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Yang–Baxter sigma models based on the CYBE

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Abstract

It is known that Yang–Baxter sigma models provide a systematic way to study integrable deformations of both principal chiral models and symmetric coset sigma models. In the original proposal and its subsequent development, the deformations have been characterized by classical r-matrices satisfying the modified classical Yang–Baxter equation (mCYBE). In this article, we propose the Yang–Baxter sigma models based on the classical Yang–Baxter equations (CYBE) rather than the mCYBE. This generalization enables us to utilize various kinds of solutions of the CYBE to classify integrable deformations. In particular, it is straightforward to realize partial deformations of the target space without loss of the integrability of the parent theory.

1. Introduction

In the recent years, it has been an intriguing subject to consider integrable deformations of type IIB superstring on the $AdS_5 \times S^5$ background (which is often abbreviated to the $AdS_5 \times S^5$ superstring). Such directions would be important to reveal the underlying common dynamics of the AdS/CFT correspondence [1].

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In the duality between the $\mathrm{AdS}_5 \times \mathrm{S}^5$ superstring and the $\mathcal{N} = 4$ super Yang–Mills theory in four dimensions, the integrability is recognized to play important roles (for a comprehensive review, see [2]). On the string-theory side, the classical integrability was argued by constructing the Lax connection on the world-sheet [3]. The existence of the Lax pair is based on the $\mathbb{Z}_4$-grading property of the supercoset of the $\mathrm{AdS}_5 \times \mathrm{S}^5$ superstring action [4]. On the gauge-theory side, it was shown that the one-loop planar dilatation operator for composite operators of adjoint scalar fields is mapped to an $SO(6)$ integrable spin-chain Hamiltonian [5]. This was only the tip of an iceberg of the whole integrable structure behind the AdS/CFT correspondence.

While, the underlying fundamental principle of the AdS/CFT duality has not been completely understood yet. In order to gain further insights for this issue, we expect that deformations of the integrable structures appearing in the AdS/CFT would be a clue to reveal it. For instance, introducing some deformation parameters enables us to take various limits of them. By doing so, we are able to discuss a family of dualities with keeping the integrable structure. One of such attempts is a q-deformation of the dynamical spin-chain model, which possesses a centrally extended $\mathfrak{su}(2|2)$ symmetry [6]. The whole symmetry algebra is naturally enhanced to an infinite-dimensional quantum affine algebra [7].

From the AdS/CFT point of view, it is desirable to figure out similar integrable deformations of the $\mathrm{AdS}_5 \times \mathrm{S}^5$ string action. However, in comparison to deformations of a quantum integrable model with a manifest algebraic symmetry, it is not so obvious to realize those of an integrable classical filed theory. This difficulty has been quite alleviated by a notion of the *Yang–Baxter sigma models* introduced by Klimcik, which are integrable deformations of principal chiral models [8–10]. In this description, integrable deformations are characterized by R-operators satisfying the *modified* classical Yang–Baxter equations (mCYBE). After these remarkable works, Delduc, Magro and Vicedo generalized the formulation to purely bosonic coset sigma models [11] and finally succeeded to an integrable deformation of the $\mathrm{AdS}_5 \times \mathrm{S}^5$ string action [12,13]. This deformation could be regarded as a classical analogue of q-deformation of Lie algebras [14–16] because it is based on the Drinfeld–Jimbo type classical r-matrix and, indeed, the Poisson symmetry algebra is a quantum group associated with the superconformal algebra, $\mathcal{U}_q(\mathfrak{psu}(2,2|4))$ [13]. The resulting deformed metric and NS–NS two-form are explicitly derived in [17].

On the other hand, an alternative integrable deformation of $\mathrm{AdS}_5 \times \mathrm{S}^5$ has been proposed in [18], where deformations are characterized by classical r-matrices satisfying the classical Yang–Baxter equation (CYBE) rather than the mCYBE.[1] Hence, this is a natural generalization of Yang–Baxter deformations of the $\mathrm{AdS}_5 \times \mathrm{S}^5$ superstring [12,13]. There are two important aspects of the deformations based on the CYBE in comparison to the mCYBE. Firstly, the CYBE allows various kinds of the skew-symmetric constant solutions, which are convenient in studying the deformations. Secondary, it is possible to consider partial deformations of the target space. The two properties of the CYBE are significant in terms of applicability because the present formulation provides a systematic way to investigate a large class of integrable deformations by using solutions of the CYBE.

In fact, as remarkable examples of deformations of CYBE-type, the Lunin–Maldacena–Frolov backgrounds [21,22] and the gravity duals of non-commutative gauge theories [23,24] have been successfully recovered [25,26]. The classical integrability of these backgrounds automatically follows from the construction. Based on these observations, it seems likely that the space of

[1] Another type of deformation is also proposed in [19,20].

solutions of the CYBE may be identified with the moduli space of a certain class of solutions of type IIB supergravity. It is referred as to the *gravity/CYBE correspondence* (for a short review, see [27]). In fact, the full supergravity solution [28] associated with a deformation argued in [18] is also obtained by a chain of string dualities such as TsT-transformations and S-dualities [29]. The deformation technique is also applicable to a Sasaki–Einstein manifold $T^{1,1}$, because it is described as a coset [30]. The resulting deformed background agrees with the one obtained in [21,31].

In this article, inspired by these developments in the $\mathrm{AdS}_5 \times \mathrm{S}^5$ superstring, we will show that integrable deformations based on the CYBE work also for bosonic principal chiral models. More precisely, our main statement is the following. Let an R-operator R be a solution of the CYBE. Then the classical action of the deformed Yang–Baxter sigma model is given by

$$S = -\frac{1}{2}\int_{-\infty}^{\infty} d\tau \int_0^{2\pi} d\sigma \,\mathrm{Tr}\left((g^{-1}\partial_- g)\frac{1}{1-\eta R}(g^{-1}\partial_+ g)\right).$$

Here g is a G-valued function on the string world-sheet with the coordinates τ and σ, where G is a Lie group and $\partial_\pm = \partial_\tau \pm \partial_\sigma$ are derivatives with respect to the light-cone coordinates. The parameter η measures the associated deformation. Note that the model is reduced to a principal chiral model when $\eta = 0$. Then, this deformed model has the following Lax pair:

$$\mathcal{L}_\pm(\lambda) = \frac{1}{1\pm\lambda}\left(1 - \frac{\lambda\eta R}{1\pm\eta R}\right)(g^{-1}\partial_\pm g), \tag{1.1}$$

where $\lambda \in \mathbb{C}$ is a spectral parameter. Thus, the resulting deformed model is also classically integrable. A similar generalization is also possible for bosonic coset sigma models, as explained in the main part of this article.

This article is organized as follows. In Section 2, we generalize Yang–Baxter sigma models and define deformed principal chiral models based on the CYBE. The Lax pair is also presented. In Section 3, we argue a similar generalization for coset sigma models. In Section 4, we explain a multi-parameter generalization of CYBE-type. Section 5 is devoted to conclusion and discussion. Appendix A explains in detail a derivation of equation of motion of the deformed coset sigma models. We also present some examples of mCYBE-type deformations in Appendix B.

2. Integrable deformations of principal chiral models

In this section we shall discuss integrable deformations of principal chiral models. In Section 2.1, after recalling the definition of the Yang–Baxter sigma models based on the modified classical Yang–Baxter equation (mCYBE) [8,9], we show that the formulation can naturally be generalized for the standard classical Yang–Baxter equation (CYBE). Then the Lax pair is explicitly presented. Finally we demonstrate an example in Section 2.2.

2.1. Yang–Baxter sigma models

Definition of the models Let G be a Lie group and $\mathfrak{g}$ be the associated Lie algebra. The action of the Yang–Baxter sigma models introduced by [8–10] is given by

$$S = -\frac{1}{2}(\gamma^{\alpha\beta} - \epsilon^{\alpha\beta})\int_{-\infty}^{\infty} d\tau \int_0^{2\pi} d\sigma \,\mathrm{Tr}\left(A_\alpha \frac{1}{1-\eta R} A_\beta\right). \tag{2.1}$$

Here $\xi^\alpha = (\xi^\tau, \xi^\sigma) = (\tau, \sigma)$ are coordinates of the two-dimensional world-sheet. We will work with the flat metric $\gamma^{\alpha\beta} = \mathrm{diag}(-1, 1)$ in the conformal gauge. Then $\epsilon^{\alpha\beta}$ is the skew-symmetric tensor normalized as $\epsilon^{\tau\sigma} = -\epsilon^{\sigma\tau} = 1$. The left-invariant one-form $A_\alpha = g^{-1}\partial_\alpha g$ is written in terms of $g(\tau, \sigma) \in G$ and A_α is a $\mathfrak{g}$-valued function. The trace is computed over the fundamental representation of $\mathfrak{g}$. A constant parameter η measures a deformation of the model. When $\eta = 0$, the action (2.1) is nothing but that of the G-principal chiral models.

An important ingredient is a classical r-matrix denoted by R, which is an $\mathbb{R}$-linear operator $R : \mathfrak{g} \to \mathfrak{g}$ and satisfies the (modified) classical Yang–Baxter equations ((m)CYBE):

$$[R(X), R(Y)] - R([R(X), Y] + [X, R(Y)]) = \omega[X, Y] \qquad \text{with} \qquad \omega = \pm 1, 0, \tag{2.2}$$

where $X, Y \in \mathfrak{g}$. When $\omega = 1$ (or -1), the R-operator satisfying (2.2) is called *non-split* (or *split*) type (respectively). In particular, when $\omega = 0$, Eq. (2.2) is reduced to the classical Yang–Baxter equation (CYBE). We assume that the R-operator does not depend on the spectral parameter and it is skew-symmetric:

$$\mathrm{Tr}(R(X)Y) = -\mathrm{Tr}(XR(Y)) \qquad \text{for} \qquad X, Y \in \mathfrak{g}. \tag{2.3}$$

Note that a classical r-matrix in the tensorial notation is associated with an R-operator by tracing out the second entry,

$$R(X) = \mathrm{Tr}_2[r_{12}(1 \otimes X)] \equiv \sum_i \big(a_i \,\mathrm{Tr}(b_i X) - b_i \,\mathrm{Tr}(a_i X)\big) \qquad \text{for} \qquad X \in \mathfrak{g}, \tag{2.4}$$

where the r-matrix is denoted symbolically as

$$r_{12} = \sum_i a_i \wedge b_i \equiv \sum_i \big(a_i \otimes b_i - b_i \otimes a_i\big) \qquad \text{with} \qquad a_i, b_i \in \mathfrak{g}. \tag{2.5}$$

For later convenience, we introduce the light-cone expressions of A_α like

$$A_\pm = A_\tau \pm A_\sigma. \tag{2.6}$$

With these notations, the Lagrangian of the action (2.1) is recast into a simple form:

$$L = \frac{1}{2}\mathrm{Tr}\,(A_- J_+) = \frac{1}{2}\mathrm{Tr}\,(A_+ J_-) \qquad \text{where} \qquad J_\pm := \frac{1}{1 \mp \eta R} A_\pm. \tag{2.7}$$

Equation of motion To obtain the equation of motion, let us take a variation of the Lagrangian (2.7). Defining a variation of $g \in G$ as $\delta g = g\epsilon$ with an infinitesimal parameter ϵ, the following relation is derived:

$$\delta A_\alpha = \partial_\alpha \epsilon + [A_\alpha, \epsilon]. \tag{2.8}$$

Then, the variation of (2.7) is evaluated as $\delta L = -\mathrm{Tr}(\mathcal{E}\epsilon)$ with $\mathcal{E}$ defined by

$$\mathcal{E} := \partial_+ J_- + \partial_- J_+ - \eta([R(J_+), J_-] + [J_+, R(J_-)]). \tag{2.9}$$

Thus, the equation of motion turns out to be $\mathcal{E} = 0$.

Zero-curvature condition The next task is to rewrite the zero-curvature condition of $A_\pm = g^{-1}\partial_\pm g$ in terms of the deformed current $J_\pm$. For this purpose, let us introduce the following quantity:

$$\mathcal{Z} := \partial_+ A_- - \partial_- A_+ + [A_+, A_-]. \tag{2.10}$$

By definition of A_α, the zero-curvature condition is nothing but $\mathcal{Z} = 0$. Plugging $A_\pm = (1 \mp \eta R) J_\pm$ with the above definition (2.10), we obtain the following expression,

$$\begin{aligned}\mathcal{Z} = {} & \partial_+ J_- - \partial_- J_+ - \eta([R(J_+), J_-] - [J_+, R(J_-)]) \\ & + [J_+, J_-] - \eta^2 \mathrm{YBE}(J_+, J_-) + \eta R(\mathcal{E})\end{aligned} \tag{2.11}$$

where the left-hand-side of the CYBE (2.2) is denoted as

$$\mathrm{YBE}(X, Y) := [R(X), R(Y)] - R([R(X), Y] + [X, R(Y)]). \tag{2.12}$$

When the R-operator is a solution of the (m)CYBE (2.2), the quantity $\mathcal{Z}$ becomes

$$\begin{aligned}\mathcal{Z} = {} & \partial_+ J_- - \partial_- J_+ - \eta([R(J_+), J_-] - [J_+, R(J_-)]) \\ & + (1 - \eta^2 \omega)[J_+, J_-] + \eta R(\mathcal{E}).\end{aligned} \tag{2.13}$$

Lax pair A novel feature of the Yang–Baxter sigma models (2.1) is that there exists the Lax pair,

$$\mathcal{L}_\pm(\lambda) = \left(\frac{1 \mp \eta^2 \omega \lambda}{1 \pm \lambda} \mp \eta R\right) J_\pm = \frac{1}{1 \pm \lambda}\left(1 \mp \frac{\eta\lambda(\eta\omega \pm R)}{1 \pm \eta R}\right) A_\pm \tag{2.14}$$

with a spectral parameter $\lambda \in \mathbb{C}$. Indeed, the equation of motion $\mathcal{E} = 0$ and the zero-curvature condition $\mathcal{Z} = 0$ are equivalent to the flatness condition of the Lax pair $\mathcal{L}_\pm(\lambda)$:

$$\partial_+ \mathcal{L}_-(\lambda) - \partial_- \mathcal{L}_+(\lambda) + [\mathcal{L}_+(\lambda), \mathcal{L}_-(\lambda)] = 0. \tag{2.15}$$

Thus, the models defined in (2.1) are classically integrable in the sense of kinematical integrability.

It should be emphasized that the Lax pair exists not only for $\omega = 1$ [8,9] but also for $\omega = -1$ (*split*-type) and $\omega = 0$ (CYBE). In particular, when the R-operator satisfies the CYBE with $\omega = 0$, the Lax pair in (2.14) turns out to be

$$\mathcal{L}_\pm(\lambda) = \left(\frac{1}{1 \pm \lambda} \mp \eta R\right) J_\pm = \frac{1}{1 \pm \lambda}\left(1 - \frac{\lambda \eta R}{1 \pm \eta R}\right) A_\pm. \tag{2.16}$$

This is nothing but the Lax connection given in (1.1). Note that, when $\eta = 0$, it reduces to the well-known Lax pair [32] of principal chiral models,

$$\mathcal{L}_\pm(\lambda) = \frac{A_\pm}{1 \pm \lambda}. \tag{2.17}$$

Let us prove the relation (2.15) is equivalent with the equation of motion $\mathcal{E} = 0$ and the zero-curvature condition $\mathcal{Z} = 0$ by a constructive way. In order to do this, we adopt the following ansatz (see also [8,9]):

$$\mathcal{L}_\pm(\lambda) = \left(\frac{F \pm G\lambda}{1 \pm \lambda} \mp \eta R\right) J_\pm. \tag{2.18}$$

Here it is supposed that the unknown variables F and G do not depend on neither spectral parameter λ nor the world-sheet coordinates (τ, σ). Under the ansatz (2.18), the flatness condition (2.15) can be rewritten as

$$\begin{aligned} 0 &= \partial_+ \mathcal{L}_-(\lambda) - \partial_- \mathcal{L}_+(\lambda) + [\mathcal{L}_+(\lambda), \mathcal{L}_-(\lambda)] \\ &= \frac{1}{1-\lambda^2}\big(F\mathcal{Z} + (F-1)(1+\eta^2\omega - \eta R(\mathcal{E}))[J_+, J_-]\big) + \frac{\lambda}{1-\lambda^2}(F-G)\mathcal{E} \\ &\quad - \frac{\lambda^2}{1-\lambda^2}\big(G\mathcal{Z} + (G-1)(G+\eta^2\omega - \eta R(\mathcal{E}))[J_+, J_-]\big). \end{aligned} \tag{2.19}$$

Provided that the equation of motion $\mathcal{E} = 0$ and the zero-curvature condition $\mathcal{Z} = 0$ hold, the possible values of F and G are obtained as follows:

$$F = 1 \qquad \text{and} \qquad G = 1,\ -\eta^2\omega. \tag{2.20}$$

Note that, when $F = G = 1$, the ansatz (2.18) is nothing but $A_\pm$ itself. Thus, the Lax pair with λ corresponds to the case that $F = 1$ and $G = -\eta^2\omega$. In fact, it agrees with the Lax pair presented in (2.14).

Inversely, given that the Lax pair (2.14) is flat, one can readily see that $\mathcal{E} = \mathcal{Z} = 0$. Thus, this completes the proof.

2.2. Example: 3D Schrödinger sigma models

Let us see an example of the deformed principal chiral models in (2.1). This is a deformation of $SL(2;\mathbb{R}) \simeq \mathrm{AdS}_3$ based on an r-matrix satisfying the CYBE with $\omega = 0$ rather than the mCYBE.[2] This model is defined on a three-dimensional Schrödinger spacetime [33–35] and hence it is often called 3D *Schrödinger sigma model*. The classical integrable structure of this model has been discussed in [36,37].

Let E, F, H be the generators of $\mathfrak{sl}(2;\mathbb{R})$ satisfying the relations,

$$[F, E] = H, \qquad [H, E] = E, \qquad [H, F] = -F. \tag{2.21}$$

With th generators, it is easy to see that the following classical r-matrix,

$$r_{12} = H \wedge F = H \otimes F - F \otimes H, \tag{2.22}$$

satisfies the CYBE in (2.2) with $\omega = 0$. We refer the r-matrix of this type as to *Jordanian-type* because it has non-zero Cartan charges. Then, we will find that the r-matrix (2.22) yields sigma models defined on the 3D Schrödinger geometry:

$$ds^2 = \frac{-2dx^+dx^- + dz^2}{z^2} - \frac{\eta^2(dx^+)^2}{4z^4}, \tag{2.23}$$

where η is a deformation parameter. Note that the scalar curvature of this metric is equal to that of AdS_3, namely,

$$R = -6. \tag{2.24}$$

In order to derive the deformed metric (2.23) from the Jordanian r-matrix (2.22), we use the fundamental representation of $\mathfrak{sl}(2;\mathbb{R})$,

[2] For an example of the mCYBE-type deformation, see Appendix B.1.

$$E = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad F = \frac{-1}{\sqrt{2}}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad H = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.25}$$

Using these matrices, we parametrize an $SL(2;\mathbb{R})$ group element as

$$g = \mathrm{e}^{2x^+E}\mathrm{e}^{2(\log z)H}\mathrm{e}^{2x^-F}. \tag{2.26}$$

Then, the left-invariant one-form $A_\pm = g^{-1}\partial_\pm g$ is evaluated as

$$A_\alpha = A_\alpha^E F + A_\alpha^F E + A_\alpha^H H \tag{2.27}$$

with the coefficients

$$\begin{aligned} A_\pm^E &= 2\partial_\pm x^- + \frac{4x^-(x^-\partial_\pm x^+ - z\partial_\pm z)}{z^2}, \\ A_\pm^F &= \frac{2\partial_\pm x^+}{z^2}, \\ A_\pm^H &= \frac{2(z\partial_\pm z - 2x^-\partial_\pm x^+)}{z^2}. \end{aligned} \tag{2.28}$$

Next, to compute the deformed current $J_\pm$, it is necessary to figure out the action of the linear R-operator associated with the Jordanian r-matrix (2.22). Taking the trace over the second sites, the R-operator and its transformation law turn out to be

$$\begin{aligned} & R(X) = \mathrm{Tr}_2[r_{12}(1\otimes X)] \qquad \text{for} \qquad X \in \mathfrak{sl}(2;\mathbb{R}), \\ \Longrightarrow \quad & R(E) = -\frac{1}{2}H, \qquad R(H) = -\frac{1}{2}F, \qquad R(F) = 0. \end{aligned} \tag{2.29}$$

In particular, note that the R-operator is nilpotent; $R^3 = 0$. This property enables us to compute the current explicitly as follows:

$$\begin{aligned} J_\pm &= \frac{1}{1\mp\eta R}A_\pm = (1 \pm \eta R + \eta^2 R^2)A_\pm \\ &= \left(A_\pm^E \mp \frac{\eta}{2}A_\pm^H + \frac{\eta^2}{4}A_\pm^F\right)F + \left(A_\pm^H \mp \frac{\eta}{2}A_\pm^F\right)H + A_\pm^F E. \end{aligned} \tag{2.30}$$

Finally, one can rewrite the Lagrangian as

$$\begin{aligned} L &= \frac{1}{2}\mathrm{Tr}[A_- J_+] \\ &= -\gamma^{\alpha\beta}\left(\frac{-2\partial_\alpha x^+\partial_\beta x^- + \partial_\alpha z\partial_\beta z}{z^2} - \frac{\eta^2\partial_\alpha x^+\partial_\beta x^+}{4z^4}\right) - \eta\epsilon^{\alpha\beta}\partial_\alpha x^+\partial_\beta\left(\frac{1}{2z^2}\right). \end{aligned} \tag{2.31}$$

The last term coupled with the anti-symmetric tensor is total derivative, and hence it can be omitted. Indeed, this is nothing but a non-linear sigma model defined on the 3D Schrödinger spacetime (2.23). The classical integrability of this model follows automatically because the Lax pair is explicitly obtained by plugging (2.29) with the expression (2.16).

It would be interesting to try to reveal the relation between the above construction and the coset construction argued in [38]. The symmetric two-form discussed in [38] would possibly be related to the classical r-matrix (2.22).

3. Integrable deformations of coset sigma models

In the next, we will extend the previous argument on deformed principal chiral models to purely bosonic sigma models defined on symmetric cosets. This is a natural generalization of [11].[3]

3.1. Yang–Baxter deformations of symmetric cosets

Symmetric cosets Recall first the definition of symmetric cosets. Let G be a Lie group and H be a subgroup of G. The associated Lie algebras of G and H are denoted by $\mathfrak{g}$ and $\mathfrak{g}^{(0)}$, respectively. The Lie algebra $\mathfrak{g}$ is a direct sum of $\mathfrak{g}^{(0)}$ and its complementary space $\mathfrak{g}^{(1)}$ as a vector space:

$$\mathfrak{g} = \mathfrak{g}^{(0)} \oplus \mathfrak{g}^{(1)}. \tag{3.1}$$

Then, the homogeneous space G/H is called *symmetric space* if $\mathfrak{g}^{(0)}$ and $\mathfrak{g}^{(1)}$ satisfy the following $\mathbb{Z}_2$-grading property,

$$[\mathfrak{g}^{(0)}, \mathfrak{g}^{(0)}] \subset \mathfrak{g}^{(0)}, \qquad [\mathfrak{g}^{(0)}, \mathfrak{g}^{(1)}] \subset \mathfrak{g}^{(1)}, \qquad [\mathfrak{g}^{(1)}, \mathfrak{g}^{(1)}] \subset \mathfrak{g}^{(0)}. \tag{3.2}$$

The pair $(\mathfrak{g}^{(0)}, \mathfrak{g}^{(1)})$ satisfying the above property is often referred as to *symmetric pair*. It is convenient to introduce a projector to the subspace $\mathfrak{g}^{(1)}$ by

$$P: \ \mathfrak{g} \longrightarrow \mathfrak{g}^{(1)}. \tag{3.3}$$

Definition of the coset models Yang–Baxter deformations of symmetric coset sigma models have been introduced in [11] and the action is given by

$$S = -\frac{1}{2}(\gamma^{\alpha\beta} - \epsilon^{\alpha\beta}) \int_{-\infty}^{\infty} d\tau \int_{0}^{2\pi} d\sigma \ \mathrm{Tr}\left(A_\alpha P \frac{1}{1 - \eta R_g \circ P} A_\beta\right). \tag{3.4}$$

The coset projector P is given in (3.3) and the dressed R-operator R_g is defined by

$$R_g(X) \equiv g^{-1} R(g X g^{-1}) g \qquad \text{with} \qquad g \in G. \tag{3.5}$$

Here η is a deformation parameter. The action is reduced to that of the undeformed coset sigma model when $\eta = 0$.

By using the light-cone notation (2.6), the Lagrangian can be rewritten as

$$L = \frac{1}{2}\mathrm{Tr}\big(A_- P(\tilde{J}_+)\big) = \frac{1}{2}\mathrm{Tr}\big(A_+ P(\tilde{J}_-)\big), \tag{3.6}$$

where we have introduced the deformed current,

$$\tilde{J}_\pm := \frac{1}{1 \mp \eta P \circ R_g} A_\pm. \tag{3.7}$$

Equation of motion The equation of motion of the model (3.6) is given by $\tilde{\mathcal{E}} = 0$ where

$$\tilde{\mathcal{E}} := \partial_+ P(\tilde{J}_-) + \partial_- P(\tilde{J}_+) + [\tilde{J}_+, P(\tilde{J}_-)] + [\tilde{J}_-, P(\tilde{J}_+)]. \tag{3.8}$$

For the detail of the derivation, see Appendix A.

[3] For an example of the mCYBE-type deformation of symmetric cosets, see Appendix B.2.

Zero-curvature condition By definition of the left-invariant one-form $A_\alpha = g^{-1}\partial_\alpha g$, it satisfies the zero-curvature condition $\tilde{\mathcal{Z}} = 0$, where

$$\tilde{\mathcal{Z}} := \partial_+ A_- - \partial_- A_+ + [A_+, A_-]. \tag{3.9}$$

Plugging the relation $A_\pm = (1 \mp \eta P \circ R_g)\tilde{J}_\pm$ with the above expression, one can recast it into the following form,

$$\tilde{\mathcal{Z}} = \partial_+ \tilde{J}_- - \partial_- \tilde{J}_+ + [\tilde{J}_+, \tilde{J}_-] + \eta R_g(\tilde{\mathcal{E}}) + \eta^2 \,\mathrm{YBE}_g(P(\tilde{J}_+), P(\tilde{J}_-)). \tag{3.10}$$

Here we have used the bookkeeping notation:

$$\mathrm{YBE}_g(X, Y) := [R_g(X), R_g(Y)] - R([R_g(X), Y] + [X, R_g(Y)]). \tag{3.11}$$

When the R-operator satisfies the (m)CYBE:

$$\mathrm{YBE}_g(X, Y) = \omega[X, Y], \tag{3.12}$$

the expression of $\tilde{\mathcal{Z}}$ further reduces to

$$\tilde{\mathcal{Z}} = \partial_+ \tilde{J}_- - \partial_- \tilde{J}_+ + [\tilde{J}_+, \tilde{J}_-] + \eta R_g(\tilde{\mathcal{E}}) + \eta^2 \omega[P(\tilde{J}_+), P(\tilde{J}_-)]. \tag{3.13}$$

Lax pair We are ready to construct the Lax pair of the model (3.6). Indeed, both the equation of motion $\tilde{\mathcal{E}} = 0$ and the zero-curvature condition $\tilde{\mathcal{Z}} = 0$ are equivalent to the flatness condition of the Lax pair,

$$\partial_+ \tilde{\mathcal{L}}_-(\lambda) - \partial_- \tilde{\mathcal{L}}_+(\lambda) + [\tilde{\mathcal{L}}_+(\lambda), \tilde{\mathcal{L}}_-(\lambda)] = 0, \tag{3.14}$$

where $\tilde{\mathcal{L}}_\pm(\lambda)$ is defined as

$$\tilde{\mathcal{L}}_\pm(\lambda) \equiv \tilde{J}^{(0)}_\pm + \lambda^{\pm 1}\sqrt{1 + \eta^2\omega}\,\tilde{J}^{(1)}_\pm. \tag{3.15}$$

It should be noted that the above Lax pair is flat not only for $\omega = 1$ (split type) [11] but also $\omega = -1$ (non-split type) and $\omega = 0$ (CYBE-type). In particular, when the R-operator satisfies the CYBE with $\omega = 0$, the Lax pair (3.15) becomes

$$\tilde{\mathcal{L}}_\pm(\lambda) = \tilde{J}^{(0)}_\pm + \lambda^{\pm 1}\tilde{J}^{(1)}_\pm. \tag{3.16}$$

Interestingly, this is of the same form with the Lax pair of the undeformed coset sigma model, up to a formal replacement $A_\pm \to \tilde{J}_\pm$.

To find out the Lax connection (3.15), we start with the following ansatz,

$$\tilde{\mathcal{L}}_\pm(\lambda) = \tilde{J}^{(0)}_\pm + \lambda^{\pm 1}\tilde{G}\,\tilde{J}^{(1)}_\pm. \tag{3.17}$$

Here we suppose that the unknown factor $\tilde{G}$ does not depend on neither the spectral parameter λ nor the world-sheet coordinates (τ, σ). Under this ansatz, the flatness condition (3.14) can be rewritten as

$$\begin{aligned} 0 &= \partial_+ \tilde{\mathcal{L}}_-(\lambda) - \partial_- \tilde{\mathcal{L}}_+(\lambda) + [\tilde{\mathcal{L}}_+(\lambda), \tilde{\mathcal{L}}_-(\lambda)] \\ &= -\tilde{G}(\partial_- \tilde{J}^{(1)}_+ - [\tilde{J}^{(1)}_+, \tilde{J}^{(0)}_-])\lambda \\ &\quad + \tilde{G}(\partial_+ \tilde{J}^{(1)}_- + [\tilde{J}^{(0)}_+, \tilde{J}^{(1)}_-])\lambda^{-1} \\ &\quad + \partial_+ \tilde{J}^{(0)}_- - \partial_- \tilde{J}^{(0)}_+ + [\tilde{J}^{(0)}_+, \tilde{J}^{(0)}_-] + \tilde{G}^2[\tilde{J}^{(1)}_+, \tilde{J}^{(1)}_-]. \end{aligned} \tag{3.18}$$

On the other hand, due to the symmetric property (3.2), the equation of motion $\tilde{\mathcal{E}} = 0$ and the zero-curvature condition $\tilde{\mathcal{Z}} = 0$ are equivalent with the following set of the three equations:

$$0 = \partial_- \tilde{J}_+^{(1)} - [\tilde{J}_+^{(1)}, \tilde{J}_-^{(0)}], \tag{3.19}$$

$$0 = \partial_+ \tilde{J}_-^{(1)} + [\tilde{J}_+^{(0)}, \tilde{J}_-^{(1)}], \tag{3.20}$$

$$0 = \partial_+ \tilde{J}_-^{(0)} - \partial_- \tilde{J}_+^{(0)} + [\tilde{J}_+^{(0)}, \tilde{J}_-^{(0)}] + (1 + \eta^2\omega)[\tilde{J}_+^{(1)}, \tilde{J}_-^{(1)}]. \tag{3.21}$$

Comparing these relations with (3.18), one can find the following relation,

$$\tilde{G} = \pm\sqrt{1 + \eta^2\omega}. \tag{3.22}$$

The overall sign does not matter because it can be absorbed by the redefinition of the spectral parameter λ. When the plus signature is adopted, the ansatz (3.17) agrees with the Lax pair (3.15).

3.2. *Twist operator*

From the expression (3.13), it should be noted that the deformed current $\tilde{J}_\pm$ is also flat if the equation of motion is satisfied $\tilde{\mathcal{E}} = 0$ and the R-operator is a solution of the CYBE with $\omega = 0$:

$$\mathcal{Z} = \partial_+ \tilde{J}_- - \partial_- \tilde{J}_+ + [\tilde{J}_+, \tilde{J}_-] = 0. \tag{3.23}$$

In other words, the current $\tilde{J}_\pm$ is *on-shell flat current* regarding the CYBE-type deformation. Hence, one may expect that there exists a group element $\mathcal{F} \in G$, which we call *twist operator*, such that the deformed current is expressed as

$$\tilde{J}_\pm = \tilde{g}^{-1}\partial_\pm \tilde{g} \qquad \text{with} \qquad \tilde{g} \equiv \mathcal{F}^{-1} g. \tag{3.24}$$

With this notation, the flatness of $J_\pm$ is obvious. In the following, we will concretely construct such a twist operator.

For this purpose, we suppose that the sigma model (3.4) is defined on an infinitely extended world-sheet parametrized by $\sigma \in (-\infty, +\infty)$, instead of a cylinder:

$$S = -\frac{1}{2}(\gamma^{\alpha\beta} - \epsilon^{\alpha\beta}) \int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} d\sigma \ \mathrm{Tr}\left(A_\alpha P \frac{1}{1 - \eta R_g \circ P} A_\beta \right). \tag{3.25}$$

Then the field g and the current A_α are also supposed to obey the following boundary conditions on the world-sheet,

$$g(\sigma = \pm\infty) = \text{const.} \qquad \Longrightarrow \qquad A_\alpha(\sigma = \pm\infty) = 0. \tag{3.26}$$

Let us next consider the gauge transformation of $J_\pm$ defined by

$$\tilde{J}_\pm^g \equiv g\tilde{J}_\pm g^{-1} - \partial_\pm g g^{-1} \qquad \Longleftrightarrow \qquad \partial_\pm + \tilde{J}_\pm^g \equiv g(\partial_\pm + \tilde{J}_\pm)g^{-1}, \tag{3.27}$$

which is explicitly calculated as

$$\tilde{J}_\pm^g = g(\tilde{J}_\pm - A_\pm)g^{-1} = g\left(\frac{\pm\eta R_g \circ P}{1 \mp \eta R_g \circ P} A_\pm\right)g^{-1} = \pm\eta R(g P(\tilde{J}_\pm) g^{-1}). \tag{3.28}$$

Because the current $\tilde{J}_\pm$ is *on-shell* flat current as we have seen in (3.23), the gauge transformed current $\tilde{J}_\pm^g$ is also flat. By taking account of the boundary condition (3.26), this observation leads us to introduce the following twist operator by

$$\mathcal{F}(\sigma,\tau) \equiv \mathrm{Pexp}\Big[-\int_{-\infty}^{\sigma} d\sigma\, \tilde{J}^g_\sigma\Big]K, \tag{3.29}$$

where $K \in G$ is a constant element and does not depend on the world-sheet variables (τ,σ).[4] By the definition, it is easily shown that

$$\tilde{J}^g_\pm = -\partial_\pm \mathcal{F}\mathcal{F}^{-1}. \tag{3.30}$$

Plugging this expression with (3.27), one can obtain the following relation,

$$\tilde{J}_\pm = -g^{-1}(\partial_\pm \mathcal{F}\mathcal{F}^{-1})g + g^{-1}\partial_\pm g = (\mathcal{F}^{-1}g)^{-1}\partial_\pm(\mathcal{F}^{-1}g). \tag{3.31}$$

This is indeed the desired form in (3.24).

4. Multi-parameter deformations based on the CYBE

One may expect a generalization of integrable deformations based on the CYBE to the multi-parameter case. An easy way of doing this is to notice the following fact.

Let r^A_{12} and r^B_{12} be solutions of the CYBE with $\omega = 0$. Suppose that they commute each other; $[r^A_{ij}, r^B_{kl}] = 0$ for $i, j, k, l \in \{1, 2, 3\}$. Then, a linear combination of the r-matrices defined as

$$r^{(\alpha,\beta)}_{12} \equiv \alpha\, r^A_{12} + \beta\, r^B_{12} \qquad (\alpha, \beta \in \mathbb{C}), \tag{4.1}$$

is also a solution of the CYBE. As a matter of course, the associated linear R-operator $R^{(\alpha,\beta)}$ (2.4) satisfies the CYBE with $\omega = 0$ in (2.2).

Plugging the R-operator $R^{(\alpha,\beta)}$ with the action (2.1) (or (3.4)), one can obtain a two-parameter integrable deformation of principal chiral models (or coset sigma models, respectively).[5] Repeating this step, it is easy to realize further multi-parameter deformations.

5. Conclusion and discussion

In this article, we have shown that the Yang–Baxter sigma models introduced in [8,9] can naturally be extended to the CYBE case. The deformed model is defined by (2.1) and the Lax pair of the CYBE-type is obtained in (2.16). We have also argued a generalization to symmetric coset sigma models as in [11]. In this case, the action is given in (3.4) and the Lax pair is given in (3.16).

As mentioned in Introduction, these generalizations would be important from the viewpoint of applications because the CYBE has a wider class of the skew-symmetric constant solutions in general rather than the mCYBE. In particular, partial deformations of the target space manifestly preserve the classical integrability. Remarkably, multi-parameter generalizations are straightforward by following the technique explained in Section 4.

As a future direction, it would be interesting to consider a similar generalization for a two-parameter deformation of the principal chiral model, which is called *bi-Yang–Baxter sigma model*

[4] The choice of a constant element K would be crucial to find out the symmetry algebra of the deformed model. A suitable fixing of the matrix K for the 3D Schrödinger sigma model has been argued in [37].

[5] Apparently, there are three deformation parameters, α, β and η, in (2.1) or (3.4). In the present case, however, η is regarded as the overall factor of the R-operator and can be absorbed by redefinition of α, β. Hence, two of them are the independent deformation parameters.

[10]. In the recent progress, this formulation has been argued for a superstring theory on an $AdS_3 \times S^3 \times M^4$ background [39,40].

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Appendix A. Derivation of the equation of motion (3.8)

In this appendix, we shall derive the equation of motion (3.8) for the deformed coset model. The variation of the undeformed current is computed as

$$\delta A_\pm = \partial_\pm \epsilon + [A_\pm, \epsilon] \qquad \text{under} \qquad \delta g = g\epsilon. \tag{A.1}$$

To evaluate the variation of the deformed current $\delta \tilde{J}_\pm$, we need some preparation. Firstly, by the definition of the dressed R-operator R_g in (3.5), one can derive the relation,

$$\delta(R_g \circ P)(X) = (R_g \circ P)(\delta X) + [(R_g \circ P)(X), \epsilon] - R_g([P(X), \epsilon]), \tag{A.2}$$

where X is an arbitrary field and P is the coset projector defined in (3.3). Secondary, using this relation repeatedly, one can show that

$$\begin{aligned}\delta\big((R_g \circ P)^n(X)\big) = (R_g \circ P)^n(\delta X) &+ \sum_{k=0}^{n-1} (R_g \circ P)^k [(R_g \circ P)^{n-k}(X), \epsilon] \\ &- \sum_{k=0}^{n-1} (R_g \circ P)^k R_g([P \circ (R_g \circ P)^{n-1-k}(X), \epsilon])\end{aligned} \tag{A.3}$$

for a natural number n. Thirdly, multiplying $(\pm\eta)^n$ on both hand sides of the above relation and summing up n from 0 to ∞, the following expression is obtained,

$$\begin{aligned}\delta\left(\frac{1}{1 \mp \eta R_g \circ P} X\right) = \frac{1}{1 \mp \eta R_g \circ P} \\ \times \left(\delta X + \left[\frac{\pm\eta R_g \circ P}{1 \mp \eta R_g \circ P} X, \epsilon\right] \mp \eta R_g \left[P \frac{1}{1 \mp \eta R_g \circ P} X, \epsilon\right]\right).\end{aligned} \tag{A.4}$$

Finally, substituting $A_\pm$ for X, we have derived the variation of $\tilde{J}_\pm$,

$$\delta \tilde{J}_\pm = \frac{1}{1 \mp \eta R_g \circ P}\big(\partial_\pm \epsilon + [\tilde{J}_\pm, \epsilon] \mp \eta R_g [P(\tilde{J}_\pm), \epsilon]\big). \tag{A.5}$$

Here we have used the definition of the deformed current in (3.7).

We are now ready to evaluate the variation of the action in (3.4). Using the above formula (A.5) and noting the skew-symmetric property of the R-operator (2.3), one can find the following relation:

$$\delta L = -\frac{1}{2} \mathrm{Tr}\big[\big(\partial_+ P(\tilde{J}_-) + \partial_- P(\tilde{J}_+) + [\tilde{J}_+, P(\tilde{J}_-)] + [\tilde{J}_-, P(\tilde{J}_+)]\big)\epsilon\big]. \tag{A.6}$$

This is nothing but the equation of motion $\tilde{\mathcal{E}} = 0$ with $\tilde{\mathcal{E}}$ given in (3.8).

Appendix B. Examples

Here we present examples of deformations based on the mCYBE for both principal chiral models and symmetric coset sigma models.

B.1. A squashed sigma model

We consider a sigma model defined on a deformed S^3, called squashed three-sphere. This model is referred as to a *squashed sigma model* or an *anisotropic principal chiral model* in [41–43]. The classical integrable structure has been studied in [44–47].

Let $T^{\pm}, T^3$ be the generators of $\mathfrak{su}(2)$ satisfying the relations:

$$[T^3, T^{\pm}] = \pm i T^{\pm}, \qquad [T^+, T^-] = i T^3. \tag{B.1}$$

Now we consider the following classical r-matrix,

$$r_{12} = 2i T^+ \wedge T^- = 2i (T^+ \otimes T^- - T^- \otimes T^+). \tag{B.2}$$

We refer this type of r-matrix as to Drinfeld–Jimbo type. The associated R-operator satisfies the mCYBE (2.2) with $\omega = 1$ (split-type). The resulting deformed spacetime turns out to be a squashed three-sphere:

$$ds^2 = \frac{1}{4}\left(d\theta^2 + \sin^2\theta d\phi^2 + (1+\eta^2)(d\psi + \cos\theta d\phi)^2\right) \tag{B.3}$$

with a deformation parameter η. The scalar curvature is

$$R = 6 - 2\eta^2. \tag{B.4}$$

To derive the metric of the squashed S^3 in (B.3) from the r-matrix of Drinfeld–Jimbo type (B.2), we introduce the fundamental representation of $\mathfrak{su}(2)$ as follows:

$$T^1 = -\frac{i}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad T^2 = -\frac{i}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad T^3 = -\frac{i}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{B.5}$$

It is also convenient to introduce the light-cone notation,

$$T^+ = \frac{T^1 - iT^2}{\sqrt{2}} = \frac{-i}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad T^- = \frac{T^1 + iT^2}{\sqrt{2}} = \frac{-i}{\sqrt{2}}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{B.6}$$

By using these generators, an $SU(2)$ group element g can be represented by

$$g = \mathrm{e}^{\phi T^3}\mathrm{e}^{\theta T^2}\mathrm{e}^{\psi T^3}. \tag{B.7}$$

Then, the left-invariant current reads

$$A_{\pm} = g^{-1}\partial_{\pm} g = A^+_{\pm} T^- + A^-_{\pm} T^+ + A^3_{\pm} T^3 \tag{B.8}$$

with the coefficients

$$
\begin{aligned}
A^-_\pm &= \frac{i}{\sqrt{2}} e^{-i\psi}(\partial_\alpha\theta + i\partial_\alpha\phi\sin\theta),\\
A^+_\pm &= \frac{-i}{\sqrt{2}} e^{i\psi}(\partial_\alpha\theta - i\sin\theta\partial_\alpha\phi),\\
A^3_\pm &= \partial_\alpha\psi + \cos\phi\partial_\alpha\theta.
\end{aligned}
\tag{B.9}
$$

Let us next evaluate the deformed current $J_\pm$. Unlike the Jordanian r-matrix in Section 2.2, the R-operator of Drinfeld–Jimbo type is not nilpotent but diagonally acts on the Chevalley–Serre generators:

$$
\begin{aligned}
& R(X) = \mathrm{Tr}_2[r_{12}(1\otimes X)] \qquad \text{for} \qquad X\in\mathfrak{su}(2),\\
\Longrightarrow\quad & R(T^\pm) = \mp iT^\pm, \qquad R(T^3)=0.
\end{aligned}
\tag{B.10}
$$

By taking this into account, the deformed current is evaluated as follows:

$$
J_\pm = \frac{1}{1\mp\eta R}A_\pm = \frac{1}{1\pm i\eta}A^-_\pm T^+ + \frac{1}{1\mp i\eta}A^+_\pm T^- + A^3_\pm T^3. \tag{B.11}
$$

Finally, with the deformed current $J_\pm$, the Lagrangian is rewritten as[6]

$$
\begin{aligned}
L &= -\frac{1+\eta^2}{2}\mathrm{Tr}[A_-J_+] = -\frac{1+\eta^2}{2}\mathrm{Tr}[A_+J_-]\\
&= -\frac{\gamma^{\alpha\beta}}{4}\Big(\partial_\alpha\theta\partial_\beta\theta + \sin^2\theta\partial_\alpha\phi\partial_\beta\phi + (1+\eta^2)(\partial_\alpha\psi+\cos\theta\partial_\alpha\phi)(\partial_\alpha\psi+\cos\theta\partial_\alpha\phi)\Big)\\
&\quad + \frac{\epsilon^{\alpha\beta}}{2}\eta\,\partial_\alpha\phi\,\partial_\beta(\cos\theta).
\end{aligned}
\tag{B.12}
$$

Note that the last anti-symmetric term is total derivative and hence it can be dropped off. As a result, this is nothing but a sigma model on a squashed three-sphere (B.3).

B.2. A deformed sigma model on $SO(4)/SO(3)$

The next is a typical example of deformed coset models based on the mCYBE. Here we consider a deformation of a symmetric coset representation of $\mathrm{S}^3 \simeq SO(4)/SO(3)$. Interestingly, the resulting background is different from the deformation of $\mathrm{S}^3 \simeq SU(2)$. That is, the resulting deformed background depends on the representation of S^3.

To describe the Lie algebra $\mathfrak{so}(4) \simeq \mathfrak{su}(2)\oplus\mathfrak{su}(2)$, we prepare two sets of $\mathfrak{su}(2)$ algebras generated by A^i and B^i with $i=1,2,3$ satisfying the relations:

$$
[A^i, A^j] = \epsilon^{ij}{}_k A^k, \qquad [B^i, B^j] = \epsilon^{ij}{}_k B^k, \qquad [A^i, B^j] = 0. \tag{B.13}
$$

Here the structure constants $\epsilon^{ij}{}_k$ are totally anti-symmetric and normalized as $\epsilon^{12}{}_3 = 1$.

We are concerned here with a classical r-matrix of the Drinfeld–Jimbo type,

$$
r_{12} = i(A^+\wedge A^- + B^+\wedge B^-). \tag{B.14}
$$

Here $a\wedge b \equiv a\otimes b - b\otimes a$ and we have introduced the following notation,

$$
A^\pm = -iA^1 \pm A^2, \qquad B^\pm = -iB^1 \pm B^2. \tag{B.15}
$$

[6] Here we have normalized the overall factor in front of the action so that it agrees with (B.3).

It is easy to see that the associated R-operator is a solution of the mCYBE (2.2) with $\omega = 1$. Thus it is of non-split type. The resulting deformed metric turns out to be

$$ds^2 = \frac{1}{4}\left(\frac{d\theta^2 - \eta^2 \sin^2\theta (d\phi - d\psi)^2/4}{1+\eta^2\cos^2\frac{\theta}{2}} + \sin^2\theta d\phi^2 + (d\psi + \cos\theta\phi)^2\right). \tag{B.16}$$

This background is not identical with the squashed three spheres (B.3). In fact, the scalar curvature of the above metric is given by

$$R = 6 + \frac{2\eta^2(1+\eta^2\cos\theta\cos^2\frac{\theta}{2})}{1+\eta^2\cos^2\frac{\theta}{2}}. \tag{B.17}$$

Hence the two backgrounds are not related to each other through a coordinate transformation.

To derive the metric (B.16) from the r-matrix (B.14), we need to recall the symmetric structure of $\mathfrak{so}(4) = \mathfrak{so}(4)^{(0)} \oplus \mathfrak{so}(4)^{(1)}$. These subspaces are spanned by the following generators:

$$\begin{aligned} \mathfrak{so}(4)^{(0)} &= \mathrm{span}\{\ J^i = A^i + B^i \mid i = 1,2,3\ \}, \\ \mathfrak{so}(4)^{(1)} &= \mathrm{span}\{\ K^i = A^i - B^i \mid i = 1,2,3\ \}, \end{aligned} \tag{B.18}$$

respectively. Indeed, they enjoy the $\mathbb{Z}_2$-grading property (3.2) as follows:

$$[J^i, J^j] = \epsilon^{ij}{}_k J^k, \qquad [K^i, J^j] = \epsilon^{ij}{}_k K^k, \qquad [K^i, K^j] = \epsilon^{ij}{}_k J^k. \tag{B.19}$$

Then, the coset projector $P : \mathfrak{so}(4) \to \mathfrak{so}(4)^{(1)}$ in (3.3) is defined as

$$P(X) = -K^1\,\mathrm{Tr}[K^1 X] - K^2\,\mathrm{Tr}[K^2 X] - K^3\,\mathrm{Tr}[K^3 X] \qquad \text{for} \qquad X \in \mathfrak{so}(4). \tag{B.20}$$

Here the trace can be computed on the 4×4 fundamental representation of $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$,

$$\begin{aligned} A^1 &= \tfrac{i}{2}\begin{pmatrix} 0&1&0&0\\ 1&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix}, \qquad A^2 = \tfrac{i}{2}\begin{pmatrix} 0&i&0&0\\ -i&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix}, \qquad A^3 = \tfrac{i}{2}\begin{pmatrix} 1&0&0&0\\ 0&-1&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix}, \\ B^1 &= \tfrac{i}{2}\begin{pmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&1\\ 0&0&1&0 \end{pmatrix}, \qquad B^2 = \tfrac{i}{2}\begin{pmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&i\\ 0&0&-i&0 \end{pmatrix}, \qquad B^3 = \tfrac{i}{2}\begin{pmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&1&0\\ 0&0&0&-1 \end{pmatrix}. \end{aligned} \tag{B.21}$$

In the next, a group element is parametrized as follows:

$$g = \mathrm{e}^{\phi A^3}\mathrm{e}^{\theta A^2}\mathrm{e}^{\psi A^3} \in SU(2)\times SU(2). \tag{B.22}$$

Then, the left-invariant current $A_\pm = g^{-1}\partial_\pm g$ reads

$$\begin{aligned} A_\pm = {} & (\sin\psi\,\partial_\pm\theta - \sin\theta\cos\psi\,\partial_\pm\phi)A^1 + (\cos\psi\,\partial_\pm\theta + \sin\theta\sin\psi\,\partial_\pm\phi)A^2 \\ & + (\partial_\pm\psi + \cos\theta\,\partial_\pm\phi)A^3. \end{aligned} \tag{B.23}$$

To evaluate the deformed coset action in (3.4), we need to find the projected deformed current $P(\tilde{J}_\pm)$ rather than the deformed current in (3.7) itself. The current $P(\tilde{J}_\pm)$ is determined by solving the following equations:

$$(1 \mp \eta P \circ R_g)P(\tilde{J}_\pm) = P(A_\pm). \tag{B.24}$$

Note that these equations are obtained from the definition of the deformed current (3.7) by inverting the operator $(1 \mp \eta R_g \circ P)$ and multiplying the projector P from the left.

Since the above equations (B.24) are valued in $\mathfrak{so}(4)^{(1)}$, there are three independent equations with respect to K^1, K^2 and K^3. Solving the three equations, we obtain the following expression:

$$\begin{aligned}
P(\tilde{J}_\pm) = \frac{-1}{4(1+\eta^2\cos^2\frac{\theta}{2})}\Big[\Big(&((2+\eta^2\cos^2\tfrac{\theta}{2})\cos\psi \pm \eta\sin\psi)\sin\theta\partial_\pm\phi \\
&+ 2(\pm\eta\cos^2\tfrac{\theta}{2}\cos\psi - \sin\psi)\partial_\pm\theta \pm \eta(\pm\eta\cos^2\tfrac{\theta}{2}\cos\psi - \sin\psi)\sin\theta\partial_\pm\psi\Big)K_1 \\
&- \Big(((2+\eta^2\cos^2\tfrac{\theta}{2})\sin\psi \mp \eta\cos\psi)\sin\theta\partial_\pm\phi \\
&+ 2(\pm\eta\cos^2\tfrac{\theta}{2}\sin\psi + \cos\psi)\partial_\pm\theta \pm \eta(\pm\eta\cos^2\tfrac{\theta}{2}\sin\psi + \cos\psi)\sin\psi\partial_\pm\psi\Big)K_2 \\
&- \Big(((2+\eta^2)\cos\theta + \tfrac{\eta^2}{2}(1+\cos^2\theta))\partial_\pm\phi \\
&\mp \eta\sin\theta\partial_\pm\theta + 2(1+\eta^2\cos^4\tfrac{\theta}{2})\partial_\pm\psi\Big)K_3\Big].
\end{aligned} \tag{B.25}$$

Finally, with the above expression of $P(J_\pm)$, the Lagrangian can be rewritten as

$$\begin{aligned}
L &= -\operatorname{Tr}[A_+P(\tilde{J}_-)] = -\operatorname{Tr}[A_-P(\tilde{J}_+)] \\
&= -\frac{1}{4}\gamma^{\alpha\beta}\left[\frac{\partial_\alpha\theta\partial_\beta\theta - \eta^2\sin^2\theta(\partial_\alpha\phi - \partial_\alpha\psi)(\partial_\beta\phi - \partial_\beta\psi)/4}{1+\eta^2\cos^2\frac{\theta}{2}}\right. \\
&\qquad \left. + \sin^2\theta\partial_\alpha\phi\partial_\beta\phi + (\partial_\alpha\psi + \cos\theta\partial_\alpha\phi)(\partial_\beta\psi + \cos\theta\partial_\beta\phi)\right] \\
&\qquad - \frac{\eta\sin\theta}{4\left(1+\eta^2\cos^2\frac{\theta}{2}\right)}\epsilon^{\alpha\beta}\partial_\alpha\theta(\partial_\beta\phi - \partial_\beta\psi).
\end{aligned} \tag{B.26}$$

Note that the anti-symmetric two-form in the last line is total derivative and hence it can be ignored. Thus, this metric agrees with (B.16).

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