1+2\xi_2} + \frac{\delta^2}{(a_1a_2b_1b_2)^2}e^{2\xi_1+2\xi_2}, \tag{4.7b}$$

where

$$\xi_i = k_i(y - \tilde{c}_i\tau - y_{i0}), \quad \tilde{c}_i = \frac{3\kappa^4}{1-(\kappa k_i)^2}, \quad (i = 1, 2), \tag{4.7c}$$

$$a_i = \sqrt{\frac{1 - \frac{(\kappa k_i)^2}{4}}{1-(\kappa k_i)^2}}, \quad b_i = \frac{1-\kappa k_i}{1+\frac{\kappa k_i}{2}}, \quad (i = 1, 2), \tag{4.7d}$$

$$\delta = \frac{(k_1-k_2)^2[(k_1^2-k_1k_2+k_2^2)\kappa^2-3]}{(k_1+k_2)^2[(k_1^2+k_1k_2+k_2^2)\kappa^2-3]}, \quad \nu = \frac{(2k_1^4-k_1^2k_2^2+2k_2^4)\kappa^2-6(k_1^2+k_2^2)}{(k_1+k_2)^2[(k_1^2+k_1k_2+k_2^2)\kappa^2-3]}. \tag{4.7e}$$

Figure 5 depicts the time evolution of the two-soliton solution as well as its limiting profile in the peakon limit. The asymptotic analysis shows that the phase shifts of solitons are given by

$$\Delta_1 = -\frac{1}{\kappa k_1}\ln\left[\frac{(k_1-k_2)^2\{(k_1^2-k_1k_2+k_2^2)\kappa^2-3\}}{(k_1+k_2)^2\{[(k_1^2+k_1k_2+k_2^2)\kappa^2-3\}}\right] - \ln\left[\frac{\left(1+\frac{\kappa k_2}{2}\right)(1+\kappa k_2)}{\left(1-\frac{\kappa k_2}{2}\right)(1-\kappa k_2)}\right], \tag{4.8a}$$

$$\Delta_2 = \frac{1}{\kappa k_2}\ln\left[\frac{(k_1-k_2)^2\{(k_1^2-k_1k_2+k_2^2)\kappa^2-3\}}{(k_1+k_2)^2\{[(k_1^2+k_1k_2+k_2^2)\kappa^2-3\}}\right] + \ln\left[\frac{\left(1+\frac{\kappa k_1}{2}\right)(1+\kappa k_1)}{\left(1-\frac{\kappa k_1}{2}\right)(1-\kappa k1)}\right]. \tag{4.8b}$$

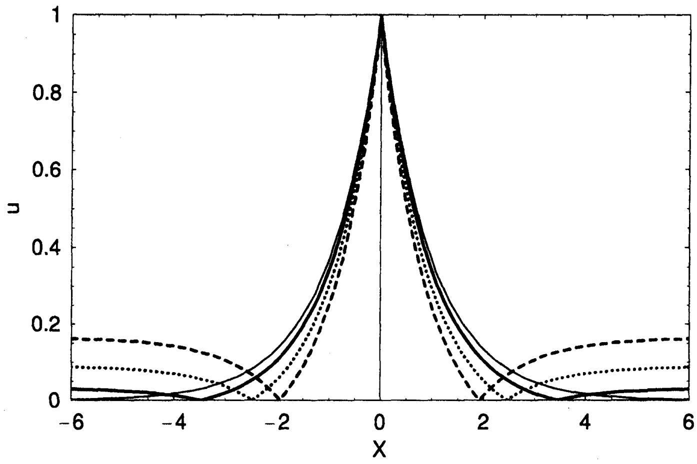

Figure 4. The peakon limit of the singular soliton with $c = 1$. $\kappa = 0.3$ (dashed curve), $\kappa = 0.2$ (dotted curve), $\kappa = 0.1$ (bold solid curve), $\kappa = 0.01$ (thin solid curve).

It is interesting that the above formulas coincide formally with those of the two-soliton solution of the DP equation. In the latter case, the parameter κ^3 is the coefficient of the linear dispersive term u_x. We can see that there exists a critical curve along which $\Delta_1 = \Delta_2$ and beyond which $\Delta_1 < \Delta_2$, implying that the phase shift of the small soliton is greater than that of the large soliton. Such a phenomenon has never been observed in the interaction process of solitons for the Korteweg-de Vries and SWW equations.

In the peakon limit, formulas (4.8a) and (4.9b) reduce respectively to

$$\Delta_1 = \ln\left[\frac{c_1(c_1+c_2)}{(c_1-c_2)^2}\right], \tag{4.9a}$$

$$\Delta_2 = \ln\left[\frac{(c_1-c_2)^2}{c_2(c_1+c_2)}\right]. \tag{4.9b}$$

This result reproduces the formulas for the phase shift of the two-pekaon solution of the Novikov equation. We recall that they coincide formally with the the corresponding formulas for the two-peakon solution of the DP equation.

4.3. N-soliton solution

The asymptotic analysis of the N-soliton solution reveals that the phase shift of the $i$the soliton is given by

$$\Delta_i = \frac{1}{\kappa k_i}\sum_{j=1}^{i-1}\ln\left[\frac{(k_i-k_j)^2\{(k_i^2-k_ik_j+k_j^2)\kappa^2-3\}}{(k_i+k_j)^2\{[(k_i^2+k_ik_j+k_j^2)\kappa^2-3\}}\right]$$

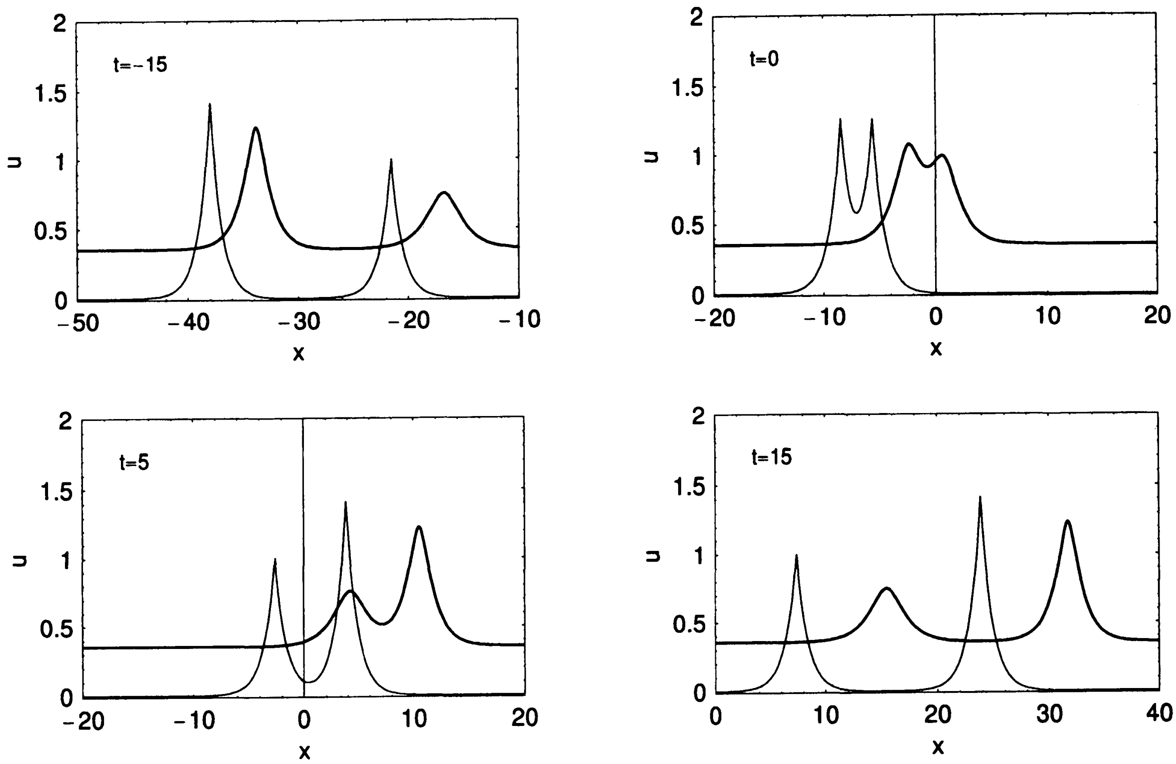

Figure 5. The profile of the smooth two-soliton solution ($\kappa = 0.5$, bold solid curve) and its peakon limit ($\kappa = 0.01$, thin solid curve) with $c_1 = 2, c_2 = 1$ and $y_{10} = y_{20} = 0$.

$$-\frac{1}{\kappa k_i}\sum_{j=i+1}^{N}\ln\left[\frac{(k_i-k_j)^2\{(k_i^2-k_ik_j+k_j^2)\kappa^2-3\}}{(k_i+k_j)^2\{(k_i^2+k_ik_j+k_j^2)\kappa^2-3\}}\right]+\sum_{j=1}^{i-1}\ln\left[\frac{\left(1+\frac{\kappa k_i}{2}\right)(1+\kappa k_i)}{\left(1-\frac{\kappa k_i}{2}\right)(1-\kappa k_i)}\right]$$
$$-\sum_{j=i+1}^{N}\ln\left[\frac{\left(1+\frac{\kappa k_i}{2}\right)(1+\kappa k_i)}{\left(1-\frac{\kappa k_i}{2}\right)(1-\kappa k_i)}\right],\qquad (i=1,2,...,N). \qquad (4.10)$$

The peakon limit of (4.10) can be carried out straightforwardly to give the formulas

$$\Delta_i=\sum_{j=1}^{i-1}\ln\left[\frac{(c_i-c_j)^2}{c_i(c_i+c_j)}\right]-\sum_{j=i+1}^{N}\ln\left[\frac{(c_i-c_j)^2}{c_i(c_i+c_j)}\right],\qquad (i=1,2,...,N), \qquad (4.11)$$

which reproduce the corresponding formulas for the N-peakon solution of the Novikov equation and they coincide formally with those of the DP equation.

5. Summary

- We have constructed the smooth and singular multisoliton solutions of the Novikov equation.

• The structure of the tau-functions associated with the N-soliton solution is essentially the same as that of a model equation for shallow-water waves introduced Hirota and Satsuma.
• The peakon limit of both smooth and singular solitons recovers the peakon when the background field tends to zero.
• The formula for the phase shift coincides formally with that of the N-soliton solution of the DP equation.

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