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“Intensive and Extensive Margins of Fertility, Capital Accumulation, and Economic Welfare”

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Intensive and Extensive Margins of Fertility, Capital Accumulation, and Economic Welfare

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Abstract

This paper investigates the impact of low fertility on long-term capital accumulation and economic welfare. We find that the impact differs according to whether the low fertility arises from a decrease in the intensive or extensive margin of fertility. We show that an increase in the intensive margin of fertility decreases the capital stock and economic welfare. Conversely, we identify a U-shaped relationship between the extensive margin of fertility and the capital stock because of the existence of two opposing effects, such that the decline in fertility may reduce economic welfare. Furthermore, we show that an intragenerational income redistribution policy can eliminate the welfare loss resulting from the incomplete market.

JEL Classification No.: H23, J13, O41.

Keywords: Childlessness, Economic growth, Extensive margin of fertility, Income redistribution, Intensive margin of fertility, Overlapping generations.

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1 Introduction

Declines in fertility rates have been widely observed in most developed economies, and many economists have examined this phenomenon. In the field of economic growth theory, the Solow–Swan model is considered to provide a standard view of the relationship between population growth and the macroeconomy. According to this model, lower population growth leads to a higher capital–labor ratio, and thus, a higher per capita income level in the steady state. This result is also supported by other neoclassical growth models with microfoundations; for example, Diamond's (1965) overlapping-generations model and the optimal growth model of Blanchard and Fischer (1989).

Using these models, we typically find that lower population growth increases an agent's welfare in the steady state insofar as the economy is dynamically efficient, that is, when the interest rate exceeds the population growth rate.[1] Furthermore, in the literature focusing on very long-term development from Malthusian stagnation to sustainable growth, the relationship between declining fertility and sustainable growth has been emphasized.[2]

To our knowledge, almost all theoretical studies have investigated the impact of low fertility on the macroeconomy by assuming that all (female) agents have children in their lifetime, and that low fertility is therefore described as a reduction in the number of children each mother has (that is, the *intensive margin of fertility)*. However, there is another important factor that has contributed to the recent declining birthrate in low-fertility economies: namely, the *extensive margin of fertility*, that is, an increase in the proportion of women who do not have any children in their lifetime, which we call the 'rate of definitive childlessness,' or more simply, the 'childlessness rate.'

In recent decades, such childlessness rates have increased in many developed economies. The OECD Family Database uses the proportion of childless women at 45 years of age as a

[1] In other words, higher population growth decreases long-term economic welfare because of a reduction in the capital–labor ratio. In neoclassical growth theory, this is often referred to as the 'capital dilution effect.' If we also consider the possibility of income transfers between generations in a Diamond-type overlapping generations framework, and take the 'intergenerational transfer effect' into account, the relationship between population growth and economic welfare will change. Samuelson (1975) investigated this issue, and this has been subsequently developed by Deardorff (1976), Michel and Pestieau (1993), and de la Croix et al. (2012).

[2] There are many approaches for describing the long-term economic development process and population dynamics. Galor (2005) and Doepke (2008) provided summaries of key research in this field.

measure of the childlessness rate. Table 1 compares the childlessness rate of women born in 1950 (let us call them the 1950 cohort) with the 1965 cohort. We observe two characteristics of the data. First, the childlessness rate of the 1965 cohort is higher than that of the 1950 cohort in many countries. In fact, the childlessness rate decreased only in Portugal, Sweden, and the United States. Second, the childlessness rate differs between countries. For instance, the rate is less than 5% in Bulgaria and Portugal, but greater than 20% in Austria, England and Wales, Germany, Italy, and Japan.

In this paper, we construct a model where agents face uncertainty about whether they will have children, and consider the situation where a declining birthrate results from either each mother having fewer children or a rise in the childlessness rate. We then explore the effects of low fertility on the steady-state capital–labor ratio and on economic welfare in the context of both the intensive and extensive margins of fertility. In the present analysis, we treat fertility choice (that is, whether to have children, and how many children to have, if any) as exogenous.

We find that the result of the analysis with the extensive margin of fertility qualitatively differs from that based on the intensive margin of fertility. We show that in the context of the intensive margin of fertility, an increase in the fertility rate reduces the steady-state capital–labor ratio. This result is consistent with the preceding literature. By contrast, we obtain a U-shaped relationship between the fertility rate and the steady-state capital–labor ratio when parents confer positive bequests on their children. To understand this result intuitively, consider the situation where the childlessness rate falls permanently, that is, the extensive margin of fertility expands. In this case, two opposing effects exist. The first effect is where the decrease in the childlessness rate increases the ratio of the current to the future working population, *ceteris paribus*, and leads to a fall in capital per capita in the next period. Roughly speaking, this effect is similar to the capital dilution effect.

The second effect is where by decreasing the childlessness rates of future generations, agents expect that their children are more likely to have children (that is, their grandchildren). When their children do not have their own children, they spend their entire income on their own consumption. But when they have children, they spend their income on their children as well as their own consumption, and their budget constraint is tighter than childless agents. In response to an increase in the latter possibility, altruistic agents choose to increase income transfers to their children, which leads to an increase in capital accumulation.

Moreover, such changes lead to different implications for the relationship between fertility and economic welfare in the long run. We show that economic welfare increases when the decrease in the fertility rate results from the intensive margin of fertility (unless parents display an extremely high preference for the number of children). Alternatively, we also show that economic welfare may decrease in response to the declining fertility rate brought about by an increase in the childlessness rate.

The literature focusing on the distinction between the intensive and extensive margin of fertility has recently grown. Aaronson et al. (2011) examined the impact of improved schooling opportunities on both margins of fertility in the course of the demographic transition. Gobbi (2013) considered the impact of the gender wage gap and the preference for children on fertility and childlessness. Baudin et al. (2015) investigated the relationship between fertility, childlessness, marriage, and education using an endogenous fertility model. All of these argued that distinguishing between the two fertility margins is important because they have different effects. The present study illustrates this in the context of economic growth. In addition, these preceding studies explore the impact of economic shocks on the childlessness rate. This study, on the other hand, examines the impact of the childlessness rate on economic variables. In this sense, the present study and these existing studies are complementary.

In addition, using this model, we investigate the effect of an income redistribution policy on economic welfare. We show that the marginal utility of income of those who have children is higher than those who are childless in equilibrium, and that such inequality in marginal utility generates a welfare loss. When implementing *intragenerational* income redistribution from agents without children to those with children, we expect the gap in the marginal utility of income between the two groups to narrow, resulting in a reduction in welfare loss. We rigorously analyze this issue, and derive in a simple way the income redistribution rule under which we can eliminate the welfare loss.

The remainder of this paper is organized as follows. Section 2 introduces the model. Section 3 derives a dynamic representation of the economy in which we show that a steady-state equilibrium exists, is unique, and is saddle-point stable. Section 4 analyzes the effect of a change in the rate of population growth on the capital–labor ratio in the steady state, and Section 5 explores its welfare implications. We investigate the optimal income redistribution rule in Section 6. Section 7 concludes the paper.

2 Model

Firms act competitively and produce a single good using labor and capital. The aggregate production function has Cobb–Douglas technology, $Y_t = F(K_t, L_t) \equiv AK_t^{\alpha} L_t^{1-\alpha}$, where K_t, L_t, and Y_t represent the total capital stock, total labor input, and total output, respectively. Defining $y_t \equiv Y_t/L_t$ and $k_t \equiv K_t/L_t$, it is transformed as $y_t = f(k_t) \equiv Ak_t^{\alpha}$. We assume that capital fully depreciates during the production process. w_t and r_t denote the wage rate and the gross interest rate, respectively, and the optimal conditions for a representative firm are given by:

$$w_t = (1-\alpha) Ak_t^{\alpha}, \tag{1}$$

$$r_t = \alpha Ak_t^{\alpha-1}. \tag{2}$$

We consider a three-period overlapping-generations model, where agents live in 'child,' 'young,' and 'old' periods. Children do not contribute to economic activity. Young agents have one unit of labor endowment and work inelastically, and they retire when they become old. We refer to young agents in period t as generation-t. All agents are ex ante identical, but they face uncertainty about having children, and π denotes the probability of having children at the beginning of the young period. No insurance to hedge this risk exists in the economy.[3] Let us refer to agents who have children as state-κ agents, and those without children as state-χ agents. Each state-κ young agent is assumed to have n children. We treat the number of children in a household as a real number, and for simplicity, we assume that the n children born from the same parent (that is, siblings) face the same state in the young period. That is, with probability π (respectively, $1-\pi$), they become state-κ young agents (respectively, state-χ young agents).[4] Thus, the state-κ young agents are classified into two groups when they become old, depending on whether they have grandchildren (put differently, whether their own children are state-κ or state-χ). We denote state-κ young agents who have (respectively,

[3] We can justify this assumption from the viewpoint of reality.

[4] If becoming a state-κ or state-χ young agent were an independent event across children in each household, each state-κ young agent would have πn units of state-κ children and $(1-\pi)n$ units of state-χ children when he/she becomes old. In this case, the idiosyncratic uncertainty about having children is eliminated at the household level. By contrast, we consider the situation where the number of children a household has is so small that the law of large numbers does not apply at the household level.

do not have) grandchildren as state-$\kappa\kappa$ old agents (respectively, state-$\kappa\chi$ old agents). The (conditional) probability that a state-κ young agent will become a state-$\kappa\kappa$ (respectively, state-$\kappa\chi$) old agent is π (respectively, $1-\pi$). In sum, while agents are identical in their childhood, they are classified into two groups (state-χ and state-κ) when young, and into three groups (state-χ, state-$\kappa\chi$, and state-$\kappa\kappa$) when old.

The cohort size is large, so that although each agent faces uncertainty, there is no aggregate uncertainty. In the aggregate economy, $n\pi$ represents the rate of population growth. Denoting the cohort size of generation-t by N_t, the population dynamics are given by:

$$N_{t+1} = n\pi N_t.$$

Let V_t^{χ} and V_t^{κ} denote the maximized utility of state-χ and state-κ young agents of generation-t, respectively. Furthermore, EV_t represents the maximized expected utility before uncertainty is resolved, that is, $EV_t = (1-\pi) V_t^{\chi} + \pi V_t^{\kappa}$. EV_t is interpreted as the ex ante welfare of generation-t, upon which we mainly focus.[5]

Consider the lifetime utility of a state-ij agent (those whose states are i and ij when young and old, respectively), where:

$$i \in \{\chi, \kappa\}, \quad j \in \begin{cases} \emptyset, & i = \chi \\ \{\chi, \kappa\}, & i = \kappa \end{cases}$$

Agents derive utility from their own consumption when young and old (c_t^i and d_{t+1}^{ij}, respectively), the number of their children (n^i), and the welfare of their children (V_{t+1}^j) (if they have children). Based on Razin and Ben-Zion (1975) and Lucas (2002), the lifetime utility of a state-ij agent is described as:

$$u\left(c_t^i\right) + \beta u\left(d_{t+1}^{ij}\right) + v\left(n^i\right) + I^i \eta V_{t+1}^j, \tag{3}$$

where $u'(\cdot) > 0$, $v'(\cdot) > 0$, $u''(\cdot) < 0$ and $v''(\cdot) < 0$, and β is a discount factor. $\eta \in (0,1)$ is the parents' preferences toward their representative children (a measure of altruism),[6] and I^i is an indicator function which takes the following value:

[5] From the viewpoint of ex post welfare, we can also interpret $\pi V_{\kappa} + (1-\pi) V_{\chi}$ as utilitarian preferences in the steady state. (Note that the ratio of state-κ agents to state-χ agents is π to $1-\pi$.)

[6] As another possible formulation of utility function, we could consider:

$$I^i = \begin{cases} 0, & i = \chi \\ 1, & i = \kappa \end{cases}$$

As described later, to guarantee that state-κ adults obtain nonnegative utility from their children, that is, to guarantee $V_{t+1}^j \geq 0$, we also assume $u(\cdot) \geq 0$ and $v(0) = 0$.

Let us consider the state-χ young agents' problem. From (3), the utility function is given by $u(c_t^\chi) + \beta u(d_{t+1}^\chi)$, where c_t^χ and d_{t+1}^χ denote the consumption of generation-t state-χ agents when young and old, respectively. They consume all their disposable income in their lifetime. The budget constraints for state-χ agents are expressed as:

$$c_t^\chi + s_t^\chi = w_t, \tag{4}$$

$$d_{t+1}^\chi = r_{t+1}(s_t^\chi + x_t^\chi), \tag{5}$$

where s_t^χ and x_t^χ are the savings of young state-χ agents and the wealth inherited from their parents, respectively. Their parents (generation-$t-1$) leave their bequest when old (in period t), and children receive it with interest after their parents die.[7] The optimal condition for intertemporal consumption is given by:

$$u'(c_t^\chi) = \beta r_{t+1} u'(d_{t+1}^\chi). \tag{6}$$

We express V_t^χ as $V_t^\chi \equiv V_\chi(x_t^\chi)$,[8] and define $\left.\frac{dV_t^\chi}{dx_t^\chi}\right|_{dw_t = dr_{t+1} = 0}$ as $V_\chi'(x_t^\chi)$. This is interpreted as the marginal utility of income of a state-χ young agent, and we obtain:

$$V_\chi'(x_t^\chi) = u'(c_t^\chi). \tag{7}$$

$$u(c_t^i) + \beta u\left(d_{t+1}^{ij}\right) + \widehat{\eta}(n^i)^\theta V_{t+1}^j,$$

where $\widehat{\eta} > 0$ and $\theta \in (0,1)$. This functional form is based on Becker and Barro (1988) and Barro and Becker (1989). In Section 4, we show that even when using this utility function, Proposition 3, the main result, remains valid. (In this regard, refer to footnote 17 for details.)

[7] Although we consider the situation where children receive transfers in period t (inter-vivos transfers), we also obtain (32), (38), and (40). Thus, the main results of the present analysis are unchanged.

[8] Exactly, V_t^χ should be represented as a function of x_t^χ, w_t, and r_{t+1}; that is, $V_t^\chi \equiv V_\chi(x_t^\chi, w_t, r_{t+1})$. For notational simplicity, we omit w_t and r_{t+1} when there is no risk of misunderstanding.

The process to derive (7) is given in Appendix A.

Next, we consider the state-κ agents' problem. When young, the state-κ agents face uncertainty regarding their state in the old period, and choose their consumption and savings when young, denoted by c_t^{κ} and s_t^{κ}, respectively. We assume that ζw_t units of final goods are needed to raise one child in period t. The budget constraint for the young period is given by:

$$c_t^{\kappa} + s_t^{\kappa} + n\zeta w_t = w_t. \tag{8}$$

If agents become state-$\kappa\chi$ when old (that is, their children become state-χ in period $t+1$), their lifetime utility is expressed as:

$$u\left(c_t^{\kappa}\right) + \beta u\left(d_{t+1}^{\kappa\chi}\right) + v\left(n\right) + \eta V_{t+1}^{\chi},$$

where $d_{t+1}^{\kappa\chi}$ denotes consumption in the old period.

When old, they allocate disposable income between their own consumption and bequests for their children. The budget constraint in the old period is expressed as:

$$d_{t+1}^{\kappa\chi} + nx_{t+1}^{\chi} = r_{t+1}\left(s_t^{\kappa} + x_t^{\kappa}\right), \tag{9}$$

where x_t^{κ} is the wealth inherited from the parents, and x_{t+1}^{χ} is the bequest to their state-χ children.

On the other hand, if state-κ agents become state-$\kappa\kappa$ old, the lifetime utility is:

$$u\left(c_t^{\kappa}\right) + \beta u\left(d_{t+1}^{\kappa\kappa}\right) + v\left(n\right) + \eta V_{t+1}^{\kappa},$$

where $d_{t+1}^{\kappa\kappa}$ denotes consumption in the old period, and the budget constraint is given by:

$$d_{t+1}^{\kappa\kappa} + nx_{t+1}^{\kappa} = r_{t+1}\left(s_t^{\kappa} + x_t^{\kappa}\right), \tag{10}$$

where x_{t+1}^{κ} represents the bequest to their state-κ children. We assume that parents cannot leave debt to their children; that is, the following nonnegative constraints are imposed for any t:

$$x_t^{\chi} \geq 0, \quad x_t^{\kappa} \geq 0. \tag{11}$$

Consequently, V_t^{κ} is expressed as:

$$V_t^\kappa = \max\left\{u\left(c_t^\kappa\right) + v\left(n\right) + \beta\left[\left(1-\pi\right)u\left(d_{t+1}^{\kappa\chi}\right) + \pi u\left(d_{t+1}^{\kappa\kappa}\right)\right] + \eta\left[\left(1-\pi\right)V_{t+1}^{\chi} + \pi V_{t+1}^{\kappa}\right]\right\},$$

which is maximized with respect to $c_t^\kappa, s_t^\kappa, d_{t+1}^{\kappa\chi}, d_{t+1}^{\kappa\kappa}, x_{t+1}^{\chi}$, and x_{t+1}^{κ} subject to (8)–(11). Because $\pi\eta < 1$ holds, V_t^κ is finite-valued.[9] We express V_t^κ as $V_t^\kappa \equiv V_\kappa\left(x_t^\kappa\right)$, and define $\left.\frac{dV_t^\kappa}{dx_t^\kappa}\right|_{dw_t = dr_{t+1} = 0}$ as $V_\kappa'\left(x_t^\kappa\right)$. This is interpreted as the marginal utility of income for a state-κ young agent.

The optimal conditions for state-κ agents are summarized as follows:[10]

$$u'\left(c_t^\kappa\right) = \beta r_{t+1}\left[\left(1-\pi\right)u'\left(d_{t+1}^{\kappa\chi}\right) + \pi u'\left(d_{t+1}^{\kappa\kappa}\right)\right], \tag{12}$$

$$\beta n u'\left(d_{t+1}^{\kappa\chi}\right) \geq \eta V_\chi'\left(x_{t+1}^{\chi}\right) = \eta u'\left(c_{t+1}^{\chi}\right), \tag{13}$$

$$\beta n u'\left(d_{t+1}^{\kappa\kappa}\right) \geq \eta V_\kappa'\left(x_{t+1}^{\kappa}\right) = \eta u'\left(c_{t+1}^{\kappa}\right), \tag{14}$$

$$V_\kappa'\left(x_t^\kappa\right) = \beta r_{t+1}\left[\left(1-\pi\right)u'\left(d_{t+1}^{\kappa\chi}\right) + \pi u'\left(d_{t+1}^{\kappa\kappa}\right)\right] = u'\left(c_t^\kappa\right). \tag{15}$$

We provide the process for the derivation of these equations in Appendix A. (12) represents the optimal choice of c_t^κ, $d_{t+1}^{\kappa\chi}$, $d_{t+1}^{\kappa\kappa}$, and s_t^κ. (13) and (14) correspond to the optimal choice concerning the bequests for children, x_{t+1}^{χ} and x_{t+1}^{κ}, respectively. If $x_{t+1}^{\chi} > 0$ and $x_{t+1}^{\kappa} > 0$ are chosen, (13) and (14) hold as equalities, respectively. In this case, (12) can be written as:

[9] This is confirmed by rewriting the utility of a state-κ agent as:

$$\sum_{i=0}^{\infty}\left(\eta\pi\right)^i\left\{u\left(c_{t+i}^\kappa\right) + v\left(n\right) + \beta\left[\left(1-\pi\right)u\left(d_{t+1+i}^{\kappa\chi}\right) + \pi u\left(d_{t+1+i}^{\kappa\kappa}\right)\right] + \eta\left(1-\pi\right)V_{t+1+i}^{\chi}\right\}.$$

[10] In addition, we also obtain the transversality conditions with regard to x_{t+1}^{χ} and x_{t+1}^{κ}. Denoting the shadow prices of x_{t+1}^{χ} and x_{t+1}^{κ} (evaluated at period 0) by $\lambda_{t+1}^{\kappa\chi}$ and $\lambda_{t+1}^{\kappa\kappa}$, the transversality conditions are respectively expressed as follows:

$$\lim_{T\to\infty}\lambda_{T+1}^{\kappa\chi}x_{T+1}^{\chi} = 0, \quad \lim_{T\to\infty}\lambda_{T+1}^{\kappa\kappa}x_{T+1}^{\kappa} = 0.$$

Considering (9) and (10) from the viewpoint of generation-T state-k young agents, the shadow prices of x_{t+1}^{χ} and x_{t+1}^{κ} are evaluated as $\beta\left(1-\pi\right)u'\left(d_{T+1}^{\kappa\chi}\right)$ and $\beta\pi u'\left(d_{T+1}^{\kappa\kappa}\right)$, respectively. Moreover, as confirmed by the Bellman equation, the current state-κ young agents discount V_κ of the next generation at the rate of $\eta\pi$. Thus, $\lambda_{T+1}^{\kappa\chi} = \beta\left(1-\pi\right)\left(\eta\pi\right)^T u'\left(d_{T+1}^{\kappa\chi}\right)$ and $\lambda_{T+1}^{\kappa\kappa} = \beta\pi\left(\eta\pi\right)^T u'\left(d_{T+1}^{\kappa\kappa}\right)$ hold. Using these equations, the above transversality conditions are respectively expressed as:

$$\lim_{T\to\infty}\left(\eta\pi\right)^T u'\left(d_{T+1}^{\kappa\chi}\right)x_{T+1}^{\chi} = 0, \quad \lim_{T\to\infty}\left(\eta\pi\right)^T u'\left(d_{T+1}^{\kappa\kappa}\right)x_{T+1}^{\kappa} = 0.$$

$$u'\left(c_t^\kappa\right) = \frac{\eta r_{t+1}}{n}\left[\left(1-\pi\right)u'\left(c_{t+1}^\chi\right) + \pi u'\left(c_{t+1}^\kappa\right)\right]. \tag{16}$$

(15) represents the envelope condition of a state variable, x_t^κ, which is known as the Benveniste–Sheinkman condition. This indicates that the marginal utility of income of a state-κ young agent is $u'\left(c_t^\kappa\right)$.

There are three kinds of markets: namely, goods, labor, and capital markets. In the capital market equilibrium, total savings equal the aggregate capital stock in the next period. Total savings in period t consist of savings by generation-t agents (young agents) and bequests by generation-$t-1$ agents (old agents). Noting that the population sizes of generation-$t-1$ state-$\kappa\chi$ and state-$\kappa\kappa$ agents are $\left(1-\pi\right)\pi N_{t-1}$ and $\pi^2 N_{t-1}$, respectively, the following equation holds:

$$N_t\left(\pi s_t^\kappa + \left(1-\pi\right)s_t^\chi\right) + \pi N_{t-1}\left(\pi n x_t^\kappa + \left(1-\pi\right)n x_t^\chi\right) = K_{t+1}.$$

Taking into account that $L_t = N_t$ holds in the labor market, the capital market equilibrium condition is rewritten as:

$$\pi\left(s_t^\kappa + x_t^\kappa\right) + \left(1-\pi\right)\left(s_t^\chi + x_t^\chi\right) = n\pi k_{t+1}. \tag{17}$$

Walras' law guarantees equilibrium in the goods market.

3 Equilibrium Dynamics

We now explore the dynamic equilibrium system of the economy. First, we investigate the properties of the steady state. We define the steady state as the situation where the capital–labor ratio, k_t; factor prices, w_t and r_t; and an agent's lifetime consumption profile, savings, and bequests in all states are constant over time. We omit the time subscript when variables are time invariant, for example, $k_t = k_{t+1} \equiv k$.

We specify $u\left(c\right)$ as $u\left(c\right) = \log c$ so as to obtain an analytical solution.[11] Using (4)–(6), the

[11] As stated earlier, we presuppose $u\left(c\right) \geq 0$. This means that $c \geq 1$ must hold. Generally, this condition is satisfied when A is large.

optimal behavior of a state-χ agent is derived:

$$c^{\chi} = \frac{1}{1+\beta}\left(w + x^{\chi}\right), \tag{18}$$

$$s^{\chi} = \frac{1}{1+\beta}\left(\beta w - x^{\chi}\right), \tag{19}$$

$$d^{\chi} = \frac{\beta r}{1+\beta}\left(w + x^{\chi}\right). \tag{20}$$

On the other hand, the optimal conditions of a state-κ agent are represented as the budget constraints (8)–(10) and the following equations:

$$\frac{1}{c^{\kappa}} = \beta r\left[(1-\pi)\frac{1}{d^{\kappa\chi}} + \pi\frac{1}{d^{\kappa\kappa}}\right], \tag{21}$$

$$c^{\chi}\begin{cases} = \frac{\eta}{\beta n}d^{\kappa\chi}, & \text{if } x^{\chi} > 0 \\ > \frac{\eta}{\beta n}d^{\kappa\chi}, & \text{if } x^{\chi} = 0 \end{cases}, \tag{22}$$

$$c^{\kappa}\begin{cases} = \frac{\eta}{\beta n}d^{\kappa\kappa}, & \text{if } x^{\kappa} > 0 \\ > \frac{\eta}{\beta n}d^{\kappa\kappa}, & \text{if } x^{\kappa} = 0 \end{cases}, \tag{23}$$

which are derived from (12)–(14), respectively. However, it is excessively difficult to calculate the solution because of (21). Instead, it is useful to introduce the following the three variables q, ϕ, and ρ when we examine the dynamic properties of the model:

$$q \equiv \frac{x^{\chi}}{w}, \tag{24}$$

$$\phi \equiv \frac{c^{\kappa}}{c^{\chi}} = \frac{V_{\chi}'\left(x^{\chi}\right)}{V_{\kappa}'\left(x^{\kappa}\right)}, \tag{25}$$

$$\rho \equiv \frac{n}{r}. \tag{26}$$

q represents the level of a parent's income transfer to a state-χ young agent as a fraction of the wage rate, and $q \geq 0$ must hold. ϕ represents the ratio of the marginal utility of income of a state-χ young agent to that of a state-κ young agent, and $\phi < 1$ indicates that the marginal utility of income of state-κ agents is higher than that of state-χ agents. ρ is defined as the ratio of the number of children each state-κ agent has to the interest rate. Furthermore, to simplify the analysis, we assume $\zeta = 0$.[12] Using (24), the optimal choice of a state-χ young

[12]In this regard, refer to footnotes 14 and 16. Note that the qualitative features of the main results do not change even when we explicitly incorporate the cost of childrearing, as long as we treat the fertility behavior as exogenous.

agent, (18)–(20) is expressed as:

$$c^{\chi} = \frac{1+q}{1+\beta} w, \tag{27}$$

$$s^{\chi} = \frac{\beta - q}{1+\beta} w, \tag{28}$$

$$d^{\chi} = \beta r c^{\chi} = \frac{\beta}{1+\beta} (1+q) rw, \tag{29}$$

and by using (8), (25), and (27), a state-κ young agent's optimal choice is expressed as follows:

$$c^{\kappa} = \phi \frac{1+q}{1+\beta} w, \tag{30}$$

$$s^{\kappa} = w - c^{\kappa} = \left(1 - \phi \frac{1+q}{1+\beta}\right) w. \tag{31}$$

A state-κ agent decides the optimal level of bequest in the old period. In some situations, the bequest may be operative ($x > 0$), and in others, the nonnegative constraints may bind ($x = 0$). In this regard, we obtain the following result:

Lemma 1 *In the steady state, either case may hold: (i) both x^{χ} and x^{κ} are positive; (ii) both x^{χ} and x^{κ} are zero.*

The proof of Lemma 1 is given in Appendix B. We define cases (i) and (ii) as the *interior solution case* and the *corner solution case*, respectively.[13] Lemma 1 identifies the case where neither $x^{\chi} > 0$ and $x^{\kappa} = 0$ nor $x^{\chi} = 0$ and $x^{\kappa} > 0$ hold. Put differently, when $q > 0$ (respectively, $q = 0$) holds, the interior solution case (respectively, corner solution case) is realized in the steady state. The intuition of this lemma is as follows. While state-χ agents spend their entire income on their own consumption, state-κ agents spend a fraction of their income on

[13]In the interior solution case, our model has a feature in common with Michel and Pestieau (1998, 2005) in the sense that the agents who leave bequests and those who do not coexist. They considered the situation where parents are heterogeneous concerning their preferences toward bequests to their children. We can thus regard our work as being complementary to their analysis. We consider the situation where agents in the same cohort are ex ante identical, and all children receive a positive bequest from their parents. That is, only childless agents do not leave bequests. Furthermore, in contrast to our work, there is no uncertainty and markets are complete in Michel and Pestieau (1998, 2005).

their children. This means that the budget constraint of a state-κ households is tighter than a state-χ households. In response to this, altruistic parents will leave a larger bequest for state-κ children (x^{κ}) than for state-χ children (x^{χ}).[14] Thus, $x^{\chi} > 0$ and $x^{\kappa} = 0$ cannot be the optimal solution. On the other hand, the case where $x^{\chi} = 0$ and $x^{\kappa} > 0$ is not always excluded. In the present model, we just happen to exclude this case for a technical reason when we assume $\zeta = 0$. (In this regard, refer to footnote 16.) We now consider where the interior solution case takes place. In the interior solution case, the optimal condition for the intertemporal consumption, (16), yields the following equation:

$$\rho = \eta \left(\pi + (1 - \pi) \phi\right). \tag{32}$$

(13) and (14) hold as equalities, so that we obtain a state-κ agent's consumption in the old period as:

$$d^{\kappa\chi} = \frac{\beta n}{\eta} c^{\chi} = \frac{n}{\eta} \frac{\beta}{1+\beta} (1+q) w, \tag{33}$$

$$d^{\kappa\kappa} = \frac{\beta n}{\eta} c^{\kappa} = \frac{n}{\eta} \frac{\beta}{1+\beta} \phi (1+q) w. \tag{34}$$

Furthermore, by using (9) and (10) together with (33) and (34), x^{κ} is derived as:

$$x^{\kappa} = x^{\chi} + \frac{1}{n} \left(d^{\kappa\chi} - d^{\kappa\kappa}\right) = \left(q + \frac{1}{\eta} \frac{\beta}{1+\beta} (1-\phi)(1+q)\right) w. \tag{35}$$

Let us consider the market equilibrium condition, (17). Using $x^{\chi} = qw$, (28), (31), and (35), we can calculate $s^{\chi} + x^{\chi}$ and $s^{\kappa} + x^{\kappa}$ as follows:

$$s^{\chi} + x^{\chi} = \frac{1+q}{1+\beta} \beta w. \tag{36}$$

$$s^{\kappa} + x^{\kappa} = \frac{1+q}{1+\beta} \left(\frac{\beta + \eta}{\eta} (1-\phi) + \beta\right) w. \tag{37}$$

Substituting (36) and (37) into (17), and noting $\frac{k}{w} = \frac{\alpha}{1-\alpha} \frac{1}{r}$, we obtain:

$$\frac{1+q}{1+\beta} \left(\frac{\beta+\eta}{\eta} (1-\phi) + \beta + \frac{1-\pi}{\pi} \beta\right) = \frac{\alpha}{1-\alpha} \rho. \tag{38}$$

[14] In the case of $\zeta > 0$, this effect will be amplified because state-κ agent spend a larger fraction of their income on their children than in the case of $\zeta = 0$.

Furthermore, observe that when using (9), $s^{\kappa}+x^{\kappa}$ is also represented as:

$$s^{\kappa}+x^{\kappa}=\frac{1}{r}\left(d^{\kappa\chi}+nx^{\chi}\right)=\rho\left(\frac{1}{\eta}\frac{\beta}{1+\beta}\left(1+q\right)+q\right)w. \tag{39}$$

Combing this equation with (37), we derive the following relationship:

$$\frac{1+q}{1+\beta}\left(\frac{\beta+\eta}{\eta}\left(1-\phi\right)+\beta\right)=\rho\left(\frac{1}{\eta}\frac{\beta}{1+\beta}\left(1+q\right)+q\right). \tag{40}$$

Three equations summarize the steady state of the interior solution case, (32), (38), and (40), and we obtain the steady-state equilibrium by solving these with respect to three unknown variables, q, ϕ, and ρ. Of course, remember that $q>0$ must hold for the interior solution. Furthermore, using (38) and (40), we derive the relationship between ϕ and ρ by eliminating q:

$$\phi=\left(1+\frac{\beta\eta}{\beta+\eta}\right)\left(1-\alpha\rho\right)+\frac{1-\pi}{\pi}\frac{\beta\eta}{\beta+\eta}\left(1-\alpha\right). \tag{41}$$

Finally, we represent the steady state by two equations, (32) and (41), with respect to ϕ and ρ. Note that we interpret (32) and (41) as reflecting the Euler equation and the market-clearing condition, respectively. On a $\rho-\phi$ plane, (32) is drawn as an upward-sloping line that passes the points $(\rho,\phi)=(\pi\eta,0)$ and $(\eta,1)$, and (41) is a downward-sloping line that passes $\left(1,(1-\alpha)\left(1+\frac{1}{\pi}\frac{\beta\eta}{\beta+\eta}\right)\right)$, as depicted in Fig. 1.

Let ϕ^*, ρ^*, and q^* denote the values of ϕ, ρ, and q in the steady state, respectively. In the interior solution case, $q^*>0$ holds, and ϕ^* and ρ^* are obtained as the intersection of (32) and (41). Here, let us define π^+ as:

$$\pi^{+}\equiv\frac{\beta}{1+\beta}\frac{1-\alpha}{\alpha}\frac{1}{\eta}. \tag{42}$$

The following important property holds:

Proposition 1 *(i) When $\pi>\pi^+$, the interior solution case holds, that is, parents leave positive bequests to their children. In this case, $\phi^*<1$ and $\rho^*\leq\eta$ hold ($\rho^*=\eta$ holds when $\pi=1$). Furthermore, $x^{\kappa}>x^{\chi}>0$ holds.*

(ii) When $\pi\leq\pi^+$, the corner solution case holds, that is, parents do not leave a bequest, $x^{\kappa}=x^{\chi}=0$. In this case, $\phi^*=1$ and $\rho^*=\eta\frac{\pi^+}{\pi}$ hold.

Proof. First, we examine the intersection of (32) and (41). As shown in Fig. 1, $\phi < 1$ holds at the intersection if and only if the point $(\rho, \phi) = (\eta, 1)$ lies above the locus of (41). That is:

$$1 > \left(1 + \frac{\beta\eta}{\beta + \eta}\right)(1 - \alpha\eta) + \frac{1 - \pi}{\pi}\frac{\beta\eta}{\beta + \eta}(1 - \alpha).$$

This is equivalent to $\pi > \pi^+$. Furthermore, we confirm from (32) that when $\pi < 1$, $\rho < \eta$ holds at the intersection if and only if $\phi < 1$. (We also observe that $\rho = \eta$ always holds when $\pi = 1$.) In sum, we obtain the following properties regarding the intersection.

(i) $\phi < 1$ and $\rho \leq \eta$ hold when $\pi > \pi^+$ ($\rho = \eta$ holds when $\pi = 1$);

(ii) $\phi = 1$ and $\rho = \eta$ hold when $\pi = \pi^+$;

(iii) $\phi > 1$ and $\rho > \eta$ hold when $\pi < \pi^+$.

Next, we show that $q > 0$ holds if and only if $\pi > \pi^+$. From (38) and (40), we derive:

$$\frac{1 - \alpha}{\alpha}\left(\frac{1}{\eta}\frac{\beta}{1 + \beta}(1 + q) + q\right) = \frac{\frac{\beta+\eta}{\eta}(1 - \phi) + \beta}{\frac{\beta+\eta}{\eta}(1 - \phi) + \frac{\beta}{\pi}}. \tag{43}$$

We examine the property of q which satisfies (43). We confirm from (43) that q increases as π increases and decreases as ϕ increases, and that $q = 0$ holds when $\pi = \pi^+$ and $\phi = 1$. Items (i) to (iii) above indicate that $q^* > 0$ holds if and only if $\pi > \pi^+$, and that $\phi^* < 1$ and $\rho^* \leq \eta$ hold ($\rho = \eta$ holds when $\pi = 1$). Furthermore, because $\phi^* < 1$ holds, we confirm from (35) that $x^\kappa > x^\chi > 0$ holds, and thus, Proposition 1 (i) is proved.

Conversely, when $\pi < \pi^+$, the level of q satisfying (43) becomes negative. Taking the nonnegative constraint on q into account, a corner solution ($q^* = 0$) is attained when $\pi \leq \pi^+$. Because $x^\chi = x^\kappa = 0$, $d_{t+1}^{\kappa\chi} = d_{t+1}^{\kappa\kappa}$ holds immediately from (9) and (10). Using (4)–(6), (8)–(10), and (12), we obtain:

$$c^\chi = c^\kappa = \frac{1}{1 + \beta}w, \tag{44}$$

$$s^\chi = s^\kappa = \frac{\beta}{1 + \beta}w, \tag{45}$$

$$d^\chi = d^{\kappa\chi} = d^{\kappa\kappa} = \frac{\beta}{1 + \beta}rw. \tag{46}$$

Substituting (45) into (17), and taking $\frac{k}{w} = \frac{\alpha}{1-\alpha}\frac{1}{r}$ and (42) into account, we obtain the following result as the corner solution:

$$\rho = \eta \frac{\pi^{+}}{\pi}. \tag{47}$$

Here, the proof of Proposition 1 (ii) is completed. ■

Proposition 1 points out that $\pi^{+} < 1$ is required to guarantee the existence of the interior solution. We assume $\pi^{+} < 1$ throughout the present analysis. Broadly speaking, this assumption means that we consider the situation where η is not very small. As discussed earlier, parents discount the future generation's welfare at the rate of $\eta\pi$. Proposition 1 signifies that parents leave positive bequests to their children when η and π are not too low.

In particular, Proposition 1 (i) states that parents leave positive bequests to their children when $\pi > \pi^{+}$. We can see that $x^{\kappa} > x^{\chi} > 0$ holds, and explained the reason when we interpreted Lemma 1. That is, the budget constraint of a state-κ household is tighter than a state-χ household. In response to this, altruistic parents choose $x^{\kappa} > x^{\chi}$.[15] Here, it is important to note that although state-κ agents receive a larger amount of income transfer than state-χ agents, $\phi^{*} < 1$ continues to hold in equilibrium; that is, the marginal utility of income of the state-κ agents ($V_{\kappa}'(x^{\kappa})$) is still higher than that of the state-χ agents ($V_{\chi}'(x^{\chi})$). This result is a consequence of the market incompleteness. State-κ young agents have to decide s^{κ} before knowing whether they will become state-$\kappa\chi$ or state-$\kappa\kappa$ old agents. Recall also that there is no insurance to hedge this risk. In such a situation, $d^{\kappa\chi} > d^{\kappa\kappa}$ holds from (9) and (10). In the interior solution case, as shown in (22) and (23), the optimal intergenerational income transfer is implemented to satisfy,

[15] As seen from (3), we assume the additively separable utility with respect c and n. If we consider the case where the utility is not additively separable with respect to c and n, the marginal utility of consumption is influenced by a change in n. Suppose that the marginal utility of consumption increases as n increases. In this case, compared to the additively separable utility case, the marginal utility of consumption of state-κ agents is much higher, while that of state-χ agents is lower, and thus, $x^{\kappa} > x^{\chi}$ becomes easier to hold. On the hand, suppose that the marginal utility of consumption decreases as n increases. In this case, compared to the additively separable utility case, the marginal utility of consumption of state-κ agents becomes lower, while that of state-χ agents becomes higher. If this effect is sufficiently strong, it is possible that the marginal utility of the state-κ agents is lower than that of the state-χ agents, even though the budget constraint of the state-κ agents is tighter, and thus $x^{\kappa} < x^{\chi}$ arises.

$$\frac{d^{\kappa\kappa}}{d^{\kappa\chi}} = \frac{c^{\kappa}}{c^{\chi}}.$$

Thus, $\phi^* < 1$ holds. If there were an insurance contract to cover this uncertainty, the state-κ young agents would purchase the insurance to equate the marginal utility of consumption of state-$\kappa\chi$ with that of state-$\kappa\kappa$. That is, $d^{\kappa\chi} = d^{\kappa\kappa}$ would hold, and thus, $c^{\chi} = c^{\kappa}$ (or equivalently, $\phi^* = 1$) would hold.

Proposition 1 (ii) states that parents choose not to leave bequests to their children when $\pi < \pi^+$. In this case, the young agents' lifetime income is w, regardless of their state. Because $\zeta = 0$ is assumed, the lifetime budget constraint of a state-κ agent is identical to that of a state-χ agent, and consequently, $\phi^* = 1$ is realized.[16]

We obtain the stability of the steady state by examining the transition dynamics. Similarly to the definition of q, ϕ, and ρ, let us define q_t, ϕ_t, and ρ_t as follows:

$$q_t \equiv \frac{x_t^{\chi}}{w_t}, \quad \phi_t \equiv \frac{c_t^{\kappa}}{c_t^{\chi}}, \quad \rho_t \equiv \frac{n}{r_t}.$$

In the log-utility function case, we obtain the following result:

Proposition 2 *: $q_t = q^*$ and $\phi_t = \phi^*$ hold for any t. The level of ρ_t converges to ρ^* monotonically, which means that the capital stock k_t also converges to its steady state monotonically.*

The proof is in Appendix C. Proposition 2 indicates that in a transition process, the control variables q_t and ϕ_t immediately jump to the steady-state values, and the state variable ρ_t converges from the initial value ρ_0 to the steady-state value. Remembering that ρ_t and r_t has a negative relationship by definition of ρ_t, and that r_t is a decreasing function of k_t, ρ_t has a positive correlation with k_t. That is, k_t converges from the initial value k_0 to the steady-state value.

[16] If we explicitly consider the childrearing cost ($\zeta > 0$), $\phi^* < 1$ (that is, $c^{\chi} > c^{\kappa}$) will hold when $x^{\kappa} = x^{\chi} = 0$ is chosen. In other words, $u'(c^{\kappa}) > u'(c^{\chi})$ holds. In this case, it is possible that an altruistic parent decides to leave a positive bequest for only state-κ agents (that is, $x^{\kappa} > 0$ and $x^{\chi} = 0$). In this case, there exist two thresholds $\widehat{\pi}^-$ and $\widehat{\pi}^+$, where $x^{\kappa} = x^{\chi} = 0$ when $\pi \in [0, \widehat{\pi}^-]$, $x^{\kappa} > x^{\chi} = 0$ when $\pi \in (\widehat{\pi}^-, \widehat{\pi}^+]$, and $x^{\kappa} > x^{\chi} > 0$ when $\pi \in (\widehat{\pi}^+, 1]$.

However, as will be explained in Section 4, to obtain the main result, the situation where $\phi^* < 1$ holds when $x^{\kappa} > 0$ is important, and whether x^{χ} is positive or zero does not matter. Thus, we assume $\zeta = 0$ to help keep the analysis as simple as possible.

Furthermore, we can see that the interest rate is higher than the population growth rate in the steady state. As argued in Proposition 1, $\rho^* \leq \eta$ (or equivalently, $\frac{\pi n}{r} \leq \eta\pi$) holds, and recall that we assume $\eta\pi < 1$. Thus, $\pi n < r$ holds in the steady state.

4 Population Growth and the Steady-State Capital Stock

Let us examine the effect of population growth on the capital stock per capita at the steady state. The rate of population growth, πn, rises (i) when n (the intensive margin of fertility) increases, or (ii) when π (the extensive margin of fertility) increases.

To begin, we consider the interior solution case (that is, $\pi^+ < \pi \leq 1$). As stated in Section 3, the steady state is expressed as the intersection of (32) and (41). Analyzing (32) and (41), we obtain the following proposition.[17]

Proposition 3 *: In the interior solution case ($\pi^+ < \pi < 1$), the following hold:*

(i) As n increases, k^* decreases.

(ii) There is a U-shaped relationship between k^* and π. That is, there exists $\widehat{\pi} \in (\pi^+, 1)$ such that $\frac{dk^*}{d\pi} < 0$ for any $\pi \in (\pi^+, \widehat{\pi})$ and $\frac{dk^*}{d\pi} > 0$ for any $\pi \in (\widehat{\pi}, 1)$.

Proof. (i) Observe that the graphs of (32) and (41) remain unchanged when n rises. This implies that the steady state ρ^* is independent of n, that is, $\frac{d\rho^*}{dn} = 0$. Noting that $\rho \equiv \frac{n}{\alpha A}k^{1-\alpha}$, we immediately confirm that k decreases as n increases.

(ii) Differentiating (32) with respect to π yields:

$$\frac{d\rho^*}{d\pi} = \eta\,(1 - \phi^*) + \eta\,(1 - \pi)\,\frac{d\phi^*}{d\pi}, \tag{48}$$

and differentiating (41) with respect to π yields:

$$\frac{d\phi^*}{d\pi} = -\alpha\frac{\beta + \eta + \beta\eta}{\beta + \eta}\frac{d\rho^*}{d\pi} - \frac{1}{\pi^2}\frac{\beta\eta}{\beta + \eta}\,(1 - \alpha)\,. \tag{49}$$

[17] Even if we use the utility function based on Becker and Barro (1988) and Barro and Becker (1989) (refer to footnote 6 on this point), Proposition 3 remains valid, as rigorously discussed in Appendix D.

From (48) and (49), we obtain:

$$\left[1+\alpha\frac{\beta+\eta+\beta\eta}{\beta+\eta}\eta\left(1-\pi\right)\right]\frac{d\phi^*}{d\pi}=-\left[\alpha\frac{\beta+\eta+\beta\eta}{\beta+\eta}\eta\left(1-\phi^*\right)+\left(1-\alpha\right)\frac{\beta\eta}{\beta+\eta}\frac{1}{\pi^2}\right], \tag{50}$$

$$\left[1+\alpha\frac{\beta+\eta+\beta\eta}{\beta+\eta}\eta\left(1-\pi\right)\right]\frac{d\rho^*}{d\pi}=\eta\left(1-\phi^*\right)-\eta\frac{1-\pi}{\pi^2}\frac{\beta\eta}{\beta+\eta}\left(1-\alpha\right). \tag{51}$$

The bracket on the left-hand side, $1+\alpha\frac{\beta+\eta+\beta\eta}{\beta+\eta}\eta\left(1-\pi\right)$, is positive. We can see from Proposition 1 that $\phi^*\leq 1$ holds, so that the sign of the right-hand side of (50) is negative. Thus, we confirm that $\frac{d\phi^*}{d\pi}<0$ holds.

Taking into account that both ϕ^* and $\frac{1-\pi}{\pi^2}$ are decreasing functions of π, we observe that the right-hand side of (51) is an increasing function of π (and of course, it is also continuous). Moreover, as shown in Proposition 1, $\phi^*=1$ when $\pi=\pi^+$ and $\phi^*<1$ when $\pi=1$. Using this, we find that the right-hand side of (51) is negative when $\pi=\pi^+$ and positive when $\pi=1$, which indicates that there exists $\widehat{\pi}\in(\pi^+,1)$ such that $\frac{d\rho^*}{d\pi}<0$ for any $\pi\in(\pi^+,\widehat{\pi})$ and $\frac{d\rho^*}{d\pi}>0$ for any $\pi\in(\widehat{\pi},1)$. That is, we obtain a U-shaped relationship between ρ^* and π. By definition of $\rho\equiv\frac{n}{r}=\frac{n}{A\alpha}k^{1-\alpha}$, ρ^* and k^* has a positive relationship. Therefore, k^* has a U-shaped relationship with π. ■

Proposition 3 (i) states that when a decrease in the population growth rate is brought about by a decrease in the number of children each state-κ adult bears, k^* increases. As shown by (9) and (10), we interpret that n represents the cost of bequests, x^χ and x^κ, and that a decrease in n infers a reduction in the cost of x^χ and x^κ, which leads to an increase in k^*.

In contrast, Proposition 3 (ii) maintains that the impact of the extensive margin of fertility on k^* differs from the impact of the intensive margin. The U-shaped relationship means that for a high value of π, k^* *decreases* as π falls. This result is novel; therefore, let us explain the mechanism in detail. To begin, we explain it using an illustration. The graph of (32) rotates counterclockwise around a fixed point $(\rho,\phi)=(\eta,1)$ as π rises. In other words, it shifts downward in the region where $\rho<\eta$. On the other hand, the graph of (41) shifts downward in parallel when π rises. Fig. 2 (a) shows the situation. As shown, ϕ^* always falls, but the impact of an increase in π on ρ^* (and thus, k^*) is ambiguous.[18] Fig. 2 (b) depicts the situation where π rises marginally from $\pi=\pi^+$. In this case, the intersection moves from point A to point A', and thus, ρ decreases as π rises. By contrast, Fig. 2 (c) illustrates the case where π *falls*

[18] As described in the proof for Proposition 3 (ii), $\frac{d\phi^*}{d\pi}<0$ holds true.

marginally from $\pi = 1$. The intersection moves from point B to point B', and in this case, ρ decreases as π falls.

Next, let us interpret this outcome. The key point is that there are two countervailing effects. First, the graph of the Euler equation, (32), shifts downward in response to *an increase* in π, which leads to an increase in k. This effect implies that an increase in the population growth rate has a positive impact on capital accumulation. As shown in Proposition 1 (i), $\phi^* < 1$ holds in the interior solution case. Recall that this is equivalent to $V'_\chi(x^\chi) < V'_\kappa(x^\kappa)$, that is, the marginal utility of income of state-κ agents is higher than that of state-χ agents. This plays an important role in this result. When π increases, children are more likely to be state-κ adults when they grow up, and altruistic parents anticipate that their children's (expected) marginal utility of income will rise. Accordingly, they will increase income transfers to their children, and this increases k.[19] Moreover, we can see from (50) that the gap in the marginal utility of income between state-κ and state-χ agents widens as π increases. Thus, this effect becomes larger as π becomes higher.

On the other hand, when $\pi \leq \pi^+$, parents do not make an income transfer to their children ($x^\kappa = x^\chi = 0$). In this case, $V'_\chi(x^\chi) = V'_\kappa(x^\kappa)$ holds, and thus, the magnitude of this positive effect is zero.

Second, the graph of the market-equilibrium condition, (41), shifts upward in response to *a decrease* in π, and this raises k. This effect indicates that an increase in the population growth rate has a negative impact on capital accumulation. To understand this better, consider the contribution of s^χ to capital accumulation. Note that state-χ agents do not have children, and that the population ratio of state-χ agents to the next generation's cohort size is $1-\pi$ to $n\pi$. Thus, an increase in s^χ by one unit raises k_{t+1} by $\frac{1-\pi}{n\pi}$ units, and this effect becomes large as π decreases.

In summary, we find two countervailing effects regarding the impact of the extensive margin of fertility on the capital stock. When π is close to π^+, the positive (first) effect is negligible, and is dominated by the negative (second) effect. Thus, an increase in the fertility rate decreases k. On the other hand, when π is close to one, the negative effect is negligible, and is

[19] Put differently, market incompleteness is an important factor in deriving the result. If the market were complete, $\phi^* = 1$ (or equivalently, $V'_\kappa(x^\kappa) = V'_\chi(x^\chi)$) would hold. In this case, the children's expected marginal utility $\pi V'_\kappa(x^\kappa) + (1-\pi) V'_\chi(x^\chi)$ does not change even when π changes, such that the parents do not have any incentive to increase the bequest.

dominated by the positive effect. In this case, the increase in the fertility rate π decreases k. Consequently, the relationship between π and k is drawn as a U-shaped curve, as depicted in Fig. 3. Parenthetically, as shown in Proposition 1, the levels of k are the same when $\pi = \pi^+$ and $\pi = 1$.

Finally, let us note the relationship between the fertility rate and k in the corner solution case ($\pi \leq \pi^+$). Recall that $\rho^* = \eta \frac{\pi^+}{\pi}$ holds, and we obtain the following result.

Proposition 4 *: In the corner solution case ($\pi < \pi^+$), the following hold:*

(i) As n increases, k^* decreases.

(ii) As π increases, k^* decreases.

Proof. $\rho^* = \eta \frac{\pi^+}{\pi}$ is equivalent to $r^* = \frac{1}{\eta \pi^+} n\pi$. From (2), r and k has a negative relationship. Thus, k^* decreases as n or π rises. ■

When $x^\kappa = x^\chi = 0$, only the fertility rate $n\pi$ matters for k^*, and whether the increase in fertility is brought about by the intensive margin or the extensive margin does not matter. This is because, as we have stated earlier, when π rises, the positive effect is missing, and only the negative effect exists.

5 Population Growth and Steady-State Welfare

We now explore the relationship between population growth and long-term economic welfare. As we obtain a new insight in the interior solution case, we only pay attention to this (that is, $\pi > \pi^+$) in the remainder of the analysis.

We focus on the ex ante welfare $EV = \pi V_\kappa + (1 - \pi) V_\chi$, where V_χ and V_κ represent the levels of V_t^χ and V_t^κ in the steady state, respectively. For notational convenience, we define U_κ as $U_\kappa \equiv u(c^\kappa) + \beta [\pi u(d^{\kappa\kappa}) + (1 - \pi) u(d^{\kappa\chi})]$, which represents the expected utility of the state-κ young agents stemming from their own consumption, and we express V_χ and V_κ as, respectively:

$$V_\chi = u\left(c^\chi\right) + \beta u\left(d^\chi\right), \tag{52}$$

$$V_\kappa = U_\kappa + v\left(n\right) + \eta EV. \tag{53}$$

Using (53), EV is represented as:

$$EV = \frac{1}{1-\eta\pi}\left[\left(1-\pi\right)V_\chi + \pi\left(U_\kappa + v\left(n\right)\right)\right]. \tag{54}$$

5.1 Effect of the Intensive Margin of Fertility on Economic Welfare

We consider the effect of a permanent increase in n on EV. We obtain the following equation from (54):

$$\frac{dEV}{dn} = \frac{1}{1-\eta\pi}\left[\left(1-\pi\right)\frac{dV_\chi}{dn} + \pi\left(\frac{dU_\kappa}{dn} + v'\left(n\right)\right)\right]. \tag{55}$$

$\frac{dV_\chi}{dn}$ and $\frac{dU_\kappa}{dn}$ represent the change in the utility stemming from consumption of state-χ and state-κ agents, respectively, and we obtain the following property.

Lemma 2 $\frac{dV_\chi}{dn} = \frac{dU_\kappa}{dn} = -\frac{1}{n}\frac{1}{1-\alpha}\left[\left(1+\beta\right)\alpha - \left(1-\alpha\right)\beta\right] < 0$ *holds.*

The proof of Lemma 2 is given in Appendix E. As we explained in Section 3, the interest rate r exceeds the population growth rate $n\pi$. This lemma argues that the utility stemming from consumption falls in the steady state when n increases. This results comes from a contraction in the consumption possibility set because of $\frac{dk}{dn} < 0$, and this result indicates that the present model is consistent with the result obtained from the preceding literature. On the other hand, because the number of children is incorporated into the utility function, a rise in n has a direct positive impact on the utility. The term $v'\left(n\right) = \theta\frac{1}{1+n} > 0$ represents this effect. Applying Lemma 2 to (55), we obtain:

$$\frac{dEV}{dn} = \frac{1}{1-\eta\pi}\left\{-\frac{1}{n}\frac{1}{1-\alpha}\left[\left(1+\beta\right)\alpha - \left(1-\alpha\right)\beta\right] + \pi\theta\frac{1}{1+n}\right\}. \tag{56}$$

The first term in (56) represents the welfare loss stemming from a decline in consumption, and the second term in (56) indicates the welfare gain stemming from having more children. We confirm that unless θ is extremely high, the negative welfare effect outweighs the positive welfare effect, and consequently, EV falls as n increases.

5.2 Effect of the Extensive Margin of Fertility on Economic Welfare

We consider the effect of a permanent increase in π on EV. From (54), we obtain:

$$\begin{aligned}\frac{dEV}{d\pi} &= \frac{\eta}{1-\eta\pi}EV + \frac{1}{1-\eta\pi}\left(U_\kappa + v(n) - V_\chi + (1-\pi)\frac{dV_\chi}{d\pi} + \pi\frac{dU_\kappa}{d\pi}\right) \\ &= \frac{1}{1-\eta\pi}(V_\kappa - V_\chi) + \frac{1}{1-\eta\pi}\left((1-\pi)\frac{dV_\chi}{d\pi} + \pi\frac{dU_\kappa}{d\pi}\right). \end{aligned} \tag{57}$$

Here, noting that:

$$\frac{dV_\chi}{d\pi} = u'(c^\chi)\frac{dc^\chi}{d\pi} + \beta u'(d^\chi)\frac{dd^\chi}{d\pi}, \tag{58}$$

$$\frac{dU_\kappa}{d\pi} = \beta\left(u(d^{\kappa\kappa}) - u(d^{\kappa\chi})\right) + \left\{u'(c^\kappa)\frac{dc^\kappa}{d\pi} + \beta\left(\pi u'(d^{\kappa\kappa})\frac{dd^{\kappa\kappa}}{d\pi} + (1-\pi)u'(d^{\kappa\chi})\frac{dd^{\kappa\chi}}{d\pi}\right)\right\}, \tag{59}$$

and substituting (58) and (59) into the above equation, we obtain the following:

$$\begin{aligned}\frac{dEV}{d\pi} &= \frac{1}{1-\eta\pi}\left\{V_\kappa - V_\chi + \pi\beta\left(u(d^{\kappa\kappa}) - u(d^{\kappa\chi})\right)\right\} \\ &\quad + \frac{1}{1-\eta\pi}\left\{(1-\pi)\left[u'(c^\chi)\frac{dc^\chi}{d\pi} + \beta u'(d^\chi)\frac{dd^\chi}{d\pi}\right]\right. \\ &\qquad \left. + \pi\left[u'(c^\kappa)\frac{dc^\kappa}{d\pi} + \beta\left(\pi u'(d^{\kappa\kappa})\frac{dd^{\kappa\kappa}}{d\pi} + (1-\pi)u'(d^{\kappa\chi})\frac{dd^{\kappa\chi}}{d\pi}\right)\right]\right\}. \end{aligned} \tag{60}$$

A change in π affects EV through the following channels. First, an increase in π raises (respectively, reduces) the probability of becoming state-κ (respectively, state-χ) young agents. $V_\kappa - V_\chi$ captures this effect. In addition, as is captured by (59), if they become state-κ young agents (with probability π), they are more (respectively, less) likely to become state-$\kappa\kappa$ old agents and consume $d^{\kappa\kappa}$ (respectively, state-$\kappa\chi$ old agents and consume $d^{\kappa\chi}$) because their children are also more likely to become state-κ young agents in the next period. The term $\pi\beta\left(u(d^{\kappa\kappa}) - u(d^{\kappa\chi})\right)$ represents this effect. In the steady state, each future state-κ young agent will face the same situation, and the welfare of future generations is discounted at the rate of $\eta\pi$. Consequently, the effect of a change of agents' states on ex ante welfare is obtained as:

$$\sum_{t=0}^{\infty}(\eta\pi)^t\left\{V_\kappa - V_\chi + \pi\beta\left(u(d^{\kappa\kappa}) - u(d^{\kappa\chi})\right)\right\},$$

and this is equivalent to the first term of (60). We call this the ‘state effect.’

Moreover, an increase in π changes the consumption profiles of agents in each state. The terms for (60) in the second curly brackets represent this effect. We call this the ‘allocation effect.’

State Effect. The state effect is positive if and only if $V_\kappa - V_\chi + \pi\beta \left(u\left(d^{\kappa\kappa}\right) - u\left(d^{\kappa\chi}\right)\right) > 0$. Concerning this, we obtain $V_\kappa - V_\chi > 0$ holds. Roughly speaking, the state-χ agent's optimal decision is attainable for a state-κ agent (that is, choosing $c^\kappa = c^\chi$ and $d^{\kappa\kappa} = d^{\kappa\chi} = d^\chi$ is feasible for the state-κ agent) because $x^\kappa \geq x^\chi$, and furthermore, the state-κ agent derives positive utility from their children. Thus, the state-κ agent's utility V_κ is greater than V_χ. (A rigorous proof is given in Appendix F). On the other hand, $u\left(d^{\kappa\kappa}\right) - u\left(d^{\kappa\chi}\right) < 0$ holds because $d^{\kappa\kappa} < d^{\kappa\chi}$.[20] This negative effect comes from a state-κ agent's consumption being lower as π increases. Thus, in general, the sign of the state effect is ambiguous. In this regard, the following property holds:

Lemma 3 *There exists $\overline{\eta} \in (0,1)$ such that the state effect is positive for any $\eta \geq \overline{\eta}$.*

Proof. From (53) and (54), we obtain:

$$V_\kappa - V_\chi = \frac{1}{1-\eta\pi}\left(U_\kappa + v\left(n\right) - \left(1-\eta\right)V_\chi\right). \tag{61}$$

Suppose that $\eta = 1$. In this case, the following equalities hold. (The first equality comes from (61).)

$$\begin{aligned}
& V_\kappa - V_\chi + \pi\beta\left(u\left(d^{\kappa\kappa}\right) - u\left(d^{\kappa\chi}\right)\right) \\
&= \frac{1}{1-\pi}\left(U_\kappa + v\left(n\right)\right) + \pi\beta\left(u\left(d^{\kappa\kappa}\right) - u\left(d^{\kappa\chi}\right)\right) \\
&= \frac{1}{1-\pi}\left\{u\left(c^\kappa\right) + \beta\left[\pi u\left(d^{\kappa\kappa}\right) + \left(1-\pi\right)u\left(d^{\kappa\chi}\right)\right] + \pi\left(1-\pi\right)\beta\left(u\left(d^{\kappa\kappa}\right) - u\left(d^{\kappa\chi}\right)\right) + v\left(n\right)\right\} \\
&= \frac{1}{1-\pi}\left\{u\left(c^\kappa\right) + \beta\left[\pi\left(2-\pi\right)u\left(d^{\kappa\kappa}\right) + \left(1-\pi\right)^2 u\left(d^{\kappa\chi}\right)\right] + v\left(n\right)\right\} > 0.
\end{aligned}$$

[20] From (33) and (34), $d^{\kappa\kappa} = \phi d^{\kappa\chi}$ holds in the interior solution case. Recall that $\phi^* < 1$ holds in equilibrium.

Because c^κ, $d^{\kappa\kappa}$, $d^{\kappa\chi}$, c^χ, and d^χ are continuous with respect to η, the lemma holds true. ■

This lemma indicates that the state effect becomes positive when η is large. When the parents place more weight on the welfare of their children, agents derive higher utility from their children, and thus the positive effect outweighs the negative consumption effect.

Allocation Effect. Let us focus on the second term in (60):

$$(1-\pi)\left[u'\left(c^\chi\right)\frac{dc^\chi}{d\pi}+\beta u'\left(d^\chi\right)\frac{dd^\chi}{d\pi}\right]+\pi\left[u'\left(c^\kappa\right)\frac{dc^\kappa}{d\pi}+\beta\left(\pi u'\left(d^{\kappa\kappa}\right)\frac{dd^{\kappa\kappa}}{d\pi}+(1-\pi)u'\left(d^{\kappa\chi}\right)\frac{dd^{\kappa\chi}}{d\pi}\right)\right]. \tag{62}$$

The first and second terms represent the allocation effects relating to state-χ and state-κ agents, respectively. We define σ_c^χ and σ_c^κ as $\sigma_c^\chi \equiv u'\left(c^\chi\right)\frac{dc^\chi}{d\pi}$ and $\sigma_c^\kappa \equiv u'\left(c^\kappa\right)\frac{dc^\kappa}{d\pi}$, respectively. These are then expressed as:

$$\sigma_c^\chi \equiv u'\left(c^\chi\right)\frac{dc^\chi}{d\pi}=\frac{1}{c^\chi}\frac{dc^\chi}{d\pi}=\frac{1}{1+q}\frac{dq}{d\pi}+\frac{\alpha}{1-\alpha}\frac{1}{\rho}\frac{d\rho}{d\pi}, \tag{63}$$

$$\sigma_c^\kappa \equiv u'\left(c^\kappa\right)\frac{dc^\kappa}{d\pi}=\frac{1}{c^\kappa}\frac{dc^\kappa}{d\pi}=\sigma_c^\chi+\frac{1}{\phi}\frac{d\phi}{d\pi}. \tag{64}$$

The last equality of (63) comes from (27) and $\frac{1}{w}\frac{dw}{d\pi}=\frac{\alpha}{1-\alpha}\frac{1}{\rho}\frac{d\rho}{d\pi}$. Moreover, in the interior solution case, from (29), (33), and (34), we obtain the following equations, respectively:

$$\begin{aligned}
u'\left(d^\chi\right)\frac{dd^\chi}{d\pi}&=\sigma_c^\chi+\frac{1}{r}\frac{dr}{d\pi},\\
u'\left(d^{\kappa\chi}\right)\frac{dd^{\kappa\chi}}{d\pi}&=\sigma_c^\chi,\\
u'\left(d^{\kappa\kappa}\right)\frac{dd^{\kappa\kappa}}{d\pi}&=\sigma_c^\kappa.
\end{aligned}$$

Using these results, (62) is rewritten as:

$$(1-\pi)\left[(1+\beta)\sigma_c^\chi+\beta\frac{1}{r}\frac{dr}{d\pi}\right]+\pi\left[(1+\beta)\sigma_c^\kappa+\beta(1-\pi)\left(\sigma_c^\chi-\sigma_c^\kappa\right)\right]. \tag{65}$$

We interpret the allocation effect by using (65). Note that the lifetime income of state-χ and state-κ agents, evaluated in their young periods, are $w+x^\chi$ and $w+x^\kappa$, respectively, and recall that the marginal utility of income of state-χ and state-κ agents is expressed as

$V_\chi'(x^\chi) = u'(c^\chi)$ and $V_\kappa'(x^\kappa) = u'(c^\kappa)$, respectively. The effects of a change in the lifetime income on welfare (we are permitted to refer to them as income effects), $V_\chi'(x^\chi)\frac{d}{d\pi}(w+x^\chi)$ and $V_\kappa'(x^\kappa)\frac{d}{d\pi}(w+x^\kappa)$, are represented as:

$$(1+\beta)\,\sigma_c^\chi = V_\chi'(x^\chi)\frac{d}{d\pi}(w+x^\chi), \tag{66}$$
$$(1+\beta)\,\sigma_c^\kappa = V_\kappa'(x^\kappa)\frac{d}{d\pi}(w+x^\kappa) - \left(1+\beta+\frac{\beta}{\eta}\right)V_\kappa'(x^\kappa)\left(\frac{dc^\chi}{d\pi}-\frac{dc^\kappa}{d\pi}\right). \tag{67}$$

We provide the process for the derivation of (66) and (67) in Appendix G.

Using (66), we observe that the allocation effect concerning a state-χ young agent is composed of the income effect and the effect of a change in the interest rate. That is:

$$V_\chi'(x^\chi)\frac{d}{d\pi}(w+x^\chi)+\beta\frac{1}{r}\frac{dr}{d\pi}.$$

On the other hand, the allocation effect of state-κ young agents is expressed as:

$$V_\kappa'(x^\kappa)\frac{d}{d\pi}(w+x^\kappa) - \left(1+\beta+\frac{\beta}{\eta}\right)V_\kappa'(x^\kappa)\left(\frac{dc^\chi}{d\pi}-\frac{dc^\kappa}{d\pi}\right)+\beta(1-\pi)(\sigma_c^\chi-\sigma_c^\kappa). \tag{68}$$

The first term represents the income effect.

Remember that the market is incomplete because there is no insurance available to hedge a household's risk about having children. As discussed in Proposition 1, if such insurance existed and the market was consequently complete, $c^\chi = c^\kappa$ (or equivalently, $\phi^* = 1$) would hold. Thus, in a complete market economy, the second and third terms of (68) would vanish.

The second and third terms are interpreted as that they represent the welfare loss resulting from the incompleteness of the market. As shown in the proof of Proposition 3, $\frac{d\phi^*}{d\pi} < 0$ holds, which means that the consumption gap between c^χ and c^κ widens as π increases. We obtain $\frac{dc^\chi}{d\pi} - \frac{dc^\kappa}{d\pi} > 0$ as long as the capital income share, α, takes an economically reasonable value.[21]

[21] Through a numerical calculation, we observe that when $\beta \leq 1$ and $\eta \leq 1$, $\frac{dc^\chi}{d\pi} - \frac{dc^\kappa}{d\pi} > 0$ holds as long as α is less than, roughly speaking, 0.9. By contrast, we also observe that when α exceeds 0.9, $\frac{dc^\chi}{d\pi} - \frac{dc^\kappa}{d\pi} > 0$ does not hold in a narrow range of π. (For example, when $\beta = \eta = 1$ and $\alpha = 0.95$, it does not hold when $(\pi^+ \approx)\, 0.03 < \pi < 0.15$.) As α approaches to unity, as seen in (1), w converges to zero. In the interior solution case, $x^\chi > 0$ holds, and as shown in (24), q diverges to infinity. We guess that this might cause the irregular result. Besides, the situation where $\alpha > 0.9$ is not an economically important case, so we focus only on that where $\frac{dc^\chi}{d\pi} - \frac{dc^\kappa}{d\pi} > 0$ holds.

Considering that $u'(c^{\chi}) < u'(c^{\kappa})$ (or equivalently, $\phi^* < 1$) holds in equilibrium, such a change has a negative impact on welfare.[22]

Let us examine the sign of the allocation effect (65). To do this, let us rewrite (65) using (63), (64), and $\frac{1}{r}\frac{dr}{d\pi} = -\frac{1}{\rho}\frac{d\rho}{d\pi}$ as:

$$(1+\beta)\frac{1}{1+q}\frac{dq}{d\pi} + (1+\beta\pi)\frac{\pi}{\phi}\frac{d\phi}{d\pi} + \left((1+\beta)\frac{\alpha}{1-\alpha} - \beta(1-\pi)\right)\frac{1}{\rho}\frac{d\rho}{d\pi}. \tag{69}$$

Recall that we summarized the steady state of the dynamical system as three equations, (32), (38), and (40). Using these equations, we can decompose (69) into two terms: one is a term whose sign is the same as that of $\frac{d\rho}{d\pi}$ (or equivalently, $\frac{dk}{d\pi}$), and the other is a term not linked by $\frac{d\rho}{d\pi}$:

$$T_1\frac{d\rho}{d\pi} + T_2\frac{1}{\pi^2}, \tag{70}$$

where:

$$T_1 = \left(\frac{1+\beta}{1-\alpha} - \beta(1-\pi) - \frac{\beta+\eta+\beta\eta}{\eta}(1-\alpha)(1+q)\right)\frac{1}{\rho} - \frac{\beta+\eta+\beta\eta}{\beta+\eta}\alpha(1+\beta\pi)\frac{\pi}{\phi} > 0,$$
$$T_2 = (1-\alpha)\beta\left((1+q)\frac{1}{\rho} - \frac{\eta}{\beta+\eta}(1+\beta\pi)\frac{\pi}{\phi}\right) \geq 0.$$

[22] When $\frac{dc^{\chi}}{d\pi} - \frac{dc^{\kappa}}{d\pi} > 0$, the second term is negative. Alternatively, the third term is positive when $\frac{d\phi}{d\pi} < 0$ (note that we obtain $\frac{1}{\phi}\frac{d\phi}{d\pi} = \sigma_c^{\kappa} - \sigma_c^{\chi}$ from the definition of $\phi \equiv \frac{c^{\kappa}}{c^{\chi}}$). However, the third term is always dominated by the second term. To show this, we rewrite the third term as $\beta(1-\pi)(\sigma_c^{\chi} - \sigma_c^{\kappa}) = \beta(1-\pi)V_{\kappa}'(x^{\kappa})\left(\phi\frac{dc^{\chi}}{d\pi} - \frac{dc^{\kappa}}{d\pi}\right)$. The third term is dominated by the second term if and only if:

$$(1+\beta+\frac{\beta}{\eta} - \beta(1-\pi))\frac{dc^{\kappa}}{d\pi} < (1+\beta+\frac{\beta}{\eta} - \beta\phi(1-\pi))\frac{dc^{\chi}}{d\pi}.$$

From $c^{\kappa} \equiv \phi c^{\chi}$, we obtain $\frac{dc^{\kappa}}{d\pi} = \frac{d\phi}{d\pi}c^{\chi} + \frac{dc^{\chi}}{d\pi}\phi$. Substituting this into the left-hand side of the necessary and sufficient condition, we obtain:

$$(1+\pi\beta+\frac{\beta}{\eta})\frac{1}{1-\phi}\frac{d\phi}{d\pi} < (1+\beta+\frac{\beta}{\eta})\frac{1}{c^{\chi}}\frac{dc^{\chi}}{d\pi}.$$

From $c^{\chi} - c^{\kappa} \equiv (1-\phi)c^{\chi}$, we obtain:

$$\frac{1}{c^{\chi}-c^{\kappa}}\left(\frac{dc^{\chi}}{d\pi} - \frac{dc^{\kappa}}{d\pi}\right) = -\frac{1}{1-\phi}\frac{d\phi}{d\pi} + \frac{1}{c^{\chi}}\frac{dc^{\chi}}{d\pi} > 0.$$

Considering this inequality, we confirm that the necessary and sufficient condition holds true.

We provide the process of the calculation to derive (70) and the proof for the signs of T_1 and T_2 in Appendix H. To understand (70), we focus on (49):

$$\frac{d\phi^*}{d\pi} = -\alpha \frac{\beta + \eta + \beta\eta}{\beta + \eta} \frac{d\rho^*}{d\pi} - \frac{1}{\pi^2} \frac{\beta\eta}{\beta + \eta} (1 - \alpha) .$$

The first term of (49) represents the effect of a change in the factor prices w and r, which is caused by a change in k, on $c^\kappa/c^\chi (\equiv \phi^*)$. In contrast, the second term of (49) represents that a change in the ratio of the population size of state-χ agents to state-κ agents, $\frac{1-\pi}{\pi}$, also affects c^κ/c^χ directly. (Note that $\frac{d}{d\pi}\left(\frac{1-\pi}{\pi}\right) = -\frac{1}{\pi^2}$.)

Note that the term $T_2 \frac{1}{\pi^2}$ in (70) is generated from the second term of (49). To see this, consider a special case where $\beta = 0$, that is, the situation where agents derive no utility from consumption in the old period, and thus only state-κ agents save to leave bequests for their children. In this case, $T_2 = 0$ holds. Furthermore, because $q = \frac{\alpha}{1-\alpha}$ holds, as confirmed by (38) and (40), (70) is expressed as:

$$\alpha \left(\frac{1}{1-\alpha} \frac{1}{\rho} - \frac{\pi}{\phi} \right) \frac{d\rho}{d\pi}.$$

Thus, the sign of the allocation effect is the same as the sign of $\frac{d\rho}{d\pi}$ when $\beta = 0$.[23]

Alternatively, when $\beta \neq 0$, not only state-κ, but also state-χ young agents save. In this case, the ratio of the population size of state-χ agents to that of state-κ agents, $\frac{1-\pi}{\pi}$, also affects the level of k^* through the change in c^κ/c^χ. The term $T_2 \frac{1}{\pi^2}$ represents the effect of a change in the population ratio on the ex ante welfare.

In sum, the allocation effect is as follows:

Lemma 4 *(i) When $\beta > 0$, the allocation effect is (a) positive when $\frac{dk^*}{d\pi} \geq 0$, and (b) ambiguous when $\frac{dk^*}{d\pi} < 0$.*

(ii) When $\beta = 0$, the allocation effect is positive if and only if $\frac{dk^*}{d\pi} > 0$.

Finally, we summarize the impact of a change in π on EV. The state effect is not always positive, and neither is the allocation effect. Accordingly, in general, the relationship between the extensive margin of fertility and economic welfare is not clear. Nevertheless, Lemma 3

[23] $\frac{1}{1-\alpha}\frac{1}{\rho} > \frac{\pi}{\phi}$ is confirmed as follows. Because $\phi = 1 - \alpha\rho$ holds from (41), the inequality is expressed as $1 - \alpha\rho > (1-\alpha)\rho\pi$. Subtracting $(1-\alpha\rho)\pi$ from both sides, we obtain $(1-\alpha\rho)(1-\pi) + \pi > \rho\pi$. Taking (32) into account, this is equivalent to $\frac{\rho}{\eta} > \rho\pi$. Because $\eta\pi < 1$, $\frac{1}{1-\alpha}\frac{1}{\rho} > \frac{\pi}{\phi}$ holds true.

indicates that the state effect is positive when η is high. In addition, Lemma 4 argues that the allocation effect is also positive when $\frac{dk^*}{d\pi} \geq 0$ holds. In this regard, Section 4 elucidates that $\frac{dk^*}{d\pi} \geq 0$ holds when π is high. Consequently, when both η and π are high, a rise in the population growth rate brought about by a rise in the extensive margin of fertility raises ex ante economic welfare.

6 Optimal Income Transfer Policy Rule

As discussed in Section 3, $u'(c^\kappa) > u'(c^\chi)$ holds in the steady state because of market incompleteness, and we have shown in Section 5, this accounts for the welfare losses. One could expect that we could mitigate or even eliminate this welfare loss by transferring some income from state-χ to state-κ young agents. (For example, we can regard a child allowance policy as an example of this type of income transfer.) We now explore the optimal income transfer rule under which the welfare loss is canceled out.

Suppose that the government collects τw_t units of income from each young agent as a labor income tax (τ represents the tax rate), and that the tax revenue is transferred to the state-κ young agents in a lump-sum fashion. Assuming a balanced budget, the net income transfer which each state-κ young agent receives is $\frac{1-\pi}{\pi}\tau w_t$. Thus, the budget constraints for state-χ and state-κ agents in the young period are modified as follows, respectively:

$$
\begin{aligned}
c^\chi + s^\chi &= (1-\tau)\, w, \\
c^\kappa + s^\kappa &= \left(1 + \frac{1-\pi}{\pi}\tau\right) w.
\end{aligned}
$$

Accordingly, the steady-state allocation is derived as follows:

$$
\begin{aligned}
c^{\chi} &= \frac{1-\tau+q}{1+\beta}w, \\
s^{\chi} &= \frac{\beta(1-\tau)-q}{1+\beta}w, \\
d^{\chi} &= \beta\frac{1-\tau+q}{1+\beta}rw, \\
c^{\kappa} &= \phi\frac{1-\tau+q}{1+\beta}w, \\
d^{\kappa\chi} &= \frac{\beta}{\eta}c^{\chi} = \frac{1}{\eta}\frac{\beta}{1+\beta}(1-\tau+q)\,w, \\
d^{\kappa\kappa} &= \frac{\beta}{\eta}c^{\kappa} = \phi\frac{1}{\eta}\frac{\beta}{1+\beta}(1-\tau+q)\,w, \\
s^{\kappa} &= \left(1+\frac{1-\pi}{\pi}\tau\right)w - c^{\kappa} = \left(1+\frac{1-\pi}{\pi}\tau-\phi\frac{1-\tau+q}{1+\beta}\right)w, \\
x^{\kappa} &= x^{\chi} + \left(d^{\kappa\chi}-d^{\kappa\kappa}\right) = \left(q+\frac{1}{\eta}\frac{\beta}{1+\beta}(1-\phi)(1-\tau+q)\right).
\end{aligned}
$$

Using these results and a similar procedure to derive (38) and (40), and noting that the market-equilibrium condition (17) remains the same, we obtain the following equations:

$$
\frac{1-\tau+q}{1+\beta}\left[\frac{\beta+\eta}{\eta}(1-\phi)+\frac{\beta}{\pi}\right]+\frac{\tau}{\pi} = \frac{\alpha}{1-\alpha}\rho, \tag{71}
$$

$$
\frac{1-\tau+q}{1+\beta}\left[\frac{\beta+\eta}{\eta}(1-\phi)+\beta\right]+\frac{\tau}{\pi} = \rho\left[\frac{1}{\eta}\frac{\beta}{1+\beta}(1-\tau+q)+q\right]. \tag{72}
$$

The Euler equation (32) remains valid because this policy does not distort prices, that is:

$$
\rho = \eta\left[\pi+(1-\pi)\,\phi\right].
$$

The steady-state solution is calculated from (32), (71), and (72).

Recall that if a market is complete, $u'(c^{\kappa}) = u'(c^{\chi})$, or equivalently, $\phi = 1$ holds. From (32), we confirm that $\rho = \eta$ also holds under a complete market.[24] That is, the efficient allocation is characterized by $\phi = 1$ and $\rho = \eta$.

[24] That is, the capital–labor ratio $k = (\eta\alpha A)^{\frac{1}{1-\alpha}}$ is independent of π.

We examine whether we can achieve the efficient allocation in the steady-state equilibrium with an incomplete market when τ is appropriately chosen. Substituting $\phi = 1$ and $\rho = \eta$ into (71) and (72), we obtain the following equations, respectively:

$$\beta q + \tau + \beta = \frac{\alpha}{1-\alpha}(1+\beta)\eta\pi, \tag{73}$$

$$\tau = \eta\pi q. \tag{74}$$

Of course, (32) is consistent with $\phi = 1$ and $\rho = \eta$. Let τ^* denote the level of τ which satisfies (73) and (74). We obtain τ^*, the optimal income transfer rule, in the following simple form:

$$\tau^* = \frac{\eta\pi}{\beta + \eta\pi}\left[\frac{\alpha}{1-\alpha}(1+\beta)\eta\pi - \beta\right] = \frac{\beta\eta\pi}{\beta + \eta\pi}\left(\frac{\pi}{\pi^+} - 1\right). \tag{75}$$

The last equality comes from the definition of $\pi^+ \equiv \frac{1-\alpha}{\alpha}\frac{\beta}{1+\beta}\frac{1}{\eta}$. From (75), we confirm $\tau^* > 0$ because $\pi^+ < \pi$ holds in the interior solution case. Fig. 4 illustrates (73) and (74) on a $q-\tau$ plane, drawn as downward- and upward-sloping lines, respectively, and the intersection indicates τ^*. Using this result, we can obtain the properties of τ^*. First, consider the case where π rises. The graph of (73) in Fig. 4 shifts upward, and the slope of (74) becomes steeper, and consequently, τ^* rises, as depicted in Fig. 5.[25] As observed in the previous section, the gap in the marginal utility of income between state-χ and state-κ agents widens (put differently, ϕ becomes smaller) as π increases. This is the intuitive reason why a higher level of τ is needed to attain $\phi = 1$.

From (73) and (74), we immediately confirm that the impact of η on τ^* is qualitatively the same as the impact of π on τ^*. That is, τ^* becomes higher as η rises. On the other hand, when β increases, we confirm from (75) that τ falls.[26]

---

[25] From (73) and (74), q is calculated as:

$$q = \frac{\alpha}{1-\alpha}(1+\beta)\frac{\eta\pi}{\beta+\eta\pi} - \frac{\beta}{\beta+\eta\pi}.$$

Although the effect of a rise in π on q appears ambiguous in Fig. 5, we easily confirm that q necessarily increases when π increases.

[26] From (75), the following is obtained:

$$\frac{\partial\tau^*}{\partial\beta} = \frac{\eta\pi}{(\beta+\eta\pi)^2}\left[\frac{\alpha}{1-\alpha}(\eta\pi - 1) - 1\right].$$

7 Concluding Remarks

This paper investigated the impact of the intensive and extensive margins of fertility on long-term capital accumulation and economic welfare, for which we argued that disentangling the intensive and extensive margins of fertility is important. We showed that when the low fertility rate is brought about by a fall in the number of the children each parent has (the intensive margin), a negative relationship between the population growth rate and the capital–labor ratio is obtained. This result is consistent with existing studies in this area. However, when the low fertility rate is brought about by a rise in the childlessness rate (extensive margin), we showed the relationship between the population growth rate and the capital–labor ratio exhibits a U-shaped relationship. This result largely differs from the effect of the intensive margin of fertility on capital accumulation.

In terms of the welfare analysis, we showed that a rise in the intensive margin of fertility reduces ex ante welfare (unless agents have an extremely high preference toward the number of children). On the other hand, we also showed that the relationship between the extensive margin of fertility and ex ante economic welfare is not determined a priori, but showed that they exhibit a positive relationship when agents' preferences toward their children are not small and the childlessness rate is not very high. In addition, market incompleteness is an important characteristic of our present model, and we showed that this incompleteness accounts for the welfare loss. In this regard, we presented the optimal income transfer rule under which the welfare loss may be eliminated.

This paper attempted to show how easily the impact of a declining birthrate on the macroeconomy can be changed by simply modeling the population dynamics. For this reason, we omitted many interesting features. For example, it would be very interesting to reconsider this issue considering the probability of having children to be endogenous.[27] Delayed birth timing is also an important factor in explaining childlessness.[28] We defer these interesting developments

Because $\eta\pi < 1$, we confirm that $\frac{\partial \tau^*}{\partial \beta} < 0$ holds.

[27] It is often pointed out that childlessness appears to be related to educational attainment (for example, OECD Family Database (2010)). However, there is some controversy. Monstad et al. (2008), for example, argued that although the data show a statistically significant correlation, such that women with more education are more commonly childless, they do not find evidence of a causal relationship between being an educated woman and childlessness.

[28] For models of delayed birth timing, see d'Albis et al. (2010) and Momota and Horii (2013).

to future research.

Appendix A

We first provide the process for the derivation of (7), and then describe the derivation of (12)–(15).

Derivation of (7) The optimal conditions for a state-χ young agent are expressed as (4)–(6); that is:

$$\begin{aligned} c_t^\chi + s_t^\chi &= w_t, \\ d_{t+1}^\chi &= r_{t+1}\left(s_t^\chi + x_t^\chi\right), \\ u'\left(c_t^\chi\right) &= \beta r_{t+1} u'\left(d_{t+1}^\chi\right). \end{aligned}$$

From these equations, c_t^χ, s_t^χ, and d_{t+1}^χ are expressed as the function of x_t^χ, w_t, and r_{t+1}; for example, $c_t^\chi = c^\chi\left(x_t^\chi, w_t, r_{t+1}\right)$. Households consider w_t and r_{t+1} as given. Because we focus only on the household's optimal problem in this appendix, let us express the optimal solution only as a function of x_t^χ; that is, $c_t^\chi = c^\chi\left(x_t^\chi\right)$, $s_t^\chi = s^\chi\left(x_t^\chi\right)$, and $d_{t+1}^\chi = d^\chi\left(x_t^\chi\right)$. Using these, V_t^χ is expressed as:

$$V_t^\chi = V_\chi\left(x_t^\chi\right) \equiv u\left(c^\chi\left(x_t^\chi\right)\right) + \beta u\left(d^\chi\left(x_t^\chi\right)\right).$$

Using this, we obtain the following:

$$V_\chi'\left(x_t^\chi\right) = u'\left(c^\chi\left(x_t^\chi\right)\right)\frac{dc_t^\chi}{dx_t^\chi} + \beta u'\left(d^\chi\left(x_t^\chi\right)\right)\frac{dd_{t+1}^\chi}{dx_t^\chi} = u'\left(c^\chi\left(x_t^\chi\right)\right)\left(\frac{dc_t^\chi}{dx_t^\chi} + \frac{1}{r_{t+1}}\frac{dd_{t+1}^\chi}{dx_t^\chi}\right). \tag{76}$$

(Strictly speaking, $\frac{dc_t^\chi}{dx_t^\kappa}$ should be written as $\left.\frac{dc_t^\chi}{dx_t^\kappa}\right|_{dw_t=dr_{t+1}=0}$, and this is applied to others similarly.) Under the condition of $dw_t = 0$ and $dr_{t+1} = 0$, we obtain the following equations from (4) and (5):

$$\begin{aligned} \frac{dc_t^\chi}{dx_t^\chi} + \frac{ds_t^\chi}{dx_t^\chi} &= 0, \\ \frac{dd_{t+1}^\chi}{dx_t^\chi} &= r_{t+1}\left(\frac{ds_t^\chi}{dx_t^\chi} + 1\right). \end{aligned}$$

By applying these equations to (76), we obtain (7):

$$V_\chi'\left(x_t^\chi\right)=u'\left(c_t^\chi\right).$$

Derivation of (12)–(15) A state-κ young agent chooses $c_t^\kappa, s_t^\kappa, d_{t+1}^{\kappa\chi}, d_{t+1}^{\kappa\kappa}, x_{t+1}^\chi$, and x_{t+1}^κ to maximize:

$$u\left(c_t^\kappa\right)+v\left(n\right)+\beta\left[\left(1-\pi\right)u\left(d_{t+1}^{\kappa\chi}\right)+\pi u\left(d_{t+1}^{\kappa\kappa}\right)\right]+\eta\left[\left(1-\pi\right)V_\chi\left(x_{t+1}^\chi\right)+\pi V_\kappa\left(x_{t+1}^\kappa\right)\right],$$

subject to the constraints (8)–(11). The first-order conditions are derived as:

$$u'\left(c_t^\kappa\right)=\beta r_{t+1}\left[\left(1-\pi\right)u'\left(d_{t+1}^{\kappa\chi}\right)+\pi u'\left(d_{t+1}^{\kappa\kappa}\right)\right],\tag{77}$$

$$\beta n u'\left(d_{t+1}^{\kappa\chi}\right)\geq\eta V_\chi'\left(x_{t+1}^\chi\right),\tag{78}$$

$$\beta n u'\left(d_{t+1}^{\kappa\kappa}\right)\geq\eta V_\kappa'\left(x_{t+1}^\kappa\right).\tag{79}$$

If $x_{t+1}^\chi>0$ and $x_{t+1}^\kappa>0$ are chosen, (78) and (79) hold as equalities, respectively. Now we show that (12)–(15) can be obtained from (77)–(79). First of all, (77) is the same as (12), and using (7), we obtain (13) from (78).

Let us show that $V_\kappa'\left(x_t^\kappa\right)=u'\left(c_t^\kappa\right)$ holds. By using (77)–(79) and the budget constraints (8)–(10), the optimal solution is expressed as a function of x_t^κ, w_t, and r_{t+1}, for example, $c_t^\kappa=c^\kappa\left(x_t^\kappa,w_t,r_{t+1}\right)$.

Households consider w_t and r_{t+1} as given. As has been explained earlier, we express the optimal solution only as a function of x_t^κ; that is, $c_t^\kappa=c^\kappa\left(x_t^\kappa\right)$, $s_t^\kappa=s^\kappa\left(x_t^\kappa\right)$, $d_{t+1}^{\kappa\chi}=d^{\kappa\chi}\left(x_t^\kappa\right)$, $d_{t+1}^{\kappa\kappa}=d^{\kappa\kappa}\left(x_t^\kappa\right)$, $x_{t+1}^\chi=x^\chi\left(x_t^\kappa\right)$, and $x_{t+1}^\kappa=x^\kappa\left(x_t^\kappa\right)$. Using these, V_t^κ is expressed as:

$$\begin{aligned}V_t^\kappa=V_\kappa\left(x_t^\kappa\right)\equiv u\left(c^\kappa\left(x_t^\kappa\right)\right)+v\left(n\right)+\beta\left[\left(1-\pi\right)u\left(d^{\kappa\chi}\left(x_t^\kappa\right)\right)+\pi u\left(d^{\kappa\kappa}\left(x_t^\kappa\right)\right)\right]&\\+\eta\left[\left(1-\pi\right)V_\chi\left(x^\chi\left(x_t^\kappa\right)\right)+\pi V_\kappa\left(x^\kappa\left(x_t^\kappa\right)\right)\right].&\end{aligned}$$

Using this, we obtain the following:

$$\begin{aligned}V_\kappa'\left(x_t^\kappa\right)=u'\left(c_t^\kappa\right)\frac{dc_t^\kappa}{dx_t^\kappa}+\beta\left[\left(1-\pi\right)u'\left(d_{t+1}^{\kappa\chi}\right)\frac{dd_{t+1}^{\kappa\chi}}{dx_t^\kappa}+\pi u'\left(d_{t+1}^{\kappa\kappa}\right)\frac{dd_{t+1}^{\kappa\kappa}}{dx_t^\kappa}\right]&\\+\eta\left[\left(1-\pi\right)V_\chi'\left(x_{t+1}^\chi\right)\frac{dx_{t+1}^\chi}{dx_t^\kappa}+\pi V_\kappa'\left(x_{t+1}^\kappa\right)\frac{dx_{t+1}^\kappa}{dx_t^\kappa}\right].&\end{aligned}\tag{80}$$

(Strictly speaking, $\frac{dc_t^\kappa}{dx_t^\kappa}$ should be written as $\left.\frac{dc_t^\kappa}{dx_t^\kappa}\right|_{dw_t=dr_{t+1}=0}$, and this is applied to others similarly.) Under the condition of $dw_t = 0$ and $dr_{t+1} = 0$, we obtain the following equations from (8)–(10):

$$\begin{aligned}
\frac{dc_t^\kappa}{dx_t^\kappa} + \frac{ds_t^\kappa}{dx_t^\kappa} &= 0, \\
\frac{dd_{t+1}^{\kappa\chi}}{dx_t^\kappa} + n\frac{dx_{t+1}^\chi}{dx_t^\kappa} &= r_{t+1}\left(\frac{ds_t^\kappa}{dx_t^\kappa} + 1\right), \\
\frac{dd_{t+1}^{\kappa\kappa}}{dx_t^\kappa} + n\frac{dx_{t+1}^\kappa}{dx_t^\kappa} &= r_{t+1}\left(\frac{ds_t^\kappa}{dx_t^\kappa} + 1\right).
\end{aligned}$$

Applying these equations to (80) yields:

$$\begin{aligned}
V_\kappa'\left(x_t^\kappa\right) = {} & \frac{ds_t^\kappa}{dx_t^\kappa}\left[-u'\left(c_t^\kappa\right) + \beta r_{t+1}\left(\left(1-\pi\right)u'\left(d_{t+1}^{\kappa\chi}\right) + \pi u'\left(d_{t+1}^{\kappa\kappa}\right)\right)\right] + \beta r_{t+1}\left(\left(1-\pi\right)u'\left(d_{t+1}^{\kappa\chi}\right) + \pi u'\left(d_{t+1}^{\kappa\kappa}\right)\right) \\
& + \left(1-\pi\right)\left[\eta V_\chi'\left(x_{t+1}^\chi\right) - \beta n u'\left(d_{t+1}^{\kappa\chi}\right)\right]\frac{dx_{t+1}^\chi}{dx_t^\kappa} + \pi\left[\eta V_\kappa'\left(x_{t+1}^\kappa\right) - \beta n u'\left(d_{t+1}^{\kappa\kappa}\right)\right]\frac{dx_{t+1}^\kappa}{dx_t^\kappa}.
\end{aligned}$$

From (78) and (79), we observe that $\eta V_j'\left(x_{t+1}^j\right) - \beta n u'\left(d_{t+1}^{\kappa j}\right) = 0$ holds ($j = \chi,\ \kappa$) when $x_{t+1}^j > 0$, and that when $\eta V_j'\left(x_{t+1}^j\right) - \beta n u'\left(d_{t+1}^{\kappa j}\right) > 0$, $x_{t+1}^j = 0$ (that is, $\frac{dx_{t+1}^j}{dx_t^\kappa} = 0$) holds. Put differently, $\left[\eta V_\chi'\left(x_{t+1}^j\right) - \beta n u'\left(d_{t+1}^{\kappa j}\right)\right]\frac{dx_{t+1}^j}{dx_t^\kappa} = 0$ holds. Furthermore, by using (77), we obtain (15):

$$V_\kappa'\left(x_t^\kappa\right) = \beta r_{t+1}\left[\left(1-\pi\right)u'\left(d_{t+1}^{\kappa\chi}\right) + \pi u'\left(d_{t+1}^{\kappa\kappa}\right)\right] = u'\left(c_t^\kappa\right).$$

Finally, using this and (79), we obtain (14).

Appendix B

In proving Lemma 1, we show that the following two cases never take place: (a) $x^\kappa > 0$ and $x^\chi = 0$, and (b) $x^\chi > 0$ and $x^\kappa = 0$.

Suppose that case (a) occurs. In this case, (13) and (14) hold as an inequality and an equality, respectively. Thus, noting that $\phi \equiv c^\kappa / c^\chi$, the following inequality holds:

$$\phi d^{\kappa\chi} < d^{\kappa\kappa}. \tag{81}$$

At the same time, we obtain the following equation from (9) and (10):

$$d^{\kappa\chi} - d^{\kappa\kappa} = n\left(x^{\kappa} - x^{\chi}\right) > 0, \tag{82}$$

where the last inequality holds by the definition of case (a). Note that the following condition is necessary in order for (81) to be consistent with (82):

$$\phi < 1. \tag{83}$$

When $q = 0$ (that is, $x^{\chi} = 0$), we obtain from (8) and (9), and (30):

$$d^{\kappa\chi} = r\left(s^{\kappa} + x^{\kappa}\right) = r\left(w + x^{\kappa} - \frac{w}{1+\beta}\phi\right) > rw\left(1 - \frac{1}{1+\beta}\phi\right). \tag{84}$$

Note that (12) is expressed as:

$$1 = \beta r\left[(1-\pi)\frac{c^{\kappa}}{d^{\kappa\chi}} + \pi\frac{c^{\kappa}}{d^{\kappa\kappa}}\right]. \tag{85}$$

Taking into account that (14) holds as an equality, and applying (84) to the above equation, we obtain the following inequality:

$$1 < \beta r\left[(1-\pi)\frac{\frac{1}{1+\beta}\phi w}{\left(1 - \frac{1}{1+\beta}\phi\right) rw} + \pi\frac{\eta}{\beta n}\right],$$

or equivalently:

$$1 < (1-\pi)\frac{\frac{\beta}{1+\beta}\phi}{\left(1 - \frac{1}{1+\beta}\phi\right)} + \pi\frac{\eta}{\rho}. \tag{86}$$

Note that $\frac{\beta}{1+\beta}\phi / \left(1 - \frac{1}{1+\beta}\phi\right) < 1$ holds when $\phi < 1$, so that the following inequality is necessary to satisfy (86):

$$\rho < \eta. \tag{87}$$

On the other hand, considering (13) holds as an inequality, and applying (84) and (27) to (13), we obtain:

$$r\left(w + x^{\kappa} - \frac{1}{1+\beta}\phi w\right) \leq \frac{\beta n}{\eta}\frac{1}{1+\beta}w,$$

or equivalently:

$$x^{\kappa} < \left[\frac{\phi}{1+\beta} + \frac{\rho}{\eta}\frac{\beta}{1+\beta} - 1\right] w \leq 0. \tag{88}$$

The last inequality comes by using (83) and (87). Observe that (88) contradicts the assumption $x^{\kappa} > 0$. Therefore, case (a) never takes place.

We can show that case (b) never occurs using a similar argument. In contrast to case (a), (13) and (14) hold as an equality and an inequality, respectively. In this case, the following inequality holds:

$$\phi d^{\kappa\chi} > d^{\kappa\kappa}. \tag{89}$$

Here, (82) is modified as:

$$d^{\kappa\chi} - d^{\kappa\kappa} = n\left(x^{\kappa} - x^{\chi}\right) < 0. \tag{90}$$

Thus, the following condition must hold:

$$\phi > 1. \tag{91}$$

When $x^{\kappa} = 0$, we obtain the following equation by using (8) and (10):

$$d^{\kappa\kappa} = r\left(w - c^{\kappa}\right) = rw\left(1 - \frac{1+q}{1+\beta}\phi\right), \tag{92}$$

where the last equality comes from (30). Using (13) and (92), (85) is rewritten as:

$$1 = \beta r\left[(1-\pi)\frac{\eta}{\beta n}\phi + \pi\frac{\frac{1+q}{1+\beta}\phi w}{\left(1 - \frac{1+q}{1+\beta}\phi\right) rw}\right],$$

or equivalently:

$$1 = (1-\pi)\frac{\eta}{\rho}\phi + \pi\frac{\beta\frac{1+q}{1+\beta}\phi}{1 - \frac{1+q}{1+\beta}\phi}. \tag{93}$$

Because $\beta\frac{1+q}{1+\beta}\phi > 1 - \frac{1+q}{1+\beta}\phi$ holds when $q > 0$ and $\phi > 1$, the following condition is necessary to satisfy (93):

$$\phi\eta < \rho. \tag{94}$$

On the other hand, from (13) and (90), we obtain:

$$c^{\chi} = \frac{\eta}{\beta n} d^{\kappa\chi} = \frac{\eta}{\beta n}\left(d^{\kappa\kappa} - nx^{\chi}\right).$$

Applying (27) and (92) to the above equation, we derive the following equation:

$$\frac{\beta}{1+\beta}(1+q) + \eta q = \frac{\eta}{\rho}\left(1 - \frac{1+q}{1+\beta}\phi\right) < \frac{1}{\phi}\left(1 - \frac{1+q}{1+\beta}\phi\right),$$

where the last inequality comes from (94). We obtain the following result from the above inequality:

$$(1+\eta)\, q < \frac{1}{\phi} - 1 < 0, \tag{95}$$

where the last inequality comes from (91). Observe that (95) contradicts the assumption $x^{\chi} > 0$ (that is, $q > 0$). Therefore, case (b) never takes place.

Appendix C

We prove Proposition 2. The proof is quite easy in the corner solution case ($x^{\chi} = x^{\kappa} = 0$). First, $q = 0$ and $\phi = 1$ immediately hold from $x^{\chi} = 0$ and (44), respectively. Second, from (45) and (17), we obtain:

$$n\pi k_{t+1} = \frac{\beta}{1+\beta} w_t.$$

Taking (1) into account, we confirm that the capital stock k_t converges to its steady state monotonically.

Next, we examine the interior solution case ($x^{\kappa} > x^{\chi} > 0$). First, c_t^{χ}, s_t^{χ}, c_t^{κ}, s_t^{κ}, $d_t^{\kappa\chi}$, $d_t^{\kappa\kappa}$, x_t^{κ}, $s_t^{\chi} + x_t^{\chi}$, and $s_t^{\kappa} + x_t^{\kappa}$ are obtained only by replacing q, ϕ, ρ, and w in (27)–(31) and (33)–(37) with q_t, ϕ_t, ρ_t, and w_t, respectively. From the Euler equation (16), the following is derived:

$$\frac{1}{\phi_t} = \frac{\eta}{n} r_{t+1} \frac{(1+q_t) w_t}{(1+q_{t+1}) w_{t+1}} \left((1-\pi) + \pi \frac{1}{\phi_{t+1}} \right).$$

Observe that this coincides with (32) in the steady state. Note that $\frac{w_{t+1}}{r_{t+1}} = \frac{1-\alpha}{\alpha} k_{t+1}$ holds, and we rewrite the above equation as:

$$\frac{1}{\phi_t} n k_{t+1} = \eta \frac{\alpha}{1-\alpha} \frac{1+q_t}{1+q_{t+1}} \left(1 - \pi + \pi \frac{1}{\phi_{t+1}} \right) w_t. \tag{96}$$

From the market-equilibrium condition (17), we obtain the following, which corresponds to (38):

$$\frac{1+q_t}{1+\beta} \left(\frac{\beta+\eta}{\eta} (1-\phi_t) + \frac{\beta}{\pi} \right) w_t = n k_{t+1}. \tag{97}$$

Using (9) and a similar argument to derive (39), we obtain:

$$s_t^{\kappa} + x_t^{\kappa} = \frac{1}{r_{t+1}} \left(d_{t+1}^{\kappa\chi} + n x_{t+1}^{\chi} \right) = n \frac{w_{t+1}}{r_{t+1}} \left(\frac{1}{\eta} \frac{\beta}{1+\beta} (1+q_{t+1}) + q_{t+1} \right).$$

Using this, we obtain the following equation, which corresponds to (40):

$$\frac{1-\alpha}{\alpha} n k_{t+1} \left(\frac{1}{\eta} \frac{\beta}{1+\beta} (1+q_{t+1}) + q_{t+1} \right) = \frac{1+q_t}{1+\beta} \left(\frac{\beta+\eta}{\eta} (1-\phi_t) + \beta \right) w_t. \tag{98}$$

Thus, the transitional dynamics are described by (96), (97), and (98): the difference system with regard to q_t, ϕ_t, and k_t.

Dividing both sides of (97) by (98) and solving it for q_{t+1}, we obtain:

$$q_{t+1} = \frac{1}{\beta+\eta+\beta\eta} \left(\pi \frac{\alpha (1+\beta) \eta}{1-\alpha} \frac{(\beta+\eta+\beta\eta) - (\beta+\eta) \phi_t}{\beta\eta + \pi (\beta+\eta)(1-\phi_t)} - \beta \right). \tag{99}$$

Dividing both sides of (97) by (96), we derive the following equation:

$$\frac{1}{\phi_t} \frac{1}{1+\beta} \left(\frac{\beta+\eta}{\eta} (1-\phi_t) + \frac{\beta}{\pi} \right) = \eta \frac{\alpha}{1-\alpha} \frac{1}{1+q_{t+1}} \left(1 - \pi + \pi \frac{1}{\phi_{t+1}} \right).$$

Substituting (99) into the above equation, we obtain the first-order difference equation with respect to ϕ_t:

$$\phi_{t+1} = \frac{\pi^2 \alpha\eta (\beta+\eta+\beta\eta) \phi_t}{\{\pi (\beta+\eta+\alpha\beta\eta) + (1-\alpha) \beta\eta\} - \pi \{(\beta+\eta)(1+(1-\pi)\alpha\eta) + (1-\pi)\alpha\beta\eta^2\} \phi_t},$$

or equivalently:

$$\frac{1}{\phi_{t+1}} = \frac{\pi\left(\beta+\eta+\alpha\beta\eta\right)+\left(1-\alpha\right)\beta\eta}{\pi^2\alpha\eta\left(\beta+\eta+\beta\eta\right)}\frac{1}{\phi_t} - \frac{\left(\beta+\eta\right)\left(1+\left(1-\pi\right)\alpha\eta\right)+\left(1-\pi\right)\alpha\beta\eta^2}{\pi\alpha\eta\left(\beta+\eta+\beta\eta\right)}. \quad (100)$$

Fig. 6 illustrates (100). The slope is greater than unity given the assumption $\eta\pi < 1$, so that there exists a unique steady state, $\phi_t = \phi_{t+1} \equiv \phi^*$. We confirm that $\phi^* < 1$ is guaranteed because $\pi > \pi^+ \equiv \frac{1-\alpha}{\alpha}\frac{\beta}{1+\beta}\frac{1}{\eta}$ holds in the interior solution case. Note that ϕ_t is a control variable in period t. In this regard, the following result holds:

Lemma 5 *$\phi_t = \phi^*$ is chosen for any t.*

Proof. Suppose that $\phi_s > \phi^*$ is chosen in period s. In this case, $\phi_t > 1$ holds in a finite period t. However, the optimal condition for agents is not consistent with $\phi_t > 1$.

Next, suppose that $\phi_s < \phi^*$ is chosen in period s. In this case, $\lim_{t\to\infty}\phi_t = 0$ holds. We will show that this situation violates the transversality conditions. As described in footnote 10, the transversality conditions are represented as:

$$\lim_{t\to\infty}\left(\eta\pi\right)^t\frac{x^{\chi}_{t+1}}{d^{\kappa\chi}_{t+1}} = 0, \quad (101)$$

$$\lim_{t\to\infty}\left(\eta\pi\right)^t\frac{x^{\kappa}_{t+1}}{d^{\kappa\kappa}_{t+1}} = 0. \quad (102)$$

In the interior solution case, (13) and (14) hold as equalities, so that the following equation is derived:

$$d^{\kappa\kappa}_{t+1} = \phi_{t+1}d^{\kappa\chi}_{t+1}.$$

Using (9), (10), and the above equation, we obtain:

$$d^{\kappa\chi}_{t+1}\left(1+n\frac{x^{\chi}_{t+1}}{d^{\kappa\chi}_{t+1}}\right) = d^{\kappa\kappa}_{t+1}\left(1+n\frac{x^{\kappa}_{t+1}}{d^{\kappa\kappa}_{t+1}}\right) = \phi_{t+1}d^{\kappa\chi}_{t+1}\left(1+n\frac{x^{\kappa}_{t+1}}{d^{\kappa\kappa}_{t+1}}\right),$$

and thus:

$$\frac{x^{\kappa}_{t+1}}{d^{\kappa\kappa}_{t+1}} = \frac{1}{n}\left\{\frac{1}{\phi_{t+1}}\left(1+n\frac{x^{\chi}_{t+1}}{d^{\kappa\chi}_{t+1}}\right)-1\right\}.$$

Using this equation, (102) is rewritten as:

$$\lim_{t\to\infty}(\eta\pi)^t\frac{1}{n}\left\{\frac{1}{\phi_{t+1}}\left(1+n\frac{x_{t+1}^{\chi}}{d_{t+1}^{\kappa\chi}}\right)-1\right\}=0.$$

Taking $\lim_{t\to\infty}(\eta\pi)^t=0$ into account, the following equation must hold to satisfy the above equation:

$$\lim_{t\to\infty}(\eta\pi)^t\frac{1}{\phi_{t+1}}\left(1+n\frac{x_{t+1}^{\chi}}{d_{t+1}^{\kappa\chi}}\right)=0. \tag{103}$$

Performing a simple calculation, we confirm that the coefficient of $\frac{1}{\phi_{t+1}}$ in (100) exceeds $\frac{1}{\eta\pi}$; that is:

$$\frac{\pi(\beta+\eta+\alpha\beta\eta)+(1-\alpha)\beta\eta}{\pi^2\alpha\eta(\beta+\eta+\beta\eta)}>\frac{1}{\eta\pi}.$$

The intercept of (100) is negative, and thus, we find that the growth rate of $\frac{1}{\phi_{t+1}}$ is larger than $\frac{1}{\eta\pi}$, which means that (103) is violated. Therefore, only $\phi_t=\phi^*$ is consistent with the equilibrium condition. ■

Taking Lemma 5 into account, (99) is represented as:

$$q_{t+1}=q^*\equiv\frac{1}{\beta+\eta+\beta\eta}\left(\pi\frac{\alpha(1+\beta)\eta}{1-\alpha}\frac{(\beta+\eta+\beta\eta)-(\beta+\eta)\phi^*}{\beta\eta+\pi(\beta+\eta)(1-\phi^*)}-\beta\right).$$

That is, q_t does not have any transition process either. We obtain the dynamics of k_t from (96):

$$k_{t+1}=\frac{\eta}{n}\frac{\alpha}{1-\alpha}((1-\pi)\phi^*+\pi)w_t=\frac{A\eta\alpha}{n}((1-\pi)\phi^*+\pi)k_t^{\alpha}.$$

Therefore, k_t converges from the initial value k_0 to the steady-state value.

Appendix D

In this appendix, instead of (3), we use an alternative functional form to express a state-ij agent's utility function, which is based on Becker and Barro (1988) and Barro and Becker (1989):

$$u\left(c_t^i\right)+\beta u\left(d_{t+1}^{ij}\right)+\widehat{\eta}\left(n^i\right)^{\theta}V_{t+1}^{j},$$

where $\widehat{\eta} > 0$ and $\theta \in (0,1)$. Let us specify $u(c) = \log c$. We show that the main result, the effect of both margins of fertility on the long-run capital stock, $\frac{dk^*}{dn}$ and $\frac{dk^*}{d\pi}$, is unchanged from a qualitative viewpoint, and that Proposition 3 remains valid. That is, (i) there is a U-shaped relationship between k^* and π, and (ii) k^* decreases as n increases.

State-χ agents do not have children ($n^\chi = 0$), so that their utility is expressed as $u(c_t^\chi) + \beta u\left(d_{t+1}^\chi\right)$, which is the same as the utility function appearing in Section 2. Moreover, their budget constraints are represented as (4) and (5). Thus, the optimal behavior of the state-χ agents does not change, and is represented as (6).

The utility functions of state-$\kappa\chi$ agents and state-$\kappa\kappa$ agents are expressed as, respectively:

$$
\begin{aligned}
&u(c_t^\kappa) + \beta u\left(d_{t+1}^{\kappa\chi}\right) + \widehat{\eta} n^\theta V_{t+1}^\chi, \\
&u(c_t^\kappa) + \beta u\left(d_{t+1}^{\kappa\kappa}\right) + \widehat{\eta} n^\theta V_{t+1}^\kappa.
\end{aligned}
$$

Let us define η as $\eta \equiv \widehat{\eta} n^\theta$, and rewrite the above functions as follows:

$$
\begin{aligned}
&u(c_t^\kappa) + \beta u\left(d_{t+1}^{\kappa\chi}\right) + \eta V_{t+1}^\chi, \\
&u(c_t^\kappa) + \beta u\left(d_{t+1}^{\kappa\kappa}\right) + \eta V_{t+1}^\kappa.
\end{aligned}
$$

Taking into account that n is treated as exogenous, η is considered an exogenous variable. The optimal problem of state-κ agents is expressed as:

$$
V_t^\kappa = \max\left\{u(c_t^\kappa) + \beta\left[(1-\pi)\,u\left(d_{t+1}^{\kappa\chi}\right) + \pi u\left(d_{t+1}^{\kappa\kappa}\right)\right] + \eta\left[(1-\pi)\,V_{t+1}^\chi + \pi V_{t+1}^\kappa\right]\right\},
$$

subject to (8)–(11). Thus, the optimal behavior of the state-κ agents is represented by the same equations as (12)–(14).

The firm's optimal behavior, (1) and (2), and the market equilibrium condition, (17), are unaffected by the functional form of the utility function. We assume $\zeta = 0$ as we do so in the main text. Consequently, as we obtained in Section 3, we obtain (32) and (41), that is:

$$
\begin{aligned}
\rho^* &= \eta\left(\pi + (1-\pi)\,\phi^*\right), \\
\phi^* &= \left(1 + \frac{\beta\eta}{\beta+\eta}\right)(1-\alpha\rho^*) + \frac{1-\pi}{\pi}\frac{\beta\eta}{1+\eta}(1-\alpha).
\end{aligned}
$$

The steady state of the interior solution case is derived from these equations.

Based on (32) and (41), the effect of both margins of fertility on the long-run capital stock, $\frac{dk^*}{dn}$ and $\frac{dk^*}{d\pi}$, is examined. Thus, it is immediately confirmed that the qualitative property of $\frac{dk^*}{d\pi}$ is the same as Proposition 3. That is, there is a U-shaped relationship between k^* and π.

On the other hand, when we examine $\frac{dk^*}{dn}$, we should note that $\eta \equiv \widehat{\eta} n^{\theta}$; that is, contrary to Section 4, η varies as n changes. From (32) and (41), we obtain, respectively:

$$\frac{1}{\rho^*}\frac{d\rho^*}{dn} = \frac{1}{\eta}\frac{d\eta}{dn} + \frac{1-\pi}{\pi + (1-\pi)\phi^*}\frac{d\phi^*}{dn},$$
$$\frac{d\phi^*}{dn} = -\alpha\left(1 + \frac{\beta\eta}{\beta+\eta}\right)\frac{d\rho^*}{dn} + \left(\frac{\beta}{\beta+\eta}\right)^2\left[(1-\alpha\rho^*) + \frac{1-\pi}{\pi}(1-\alpha)\right]\frac{d\eta^*}{dn}.$$

Eliminating $\frac{d\phi^*}{dn}$ from the above equations and rearranging, we obtain:

$$\left[1 + \alpha\eta(1-\pi)\left(1 + \frac{\beta\eta}{\beta+\eta}\right)\right]\frac{1}{\rho^*}\frac{d\rho^*}{dn} = \left\{1 + \frac{1-\pi}{\rho^*}\left(\frac{\beta\eta}{\beta+\eta}\right)^2\left[(1-\alpha\rho^*) + \frac{1-\pi}{\pi}(1-\alpha)\right]\right\}\frac{1}{\eta}\frac{d\eta}{dn}. \quad (104)$$

By definition of ρ and η, $\frac{1}{\rho}\frac{d\rho}{dn} = \frac{1}{n} - \frac{1}{r}\frac{dr}{dn}$ and $\frac{d\eta}{dn} = \theta\frac{\eta}{n}$ hold, and using them, (104) is rewritten as:

$$\left[1 + \alpha\eta(1-\pi)\left(1 + \frac{\beta\eta}{\beta+\eta}\right)\right]\frac{1}{r^*}\frac{dr^*}{dn}$$
$$= (1-\theta) + (1-\pi)\left[\alpha\eta\left(1 + \frac{\beta\eta}{\beta+\eta}\right) - \theta\left(\frac{\beta\eta}{\beta+\eta}\right)^2\frac{1}{\rho^*}\left[(1-\alpha\rho^*) + \frac{1-\pi}{\pi}(1-\alpha)\right]\right], \quad (105)$$

where r^* denotes the interest rate in the steady state. We examine the sign of the right-hand side of (105). We obtain $\phi^* < 1$ in the interior solution case, so that from (41), we obtain

$$\left(1 + \frac{\beta\eta}{\beta+\eta}\right)(1-\alpha\rho^*) + \frac{1-\pi}{\pi}\frac{\beta\eta}{\beta+\eta}(1-\alpha) < 1,$$

or equivalently,

$$\left[(1-\alpha\rho^*) + \frac{1-\pi}{\pi}(1-\alpha)\right]\frac{\beta\eta}{\beta+\eta} < \alpha\rho^*.$$

Applying this to the bracket on the right-hand side of (105), we obtain:

$$\alpha\eta\left(1+\frac{\beta\eta}{\beta+\eta}\right)-\theta\left(\frac{\beta\eta}{\beta+\eta}\right)^2\frac{1}{\rho^*}\left[(1-\alpha\rho^*)+\frac{1-\pi}{\pi}(1-\alpha)\right]>\alpha\left[\eta\left(1+\frac{\beta\eta}{\beta+\eta}\right)-\theta\left(\frac{\beta\eta}{\beta+\eta}\right)\right]$$
$$=\frac{\alpha\eta}{\beta+\eta}\left[(\beta+\eta)+\beta(\eta-\theta)\right]$$
$$=\frac{\alpha\eta}{\beta+\eta}\left[\eta(1+\beta)+\beta(1-\theta)\right]>0.$$

Thus, the sign of the right-hand side of (105) is positive. Hence, $\frac{dr^*}{dn}>0$ holds. Noting that $r=A\alpha k^{\alpha-1}$, $\frac{dk^*}{dn}<0$ is confirmed.

Appendix E

First, we examine $\frac{dV_\chi}{dn}$. Substituting (27) and (29) into the utility function of state-χ, we obtain:

$$V_\chi=(1+\beta)\log w+\beta\log r+\left(\beta\log\beta+(1+\beta)\log\frac{1+q^*}{1+\beta}\right).$$

As we argued in the proof of Proposition 3 (i) that the graphs of (32) and (41) remain unchanged when n rises, and thus that ρ^* and ϕ^* remain unchanged. In this case, from (38), we obtain q^* is kept unchanged as well. That is, $\frac{d\rho^*}{dn}=\frac{d\phi^*}{dn}=\frac{dq^*}{dn}=0$ holds. Taking this into account, we obtain:

$$\frac{dV_\chi}{dn}=\left((1+\beta)\frac{1}{w}\frac{dw}{dk}+\beta\frac{1}{r}\frac{dr}{dk}\right)\frac{dk^*}{dn}.$$

Considering that $\frac{k}{w}\frac{dw}{dk}=\alpha$ and $\frac{k}{r}\frac{dr}{dk}=\alpha-1$ hold from $w=A(1-\alpha)k^\alpha$ and $r=A\alpha k^{\alpha-1}$, we obtain:

$$\frac{dV_\chi}{dn}=\left[(1+\beta)\alpha-(1-\alpha)\beta\right]\frac{1}{k^*}\frac{dk^*}{dn}.$$

Furthermore, let us explore $\frac{1}{k}\frac{dk}{dn}$ in the steady state. Recalling that $\rho\equiv\frac{n}{r}$ and $r=A\alpha k^{\alpha-1}$, we obtain:

$$\log\rho=\log n+(1-\alpha)\log k-\log A\alpha.$$

Differentiating the above equation with respect to n, and noting $\frac{d\rho^*}{dn}=0$, we derive the following:

$$\frac{n}{k^*}\frac{dk^*}{dn}=-\frac{1}{1-\alpha}. \tag{106}$$

Consequently, we obtain the following result:

$$\frac{dV_\chi}{dn} = -\frac{1}{n}\frac{1}{1-\alpha}\left[(1+\beta)\alpha - (1-\alpha)\beta\right]. \tag{107}$$

Here, note that $(1+\beta)\alpha - (1-\alpha)\beta > 0$ holds on the assumption of $\pi^+ < 1$. (Recall that $\pi^+ \equiv \frac{1-\alpha}{\alpha}\frac{\beta}{1+\beta}\frac{1}{\eta}$.) Thus, $\frac{dV_\chi}{dn} < 0$ holds.

Next, we consider $\frac{dU_\kappa}{dn}$, where $U_\kappa \equiv \log c^\kappa + \beta\left[\pi \log d^{\kappa\kappa} + (1-\pi)\log d^{\kappa\chi}\right]$. In the interior solution case, observe that the following equations hold from (25), (22), and (23), respectively:

$$\begin{aligned} c^\kappa &= \phi c^\chi, \\ d^{\kappa\chi} &= \frac{\beta n}{\eta} c^\chi, \\ d^{\kappa\kappa} &= \frac{\beta n}{\eta} \phi c^\chi. \end{aligned}$$

Substituting these into U_κ yields:

$$U_\kappa = (1+\beta)\log c^\chi + \beta \log n + (1+\beta\pi)\log\phi + \beta\log\frac{\beta}{\eta}.$$

Recalling $\frac{d\phi^*}{dn} = 0$, we obtain:

$$\frac{dU_\kappa}{dn} = (1+\beta)\frac{1}{c^\chi}\frac{dc^\chi}{dn} + \beta\frac{1}{n}.$$

Here, we examine $\frac{1}{c^\chi}\frac{dc^\chi}{dn}$ in the steady state. From (27), we obtain:

$$\log c^\chi = \log(1+q) + \log w - \log(1+\beta).$$

Differentiating the above equation with respect to n, and noting that $\frac{dq^*}{dn} = 0$ and $\frac{k}{w}\frac{dw}{dk} = \alpha$ hold, we obtain:

$$\frac{1}{c^\chi}\frac{dc^\chi}{dn} = \frac{1}{w}\frac{dw}{dn} = \frac{1}{w}\frac{dw}{dk}\frac{dk}{dn} = \frac{\alpha}{k}\frac{dk}{dn}.$$

Applying this to the above equation, we derive the following equation:

$$\frac{dU_\kappa}{dn} = \frac{1}{n}\left((1+\beta)\alpha\frac{n}{k}\frac{dk}{dn} + \beta\right).$$

Furthermore, applying (106) to the above equation, we obtain:

$$\frac{dU_\kappa}{dn} = -\frac{1}{n}\frac{1}{1-\alpha}\left[(1+\beta)\alpha - (1-\alpha)\beta\right]. \tag{108}$$

Comparing (107) with (108), we confirm that $\frac{dV_\chi}{dn} = \frac{dU_\kappa}{dn}$ holds.

Appendix F

Consider the behavior of agents (let us say generation j) in the steady state. The budget constraint for a state-κ agent is:

$$
\begin{aligned}
c_j^\kappa + s_j^\kappa &= w, \\
d_{j+1}^{\kappa\chi} + n x_{j+1}^\chi &= r\left(s_j^\kappa + x^\kappa\right), \\
d_{j+1}^{\kappa\kappa} + n x_{j+1}^\kappa &= r\left(s_j^\kappa + x^\kappa\right).
\end{aligned}
$$

Here, suppose that generation-j state-κ agents choose the same steady- state consumption profile as a state-χ agent; that is, $c_j^\kappa = c^\chi$ and $d_{j+1}^{\kappa\chi} = d_{j+1}^{\kappa\kappa} = d^\chi$. In this case, $s_j^\kappa = s^\chi$ holds. Note that such an allocation is feasible for the generation-j state-κ agents because the nonnegative constraint (11) is satisfied; that is:

$$
\begin{aligned}
x_{j+1}^\chi &= \frac{r}{n}\left(s^\chi + x^\kappa - d^\chi\right) = \frac{r}{n}\left(x^\kappa - x^\chi\right) > 0, \\
x_{j+1}^\kappa &= \frac{r}{n}\left(s^\chi + x^\kappa - d^\chi\right) = \frac{r}{n}\left(x^\kappa - x^\chi\right) > 0.
\end{aligned}
$$

The utility of the generation-j state-κ agents in this case, V_j^κ, is expressed as following:

$$
\begin{aligned}
V_j^\kappa &= u\left(c^\kappa\right) + \beta u\left(d^\kappa\right) + v\left(n\right) + \eta\left[\pi V_\kappa\left(x_{j+1}^\kappa\right) + (1-\pi) V_\chi\left(x_{j+1}^\chi\right)\right] \\
&> u\left(c^\kappa\right) + \beta u\left(d^\kappa\right) = V_\chi.
\end{aligned}
$$

The above inequality comes from the nonnegative utility of having children. When the state-κ agents choose their allocation optimally, their welfare, V_κ is no less than V_j^κ; that is, $V_\kappa \geq V_j^\kappa$. Combining this with the above inequality, we obtain $V_\kappa > V_\chi$.

Appendix G

We provide the process of derivation of (66) and (67).

Derivation of (66). Taking the logarithm of (27), and differentiating with respect to π, we obtain:

$$\frac{1}{c^{\chi}}\frac{dc^{\chi}}{d\pi} = \frac{1}{w+x^{\chi}}\frac{d}{d\pi}\left(w+x^{\chi}\right) = \frac{1}{1+\beta}\frac{1}{c^{\chi}}\frac{d}{d\pi}\left(w+x^{\chi}\right).$$

We use (27) to derive the last equality. Noting that the left-hand side of the above equation is σ_c^{χ} by definition, and that $V_{\chi}'\left(x^{\chi}\right) = u'\left(c^{\chi}\right) = \frac{1}{c^{\chi}}$, (66) is obtained:

$$\left(1+\beta\right)\sigma_c^{\chi} = V_{\chi}'\left(x^{\chi}\right)\frac{d}{d\pi}\left(w+x^{\chi}\right).$$

Derivation of (67). From $x^{\chi} = qw$ and (35), the following equation is obtained:

$$w + x^{\kappa} = \left(w+x^{\chi}\right)\left(1+\frac{1}{\eta}\frac{\beta}{1+\beta}\left(1-\phi\right)\right). \tag{109}$$

Taking the logarithm, and differentiating with respect to π, we obtain:

$$\frac{1}{w+x^{\kappa}}\frac{d\left(w+x^{\kappa}\right)}{d\pi} = \frac{1}{w+x^{\chi}}\frac{d\left(w+x^{\chi}\right)}{d\pi} + \frac{-\frac{1}{\eta}\frac{\beta}{1+\beta}}{1+\frac{1}{\eta}\frac{\beta}{1+\beta}\left(1-\phi\right)}\frac{d\phi}{d\pi},$$

and furthermore, applying (109) to the right-hand side of the above equation, the following is derived:

$$\frac{d\left(w+x^{\kappa}\right)}{d\pi} = \left(1+\frac{1}{\eta}\frac{\beta}{1+\beta}\left(1-\phi\right)\right)\frac{d\left(w+x^{\chi}\right)}{d\pi} - \frac{1}{\eta}\frac{\beta}{1+\beta}\left(w+x^{\chi}\right)\frac{d\phi}{d\pi}.$$

Divide both terms in this equation by c^{κ}. Taking into account that $V_{\kappa}'\left(x^{\kappa}\right) = u'\left(c^{\kappa}\right) = \frac{1}{c^{\kappa}} = \frac{1}{c^{\chi}}\frac{1}{\phi} = V_{\chi}'\left(x^{\chi}\right)\frac{1}{\phi}$ holds, the following equation is derived:

$$V_{\kappa}'\left(x^{\kappa}\right)\frac{d\left(w+x^{\kappa}\right)}{d\pi} = \left(1+\frac{1}{\eta}\frac{\beta}{1+\beta}\left(1-\phi\right)\right)\frac{1}{\phi}V_{\chi}'\left(x^{\chi}\right)\frac{d\left(w+x^{\chi}\right)}{d\pi} - \frac{1}{\eta}\frac{\beta}{1+\beta}\frac{w+x^{\chi}}{c^{\chi}}\frac{1}{\phi}\frac{d\phi}{d\pi}.$$

Here, we apply (66) and $c^{\chi} = \frac{1}{1+\beta}\left(w+x^{\chi}\right)$ to the right-hand side of the above equation:

$$\begin{aligned} V_{\kappa}'\left(x^{\kappa}\right)\frac{d\left(w+x^{\kappa}\right)}{d\pi} &= \left(1+\frac{1}{\eta}\frac{\beta}{1+\beta}\left(1-\phi\right)\right)\frac{1+\beta}{\phi}\sigma_c^{\chi} - \frac{\beta}{\eta}\frac{1}{\phi}\frac{d\phi}{d\pi} \\ &= \left(1+\beta\right)\sigma_c^{\chi} + \left(1+\beta+\frac{\beta}{\eta}\right)\frac{1-\phi}{\phi}\sigma_c^{\chi} - \frac{\beta}{\eta}\frac{1}{\phi}\frac{d\phi}{d\pi}. \end{aligned}$$

Note that $\sigma_c^{\kappa} = \sigma_c^{\chi} + \frac{1}{\phi}\frac{d\phi}{d\pi}$ holds from $c^{\kappa} = \phi c^{\chi}$. Applying this to the first term of the right-hand side of the above equation, we obtain:

$$V_{\kappa}'(x^{\kappa})\frac{d(w+x^{\kappa})}{d\pi} = (1+\beta)\sigma_c^{\kappa} + \left(1+\beta+\frac{\beta}{\eta}\right)\frac{1}{\phi}\left((1-\phi)\sigma_c^{\chi} - \frac{d\phi}{d\pi}\right). \tag{110}$$

We obtain the following equation by subtracting the term $\frac{1}{c^{\kappa}}\frac{dc^{\chi}}{d\pi}$ from the both sides of $\sigma_c^{\kappa} = \sigma_c^{\chi} + \frac{1}{\phi}\frac{d\phi}{d\pi}$:

$$\frac{1}{c^{\kappa}}\left(\frac{dc^{\kappa}}{d\pi} - \frac{dc^{\chi}}{d\pi}\right) = \frac{1}{\phi}\frac{d\phi}{d\pi} + \left(1-\frac{c^{\chi}}{c^{\kappa}}\right)\frac{1}{c^{\chi}}\frac{dc^{\chi}}{d\pi}.$$

Apply $V_{\kappa}'(x^{\kappa}) = \frac{1}{c^{\kappa}}$ and $c^{\kappa} = \phi c^{\chi}$ to the left- and right-hand sides, respectively, and we obtain the following equation:

$$V_{\kappa}'(x^{\kappa})\left(\frac{dc^{\kappa}}{d\pi} - \frac{dc^{\chi}}{d\pi}\right) = -\frac{1}{\phi}\left((1-\phi)\sigma_c^{\chi} - \frac{d\phi}{d\pi}\right). \tag{111}$$

Finally, combining (110) with (111), (67) is obtained:

$$(1+\beta)\sigma_c^{\kappa} = V_{\kappa}'(x^{\kappa})\frac{d}{d\pi}(w+x^{\kappa}) - \left(1+\beta+\frac{\beta}{\eta}\right)V_{\kappa}'(x^{\kappa})\left(\frac{dc^{\chi}}{d\pi} - \frac{dc^{\kappa}}{d\pi}\right).$$

Appendix H

First, the process of the calculation to derive (70) is given:

$$T_1\frac{d\rho}{d\pi} + T_2\frac{1}{\pi^2},$$

where:

$$T_1 = \left(\frac{1+\beta}{1-\alpha} - \beta(1-\pi) - \frac{\beta+\eta+\beta\eta}{\eta}(1-\alpha)(1+q)\right)\frac{1}{\rho} - \frac{\beta+\eta+\beta\eta}{\beta+\eta}\alpha(1+\beta\pi)\frac{\pi}{\phi},$$
$$T_2 = (1-\alpha)\beta\left((1+q)\frac{1}{\rho} - \frac{\eta}{\beta+\eta}(1+\beta\pi)\frac{\pi}{\phi}\right).$$

Then, we show that $T_1 > 0$ and $T_2 \geq 0$ hold.

Recall that the levels of q, ϕ, and ρ in the steady state are obtained as the solution of (32), (38), and (41). From (41), we obtain:

$$\frac{d\phi}{d\pi} = -\alpha\frac{\beta+\eta+\beta\eta}{\beta+\eta}\frac{d\rho}{d\pi} - \frac{1}{\pi^2}\frac{\beta\eta}{\beta+\eta}\left(1-\alpha\right). \tag{112}$$

Substitute (41) into (38) to eliminate ϕ, and we obtain:

$$\left(1+q\right)\left(\frac{\beta+\eta+\beta\eta}{\eta}\rho + \frac{1-\pi}{\pi}\beta\right) = \frac{1+\beta}{1-\alpha}\rho. \tag{113}$$

Differentiating (113) with respect to π yields:

$$\frac{dq}{d\pi}\left(\frac{\beta+\eta+\beta\eta}{\eta}\rho + \frac{1-\pi}{\pi}\beta\right) + \left(1+q\right)\left(\frac{\beta+\eta+\beta\eta}{\eta}\frac{d\rho}{d\pi} - \frac{\beta}{\pi^2}\right) = \frac{1+\beta}{1-\alpha}\frac{d\rho}{d\pi}.$$

Applying (113) to the first term of the left-hand side of the above equation, we obtain:

$$\left(1+\beta\right)\frac{1}{1+q}\frac{dq}{d\pi} = \left(\left(1+\beta\right) - \left(1+q\right)\left(1-\alpha\right)\frac{\beta+\eta+\beta\eta}{\eta}\right)\frac{1}{\rho}\frac{d\rho}{d\pi} + \frac{1}{\pi^2}\left(1-\alpha\right)\beta\left(1+q\right)\frac{1}{\rho}. \tag{114}$$

By substituting (112) and (114) into (69), (70) is derived.

Next, let us show that $T_1 > 0$ holds in the interior solution case ($\pi > \pi^+$). $T_1 > 0$ holds if and only if:

$$\left(\beta+\eta+\beta\eta\right)\left(\alpha\frac{1+\beta\pi}{\beta+\eta}\frac{\pi}{\phi} + \frac{\left(1-\alpha\right)\left(1+q\right)}{\eta}\frac{1}{\rho}\right) < \left(\frac{1+\beta}{1-\alpha} - \beta\left(1-\pi\right)\right)\frac{1}{\rho}.$$

Applying (113) to the left-hand side of the above inequality, the following is obtained:

$$\left(\beta+\eta+\beta\eta\right)\left(\alpha\frac{1+\beta\pi}{\beta+\eta}\frac{\pi}{\phi} + \frac{1+\beta}{\left(\beta+\eta+\beta\eta\right)\rho + \frac{1-\pi}{\pi}\beta\eta}\right) < \left(\frac{1+\beta}{1-\alpha} - \beta\left(1-\pi\right)\right)\frac{1}{\rho}. \tag{115}$$

From (41), we observe that $\phi > \left(1+\frac{\beta\eta}{\beta+\eta}\right)\left(1-\alpha\rho\right)$, and taking $\frac{1-\pi}{\pi}\beta\eta > 0$ into account, we immediately confirm that the following is larger than the left-hand side of (115):

$$\left(\beta+\eta+\beta\eta\right)\left(\alpha\frac{1+\beta\pi}{\beta+\eta}\frac{\pi}{\left(1+\frac{\beta\eta}{\beta+\eta}\right)\left(1-\alpha\rho\right)} + \frac{1+\beta}{\left(\beta+\eta+\beta\eta\right)\rho}\right).$$

We observe that this term is equivalent to $\alpha\left(1+\beta\pi\right)\frac{\pi}{1-\alpha\rho} + \frac{1+\beta}{\rho}$. Thus, the following inequality is a sufficient condition for $T_1 > 0$:

$$\alpha\left(1+\beta\pi\right)\frac{\pi}{1-\alpha\rho}+\frac{1+\beta}{\rho}<\left(\frac{1+\beta}{1-\alpha}-\beta\left(1-\pi\right)\right)\frac{1}{\rho}.$$

By solving this with respect to ρ, we express the sufficient condition as follows:

$$\rho<\frac{1}{\alpha}\frac{1}{\frac{(1-\alpha)(1+\beta\pi)\pi}{\alpha(1+\beta)-(1-\alpha)\beta(1-\pi)}+1}. \tag{116}$$

Here, note that $\frac{(1-\alpha)(1+\beta\pi)\pi}{\alpha(1+\beta)-(1-\alpha)\beta(1-\pi)}$ is an increasing function of π, assuming that $\pi^{+}<1$ (cf. Section 3), and that it takes the maximum value $\frac{1-\alpha}{\alpha}$ when $\pi=1$. Thus, we obtain:

$$\frac{\left(1-\alpha\right)\left(1+\beta\pi\right)\pi}{\alpha\left(1+\beta\right)-\left(1-\alpha\right)\beta\left(1-\pi\right)}<\frac{1-\alpha}{\alpha},$$

or equivalently:

$$1<\frac{1}{\alpha}\frac{1}{\frac{(1-\alpha)(1+\beta\pi)\pi}{\alpha(1+\beta)-(1-\alpha)\beta(1-\pi)}+1}.$$

As shown in Proposition 1, $\rho\leq\eta<1$ holds. Hence, (116), a sufficient condition for $T_1>0$, holds true.

Finally, let us show that $T_2\geq 0$ holds in the interior solution case. $T_2\geq 0$ holds if and only if:

$$\frac{\eta}{\beta+\eta}\left(1+\beta\pi\right)\frac{\pi}{\phi}\leq\left(1+q\right)\frac{1}{\rho}. \tag{117}$$

Using (38), (117) can be rewritten as:

$$\frac{\eta}{\beta+\eta}\left(1+\beta\pi\right)\pi\leq\frac{\alpha}{1-\alpha}\left(1+\beta\right)\frac{\phi}{\frac{\beta+\eta}{\eta}\left(1-\phi\right)+\frac{\beta}{\pi}},$$

or equivalently:

$$\left(1+\beta\pi\right)\left(\left(1-\phi\right)\pi+\frac{\beta\eta}{\beta+\eta}\right)\leq\frac{\alpha}{1-\alpha}\left(1+\beta\right)\phi. \tag{118}$$

Recall that ϕ is a decreasing function of π, as discussed in Section 4. Thus, the left-hand side of (118) is an increasing function of π, while the right-hand side of (118) is a decreasing function of π. Hence, if (118) holds at $\pi=1$, it is guaranteed that (118) holds true for any $\pi\in[\pi^{+},1]$. Define the level of ϕ at $\pi=1$ as $\phi_{\min}$. When $\pi=1$, (118) is expressed as:

$$1 + \frac{\beta\eta}{\beta + \eta} \leq \frac{1}{1-\alpha}\phi_{\min}. \tag{119}$$

Recall that Proposition 1 (i) argues that $\rho = \eta$ holds when $\pi = 1$. Taking (41) into account, $\phi_{\min}$ is derived as:

$$\phi_{\min} = \left(1 + \frac{\beta\eta}{\beta + \eta}\right)(1 - \alpha\eta).$$

Because $\eta < 1$, (119) holds, and thus, (118) holds for any $\pi \in [\pi^+, 1]$. Therefore, $T_2 \geq 0$ holds.

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References

[1] Aaronson, D., F. Lange, and B. Mazumder, 2014, "Fertility Transitions Along the Extensive and Intensive Margins," American Economic Review, 104(11), 3701-24.

[2] Barro, R.J. and G.S. Becker, 1989, "Fertility Choice in a Model of Economic Growth," Econometrica, 57(2), 481-501.

[3] Baudin, T., D. de la Croix, and P. Gobbi, 2015, "Fertility and Childlessness in the United States," American Economic Review, forthcoming.

[4] Becker, G.S. and R.J. Barro, 1988, "A Reformulation of the Economic Theory of Fertility," Quarterly Journal of Economics, 103(1), 1-25.

[5] Blanchard O.J. and S. Fischer, 1989, "Lectures on Macroeconomics," MIT Press, Cambridge, MA.

[6] Cabinet Office, 2005, "Perceptions and Lifestyle of the Child-rearing Generation," White Paper on the National Lifestyle 2005, Cabinet Office, Government of Japan.

[7] d'Albis, H., E. Augeraud-Véron, and K. Schubert, 2010, "Demo-economic Equilibria when Child-bearing Age is Endogenous," Journal of Mathematical Economics, 44(6), 1211-21.

[8] Deardorff A.V., 1976, "The Optimum Growth Rate for Population: Comment," International Economic Review, 17(2), 510-515.

[9] de la Croix, D., P. Pestieau, and G. Ponthière, 2012, "How Powerful is Demography? The Serendipity Theorem Revisited," Journal of Population Economics, 25(3), 899-922.

[10] Diamond, P.A., 1965, "National Debt in a Neoclassical Growth Model," American Economic Review, 55(5), 1126–50.

[11] Doepke, M., 2008, "Growth Takeoffs," In: Durlauf, S.N., Blume, L. (Eds.), The New Palgrave Dictionary of Economics, Second ed., Palgrave MacMillan, New York.

[12] Galor, O., 2005, "From Stagnation to Growth: Unified Growth Theory," In: Aghion, P., Durlauf, S. (Eds.), Handbook of Economic Growth, Elsevier, Norh-Holland, Amsterdam, 171–293.

[13] Gobbi, P.E., 2013, "A Model of Voluntary Childlessness," Journal of Population Economics, 26(3), 963-82.

[14] Lucas, R.E., 2002, "The Industrial Revolution: Past and Future," In: Lectures on Economic Growth, Harvard University Press, Cambridge, MA.

[15] Michel, P. and P. Pestieau, 1993, "Population Growth and Optimality: When Does the Serendipity Hold?" Journal of Population Economics, 6(4), 353-362.

[16] Michel, P. and P. Pestieau, 1998, "Fiscal Policy in a Growth Model with Both Altruistic and Nonaltruistic Agents," Southern Economic Journal, 64(3), 682-97.

[17] Michel, P. and P. Pestieau, 2005, "Fiscal Policy with Agents Differing in Altruism and Ability," Economica, 72(285), 121-35.

[18] Momota, A. and R. Horii, 2013, "Timing of Childbirth, Capital Accumulation, and Economic Welfare," Oxford Economic Papers, 65(2), 494-522.

[19] Monstad K., C. Propper, and K.G. Salvanes, 2008, "Education and Fertility: Evidence from a Natural Experiment," Scandinavian Journal of Economics, 110(4), 827-52.

[20] National Institute of Population and Social Security Research, 2006, "Nihon no Shourai Jinkou Suikei: Heisei 18 Nen 12 Gatsu Suikei (in Japanese) [Population Projection of Japan in December, 2006]," National Institute of Population and Social Security Research, Tokyo.

[21] OECD, 2014, "OECD Family Database," OECD, Paris (www.oecd.org/social/family/database.htm).

[22] Razin, A., and U. Ben-Zion, 1975, "An Intergenerational Model of Population Growth," American Economic Review, 65(5), 923-933.

[23] Samuelson P.A., 1975, "The Optimum Growth Rate for Popluation," International Economic Review, 16(3), 531-538.

Table 1: Definitive Childlessness.

Country	Women who were born in	
	1950	1965
Austria	12.6	21.1
Bulgaria	1.3	4.4
Czech Republic	6.6	7.2
England and Wales	13.9	20.5
Germany (Western)	14.8	20.3*
Greece	9.6	16.3
Hungary	9.1	9.6
Ireland	12.2	18.4
Italy	13.0	20.1**
Japan	18.4	24.9
Netherlands	14.6	18.3
Norway	10.0	12.1
Poland	9.5	15.5
Portugal	11.0	4.0
Romania	6.3	11.5
Slovak Republic	9.8	11.1
Slovenia	4.4	9.0
Sweden	13.9	12.9
United States	15.1	14.4

Note: The superscript `*' indicates the childlessness rate of the 1955 cohort.
The superscript `**' indicates the childlessness rate of the 1964 cohort.

Sources : Data except Japan are obtained from OECD (2014).
Data for Japan are obtained from the National Institute of Population and Social Security Research (2006).

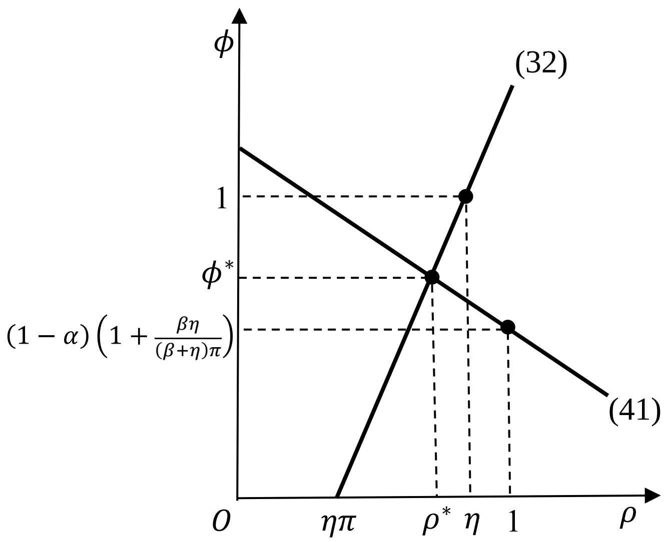
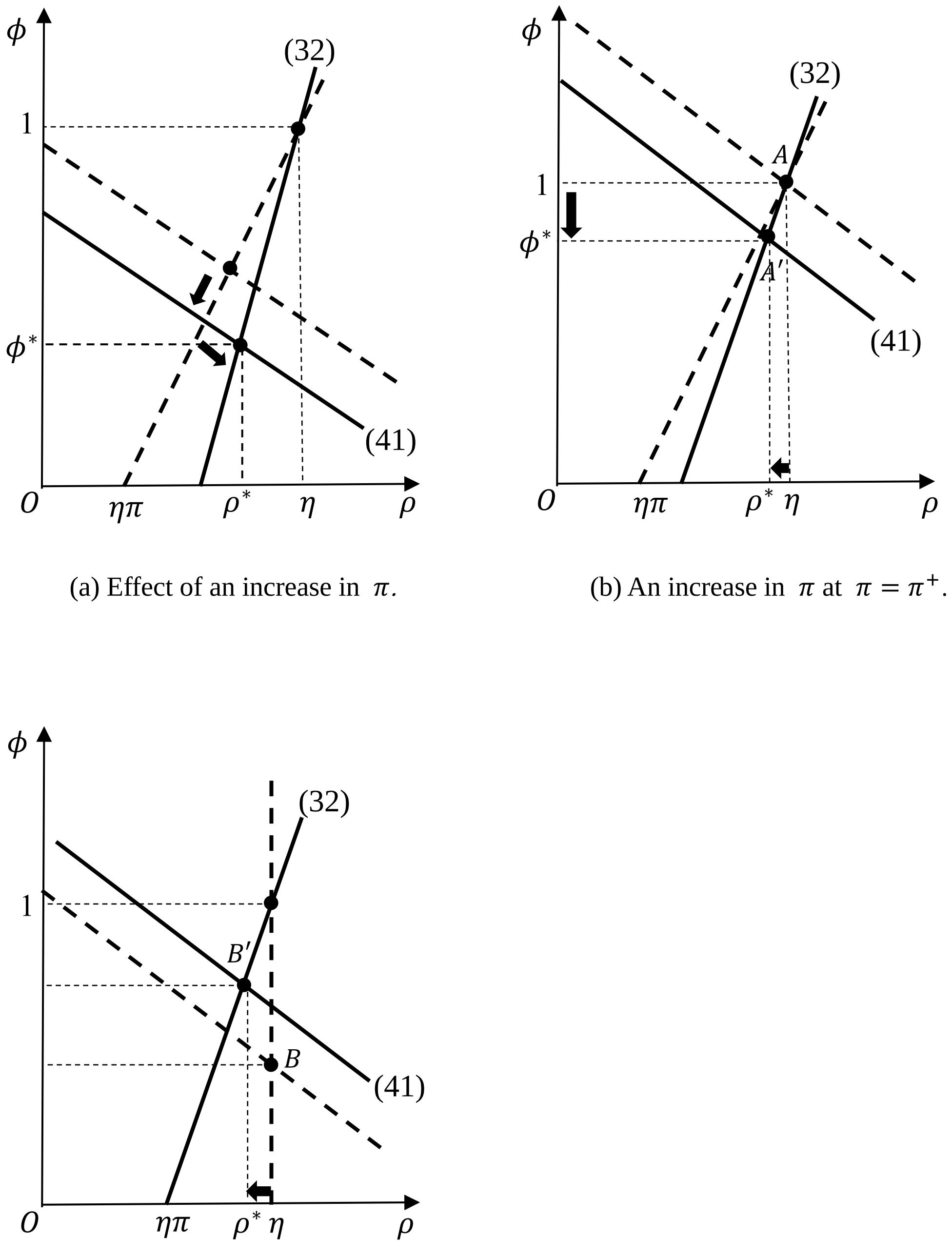


Fig. 1: Graph of (32) and (41).



(a) Effect of an increase in π.

(b) An increase in π at $\pi = \pi^+$.

(c) A decrease in π at $\pi = 1$.

Fig. 2: Effect of a change in π on steady state.

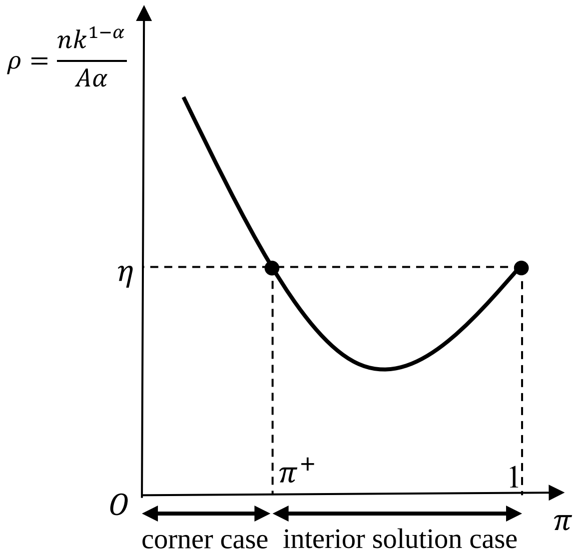


Fig. 3: Relationship between π and k.

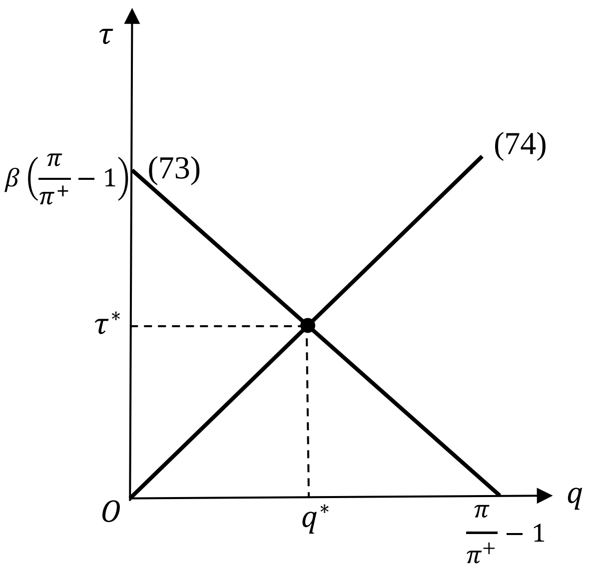


Fig. 4: Optimal income transfer τ^*.

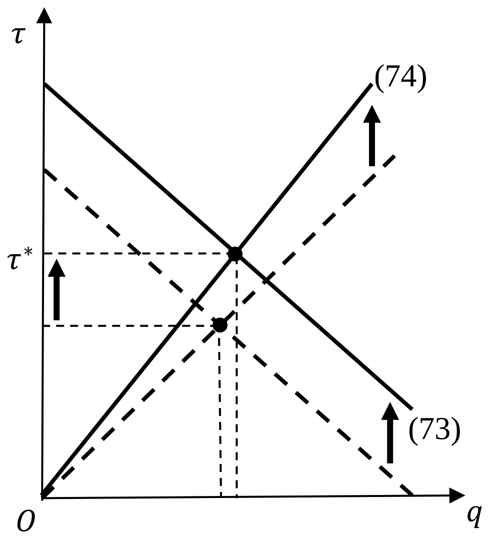


Fig. 5: Effect of an increase in $\boldsymbol{\pi}$ ***on*** $\boldsymbol{\tau^*}$**.**

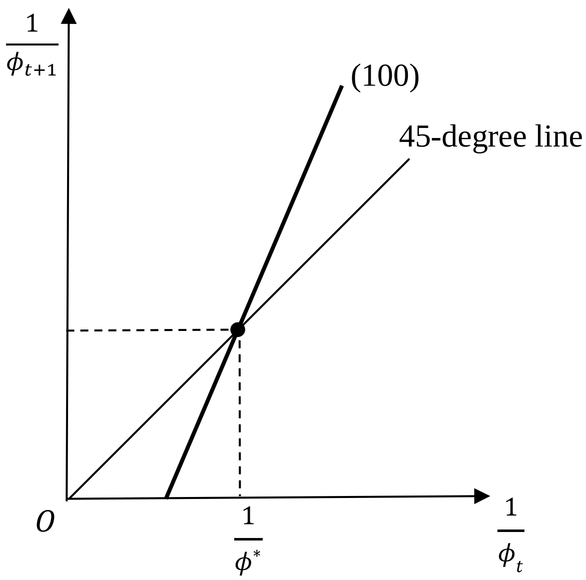


Fig. 6: Transitional dynamics.