PERVERSE COHERENT SHEAVES ON BLOW-UP. III. BLOW-UP FORMULA FROM WALL-CROSSING

HIRAKU NAKAJIMA AND KŌTA YOSHIOKA

To the memory of the late Professor Masaki Maruyama

Abstract. In earlier papers [21, 22] of this series we constructed a sequence of intermediate moduli spaces $\{\widehat{M}^m(c)\}_{m=0,1,2,\dots}$ connecting a moduli space $M(c)$ of stable torsion free sheaves on a nonsingular complex projective surface X and $\widehat{M}(c)$ on its one point blow-up $\widehat{X}$. They are moduli spaces of perverse coherent sheaves on $\widehat{X}$. In this paper we study how Donaldson-type invariants (integrals of cohomology classes given by universal sheaves) change from $\widehat{M}^m(c)$ to $\widehat{M}^{m+1}(c)$, and then from $M(c)$ to $\widehat{M}(c)$. As an application we prove that Nekrasov-type partition functions satisfy certain equations which determine invariants recursively in second Chern classes. They are generalization of the blow-up equation for the original Nekrasov deformed partition function for the pure $N = 2$ SUSY gauge theory, found and used to derive the Seiberg-Witten curves in [18].

Introduction

Let X be a nonsingular complex projective surface and $p\colon \widehat{X} \to X$ the blow-up at a point 0. Let $C = p^{-1}(0)$ be the exceptional divisor. Let $c = (r, c_1, \mathrm{ch}_2) \in H^{\mathrm{ev}}(\widehat{X})$ be a cohomological data. Let $\widehat{M}(c)$ be the moduli space of stable torsion free sheaves E on $\widehat{X}$ with $\mathrm{ch}(E) = c$ and $M(p_*(c))$ the corresponding moduli space on X. In [21, 22] we constructed a sequence of intermediate moduli spaces $\widehat{M}^m(c)$ connected by birational morphisms as

$$(*)\qquad \begin{tikzcd}[column sep=small] & \widehat{M}^m(c) \arrow[dl] \arrow[dr, "\xi_m"] & & \widehat{M}^{m+1}(c) \arrow[dl, "\xi_m^+"'] \arrow[dr, "\xi_{m+1}"] & & {} \arrow[dl] & \\ \cdots & & \widehat{M}^{m,m+1}(c) & & \widehat{M}^{m+1,m+2}(c) & & \cdots \end{tikzcd}$$

such that

(1) $\widehat{M}^m(c) \cong \widehat{M}(c)$ if m is sufficiently large, and
(2) $\widehat{M}^0(c) \cong M(p_*(c))$ if $(c_1, [C]) = 0$ under the natural homomorphism given by $E \mapsto p_*(E)$. (See Proposition 1.2 for the statement when $0 < (c_1, [C]) < r$.)

The diagram $(*)$ is an example of those often appearing in variations of GIT quotients [27], and similar to ones for moduli spaces of sheaves (by Thaddeus, Ellingsrud-Göttsche, Friedman-Qin and others) when we move polarizations. See [21, 22] for more references on earlier works.

2000 *Mathematics Subject Classification.* Primary 14D21; Secondary 16G20.

In this paper, we study how Donaldson type invariants (certain integrals of cohomology classes given by universal sheaves) change from $\widehat{M}^m(c)$ to $\widehat{M}^{m+1}(c)$. By a technical reason we restrict ourselves to the case when X is $\mathbb{P}^2$ and $\widehat{M}^m(c)$ is replaced by the moduli space of framed sheaves for which quiver description was given in [21]. We conjecture that the results are universal, i.e., independent of the choice of a surface. Moreover we have a natural $(r+1)$-dimensional torus $\widetilde{T} = (\mathbb{C}^*)^2 \times T^{r-1}$ action on $\widehat{M}^m(c)$ from the $(\mathbb{C}^*)^2$-action on $\widehat{\mathbb{P}}^2$ and the change of framing. Thus we can consider equivariant Donaldson type invariants to which we can apply Atiyah-Bott-Lefschetz fixed point formula to perform a further computation. In this sense, we think our situation is most basic.

Our first main result says that the difference of invariants is given by a variant of Mochizuki's weak wall-crossing formula [14], i.e., it is expressed as a sum of an integral over $\widehat{M}^m(c')$ with smaller c' (Theorem 1.5). Our argument closely follows Mochizuki's, once $(*)$ is understood as a variation of GIT quotients.

Summing up the weak wall-crossing formula from 0 to m, we get the formula for the difference of $\widehat{M}(c)$ and $M(p_*(c))$ by integrals over various $\widehat{M}^{m'}(c')$, as a result. We normalize the first Chern class of c' in the interval $[0, r-1]$ twisting by a line bundle in order to apply $M(p_*(c')) \cong \widehat{M}^0(c')$ for $(c_1(c'), [C]) = 0$ and its modification Proposition 1.2. Then those integrals themselves can be expressed by integrals over $M(p_*(c'))$ and ones over even smaller $\widehat{M}^{m''}(c'')$. We apply the same argument for $\widehat{M}^{m''}(c'')$. We thus do this argument recursively to give an algorithm to express $\widehat{M}(c)$ by a linear combination of integrals over $M(c_\flat)$ for various $c_\flat$. Since this algorithm is complicated (see Figure 1 for the flowchart), we do not try to write down an explicit formula in general. We instead focus on vanishing theorems for special cases when integrands are not twisted too much along C. This is our second main result. See §2.

Our motivation of study in this series is an application to the Nekrasov partition function [24]. Let us explain it briefly. The Nekrasov partition function is the generating function of an equivariant integral over $M(c)$. One of the main conjecture on it states that the leading part $\mathcal{F}_0$ of its logarithm is given by the Seiberg-Witten prepotential, a certain period integral on the Seiberg-Witten curves. The three consecutive coefficients (denoted by H, A, B) are also important for the application to the wall-crossing formula for usual or K-theoretic Donaldson invariants for projective surfaces with $p_g = 0$ [7, 8].

When the integrand is (1) 1, (2) slant products of Chern classes of universal sheaves with the fundamental classes of $\mathbb{C}^2$, or (3) the Todd class of $M(c)$, the authors proved that the partition function satisfies functional equations, called the *blow-up equations*, which determine coefficients recursively in second Chern numbers of c [18, 19, 20]. The functional equations induce a nonlinear partial differential equation for $\mathcal{F}_0$, which has been known as the *contact term equation* in the physics literature [12, 6]. In particular, the Seiberg-Witten prepotential satisfies the same equation and hence is equal to $\mathcal{F}_0$. This was our proof of the above mentioned conjecture. There are other completely independent proofs by [25, 2]. But so far H, A, B can be determined only from the blow-up equation.

Nekrasov's partition functions have more variants by replacing the integrand. Let us give three examples:

(a) We integrate Euler classes of vector bundles given by pushforward of universal sheaves. They are called the *theories with fundamental matters* in the physics literature.

(b) When we integrate the Todd classes, we can cap with powers of the first Chern classes of the same bundles. They are called 5-*dimensional Chern-Simons terms.*

(c) We can also incorporate universal sheaves to which Adams operators are applied. They are called *(higher) Casimir operators.* They give coefficients appearing in the defining equation of the Seiberg-Witten curve.

The blow-up equation was derived by analyzing relation between integrals over $M(c)$ and $\widehat{M}(c)$. Our vanishing results in §2 enable us to generalize our proof for those variants[1]. In this paper, we explain it for theories with 5-dimensional Chern-Simons terms and Casimir operators. The case for the theories with matters will be given elsewhere [23].

The paper is organized as follows. In §1 we state our results after preparing the necessary notations. In §2 we prove several versions of vanishing theorems as applications of the results in §1. In §3 we study the Nekrasov partition function for theories with 5-dimensional Chern-Simons terms. The blow-up equation is derived. This section is expository since the derivation of Nekrasov's conjecture was already given in [8] assuming the vanishing theorems.

The actual proof starts from §4. We review the quiver description of the framed moduli spaces obtained in [21] and the analysis of the wall-crossing in [22], and add a few things. The quiver description is necessary to define master spaces. In §5 we define enhanced master spaces. We follow Mochizuki's method [14], but give the construction in detail for the sake of a reader. In §6 we prove Theorem 1.5, the variant of Mochizuki's weak wall-crossing formula. Again the proof is the same as Mochizuki's.

Acknowledgments. The first named author is supported by the Grant-in-Aid for Scientific Research (B) (No. 19340006), Japan Society for the Promotion of Science. The second named author is supported by the Grant-in-aid for Scientific Research (B) (No. 18340010), (S) (No. 19104002), JSPS and Max Planck Institute for Mathematics. Both authors thank Takuro Mochizuki for explanations of his results, and the referee for many valuable comments.

1. Main result

Notations. Let $[z_0 : z_1 : z_2]$ be the homogeneous coordinates on $\mathbb{P}^2$ and $\ell_\infty = \{z_0 = 0\}$ the line at infinity. Let $p\colon \widehat{\mathbb{P}}^2 \to \mathbb{P}^2$ be the blow-up of $\mathbb{P}^2$ at $[1 : 0 : 0]$. Then $\widehat{\mathbb{P}}^2$ is the closed subvariety of $\mathbb{P}^2 \times \mathbb{P}^1$ defined by

$$\left\{([z_0 : z_1 : z_2], [z : w]) \in \mathbb{P}^2 \times \mathbb{P}^1 \mid z_1 w = z_2 z\right\},$$

where the map p is the projection to the first factor. We denote $p^{-1}(\ell_\infty)$ also by ℓ_∞ for brevity. Let C denote the exceptional divisor given by $z_1 = z_2 = 0$. Let $\mathcal{O}$ denote the structure sheaf of $\widehat{\mathbb{P}}^2$, $\mathcal{O}(C)$ the line bundle associated with the divisor C, and $\mathcal{O}(mC)$ its m^{th} tensor product $\mathcal{O}(C)^{\otimes m}$ when $m > 0$, $(\mathcal{O}(C)^{\otimes -m})^\vee$ if $m < 0$, and $\mathcal{O}$ if $m = 0$. And

[1] The proof in [25] can be generalized to those variants. See [26] for the theory with 5-dimensional Chern-Simons terms, and [13] for higher Casimir operators, but not with Todd genus.

we use the similar notion $\mathcal{O}(mC + n\ell_\infty)$ for tensor products of $\mathcal{O}(mC)$ and tensor powers of the line bundle corresponding to ℓ_∞ or its dual.

The structure sheaf of the exceptional divisor C is denoted by $\mathcal{O}_C$. If we twist it by the line bundle $\mathcal{O}_{\mathbb{P}^1}(n)$ over $C \cong \mathbb{P}^1$, we denote the resulted sheaf by $\mathcal{O}_C(n)$. Since C has the self-intersection number -1, we have $\mathcal{O}_C \otimes \mathcal{O}(C) = \mathcal{O}_C(-1)$.

For $c \in H^*(\widehat{\mathbb{P}}^2)$, its degree 0, 2, 4-parts are denoted by r, c_1, ch_2 respectively. If we want to specify c, we denote by $r(c)$, $c_1(c)$, $\operatorname{ch}_2(c)$.

For brevity, we twist the push-forward homomorphism p_* by Todd genera of $\mathbb{P}^2$ and $\widehat{\mathbb{P}}^2$ as in [22, §3.1] so that it is compatible with the Riemann-Roch formula.

We also use the following notations frequently:

- $\vee$ is the involution on the K-group given by taking the dual of a vector bundle.
- $\Delta(E) := c_2(E) - \frac{r-1}{2r} c_1(E)^2$, $\Delta(c) := -\operatorname{ch}_2(c) + \frac{1}{2r(c)} c_1(c)^2$.
- C_m denotes $\mathcal{O}_C(-m-1)$.
- $e_m := \operatorname{ch}(\mathcal{O}_C(-m-1))$.
- pt is a single point in X, $\widehat{X}$ or sometimes an abstract point. Its Poincaré dual in $H^4(X)$ or $H^4(\widehat{X})$ is also denoted by the same notation.
- For an integer N let $\underline{N} = \{1, 2, \dots, N\}$.

For a sheaf E on $\widehat{\mathbb{P}}^2$, we denote $H^1(E(-\ell_\infty))$, $H^1(E(C - \ell_\infty))$ by $V_0(E)$, $V_1(E)$ respectively (and simply by V_0, V_1 if there is no fear of confusion). In this paper we mainly treat sheaves E with $H^i(E(-\ell_\infty)) = 0 = H^i(E(C-\ell_\infty))$ for $i \neq 1$. This is clear after we will recall the quiver description of framed moduli spaces in §4: $V_\alpha(E)$ appears as a vector space on the vertex α, and any sheaf in this paper corresponds to a representation of the quiver. Under this assumption we have

$$\begin{aligned}
\dim V_0 &= \dim H^1(E(-\ell_\infty)) = -(\operatorname{ch}_2(E), [\widehat{\mathbb{P}}^2]) + \frac{1}{2}(c_1(E), [C]), \\
\dim V_1 &= \dim H^1(E(C - \ell_\infty)) = -(\operatorname{ch}_2(E), [\widehat{\mathbb{P}}^2]) - \frac{1}{2}(c_1(E), [C])
\end{aligned}$$

by Riemann-Roch.

Let $\widehat{M}$ be a moduli scheme (or stack) and q_1, q_2 be projections to the first and second factors of $\widehat{\mathbb{P}}^2 \times \widehat{M}$. For a sheaf $\mathcal{E}$ (e.g., the universal sheaf) on $\widehat{\mathbb{P}}^2 \times \widehat{M}$, let

- $\mathcal{V}_0(\mathcal{E}) := R^1 q_{2*}(\mathcal{E} \otimes q_1^* \mathcal{O}(-\ell_\infty))$,
- $\mathcal{V}_1(\mathcal{E}) := R^1 q_{2*}(\mathcal{E} \otimes q_1^* \mathcal{O}(C - \ell_\infty))$.

Let $\operatorname{Ext}^\bullet_{q_2}$ denotes the derived functor of the composite functor $q_{2*} \circ \mathcal{H}om$. We often consider $\operatorname{Ext}^\bullet_{q_2}(\mathcal{E}, C_m)$, where C_m is considered as a sheaf on $\widehat{\mathbb{P}}^2 \times \widehat{M}$ via the pull-back by q_1.

1.1. **Framed moduli spaces.** A *framed sheaf* (E, Φ) on $\mathbb{P}^2$ is a pair of

- a coherent sheaf E, which is locally free in a neighborhood of ℓ_∞, and
- an isomorphism $\Phi\colon E|_{\ell_\infty} \to \mathcal{O}_{\ell_\infty}^{\oplus r}$, where r is the rank of E.

An isomorphism of framed sheaves (E, Φ), (E', Φ') is an isomorphism $\xi\colon E \to E'$ such that $\Phi' \circ \xi|_{\ell_\infty} = \Phi$. When $r = 0$, we understand that a framed sheaf is an ordinary sheaf of rank 0 whose support does not intersect with ℓ_∞. We have the corresponding definition of a framed sheaf on the blow-up $\widehat{\mathbb{P}}^2$.

Definition 1.1. Let $m \in \mathbb{Z}_{\geq 0}$. A framed sheaf (E, Φ) on $\widehat{\mathbb{P}}^2$ is called *m-stable* if

(1) $\operatorname{Hom}(E, \mathcal{O}_C(-m-1)) = 0$,
(2) $\operatorname{Hom}(\mathcal{O}_C(-m), E) = 0$, and
(3) E is torsion free outside C.

Though it is not obvious from the definition, an m-stable sheaf must have $r > 0$. (See [21, §2.2].)

We have a smooth fine moduli scheme $\widehat{M}^m(c)$ of m-stable framed sheaves (E, Φ) with $\operatorname{ch}(E) = c \in H^*(\widehat{\mathbb{P}}^2)$ such that $(c, [\ell_\infty]) = 0$. It is of dimension $2r(c)\Delta(c)$. (See Theorem 4.2.) Let $\mathcal{E}$ be the universal sheaf on $\widehat{\mathbb{P}}^2 \times \widehat{M}^m(c)$, which is unique thanks to the framing unlike the case of ordinary moduli spaces.

As special cases with $m = 0$ and m sufficiently large, we have fine moduli schemes $M(c)$ and $\widehat{M}(c)$ of framed torsion free sheaves (E, Φ) on $\mathbb{P}^2$ and $\widehat{\mathbb{P}}^2$ respectively. For $M(c)$, we take $c \in H^*(\widehat{\mathbb{P}}^2)$ with $(c_1, [C]) = 0$. (See [21, §7] or [22, §3.1, §3.9].) They are connected by a sequence of birational morphisms as explained in the introduction. See §4.4.

In fact, $M(c)$ was studied earlier in [17, Chap. 2,3] (denoted there by $\mathcal{M}(r, n)$). We need to recall one important property: We have a projective morphism $\pi\colon M(c) \to M_0(c)$, where $M_0(c)$ is the Uhlenbeck (partial) compactification of the moduli space $M_0^{\text{reg}}(c)$ of framed locally free sheaves (E, Φ). In [loc. cit.] $M_0(c)$ was constructed via the quiver description, and bijective to

$$\bigsqcup_{c'} M_0^{\text{reg}}(c') \otimes S^{\Delta(c) - \Delta(c')}(\mathbb{C}^2)$$

set-theoretically. Here $S^n(\mathbb{C}^2)$ denotes the n^{th} symmetric product of $\mathbb{C}^2$.

For any m, we still have a projective morphism $\widehat{\pi}\colon \widehat{M}^m(c) \to M_0(p_*(c))$. This follows from the quiver description (Theorem 4.2) or [22, §3.2]. It is compatible with the diagram $(*)$ and induced from a projective morphism $\widehat{M}^{m,m+1}(c) \to M_0(p_*(c))$.

1.2. **Grassmann bundle structure.** As we mentioned above, we have $M^0(c) \cong M(p_*(c))$ when $(c_1, [C]) = 0$. For $0 < (c_1, [C]) < r$, we have a similar relation as follows. We need to consider $\widehat{M}^1(c)$ with $0 > (c_1, [C]) > -r$ instead after twisting by the line bundle $\mathcal{O}(C)$.

Proposition 1.2 ([22, §3.10])**.** *Suppose $0 < n := -(c_1, [C]) < r$. There is a variety $\widehat{N}(c, n)$ relating $\widehat{M}^1(c)$ and $\widehat{M}^1(c - ne_0)$ through a diagram*

$$\begin{tikzcd} & \widehat{N}(c,n) \arrow[dl, "f_1"'] \arrow[dr, "f_2"] & \\ \widehat{M}^1(c) & & \widehat{M}^1(c - ne_0) \end{tikzcd}$$

satisfying the followings:

(1) *f_1 is surjective and birational.*

(2) *f_2 is the Grassmann bundle $\operatorname{Gr}(n, \operatorname{Ext}^1_{q_2}(\mathcal{O}_C(-1), \mathcal{E}'))$ of n-planes in the vector bundle $\operatorname{Ext}^1_{q_2}(\mathcal{O}_C(-1), \mathcal{E}')$ of rank r over $\widehat{M}^1(c - ne_0)$.*

(3) *We have a short exact sequence*

$$0 \to (\mathrm{id}_{\widehat{\mathbb{P}}^2} \times f_2)^* \mathcal{E}' \to (\mathrm{id}_{\widehat{\mathbb{P}}^2} \times f_1)^* \mathcal{E} \to \mathcal{O}_C(-1) \boxtimes \mathcal{S} \to 0.$$

Here $\mathcal{E}$, $\mathcal{E}'$ are the universal sheaves for $\widehat{M}^1(c)$ and $\widehat{M}^1(c-ne_0)$ respectively, and $\mathcal{S}$ is the universal rank n subbundle of $\mathrm{Ext}^1_{q_2}(\mathcal{O}_C(-1), \mathcal{E}')$ over $\mathrm{Gr}(n, \mathrm{Ext}^1_{q_2}(\mathcal{O}_C(-1), \mathcal{E}'))$.

Remark that $\mathrm{Ext}^i_{q_2}(\mathcal{O}_C(-1), \mathcal{E}') = 0$ for $i = 0, 2$ by the remark after Lemma 4.11 below. Hence $\mathrm{Ext}^1_{q_2}(\mathcal{O}_C(-1), \mathcal{E}')$ is a vector bundle, and its rank is r by Riemann-Roch.

We have $(c_1(c-ne_0), [C]) = (c_1, [C]) + n = 0$. Therefore $\widehat{M}^1(c-ne_0)$ becomes $M(p_*(c))$ after crossing the wall between 0-stability and 1-stability.

1.3. **Torus action and equivariant homology groups.** Let T be the maximal torus of $\mathrm{SL}_r(\mathbb{C})$ consisting of diagonal matrices and let $\widetilde{T} = \mathbb{C}^* \times \mathbb{C}^* \times T$. We have a $\widetilde{T}$-action on $\widehat{M}^m(c)$ induced from the $\mathbb{C}^* \times \mathbb{C}^*$-action on $\widehat{\mathbb{P}}^2$ given by

$$([z_0 : z_1 : z_2], [z : w]) \mapsto ([z_0 : t_1 z_1 : t_2 z_2], [t_1 z : t_2 w])$$

and the change of the framing Φ. See [22, §5]. It was defined exactly as in the case of framed moduli spaces of torsion free sheaves, given in [18, §3]. The action is compatible with one on $M_0(c)$, i.e., $\widehat{\pi}$ is $\widetilde{T}$-equivariant. All the constructions, which we have explained so far, are canonically $\widetilde{T}$-equivariant. For example, we have the canonical $\widetilde{T}$-action on the universal sheaf $\mathcal{E}$.

Let $H^{\widetilde{T}}_*(X)$ be the $\widetilde{T}$-equivariant Borel-Moore homology group of a $\widetilde{T}$-space X with rational coefficients. Let $H^*_{\widetilde{T}}(X)$ be the $\widetilde{T}$-equivariant cohomology group with rational coefficients. They are defined for X satisfying a reasonable condition, say an algebraic variety with an algebraic $\widetilde{T}$-action. See, for example, [19, App. C]. They are modules over the equivariant cohomology group $H^*_{\widetilde{T}}(\mathrm{pt})$ of a point, isomorphic to the symmetric product of the dual of the Lie algebra, which we denote by $S(\widetilde{T})$.

The projective morphism $\widehat{\pi}\colon \widehat{M}^m(c) \to M_0(p_*(c))$ induce a homomorphism

$$\widehat{\pi}_*\colon H^{\widetilde{T}}_*(\widehat{M}^m(c)) \to H^{\widetilde{T}}_*(M_0(p_*(c))).$$

We denote this homomorphism $\widehat{\pi}_*$ by $\int_{\widehat{M}^m(c)}$, since we also use similar push-forward homomorphisms from homology groups of various moduli schemes or stacks and want to emphasize the domain.

On the other hand, the target space $M_0(p_*(c))$ is not at all important. We can compose the push-forward homomorphism for the inclusion $M_0(p_*(c)) \subset M_0(c')$ for $\Delta(c') \geq \Delta(p_*(c))$. Then $\int_{\widehat{M}^m(c)}$ takes values in $H^{\widetilde{T}}_*(M_0(c'))$. We can also make $\int_{\widehat{M}^m(c)}$ with values in $\mathfrak{S}(\widetilde{T})$, the quotient field of $S(\widetilde{T})$ as follows: Recall that $\widetilde{T}$ has the unique fixed point 0 in $M_0(p_*(c))$ [18, Prop.2.9(3)]. We compose $\int_{\widehat{M}^m(c)}$ with the inverse ι_{0*}^{-1} of the push-forward homomorphism ι_{0*} for the inclusion $\{0\} \to M_0(p_*(c))$ by using the localization theorem for the equivariant homology group, which says ι_{0*} becomes an isomorphism after taking tensor products with $\mathfrak{S}(\widetilde{T})$ over $H^*_{\widetilde{T}}(\mathrm{pt}) = S(\widetilde{T})$. This is compatible with the above inclusion. See [18, §4] for more detail.

1.4. **Weak wall-crossing formula.** We now state our first main result in this subsection.

Let $\Phi(\mathcal{E}) \in H^*_{\widetilde{T}}(\widehat{M}^m(c))$ be an equivariant cohomology class on $\widehat{M}^m(c)$ defined from a sheaf $\mathcal{E}$ on $\widehat{\mathbb{P}}^2 \times \widehat{M}^m(c)$ by taking a slant product by a cohomology class on $\widehat{\mathbb{P}}^2$, or taking a cohomology group, for example,

$$\Phi(\mathcal{E}) := \exp\left[\sum_{p=1}^{\infty}\left\{t_p \operatorname{ch}_{p+1}(\mathcal{E})/[C] + \tau_p \operatorname{ch}_{p+1}(\mathcal{E})/[\widehat{\mathbb{C}}^2]\right\}\right], \tag{1.3}$$

$$\text{or } \Phi(\mathcal{E}) := \prod_{f=1}^{N_f} e(\mathcal{V}_a(\mathcal{E}) \otimes e^{m_f}) \quad a = 0 \text{ or } 1,$$

where t_p, τ_p are variables and the exponential defines formal power series in t_p, τ_p in the first case, and $m_1, \dots, m_{N_f}$ are variables for the equivariant cohomology $H^*_{(\mathbb{C}^*)^{N_f}}(\mathrm{pt})$ of the N_f-dimensional torus of a point, and e^{m_f} is the corresponding equivariant line bundle. For $\mathcal{E}$ we typically take the universal sheaf, or its variant. For the latter $\Phi(\mathcal{E}) = \prod_{f=1}^{N_f} e(\mathcal{V}_a(\mathcal{E}) \otimes e^{m_f})$, we need to enlarge $\widetilde{T}$ to $\widetilde{T} \times (\mathbb{C}^*)^{N_f}$ but keep the notation $\widetilde{T}$ for brevity. And $e(\ )$ denotes the equivariant Euler class.

Remark 1.4. The notation N_f is taken from physics literature. It is the number of *flavors*. But we denote the rank by r, though it is denoted by N_c (number of *colors*) in physics literature.

The above examples of Φ are *multiplicative*, i.e., $\Phi(\mathcal{E} \oplus \mathcal{E}') = \Phi(\mathcal{E})\Phi(\mathcal{E}')$. This condition is useful when we will study the vanishing theorem in §2. But we do not assume it in general.

For $j \in \mathbb{Z}_{>0}$ we consider the j-dimensional torus $(\mathbb{C}^*)^j$ acting trivially on moduli schemes. We denote the 1-dimensional weight n representation of the i^{th} factor by $e^{n\hbar_i}$. The equivariant cohomology $H^*_{(\mathbb{C}^*)^j}(\mathrm{pt})$ of the point is identified with $\mathbb{C}[\hbar_1, \dots, \hbar_j]$. In the following formula we invert variables $\hbar_1, \dots, \hbar_j$. See §6.1 for the precise definition. Also we identify $\Phi(\mathcal{E})$ with the homology class $\Phi(\mathcal{E}) \cap [\widehat{M}^{m+1}(c)]$ and apply the push-forward homomorphism $\int_{\widehat{M}^{m+1}(c)}$.

Theorem 1.5.

$$\int_{\widehat{M}^{m+1}(c)} \Phi(\mathcal{E}) - \int_{\widehat{M}^m(c)} \Phi(\mathcal{E}) = \sum_{j=1}^{\infty} \int_{\widehat{M}^m(c-je_m)} \operatorname*{Res}_{\hbar_j=0} \cdots \operatorname*{Res}_{\hbar_1=0}\left[\Phi(\mathcal{E}_\flat \oplus \bigoplus_{i=1}^{j} C_m \boxtimes e^{-\hbar_i})\Psi^j(\mathcal{E}_\flat)\right],$$

where $\mathcal{E}_\flat$ is the universal sheaf for $\widehat{M}^m(c - je_m)$ and

$$\Psi^j(\mathcal{E}_\flat) := \frac{1}{j!}\frac{\prod_{1\le i_1 \ne i_2 \le j}(-\hbar_{i_1} + \hbar_{i_2})}{\prod_{i=1}^{j} e(\mathfrak{N}(\mathcal{E}_\flat, C_m) \otimes e^{-\hbar_i})\, e(\mathfrak{N}(C_m, \mathcal{E}_\flat) \otimes e^{\hbar_i})},$$

$$\mathfrak{N}(\mathcal{E}_\flat, C_m) := -\sum_{a=0}^{2}(-1)^a \operatorname{Ext}^a_{q_2}(\mathcal{E}_\flat, C_m), \quad \mathfrak{N}(C_m, \mathcal{E}_\flat) := -\sum_{a=0}^{2}(-1)^a \operatorname{Ext}^a_{q_2}(C_m, \mathcal{E}_\flat).$$

(Note that $\Psi^j(\mathcal{E}_\flat)$ depends on j, but not on $c - je_m$ if we consider $\mathcal{E}_\flat$ as a variable.)

The proof will be given in §6.4.

1.5. **Blow-up formula.** Recall that $\widehat{M}^m(c)$ is isomorphic to the framed moduli space $\widehat{M}(c)$ of torsion free sheaves on $\widehat{\mathbb{P}}^2$ if m is sufficiently large. Using Proposition 1.2, Theorem 1.5 and twist by the line bundle $\mathcal{O}(C)$, we can express $\int_{\widehat{M}(c)} \Phi(\mathcal{E})$ as a sum of various $\int_{M(c')} \Phi'(\mathcal{E})$'s for some c', Φ'. Unfortunately the procedure, which we will explain below in detail, is recursive in nature and rather cumbersome. See Figure 1 for the flowchart. In particular, we do not solve the recursion and do not give the explicit formula.

1.5.1. For a sequence $\vec{j} = (j_0, j_1, \dots, j_{m-1}) \in \mathbb{Z}^m_{\geq 0}$, we define $\Psi^{\vec{j}}_n$ recursively starting from $\Psi^{\vec{j}}_m = 1$ by

$$\Psi^{\vec{j}}_n(\bullet) := \Psi^{\vec{j}}_{n+1}(\bullet \oplus C_n \boxtimes e^{-\hbar^n_i}) \times \frac{1}{j_n!} \frac{\prod_{1\le i_1\ne i_2\le j_n}(-\hbar^n_{i_1} + \hbar^n_{i_2})}{\prod_{i=1}^{j_n} e(\mathfrak{N}(\bullet, C_n)\otimes e^{-\hbar^n_i})\, e(\mathfrak{N}(C_n,\bullet)\otimes e^{\hbar^n_i})},$$

where $\hbar^n_1$, …, $\hbar^n_{j_n}$ are variables. Then we set $\Psi^{\vec{j}} = \Psi^{\vec{j}}_0$. By Theorem 1.5 we get

$$\int_{\widehat{M}^m(c)} \Phi(\mathcal{E}) = \sum_{\vec{j}} \int_{\widehat{M}^0(c - \sum_n j_n e_n)} \underset{\vec{\hbar}=0}{\overrightarrow{\operatorname{Res}}}\, \Phi(\mathcal{E}_\flat \oplus \bigoplus_{n=0}^{m-1} \bigoplus_{i=1}^{j_n} C_n \otimes e^{\hbar^n_i}) \Psi^{\vec{j}}(\mathcal{E}_\flat), \tag{1.6}$$

where $\overrightarrow{\operatorname{Res}}_{\vec{\hbar}=0}$ is the iterated residues

$$\underset{\vec{\hbar}=0}{\overrightarrow{\operatorname{Res}}} := \underset{\hbar^0_{j_0}=0}{\operatorname{Res}} \cdots \underset{\hbar^0_1=0}{\operatorname{Res}}\ \underset{\hbar^1_{j_1}=0}{\operatorname{Res}} \cdots \underset{\hbar^1_1=0}{\operatorname{Res}} \cdots \underset{\hbar^{m-1}_{j_{m-1}}=0}{\operatorname{Res}} \cdots \underset{\hbar^{m-1}_1=0}{\operatorname{Res}}.$$

Since $\widehat{M}(c)$ is isomorphic to $\widehat{M}^m(c)$ for a sufficiently large m, an integral over $\widehat{M}(c)$ can be written in terms of integrals over $\widehat{M}^0(c')$ with various c' thanks to this formula.

Note that if $(c_1, [C]) \geq 0$, we have

$$\begin{aligned} &\dim \widehat{M}^0(c - \sum_n j_n e_n) \\ &= \dim \widehat{M}^m(c) - \sum r(2n+1)j_n - \sum j_n \left(\sum j_n + 2(c_1, [C])\right) < \dim \widehat{M}^m(c) \end{aligned} \tag{1.7}$$

if $(j_1, j_2, \dots) \neq 0$.

1.5.2. We usually consider the moduli space $\widehat{M}(c)$ with $0 \leq (c_1, [C]) < r$. This is always achieved by tensoring a power of the line bundle $\mathcal{O}(C)$. And then we can hope to relate $\int_{\widehat{M}(c)} \Phi(\mathcal{E})$ to $\int_{M(p_*(c))} \Phi(\mathcal{E})$ thanks to Proposition 1.2. But look at (1.6). The right hand side of (1.6) contains integrals over $\widehat{M}^0(c')$ ($c' = c - \sum_n j_n e_n$) for which we have $0 \leq (c_1(c'), [C])$, but not necessarily $< r$. Thus we need to tensor a line bundle again.

Since keeping track the precise form of the formula is a rather tiresome work, we *redefine* a term in the right hand side of (1.6) as $\int_{\widehat{M}^0(c)} \Phi(\mathcal{E})$, and start from it. We can assume $(c_1, [C]) \geq 0$, as we explained.

1.5.3. First consider the case $(c_1,[C]) = 0$. We have an isomorphism $\Pi\colon \widehat{M}^0(c) \cong M(p_*(c))$ by $(E,\Phi) \mapsto (p_*(E),\Phi)$, where the higher direct image sheaves $R^{>0}p_*(E)$ vanishes. Moreover its inverse is given by $(F,\Phi) \mapsto (p^*F,\Phi)$, and $L^{<0}p^*F = 0$. (See [22, Prop. 3.3 and §1].) If we denote the universal sheaf for $M(p_*(c))$ by $\mathcal{F}$, the universal sheaf $\mathcal{E}$ for $\widehat{M}^0(c)$ is equal to $(p\times\Pi)^*(\mathcal{F})$. Therefore we have

$$\int_{\widehat{M}^0(c)} \Phi(\mathcal{E}) = \int_{\widehat{M}^0(c)} \Phi((p\times\Pi)^*(\mathcal{F})).$$

Since $L^{<0}(p\times\Pi)^*(\mathcal{F})$ vanishes, this holds in the level of K-group.

We may have expressions $\mathcal{O}(C)$ or $[C]$, which do not come from $M(p_*(c))$ in the expression $\Phi(\mathcal{E})$, but we can use the projection formula to rewrite the right hand side as

$$\int_{M(p_*(c))} \Phi'(\mathcal{F}).$$

for a possibly different cohomology class $\Phi'(\bullet)$.

1.5.4. Now we may assume $(c_1,[C]) > 0$. We have

$$\int_{\widehat{M}^0(c)} \Phi(\mathcal{E}) = \int_{\widehat{M}^1(ce^{[C]})} \Phi(\mathcal{E}(-C)) \tag{1.8}$$

by the isomorphism $\widehat{M}^0(c) \ni (E,\Phi) \mapsto (E(C),\Phi) \in \widehat{M}^1(ce^{[C]})$. Two universal sheaves for $\widehat{M}^0(c)$, $\widehat{M}^1(ce^{[C]})$ are denoted by the same notation, but the latter is twisted by $\mathcal{O}(C)$ from the former under this isomorphism, and it is the reason why we have $\Phi(\mathcal{E}(-C))$.

We have $-r < (c_1(ce^{[C]}),[C]) = (c_1,[C]) - r$. If this is negative, in other words, if we have $(c_1,[C]) < r$, we go to the step which will be explained in §1.5.5. So we assume $(c_1(ce^{[C]}),[C]) \ge 0$. We now *redefine* the right hand side of (1.8) as $\int_{\widehat{M}^1(c)} \Phi(\mathcal{E})$ and return back to §1.5.1 and apply (1.6) with $m = 1$.

We repeat this procedure until all terms are integrals over $\widehat{M}^1(c')$ with $-r < (c_1(c'),[C]) < 0$, or $\widehat{M}^0(c'')$ with $c_1(c'') = 0$. From the dimension estimate (1.7), the procedure ends after finite steps.

For $\Phi(\mathcal{E}) = \prod_{f=1}^{N_f} e(\mathcal{V}_a(\mathcal{E})\otimes e^{m_f})$, this process requires a care, since $\mathcal{V}_a(\mathcal{E}(-C)) = R^1q_{2*}(\mathcal{E}(-C)\otimes q_1^*(\mathcal{O}(aC-\ell_\infty)))$ may not be a vector bundle on $M^0(c)$, so $e(\mathcal{V}_a(\mathcal{E}(-C))\otimes e^{m_f})$ does not make sense, and we cannot apply (1.6) with $m = 1$. We overcome this difficulty by replacing $e(\mathcal{V}_a(\mathcal{E}(-C))\otimes e^{m_f})$ by a product of $e(\mathcal{V}_a(\mathcal{E})\otimes e^{m_f})$ and a certain class, which is well-defined on $M^0(c)$. See the proof of Theorem 2.1 for detail.

1.5.5. Now we *redefine* the right hand side of (1.8) as $\int_{\widehat{M}^1(c)} \Phi(\mathcal{E})$ and consider it under the assumption $-r < (c_1,[C]) < 0$.

Let $n := -(c_1,[C])$. By Proposition 1.2 we have

$$\begin{aligned}\int_{\widehat{M}^1(c)} \Phi(\mathcal{E}) &= \int_{\widehat{N}(c,n)} \Phi((\mathrm{id}_{\widehat{\mathbb{P}}^2}\times f_1)^*\mathcal{E})\\ &= \int_{\widehat{N}(c,n)} \Phi\left((\mathrm{id}_{\widehat{\mathbb{P}}^2}\times f_2)^*(\mathcal{E})\oplus C_0\boxtimes\mathcal{S}\right),\end{aligned}$$

where we denote the universal bundle over $\widehat{M}^1(c-ne_0)$ by $\mathcal{E}$ for brevity. Since $f_2\colon \widehat{N}(c,n) \to \widehat{M}^1(c-ne_0)$ is the Grassmann bundle of n-planes in $\operatorname{Ext}^1_{q_2}(C_0,\mathcal{E})$, we can pushforward to $\widehat{M}^1(c-ne_0)$ to get

$$\int_{\widehat{N}(c,n)} \Phi\left((\mathrm{id}_{\widehat{\mathbb{P}}^2}\times f_2)^*(\mathcal{E})\oplus C_0\boxtimes\mathcal{S}\right) = \int_{\widehat{M}^1(c-ne_0)} {}'\Phi(\mathcal{E}),$$

where

$${}'\Phi(\bullet) := \int_{\mathrm{Gr}(n,r)} \Phi\left(\bullet\oplus(C_0\boxtimes\mathcal{S})\right)\Big|_{c(\underline{\mathbb{C}^r})=c(\operatorname{Ext}^1_{q_2}(C_0,\mathcal{E}))}.$$

We need to explain the notation. We consider the Grassmannian $\mathrm{Gr}(n,r)$ of n-planes in $\mathbb{C}^r$, and $\int_{\mathrm{Gr}(r-n,r)}$ is the pushforward $H^*_{\mathrm{GL}(r)}(\mathrm{Gr}(r-n,r)) \to H^*_{\mathrm{GL}(r)}(\mathrm{pt})$. The $\bullet$ is a variable living in the K-group $K(\widehat{\mathbb{P}}^2\times\mathrm{pt})$. The universal subbundle of the trivial bundle $\underline{\mathbb{C}^r}$ is denoted by $\mathcal{S}$. And $C_0\boxtimes\mathcal{S}$ is a sheaf on $\widehat{\mathbb{P}}^2\times\mathrm{Gr}(r,n)$. We consider $\mathrm{Gr}(r,n)$ as a moduli space and $\bullet\oplus C_0\boxtimes\mathcal{S}$ is a universal sheaf, and apply the function Φ. Finally $(\ )|_{c(\underline{\mathbb{C}^r})=c(\operatorname{Ext}^1_{q_2}(C_0,\mathcal{E}))}$ means that we substitute the Chern classes of $\operatorname{Ext}^1_{q_2}(C_0,\mathcal{E}))$ to the equivariant Chern classes of $\underline{\mathbb{C}^r}$ in $H^*_{\mathrm{GL}(r)}(\mathrm{pt})$.

We now redefine $\int_{\widehat{M}^1(c)}\Phi(\mathcal{E})$ as $\int_{\widehat{M}^1(c-ne_0)}{}'\Phi(\mathcal{E})$, and return to §1.5.1. Since $\dim\widehat{M}^1(c-ne_0) < \dim\widehat{M}^1(c)$, this procedure eventually stop.

1.6. **Example.** Consider the case $c\in H^*(\widehat{\mathbb{P}}^2)$ with $r(c)=r$, $c_1(c)=0$, $(\Delta(c),[\widehat{\mathbb{P}}^2])=1$. In Theorem 1.5 the wall-crossing term appears only in the case $m=0$, $j=0$. Therefore

$$\int_{\widehat{M}(c)}\Phi(\mathcal{E}) - \int_{\widehat{M}^0(c)}\Phi(\mathcal{E}) = \int_{\widehat{M}^0(c-e_0)} \operatorname*{Res}_{\hbar_1=0}\frac{\Phi(\mathcal{E}_\flat\oplus C_0\boxtimes e^{-\hbar_1})}{e(\mathfrak{N}(\mathcal{E}_\flat,C_0)\otimes e^{-\hbar_1})\,e(\mathfrak{N}(C_0,\mathcal{E}_\flat)\otimes e^{\hbar_1})} \tag{1.9}$$

In the quiver description Theorem 4.2 for $\widehat{M}^0(c-e_0)$ we have $V_0=\mathbb{C}$, $V_1=0$ and hence

$$\widehat{M}^0(c-e_0)\cong\mathbb{P}^{r-1}.$$

This also follows from Proposition 1.2. In fact, f_1 is an isomorphism in this case. We also see that $\mathcal{E}_\flat\cong\operatorname{Ker}\left[\mathcal{O}_{\mathbb{P}}^{\oplus r}\to\mathcal{O}_{\mathbb{P}}(1)\boxtimes\mathcal{O}_C\right]$. Then we have

$$\mathfrak{N}(\mathcal{E}_\flat,C_0)\cong\mathcal{O}_{\mathbb{P}}(-1),\quad \mathfrak{N}(C_0,\mathcal{E}_\flat)\cong\mathcal{O}_{\mathbb{P}}(1)^{\oplus 2}\oplus\mathcal{S},$$

where $\mathcal{O}_{\mathbb{P}}(1)$ is the hyperplane bundle of $\mathbb{P}=\mathbb{P}^{r-1}$ and $\mathcal{S}$ is the universal subbundle, i.e., kernel of $\mathcal{O}_{\mathbb{P}}^{\oplus r}\to\mathcal{O}_{\mathbb{P}}(1)$. This also follows from Lemmas 4.9,4.11. Therefore

$$\begin{aligned}
&e(\mathfrak{N}(\mathcal{E}_\flat,C_0)\otimes e^{-\hbar_1}) = -c_1(\mathcal{O}_{\mathbb{P}}(1))-\hbar_1,\\
&e(\mathfrak{N}(C_0,\mathcal{E}_\flat)\otimes e^{\hbar_1}) = (c_1(\mathcal{O}_{\mathbb{P}}(1))+\hbar_1)^2 e(\mathcal{S}\otimes e^{\hbar_1}) = \hbar_1^r(c_1(\mathcal{O}_{\mathbb{P}}(1))+\hbar_1).
\end{aligned}$$

For Φ, we consider a simplest nontrivial case. Let $\mu(C)$ be the cohomology class on $\widehat{M}^m(c)$ given by

$$\mu(C) := \Delta(\mathcal{E})/[C], \tag{1.10}$$

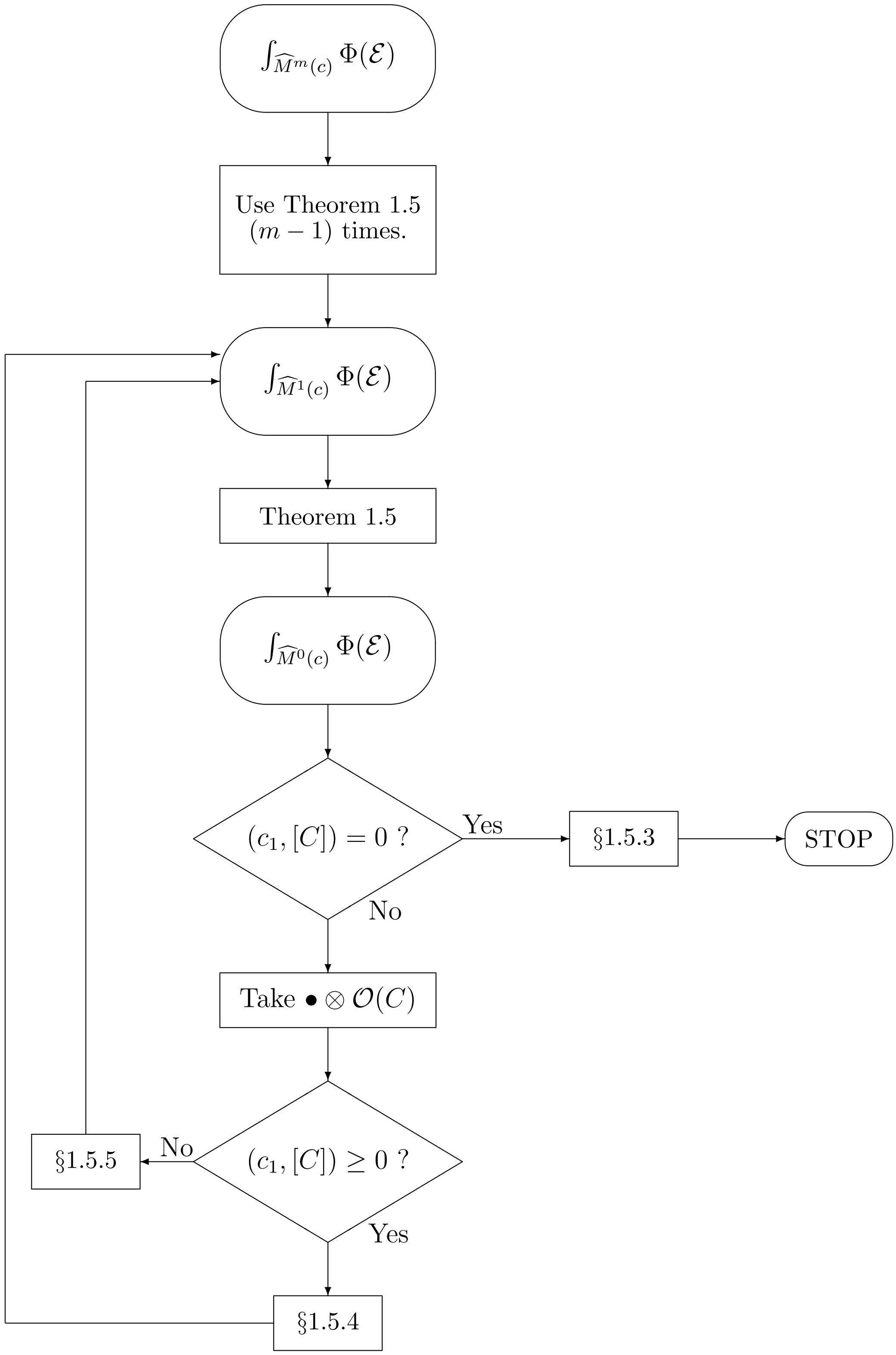

FIGURE 1. flowchart

where / denotes the slant product $/ \colon H^d_{\widetilde{T}}(\widehat{\mathbb{P}}^2 \times \widehat{M}^m(c)) \otimes H^{\widetilde{T}}_i(\widehat{\mathbb{P}}^2)) \to H^{d-i}_{\widetilde{T}}(\widehat{M}^m(c))$. This is the μ-map appearing in the usual Donaldson invariants. We have

$$\Delta(\mathcal{E}_\flat \oplus C_0 \boxtimes e^{-\hbar_1})/[C] = -c_1(\mathcal{O}_{\mathbb{P}}(1)) - \hbar_1 + \varepsilon_1 + \varepsilon_2,$$

where ε_1, ε_2 are generators of $\operatorname{Lie}(\mathbb{C}^* \times \mathbb{C}^*)$ corresponding to t_1, t_2.

We also have

$$\mathcal{V}_1(\mathcal{E}_\flat \oplus C_0 \boxtimes e^{-\hbar_1}) \cong e^{-\hbar_1}.$$

Hence

$$\prod_{f=1}^{N_f} e(\mathcal{V}_1(\mathcal{E}_\flat \oplus C_0 \boxtimes e^{-\hbar_1}) \otimes e^{m_f}) = \prod_{f=1}^{N_f} (m_f - \hbar_1).$$

We assume $2r - N_f \geq 1$ and take $\Phi(\mathcal{E}) = \mu(C)^{2r-N_f} \prod_{f=1}^{N_f} e(\mathcal{V}_1(\mathcal{E}) \otimes e^{m_f})$. Then the right hand side of (1.9) becomes

$$\int_{\mathbb{P}^{r-1}} \operatorname*{Res}_{\hbar_1=0} \frac{(-c_1(\mathcal{O}_{\mathbb{P}}(1)) - \hbar_1 + \varepsilon_1 + \varepsilon_2)^{2r-N_f}}{-\hbar_1^r (c_1(\mathcal{O}_{\mathbb{P}}(1)) + \hbar_1)^2} \prod_{f=1}^{N_f} (m_f - \hbar_1).$$

By the degree reason, this must be a constant in ε_1, ε_2, m_f. Therefore we may set all 0. Then this is equal to

$$-\int_{\mathbb{P}^{r-1}} \operatorname*{Res}_{\hbar_1=0} \hbar_1^{N_f-r} (c_1(\mathcal{O}_{\mathbb{P}}(1)) + \hbar_1)^{2r-N_f-2} = -\binom{2r-N_f-2}{r-1}.$$

If $r = 2$, $N_f = 0$, the answer is -2. This is a simplest case of the blow-up formula, which was used to define Donaldson invariants for c_2 in the unstable range.

2. Applications – Vanishing theorems

As we mentioned above, the wall-crossing formula only gives us a recursive procedure to give the blow-up formula. In this section, we concentrate on a rather special $\Phi(\mathcal{E})$ and derive certain vanishing theorems. They turn out to be enough for applications to the instanton counting.

2.1. **Theory with matters.** Let $\mu(C)$ be as in (1.10). We consider

$$\Phi(\mathcal{E}) = \prod_{f=1}^{N_f} e(\mathcal{V}_0(\mathcal{E}) \otimes e^{m_f}) \times \exp\left(t\mu(C)\right),$$

and study the coefficient $\Phi_d(\mathcal{E})$ of t^d with small d. We assume $N_f \leq 2r$ hereafter.

Theorem 2.1. *Suppose* $(c_1, [C]) = 0$. *Then*

$$\int_{\widehat{M}^m(c)} \Phi(\mathcal{E}) = \int_{M(p_*(c))} \prod_{f=1}^{N_f} e(\mathcal{V}(\mathcal{E}) \otimes e^{m_f}) + O(t^k),$$

where $k = \max(r+1, 2r - N_f)$. *Here* $\mathcal{V}(\mathcal{E}) = R^1 q_{2*}(\mathcal{E} \otimes q_1^*(\mathcal{O}(-\ell_\infty)))$ *is defined from the universal sheaf* $\mathcal{E}$ *on* $\widehat{\mathbb{P}}^2 \times M(p_*(c))$ *as in the case of* $\mathcal{V}_0(\mathcal{E})$, $\mathcal{V}_1(\mathcal{E})$.

This, in particular, means that $\Phi_d(\mathcal{E}) = 0$ for $d = 1, \dots, k-1$. When $N_f = 0$, this vanishing was shown in [18, §6] by the dimension counting argument. The key point was that $\dim M(c) = 2r\Delta$, and hence the smaller moduli spaces have codimension greater than or equal to $2r$. Once the wall-crossing formula is established as in the previous section, the remaining argument below is similar, and the bound $2r - N_f$ comes from the fact that the 'virtual fundamental class' $\prod_{f=1}^{N_f} e(\mathcal{V}(\mathcal{E}) \otimes e^{m_f}) \cap [M(p_*(c))]$ has dimension $(2r - N_f)\Delta$.

Proof. Let us compute the cohomological degrees of the both sides of the equality in Theorem 1.5, where we say $\int_{\widehat{M}^m(c)} \heartsuit$ *has degree* k if it is contained in $H^{\widetilde{T}}_{2k}(M_0(p_*(c)))$. We have

$$\begin{aligned}(2.2)\quad \deg \int_{\widehat{M}^{m+1}(c)} \Phi_d(\mathcal{E}) &= \deg \int_{\widehat{M}^m(c)} \Phi_d(\mathcal{E}) \\ &= \dim \widehat{M}^m(c) - N_f \dim V_0(c) - d = -(2r - N_f)(\mathrm{ch}_2(c), [\widehat{\mathbb{P}}^2]) - d.\end{aligned}$$

On the other hand, we can write

$$\int_{\widehat{M}^m(c - je_m)} \Phi(\mathcal{E}_\flat \oplus \bigoplus_{i=1}^{j} C_m \boxtimes e^{-\hbar_i}) \Psi^j(\mathcal{E}_\flat) = \int_{\widehat{M}^m(c-je_m)} \prod_{f=1}^{N_f} e(\mathcal{V}_0(\mathcal{E}_\flat) \otimes e^{m_f}) \cup \heartsuit$$

for some cohomology class $\heartsuit$. Therefore its degree is at most

$$\begin{aligned}&\dim \widehat{M}^m(c - je_m) - N_f \dim V_0(c - je_m) \\ = &-(2r - N_f)(\mathrm{ch}_2(c), [\widehat{\mathbb{P}}^2]) - j(m(2r - N_f) + r + j).\end{aligned}$$

Since $j(m(2r - N_f) + r + j) \geq r + 1$, it is zero if $d \leq r$.

In order to prove the vanishing for $d \leq 2r - N_f - 1$, we need a refinement of the general machinery in §1.5. We need to look at each step in the flowchart (Figure 1) more closely.

The first step §1.5.1 has no problem. In (1.6) we have

$$\Phi(\mathcal{E}_\flat \oplus \bigoplus_{n=0}^{m-1} \bigoplus_{i=1}^{j_n} C_n \otimes e^{\hbar_i^{n,1}}) = \Phi(\mathcal{E}_\flat) \prod_{n=0}^{m-1} \prod_{i=1}^{j_n} \Phi(C_n \otimes e^{\hbar_i^{n,1}}),$$

as we have a decomposition of a vector bundle $\mathcal{V}_0(\mathcal{E} \oplus \bigoplus_{n=0}^{m-1} \bigoplus_{i=1}^{j_n} C_n \otimes e^{\hbar_i^{n,1}}) = \mathcal{V}_0(\mathcal{E}) \oplus \bigoplus_{n=0}^{m-1} \bigoplus_{i=1}^{j_n} \mathcal{V}_0(C_n \otimes e^{\hbar_i^{n,1}})$, and the Euler class has a multiplicative with respect to the Whitney sum.

In §1.5.4, 1.5.5 we consider the tensor product $\mathcal{E}(-C)$, where $\mathcal{E}$ is the universal family on the moduli space of 1-stable sheaves. This causes a trouble because $R^1 q_{2*}(\mathcal{E}(-C - \ell_\infty))$ is not a vector bundle, as mentioned ealier. We need a closer look.

By [21, Lemma 7.3] the natural homomorphism $H^1(E(-\ell_\infty)) \to H^1(E(C - \ell_\infty))$ is surjective for a 0-stable framed sheaf (E, Φ). Therefore $H^1(E(-C - \ell_\infty)) \to H^1(E(-\ell_\infty))$ is surjective for a 1-stable framed sheaf (E, Φ). Let us give a direct proof since we need to understand the kernel. Suppose E is 1-stable. Since $\mathrm{Hom}(E, \mathcal{O}_C(-2)) = 0$, we have $E \otimes \mathcal{O}_C/\text{torsion} = \bigoplus \mathcal{O}_C(a_i)$ with $a_i \geq -1$. Therefore $H^1(E \otimes \mathcal{O}_C) = 0$. Since we have an exact sequence $0 = H^2(\mathrm{Tor}_1(E, \mathcal{O}_C)) \to H^1(E \otimes^L \mathcal{O}_C) \to H^1(E \otimes \mathcal{O}_C)$, it implies $H^1(E \otimes^L \mathcal{O}_C) = 0$, and hence $H^1(E(-C - \ell_\infty)) \to H^1(E(-\ell_\infty))$ is surjective.

Since the kernel of this surjective homomorphism is $H^0(E \otimes^L \mathcal{O}_C)$, we have

$$\begin{aligned} e(\mathcal{V}_0(\mathcal{E}(-C)) \otimes e^{m_f}) &= e(\mathcal{V}_0(\mathcal{E}) \otimes e^{m_f})\, e(q_{2*}(\mathcal{E} \otimes^L \mathcal{O}_C) \otimes e^{m_f}) \\ &= e(\mathcal{V}_0(\mathcal{E}) \otimes e^{m_f})\, c_a(q_{2*}([\mathcal{E}] \otimes [\mathcal{O}_C]) \otimes e^{m_f}) \end{aligned}$$

on $\widehat{M}^1(c)$, where $a = \dim H^1(E(-C-\ell_\infty)) - \dim H^1(E(-\ell_\infty)) = (c_1(E), [C]) + r$ and we replace $\mathcal{E}$, $\mathcal{O}_C$, q_{2*} by their K-theory classes and the K-theory pushforward in the last expression.

Now $\mathcal{V}_0(\mathcal{E})$ is a vector bundle over $\widehat{M}^0(c)$, $\widehat{M}^1(c)$ and master spaces from the quiver description in §4. Therefore $e(\mathcal{V}_0(\mathcal{E}) \otimes e^{m_f})$ (and also $c_a(q_{2*}([\mathcal{E}] \otimes [\mathcal{O}_C]) \otimes e^{m_f})$) are well-defined so we replace $e(\mathcal{V}_0(\mathcal{E}(-C)) \otimes e^{m_f})$ by the right hand side and continue the flow in Figure 1. We may still need to treat $\otimes\mathcal{O}(C)$ in a subsequent process in the flowchart. Then we again get $e(\mathcal{V}_0(\mathcal{E}(-C)) \otimes e^{m_f})$, so use the same procedure to replace by the right hand side.

As a result we can write

$$\int_{\widehat{M}^m(c)} \Phi_d(\mathcal{E}) = \sum_{\mathfrak{J}} \int_{\widehat{M}^0(c-c^{\mathfrak{J}})} \prod_{f=1}^{N_f} e(\mathcal{V}_0(\mathcal{E}_\flat) \otimes e^{m_f}) \Omega_d^{\mathfrak{J}}(\mathcal{E}_\flat) \tag{2.3}$$

for various $c^{\mathfrak{J}}$ with $c_1(c - c^{\mathfrak{J}}) = 0$ and cohomology classes $\Omega_d^{\mathfrak{J}}(\mathcal{E}_\flat)$. The left hand side has degree as in (2.2). On the other hand, the degree of the right hand side is at most

$$\dim M(p_*(c - c^{\mathfrak{J}})) - N_f \operatorname{rank} \mathcal{V}_0(\mathcal{E}_\flat) = (2r - N_f)(\Delta(c - c^{\mathfrak{J}}), [\widehat{\mathbb{P}}^2]).$$

If $c^{\mathfrak{J}}$ is nonzero, then it is at most $(2r - N_f)\{(\Delta(c), [\widehat{\mathbb{P}}^2]) - 1\}$, since $(\Delta(c - c^{\mathfrak{J}}), [\widehat{\mathbb{P}}^2])$ is an integer and we have (1.7). Therefore there is no contribution to the wall-crossing formula if $d < 2r - N_f$. For $c^{\mathfrak{J}} = 0$, we get $\int_{\widehat{M}^0(c)} \Phi_d(\mathcal{E})$, but it is equal to $\delta_{d0} \int_{M(p_*(c))} \prod_{f=1}^{N_f} e(\mathcal{V}(\mathcal{E}) \otimes e^{m_f})$ as $\widehat{M}^0(c) \to M(p_*(c))$ is an isomorphism and $\mu(C) = 0$ on $\widehat{M}^0(c)$. □

For a slightly modified version

$$\Phi'(\mathcal{E}) = \prod_{f=1}^{N_f} e(\mathcal{V}_1(\mathcal{E}) \otimes e^{m_f}) \times \exp\left(t\mu(C)\right),$$

the second part of the argument works, we get

Theorem 2.4. *Suppose* $(c_1, [C]) = 0$. *Then*

$$\int_{\widehat{M}^m(c)} \Phi'(\mathcal{E}) = \int_{M(p_*(c))} \prod_{f=1}^{N_f} e(\mathcal{V}(\mathcal{E}) \otimes e^{m_f}) + O(t^{2r-N_f}).$$

Moreover the coefficient of t^{2r-N_f} *is*

$$-\binom{2r - N_f - 2}{r-1} \int_{M(p_*(c)+\mathrm{pt})} \prod_{f=1}^{N_f} e(\mathcal{V}(\mathcal{E}) \otimes e^{m_f}),$$

if $N_f < 2r$.

For the last assertion, it is enough to calculate the case $(\Delta(c),[\widehat{\mathbb{P}}^2]) = 1$ by the same argument. (See the proof of Theorem 2.6 for more detail.) Hence §1.6 gives us the answer.

Next consider the $c_1 \neq 0$ case.

Theorem 2.5. *Suppose* $0 < n := (c_1,[C]) < r$. *Then*

$$\int_{\widehat{M}^m(c)} \Phi'(\mathcal{E}) = O(t^{n(r-n)}).$$

In fact, we have

$$\deg \int_{\widehat{M}^m(c)} \Phi'_d(\mathcal{E}) = (2r - N_f)\dim V_1(c) + n(r-n) - d,$$

and all terms in the right hand side of (2.3) has degrees at most $(2r - N_f)\dim V_1(c)$ as $\dim V_1(c - c^{\mathfrak{J}}) \le \dim V_1(c)$. So the same argument works.

Let us state what we observed in the above proof as a general structure theorem. Let $\Phi(\mathcal{E})$ be a multiplicative class in the universal family $\mathcal{E}$. Then

Theorem 2.6. *Let us fix* c_1 *with* $0 \le -(c_1,[C]) < r$. *There exists a class* $\Omega_j(\mathcal{E},t)$, *which is a polynomial in* $c_i(\mathcal{E})/[0]$ $(i = 2,\dots,r)$ *with coefficients in* $H^*_{\mathbb{C}^*\times\mathbb{C}^*}(\mathrm{pt})[[t]] = \mathbb{C}[\varepsilon_1,\varepsilon_2][[t]]$, *and independent of* $\Delta(c)$ *such that*

$$\int_{\widehat{M}(c)} \Phi(\mathcal{E}) \exp(t\mu(C)) = \sum_{j\ge 0} \int_{M(p_*(c)+j\,\mathrm{pt})} \Phi(\mathcal{E})\Omega_j(\mathcal{E},t). \tag{2.7}$$

Moreover $\Omega_j(\mathcal{E},t)$ *is unique if* $\Phi(\mathcal{E}) \neq 0$ *for* $H^*_{\widetilde{T}}(M(r,0,0)) = H^*_{\widetilde{T}}(\mathrm{pt}) = S(\widetilde{T})$.

For the theory with matters, the coefficients of $\Omega_j(\mathcal{E},t)$ are in $H^*_{\mathbb{C}^*\times\mathbb{C}^*\times(\mathbb{C}^*)^{N_f}}(\mathrm{pt})[[t]] = \mathbb{C}[\varepsilon_1,\varepsilon_2,m_1,\dots,m_{N_f}][[t]]$.

Proof. As in the derivation of (2.3), we obtain a formula as above, where $M(p_*(c)+j\,\mathrm{pt})$ is replaced by $\widehat{M}^0(p^*(p_*(c)+j\,\mathrm{pt}))$ and $\Omega_j(\mathcal{E},t)$ is a polynomial in Chern classes of $q_{2*}([\mathcal{E}]\otimes[\mathcal{O}_C(m)])$, $q_{2*}([\mathcal{E}]^\vee\otimes[\mathcal{O}_C(m)])$ of various m. This $\Omega_j(\mathcal{E},t)$ is independent of $\Delta(c)$, as

- $\Psi^j(\mathcal{E}_\flat)$ in Theorem 1.5 depends only on j,
- the choice, whether we perform twist by $\mathcal{O}(C)$ or not, is determined by c_1, and
- the Grassmannian in Proposition 1.2 is determined by r and c_1.

Thus it only remains to show that we can further replace Ω_j so that it is a polynomial in $c_i(\mathcal{E})/[0]$ $(i = 2,\dots,r)$.

By the Grothendieck-Riemann-Roch theorem, these classes can be expressed by $c_i(\mathcal{E})/[0] = c_i(q_{2*}([\mathcal{E}]\otimes[\mathcal{O}_C]))$. Note that $q_{2*}([\mathcal{E}]\otimes[\mathcal{O}_C]) = -\sum_{a=0}^2(-1)^a \operatorname{Ext}^a(\mathcal{O}_C(-1),\mathcal{E})$. If we cross back the wall from $\widehat{M}^0$ to $\widehat{M}^1$, we have $\operatorname{Ext}^0_{q_2}(\mathcal{O}_C(-1),\mathcal{E}) = 0 = \operatorname{Ext}^2_{q_2}(\mathcal{O}_C(-1),\mathcal{E})$ on $\widehat{M}^1$ by the remark after Lemma 4.11 below. Therefore $\operatorname{Ext}^1_{q_2}(\mathcal{O}_C(-1),\mathcal{E})$ is a vector bundle of rank r, and $c_i(\mathcal{E})/[0]$ vanishes for $i > r$. Since the difference of the integrals over $\widehat{M}^1$ and $\widehat{M}^0$ are expressed by integrals over small moduli spaces, we can eventually express Ω_j as a polynomial in $c_i(\mathcal{E})/[0]$ $(i = 2,\dots,r)$ by a recursion.

Let us show the uniqueness of $\Omega_j(\mathcal{E},t)$ by a recursion on j. Let us take the smallest possible $\Delta(c)$ with $\widehat{M}(c)\neq\emptyset$, i.e., the case when $c+(c_1,[C])e_0=(r,0,0)$ (cf. Proposition 1.2). Then we only have the term with $j=0$ in the right hand side of (2.7). In this case, $p_*(c)=(r,0,0)$ and the moduli space $M(r,0,0)$ is a single point. Therefore $\Omega_0(\mathcal{E},t)$ is determined by (2.7).

Now suppose that $\Omega_j(\mathcal{E},t)$ with $j<n$ are determined. Then we take c whose $\Delta(c)$ is n larger than the previous smallest $\Delta(c)$. Then j in the summation in (2.7) runs from 0 to n. Moreover $M(p_*(c)+n\,\mathrm{pt})=M(r,0,0)$ is a single point again. Therefore $\Omega_n(\mathcal{E},t)$ is determined from (2.7) and $\Omega_j(\mathcal{E},t)$ with $j<n$. □

Suppose that the degree of $\int_{\widehat{M}(c)}\Phi(\mathcal{E})$ is of a form $\gamma\Delta(c)+a(c_1,[C])+b$ for some constants γ, a and b, depending only on r. We further assume that $\gamma>0$, as in the case $\gamma=2r-N_f>0$ for the theory with matters. Then

$$\deg\int_{\widehat{M}(c)}\Phi(\mathcal{E})\mu(C)^d=\gamma\Delta(c)+a(c_1,[C])+b-d,$$
$$\deg\int_{M(p_*(c)+j\,\mathrm{pt})}\Phi(\mathcal{E})\Omega_j(\mathcal{E},t)\le\gamma(\Delta(p_*(c))-j)+b,$$

Since we are fixing $(c_1,[C])$ as in Theorem 2.6, the two degree cannot match if j is too large. This means that we only need to calculate finitely many $\Omega_j(\mathcal{E},t)$ to determine $\int_{\widehat{M}(c)}\Phi(\mathcal{E})\mu(C)^d$ for a fixed d. (In practice, the maximal j can be calculated explicitly.)

2.2. **K-theory version.** We derive the wall-crossing formula for a K-theoretic integration via the Grothendieck-Riemann-Roch formula.

Let $\mathrm{td}(\alpha)$ be the Todd class of a K-theory class α on various moduli spaces. The Todd class of the tangent bundle T_M of a variety M is denoted by $\mathrm{td}\,M$. We have

$$\mathrm{td}\,\widehat{M}^m(c)=\mathrm{td}(\mathfrak{N}(\mathcal{E},\mathcal{E})).$$

For integers d, l and $a=0$ or 1, we consider

$$\Phi(\mathcal{E})=\mathrm{td}(\mathfrak{N}(\mathcal{E},\mathcal{E}))\exp(lc_1(\mathcal{V}_a(\mathcal{E})))\exp(-d\,\mathrm{ch}_2(\mathcal{E})/[C]). \tag{2.8}$$

By a discussion in [20, several paragraphs preceding Def. 2.1], $-\mathrm{ch}_2(\mathcal{E})/[C]$ is the first Chern class of an equivariant line bundle up to a (rational) cohomology class in $H^2_{\widetilde{T}}(\mathrm{pt})$, which is zero if $(c_1,[C])=0$. Since this difference is immaterial in the following discussion (in particular in Theorem 2.11), we identify $-\mathrm{ch}_2(\mathcal{E})/[C]$ with the equivariant line bundle, which we denote by $\mu(C)$. (This is the same as $\mu(C)$ in the previous subsection up to a class in $H^2_{\widetilde{T}}(\mathrm{pt})$.) Then the equivariant Riemann-Roch theorem (see e.g., [9]) we have

$$\int_{\widehat{M}^m(c)}\Phi(\mathcal{E})=\tau\left(\widehat{\pi}_*(\mu(C)^{\otimes d}\otimes\det\mathcal{V}_a(\mathcal{E})^{\otimes l})\right),$$

where τ is the equivariant Todd homomorphism $\tau\colon K^{\widetilde{T}}(M_0(p_*(c)))\to H_*^{\widetilde{T}}(M_0(p_*(c)))$ for the Uhlenbeck partial compactification $M_0(p_*(c))$, and $\widehat{\pi}_*$ is the push-forward homomorphism in the equivariant K-theory.

We have

$$\mathrm{td}(\mathfrak{N}(\mathcal{E},\mathcal{E}))=\mathrm{td}(\mathfrak{N}(\mathcal{E}_1,\mathcal{E}_1))\,\mathrm{td}(\mathfrak{N}(\mathcal{E}_1,\mathcal{E}_2))\,\mathrm{td}(\mathfrak{N}(\mathcal{E}_2,\mathcal{E}_1))\,\mathrm{td}(\mathfrak{N}(\mathcal{E}_2,\mathcal{E}_2))$$

if $\mathcal{E} = \mathcal{E}_1 \oplus \mathcal{E}_2$. Then from Theorem 1.5 we get

$$\int_{\widehat{M}^{m+1}(c)} \Phi(\mathcal{E}) - \int_{\widehat{M}^m(c)} \Phi(\mathcal{E}) = \sum_{j=1}^{\infty} \int_{\widehat{M}^m(c-je_m)} \Phi(\mathcal{E}_\flat) \cup \operatorname*{Res}_{\hbar_j=0} \cdots \operatorname*{Res}_{\hbar_1=0} \Psi^j(\mathcal{E}_\flat),$$

where

$$\Psi^j(\bullet) := \frac{1}{j!}\left[\prod_{i=1}^{j} \frac{\exp\left(lc_1(\mathcal{V}_a(C_m \otimes e^{-\hbar_i})) - d\operatorname{ch}_2(C_m \otimes e^{-\hbar_i})/[C]\right)}{e^K(\mathfrak{N}(\bullet, C_m)\otimes e^{-\hbar_i})\, e^K(\mathfrak{N}(C_m,\bullet)\otimes e^{\hbar_i})} \prod_{1\le i_1\neq i_2\le j}(1-e^{\hbar_{i_1}-\hbar_{i_2}})\right].$$

Here e^K is the (Chern character of) K-theoretic Euler class:

$$e^K(\alpha) = e(\alpha)\operatorname{td}(\alpha)^{-1} = \sum_{p=0}^{\infty}(-1)^p \operatorname{ch}(\textstyle\bigwedge^p \alpha^\vee),$$

where $e(\alpha)$ is the usual Euler class as before.

Strictly speaking, we need to consider the completion $\mathbb{C}[\hbar_i^{-1}, \hbar_i]]$ for the coefficient rings of the localized equivariant homology groups of moduli spaces, as for example, $e^{-\hbar_i}$ is not allowed. Here $\mathbb{C}[\hbar_i^{-1}, \hbar_i]]$ is the algebra of formal power series $\sum a_j \hbar^j$ such that $\{j < 0 \mid a_j \neq 0\}$ is finite. The modification appears only at §6.1 and the beginning of §6.3, and the rest of the proof remains unchanged.

Observe that $\hbar_i$ appears always as a function in $x_i := e^{-\hbar_i} - 1$ in the above formula. We change the coefficient ring from $\mathbb{C}[\hbar_i^{-1}, \hbar_i]]$ to $\mathbb{C}[x_i^{-1}, x_i]]$.

We have

$$\operatorname*{Res}_{\hbar_i=0} f(e^{-\hbar_i}-1) = -\operatorname*{Res}_{x_i=0}\frac{f(x_i)}{x_i+1}$$

as $dx_i = -e^{-\hbar_i}d\hbar_i = -(x_i+1)d\hbar_i$. Therefore

Theorem 2.9.

$$\int_{\widehat{M}^{m+1}(c)} \Phi(\mathcal{E}) - \int_{\widehat{M}^m(c)} \Phi(\mathcal{E}) = \sum_{j=1}^{\infty} \int_{\widehat{M}^m(c-je_m)} \Phi(\mathcal{E}_\flat) \cup \operatorname*{Res}_{x_j=0} \cdots \operatorname*{Res}_{x_1=0} \Psi^j(\mathcal{E}_\flat),$$

where

$$\Psi^j(\bullet) := \frac{1}{j!}\prod_{i=1}^{j} \frac{\exp\left(lc_1(\mathcal{V}_a(C_m \otimes (1+x_i))) - d\operatorname{ch}_2(C_m \otimes (1+x_i))/[C]\right)}{e^K(\mathfrak{N}(\bullet, C_m)\otimes (1+x_i))\, e^K(\mathfrak{N}(C_m,\bullet)\otimes \frac{1}{1+x_i})(-(1+x_i))}$$
$$\times \prod_{1\le i_1\neq i_2\le j}\frac{x_{i_1}-x_{i_2}}{1+x_{i_1}}.$$

In view of Proposition 1.2 the following is useful to replace the K-theoretic integration on $\widehat{M}^1(c)$ by one on $\widehat{N}(c,n)$.

Lemma 2.10. *Consider the diagram in Proposition 1.2. We have*

$$\mathbf{R}f_{1*}(\mathcal{O}_{\widehat{N}(c,n)}) = \mathcal{O}_{\widehat{M}^1(c)}.$$

Proof. Since f_1 is a proper birational morphism between smooth varieties, this is a well-known result (see e.g., [11, §5.1]). $\square$

In the remaining of this section we study the vanishing theorem for small d.

Theorem 2.11. *Assume* $0 \le l \le r$.
(1) *Suppose* $(c_1, [C]) = 0$. *If* $0 \le al + d \le r$,

$$\int_{\widehat{M}(c)} \Phi(\mathcal{E}) = \int_{M(p_*(c))} \operatorname{td}(\mathfrak{N}(\mathcal{E},\mathcal{E})) \exp(lc_1(\mathcal{V}(\mathcal{E}))),$$

where $\mathcal{V}(\mathcal{E})$ *is the vector bundle defined as in Theorem 2.1.*
(2) *Suppose* $a = 1$ *and* $0 < (c_1, [C]) < r$. *If* $0 < d \le \min(r + (c_1,[C]) - l, r-1)$,

$$\int_{\widehat{M}(c)} \Phi(\mathcal{E}) = 0.$$

This result was conjectured in [8, (1.37),(1.43)].

Proof. We study factors of $\Psi^j(\mathcal{E}_\flat)$ more closely. Note that $\mathcal{E}_\flat$ is the universal sheaf for $\widehat{M}^m(c - je_m)$, hence

$$\operatorname{rank} \mathfrak{N}(C_m, \mathcal{E}_\flat) = (m+1)r + (c_1,[C]) + j, \quad \operatorname{rank} \mathfrak{N}(\mathcal{E}_\flat, C_m) = mr + (c_1,[C]) + j.$$

If $\{\alpha_a\}$, $\{\beta_b\}$ are Chern roots of $\mathfrak{N}(C_m,\mathcal{E}_\flat)$, $\mathfrak{N}(\mathcal{E}_\flat, C_m)$ respectively, we have

$$\begin{aligned} \frac{1}{e^K(\mathfrak{N}(C_m,\mathcal{E}_\flat)\otimes \frac{1}{1+x_i})} &= \frac{\exp(\sum_a \alpha_a)}{(-x_i)^{(m+1)r+(c_1,[C])+j}} \prod_a \frac{1}{1+\frac{1-e^{\alpha_a}}{x_i}}, \\ \frac{1}{e^K(\mathfrak{N}(\mathcal{E}_\flat, C_m)\otimes (1+x_i))} &= \left(1+\frac{1}{x_i}\right)^{mr+(c_1,[C])+j} \prod_b \frac{1}{1+\frac{1-e^{-\beta_b}}{x_i}}. \end{aligned} \tag{2.12}$$

On the other hand, we have

$$\exp(lc_1(\mathcal{V}_a(C_m\otimes(1+x_i)))) = \exp(lc_1(\mathcal{V}_a(C_m)))(1+x_i)^{l(m+a)}$$

as $\operatorname{rank}\mathcal{V}_a(C_m) = m+a$. Also

$$\exp(-d\operatorname{ch}_2(C_m\otimes(1+x_i))/[C]) = \exp(-d\operatorname{ch}_2(C_m)/[C])(1+x_i)^d,$$

as $\operatorname{ch}_1(C_m)/[C] = -1$.

Let us expand $\Psi^j(\mathcal{E}_\flat)$ into formal Laurent power series in x_1. Note that we have the remaining factor

$$\frac{1}{(1+x_1)}\prod_{i_2\ne 1}\frac{x_1 - x_{i_2}}{1+x_1} = \frac{\prod_{i_2\ne1}(x_1-x_{i_2})}{(1+x_1)^j}$$

from $\Psi^j(\bullet)$ in Theorem 2.9. This term can be absorbed into the second equality of (2.12) as

$$\frac{\prod_{i_2\ne1}(x_1-x_{i_2})}{(1+x_1)^j}\left(1+\frac{1}{x_1}\right)^{mr+(c_1,[C])+j} = \frac{1}{x_1}\left(1+\frac{1}{x_1}\right)^{mr+(c_1,[C])}\prod_{i_2\ne1}(1-\frac{x_{i_2}}{x_1}).$$

Note that $mr + (c_1,[C]) \ge 0$, and hence $(1+\frac{1}{x_1})^{mr+(c_1,[C])}$ is a polynomial in x_1^{-1}. We also write

$$(1+x_1)^{l(m+a)+d} = x_1^{l(m+a)+d}\left(1+\frac{1}{x_1}\right)^{l(m+a)+d}$$

as a Laurent polynomial in x_1^{-1}. Note that we have $l(m+a)+d \geq la+d \geq 0$ by our assumption. Therefore we have

$$\Psi^j(\mathcal{E}_\flat) = \frac{1}{x_1^N} f(x_1^{-1})$$

for some formal power series $f(x_1^{-1})$ in x_1^{-1} with

$$\begin{aligned} N &= (m+1)r + (c_1,[C]) + j + 1 - l(m+a) - d \\ &= m(r-l) + (r-d) + (c_1,[C]) - la + j + 1. \end{aligned}$$

Since $0 \leq r-l$ and $0 \leq m$, the first term $m(r-l)$ is nonnegative. We also have $j \geq 1$. Therefore we have $N \geq 2$ if $d+la \leq r+(c_1,[C])$. This shows that there are no wall-crossing term, i.e.,

$$\int_{\widehat{M}(c)} \Phi(\mathcal{E}) = \int_{\widehat{M}^0(c)} \Phi(\mathcal{E}).$$

If $(c_1,[C]) = 0$, we have an isomorphism $\Pi\colon \widehat{M}^0(c) \to M(p_*(c))$ given by $\Pi(E,\Phi) = (p_*(E),\Phi)$, $\Pi^{-1}(F,\Phi) = (p^*(F),\Phi)$ (see §1.5.3 for a precise statement). Therefore the tangent bundles $\mathfrak{N}(\mathcal{E},\mathcal{E})$ for $\widehat{M}(c)$ and $M(p_*(c))$ are isomorphic to each other. Vector bundles $\mathcal{V}_0(\mathcal{E})$ for $\widehat{M}(c)$ and $M(p_*(c))$ are isomorphic to each other from the description of Π in §1.5.3. We also have $\operatorname{ch}_2(\mathcal{E})/[C] = 0$. These show (1).

To show (2), we consider $\widehat{M}^0(c) \cong \widehat{M}^1(ce^{[C]})$ given by $(E,\Phi) \mapsto (E(C),\Phi)$. Then Proposition 1.2 is applicable. We then have

$$\int_{\widehat{M}^0(c)} \Phi(\mathcal{E}) = \int_{\widehat{M}^1(ce^{[C]})} \operatorname{td}(\mathfrak{N}(\mathcal{E},\mathcal{E})) \exp(lc_1(\mathcal{V}_0(\mathcal{E}))) \exp(-d \operatorname{ch}_2(\mathcal{E}(-C))/[C])$$

where $\mathcal{E}$ in the right hand side is the universal sheaf for $\widehat{M}^1(ce^{[C]})$. As we explained in the beginning of this subsection, we may replace this integral by

$$\tau(\widehat{\pi}_*(\mu(C)^{\otimes d} \otimes \det \mathcal{V}_0(\mathcal{E})^{\otimes l})).$$

The difference between $\operatorname{ch}_2(\mathcal{E}(-C))/[C]$ and $\operatorname{ch}_2(\mathcal{E})/[C]$ is immaterial, as it is a class pulled back from $M_0(p_*(c))$. By Lemma 2.10 and the projection formula, we can replace the above by the push-forward from $N(ce^{[C]},n)$ with $n = r-(c_1,[C])$, where $\mu(C)$, $\mathcal{V}_0(\mathcal{E})$ are replaced by their pull-backs by f_1. From the exact sequence in Proposition 1.2(3) we have

$$f_1^*\mathcal{V}_0(\mathcal{E}) = f_2^*\mathcal{V}_0(\mathcal{E}')$$

where $\mathcal{E}'$ is the universal sheaf for $\widehat{M}^0(ce^{[C]} - ne_0)$. We also have

$$f_1^*\mu(C) = \det \mathcal{S}$$

up to the pull-back of a line bundle from $\widehat{M}^0(ce^{[C]} - ne_0)$ by f_2. Therefore it is enough to show that

$$f_{2*}(\det \mathcal{S}^{\otimes d}) = 0 \quad \text{for } 0 < d < r.$$

Since f_2 is a Grassmann bundle of n-planes in a rank r vector bundle, this vanishing is a special case of Bott vanishing theorem [1] or a direct consequence of Kodaira vanishing theorem. $\square$

2.3. **Casimir operators.** We generalize the vanishing result in the previous subsection to the case when we integrate certain K-theoretic classes given by universal sheaves on moduli spaces.

Let ψ^p be the p^{th} Adams operation (see e.g., [5, I.§6]). We will use it for $p < 0$ defined by $\psi^p(x) = \psi^{-p}(x^\vee)$. For indeterminates $\vec{\tau} = (\cdots, \tau_{-2}, \tau_{-1}, \tau_1, \tau_2, \cdots)$ and $\vec{t} = (\cdots, t_{-2}, t_{-1}, t_1, t_2, \cdots)$ we consider a generalization of (2.8) with $a = 0$:

$$\begin{aligned}\Phi(\mathcal{E}) = \operatorname{td}(\mathfrak{N}(\mathcal{E},\mathcal{E}))\exp(lc_1(\mathcal{V}_0(\mathcal{E})) - d\operatorname{ch}_2(\mathcal{E})/[C])\\ \times\exp\left(\sum_{p\in\mathbb{Z}\setminus\{0\}}\tau_p\operatorname{ch}\left(\psi^p(\mathcal{E})/[\widehat{\mathbb{C}}^2]\right) + t_p\operatorname{ch}\left(\psi^p(\mathcal{E})\otimes\mathcal{O}_C(-1)/[\widehat{\mathbb{C}}^2]\right)\right),\end{aligned}$$

where the K-theoretic slant product $\bullet/[\widehat{\mathbb{C}}^2]$ is defined by $\bullet/[\widehat{\mathbb{C}}^2] = q_{2*}(\bullet\otimes q_1^* p^*(K_{\mathbb{C}^2}^{1/2}))$ with $p\colon \widehat{\mathbb{C}}^2 \to \mathbb{C}^2$. We have two remarks for the definition. First we define the push-forward q_{2*} by the localization formula, i.e., the sum of the fixed point contributions, since q_2 is *not* proper. This is a standard technique in instanton counting and its meaning was explained in detail in [18, §4]. Second $K_{\mathbb{C}^2}^{1/2}$ is the trivial line bundle together twisted by the square root of the character of $(\mathbb{C}^*)^2$, which we consider as a character of its double cover. Having the above two remarks in mind, we see that the above integral is essentially defined by the equivariant K-theory push-forward as before.

We expand $\int_{\widehat{M}^m(c)}\Phi(\mathcal{E})$ in t_p, τ_p and consider coefficients

$$\int_{\widehat{M}^m(c)}\Phi(\mathcal{E}) = \sum_{\vec{n},\vec{m}}\prod_{p\neq 0}\tau_p^{n_p} t_p^{m_p}\int_{\widehat{M}^m(c)}\Phi_{\vec{n},\vec{m}}(\mathcal{E}),$$

where $\vec{n} = (\cdots, n_{-1}, n_1, \cdots)$, $\vec{m} = (\cdots, m_{-1}, m_1, \cdots)$. If we set

$$\left(\frac{\partial}{\partial\vec{\tau}}\right)^{\vec{n}} := \prod_{p\neq 0}\left(\frac{\partial}{\partial\tau_p}\right)^{n_p}, \quad \vec{n}\,! = \prod_{p\neq 0} n_p!,$$

we have

$$\begin{aligned}\int_{\widehat{M}^m(c)}\Phi_{\vec{n},\vec{m}}(\mathcal{E}) &= \frac{1}{\vec{n}\,!\,\vec{m}\,!}\left(\frac{\partial}{\partial\vec{\tau}}\right)^{\vec{n}}\left(\frac{\partial}{\partial\vec{t}}\right)^{\vec{m}}\int_{\widehat{M}^m(c)}\Phi(\mathcal{E})\Bigg|_{\vec{\tau}=\vec{t}=0}\\ &= \frac{1}{\vec{n}\,!\,\vec{m}\,!}\int_{\widehat{M}^m(c)}\operatorname{td}(\mathfrak{N}(\mathcal{E},\mathcal{E}))\exp(lc_1(\mathcal{V}_0(\mathcal{E})) - d\operatorname{ch}_2(\mathcal{E})/[C])\\ &\qquad\times\operatorname{ch}\left(\bigotimes_p\left(\psi^p(\mathcal{E})/[\widehat{\mathbb{C}}^2]\right)^{\otimes n_p}\otimes\left(\psi^p(\mathcal{E})\otimes\mathcal{O}_C(-1)/[\widehat{\mathbb{C}}^2]\right)^{\otimes m_p}\right).\end{aligned}$$

Proposition 2.13. *Suppose* $(c_1, [C]) = 0$.

(1) *Assume the followings*:

(a) $0 \le d + \sum_{p<0} pn_p + pm_p$, *and*

(b) $d + \sum_{p>0} pn_p + pm_p \le r$.

Then the wall-crossing term is zero, i.e.,

$$\int_{\widehat{M}^m(c)} \Phi_{\vec{n},\vec{m}}(\mathcal{E}) = \int_{\widehat{M}^0(c)} \Phi_{\vec{n},\vec{m}}(\mathcal{E}).$$

(2) *We further assume $m_p \neq 0$ for some p. Then*

$$\int_{\widehat{M}^m(c)} \Phi_{\vec{n},\vec{m}}(\mathcal{E}) = 0.$$

Proof. (1) We note that

$$\psi^p(\mathcal{E}_\flat \oplus C_m \otimes (1 + x_i)) = \psi^p(\mathcal{E}_\flat) + \psi^p(C_m) \otimes (1 + x_i)^p,$$

as the Adams operation is a homomorphism. Then the proof exactly goes as before.

(2) Recall that $\widehat{M}^0(c) \cong M(p_*(c))$ under $(E, \Phi) \mapsto (p_*(E), \Phi)$. Then $\psi^k(\mathcal{E}) \otimes \mathcal{O}_C(-1)/[\widehat{\mathbb{C}}^2]$ vanishes, since $p_*(\mathcal{O}_C(-1)) = 0$. □

3. Partition function and Seiberg-Witten curves

In this section we explain an application of the vanishing theorems to Nekrasov partition functions. Here a reader is supposed to be familiar with [18, 19, 20, 8].

3.1. **Partition function.** Let us fix $l \in \mathbb{Z}$. We define a partition function as the generating function of integrals considered in §§2.2, 2.3:

$$Z_l^{\text{inst}}(\varepsilon_1, \varepsilon_2, \vec{a}; \Lambda, \vec{\tau}) := \sum_c (\Lambda^{2r} e^{-(r+l)(\varepsilon_1+\varepsilon_2)/2})^{(\Delta(c),[\mathbb{P}^2])} \times \int_{M(c)} \operatorname{td} M(c) \exp(l c_1(\mathcal{V}(\mathcal{E}))) \exp\left(\sum_{p \in \mathbb{Z} \setminus \{0\}} \tau_p \operatorname{ch}\left(\psi^p(\mathcal{E})/[\mathbb{C}^2]\right) \right), \tag{3.1}$$

where the rank $r = r(c)$ is fixed. Here $\vec{a} = (a_1, \dots, a_r)$ ($\sum a_\alpha = 0$) is the vector given by generators a_i of $H_T^*(\mathrm{pt})$ and ε_1, ε_2 ones of $H^*_{(\mathbb{C}^*)^2}(\mathrm{pt})$, and the integrals, more coefficients of monomials in τ_p's, take values in the quotient field $\mathfrak{S}(\widetilde{T})$ of $H^*_{\widetilde{T}}(\mathrm{pt})$ as explained in §1.3. (More precisely the quotient field of the representation ring $R(\widetilde{T}) = \mathbb{Z}[e^{\pm\varepsilon_1}, e^{\pm\varepsilon_2}, e^{\pm a_\alpha}]$ as in §2.2 since this is a K-theoretic partition function.) And the K-theoretic slant product $\bullet/[\mathbb{C}^2]$ is defined by $q_{2*}(\bullet \otimes q_1^*(K_{\mathbb{C}^2}^{1/2}))$ as in §2.3.

We have

$$\left(\frac{\partial}{\partial \vec{\tau}}\right)^{\vec{n}} Z_l^{\text{inst}}(\varepsilon_1, \varepsilon_2, \vec{a}; \Lambda, \vec{\tau} = 0) = \sum_c (\Lambda^{2r} e^{-(r+l)(\varepsilon_1+\varepsilon_2)/2})^{(\Delta(c),[\mathbb{P}^2])} \times \int_{M(c)} \operatorname{ch}\left(\bigotimes_p \left(\psi^p(\mathcal{E})/[\mathbb{C}^2]\right)^{\otimes n_p} \right) \operatorname{td} M(c) \exp(l c_1(\mathcal{V}(\mathcal{E}))). \tag{3.2}$$

Therefore $Z_l^{\text{inst}}(\varepsilon_1, \varepsilon_2, \vec{a}; \Lambda, \tau)$ gives us integrals of any tensor products of various Adams operators applied to the universal sheaves.

We consider fixed points of the $\widetilde{T}$-action on $M(c)$ as in [18, §2]: they are parametrized by r-tuples of Young diagrams $\vec{Y} = (Y_1, \dots, Y_r)$ with $|\vec{Y}| = \sum |Y_\alpha| = (\Delta(c), [\mathbb{P}^2])$ corresponding to direct sums of monomial ideals in $\mathbb{C}[x,y]$. The character of the fiber of $\mathcal{V}(\mathcal{E})$ at the fixed point $\vec{Y}$ is given by

$$\operatorname{ch}(\mathcal{V}(\mathcal{E})|_{\vec{Y}})(\varepsilon_1, \varepsilon_2, \vec{a}) = \sum_{\alpha=1}^{r} e^{a_\alpha} \sum_{s \in Y_\alpha} e^{-l'(s)\varepsilon_1 - a'(s)\varepsilon_2},$$

where $a'(s)$, $l'(s)$ are as in [18, §2]. We also have

$$\exp(lc_1(\mathcal{V}(\mathcal{E})|_{\vec{Y}})) = \exp\left[l \sum_{\alpha=1}^{r} \sum_{s\in Y_\alpha} (a_\alpha - l'(s)\varepsilon_1 - a'(s)\varepsilon_2) \right].$$

Therefore we have

$$\begin{aligned} Z_l^{\text{inst}}(\varepsilon_1, \varepsilon_2, \vec{a}; \Lambda, \vec{\tau}) &= \sum_{\vec{Y}} \frac{(\Lambda^{2r} e^{-r(\varepsilon_1+\varepsilon_2)/2})^{|\vec{Y}|}}{\prod_{\alpha,\beta} n^{\vec{Y}}_{\alpha,\beta}(\varepsilon_1, \varepsilon_2, \vec{a})} \\ &\quad \times \exp\left[l \sum_{\alpha=1}^{r} \sum_{s\in Y_\alpha} (a_\alpha - l'(s)\varepsilon_1 - a'(s)\varepsilon_2 - \frac{\varepsilon_1+\varepsilon_2}{2}) \right] \\ &\quad \times \exp\left[\sum_{p,\alpha} \tau_p \frac{e^{pa_\alpha}\left\{1 - (1-e^{-p\varepsilon_1})(1-e^{-p\varepsilon_2}) \sum_{s\in Y_\alpha} e^{-pl'(s)\varepsilon_1 - pa'(s)\varepsilon_2}\right\}}{(e^{\varepsilon_1/2} - e^{-\varepsilon_1/2})(e^{\varepsilon_2/2} - e^{-\varepsilon_2/2})} \right], \end{aligned} \tag{3.3}$$

where $n^{\vec{Y}}_{\alpha,\beta}(\varepsilon_1, \varepsilon_2, \vec{a})$ is the alternating sum of characters of exterior powers of the cotangent space of $M(r,n)$ at the fixed point $\vec{Y}$. Its explicit formula was given in [20, §1.2], where it was denoted by $n^{\vec{Y}}_{\alpha,\beta}(\varepsilon_1, \varepsilon_2, \vec{a}; \boldsymbol{\beta})$, and we put $\boldsymbol{\beta} = 1$.

We have

$$Z_l^{\text{inst}}(\varepsilon_1, \varepsilon_2, \vec{a}; \Lambda, \vec{\tau}) = Z_{-l}^{\text{inst}}(-\varepsilon_1, -\varepsilon_2, -\vec{a}; \Lambda, {}'\vec{\tau}), \tag{3.4}$$

where ${}'\vec{\tau}$ is given by ${}'\tau_p = \tau_{-p}$. This symmetry is a simple consequence of [8, the displayed formula one below (1.33)], or the Serre duality.

Let $d \in \mathbb{Z}_{\geq 0}$. We consider a similar partition function on the blow-up:

$$\begin{aligned} \widehat{Z}_{l,k,d}^{\text{inst}}(\varepsilon_1, \varepsilon_2, \vec{a}; \Lambda, \vec{\tau}, \vec{t}) &:= \sum_{c} (\Lambda^{2r} e^{-(r+l)(\varepsilon_1+\varepsilon_2)/2})^{(\Delta(c), [\mathbb{P}^2])} \\ &\quad \times \int_{\widehat{M}(c)} \operatorname{td} \widehat{M}(c) \exp(lc_1(\mathcal{V}_0(\mathcal{E})) - d \operatorname{ch}_2(\mathcal{E})/[C]) \\ &\qquad \times \exp\left(\sum_{p\in\mathbb{Z}\setminus\{0\}} \tau_p \operatorname{ch}\left(\psi^p(\mathcal{E})/[\widehat{\mathbb{C}}^2]\right) + t_p \operatorname{ch}\left(\psi^p(\mathcal{E}) \otimes \mathcal{O}_C(-1)/[\widehat{\mathbb{C}}^2]\right) \right), \end{aligned}$$

where we also fix $(c_1(c), [C]) = -k$ in this case.

This is related to the partition function (3.1) by

$$
\begin{aligned}
&\widehat{Z}^{\mathrm{inst}}_{l,k,d}(\varepsilon_1,\varepsilon_2,\vec{a},\Lambda,\vec{\tau},\vec{t})\\
&= \sum_{\substack{\vec{k}=(k_\alpha)\in\mathbb{Z}^r\\ \sum k_\alpha = k}} \frac{\left(e^{(\varepsilon_1+\varepsilon_2)(d-(r+l)/2)}\Lambda^{2r}\right)^{(\vec{k}^2)/2} e^{(d-l/2)(\vec{k},\vec{a})}}{\prod_{\vec{\alpha}\in\Delta} l^{\vec{k}}_{\vec{\alpha}}(\varepsilon_1,\varepsilon_2,\vec{a})}\\
&\qquad \times \exp\left[l\left(\frac{1}{6}(\varepsilon_1+\varepsilon_2)\sum_\alpha k_\alpha^3 + \frac{1}{2}\sum_\alpha k_\alpha^2 a_\alpha\right)\right]\\
&\qquad\quad \times Z^{\mathrm{inst}}_l(\varepsilon_1,\varepsilon_2-\varepsilon_1,\vec{a}+\varepsilon_1\vec{k};\Lambda e^{\frac{d-(r+l)/2}{2r}\varepsilon_1}, e^{-\varepsilon_1/2}(\vec{\tau}+(e^{\varepsilon_1}-1)\vec{t}))\\
&\qquad\quad\ \times Z^{\mathrm{inst}}_l(\varepsilon_1-\varepsilon_2,\varepsilon_2,\vec{a}+\varepsilon_2\vec{k};\Lambda e^{\frac{d-(r+l)/2}{2r}\varepsilon_2}, e^{-\varepsilon_2/2}(\vec{\tau}+(e^{\varepsilon_2}-1)\vec{t})),
\end{aligned} \tag{3.5}
$$

where $l^{\vec{k}}_{\vec{\alpha}}(\varepsilon_1,\varepsilon_2,\vec{a})$ is a function given in [20, (2.3)].

Let us briefly explain how this formula is proved. It is a consequence of the Atiyah-Bott-Lefschetz fixed point formula applied to the $\widetilde{T}$-action on $\widehat{M}(c)$. The fixed points are parametrized by $(\vec{k},\vec{Y}^1,\vec{Y}^2)$, where $\vec{k}\in\mathbb{Z}^r$ corresponds to a line bundle $\mathcal{O}(k_\alpha C)$, and $\vec{Y}^1$, $\vec{Y}^2$ are Young diagrams corresponding to monomial ideals in the toric coordinates at the $\mathbb{C}^*\times\mathbb{C}^*$-fixed point $p_1=([1:0:0],[1:0])$ and $p_2=([1:0:0],[0:1])$ in $\widehat{\mathbb{P}}^2$. (See [18, §3] for more detail.) The structure of the above formula, i.e., the sum over $\vec{k}$ of the product of two partition functions comes from this description of the fixed point set. The shift of variables in the partition functions come from study of tangent bundles, universal sheaves at fixed points. All these are done in [18, §3], [8, §1.7], except the expression $e^{-\varepsilon_a/2}(\vec{\tau}+(e^{\varepsilon_a}-1)\vec{t})$ $(a=1,2)$ appears for variables for the Adams operators.

If we replace the Adams operator ψ^p by the degree p part of the Chern character, the expression was given in [19, §4], where we just need to change variables as $\vec{\tau}+\varepsilon_a\vec{t}$. In our situation, $\vec{t}$ is multiplied by $\mathrm{ch}(\mathcal{O}_C(-1))|_{p_a}=e^{\varepsilon_a}-1$ instead of ε_a. The factor $e^{-\varepsilon_a/2}$ appears as the 'square root' of $K_{\mathbb{C}^2}\otimes K_{\widehat{\mathbb{C}}^2}^{-1}$ at the fixed point p_a, since the K-theoretic slant product $\bullet/[\widehat{\mathbb{C}}^2]$ was defined as $q_{2*}(\bullet\otimes q_1^*p^*(K_{\mathbb{C}^2}^{1/2}))$, not as '$q_{2*}(\bullet\otimes q_1^*(K_{\widehat{\mathbb{C}}^2}^{1/2}))$'. (We avoid $K_{\widehat{\mathbb{C}}^2}^{1/2}$, which cannot be defined.)

We have

$$
\widehat{Z}^{\mathrm{inst}}_{l,k,d}(\varepsilon_1,\varepsilon_2,\vec{a};\Lambda,\vec{\tau}=0,\vec{t}) = \widehat{Z}^{\mathrm{inst}}_{-l,k,r-d}(-\varepsilon_1,-\varepsilon_2,-\vec{a};\Lambda,\vec{\tau}=0,-'\vec{t}).
$$

This is proved exactly as in [8, the last displayed formula in §1.7.1] and (3.4), or the Serre duality.

We define the perturbation part, see [20, §4.2] and [8, §1.7.2] for more details. We set

$$\begin{aligned}\gamma_{\varepsilon_1,\varepsilon_2}(x;\Lambda) :=\ & \frac{1}{2\varepsilon_1\varepsilon_2}\left(-\frac{1}{6}\left(x+\frac{1}{2}(\varepsilon_1+\varepsilon_2)\right)^3 + x^2\log\Lambda\right) \\ & + \sum_{n\geq 1}\frac{1}{n}\frac{e^{-nx}}{(e^{n\varepsilon_1}-1)(e^{n\varepsilon_2}-1)}, \\ \widetilde{\gamma}_{\varepsilon_1,\varepsilon_2}(x;\Lambda) :=\ & \gamma_{\varepsilon_1,\varepsilon_2}(x;\Lambda) + \frac{1}{\varepsilon_1\varepsilon_2}\left(\frac{\pi^2 x}{6}-\zeta(3)\right) \\ & + \frac{\varepsilon_1+\varepsilon_2}{2\varepsilon_1\varepsilon_2}\left(x\log\Lambda + \frac{\pi^2}{6}\right) + \frac{\varepsilon_1^2+\varepsilon_2^2+3\varepsilon_1\varepsilon_2}{12\varepsilon_1\varepsilon_2}\log\Lambda\end{aligned}$$

for (x,Λ) in a neighborhood of $\sqrt{-1}\mathbb{R}_{>0}\times\mathbb{R}_{>0}$.

We define the full partition function by

$$\begin{aligned}& Z_l(\varepsilon_1,\varepsilon_2,\vec{a};\Lambda,\vec{\tau}) := \exp\left[-\sum_{\alpha\neq\beta}\widetilde{\gamma}_{\varepsilon_1,\varepsilon_2}(a_\alpha-a_\beta;\Lambda) - l\sum_{\alpha=1}^{r}\frac{a_\alpha^3}{6\varepsilon_1\varepsilon_2}\right] Z_l^{\mathrm{inst}}(\varepsilon_1,\varepsilon_2,\vec{a};\Lambda,\vec{\tau}), \\ & \widehat{Z}_{l,k,d}(\varepsilon_1,\varepsilon_2,\vec{a};\Lambda,\vec{\tau},\vec{t}) \\ & \quad := \exp\left[-\sum_{\alpha\neq\beta}\widetilde{\gamma}_{\varepsilon_1,\varepsilon_2}(a_\alpha-a_\beta;\Lambda) - l\sum_{\alpha=1}^{r}\frac{a_\alpha^3}{6\varepsilon_1\varepsilon_2}\right]\widehat{Z}_{l,k,d}^{\mathrm{inst}}(\varepsilon_1,\varepsilon_2,\vec{a};\Lambda,\vec{\tau},\vec{t}).\end{aligned}$$

Using the difference equation satisfied by the perturbation terms (see [20, §4.2] and [8, §1.7.2]), we can absorb the factors in (3.5) coming from line bundles $\mathcal{O}(k_\alpha C)$ into the partition function to get

(3.6)

$$\begin{aligned}& \widehat{Z}_{l,k,d}(\varepsilon_1,\varepsilon_2,\vec{a};\Lambda,\vec{\tau},\vec{t}) \\ & = \exp\left[\left\{-\frac{(4\,(d+l\,(-1/2+k/r))-r)\,(r-1)}{48}+\frac{k^3 l}{6r^2}\right\}(\varepsilon_1+\varepsilon_2)\right] \\ & \times \sum_{\{\vec{l}\}=-\frac{k}{r}} Z_l(\varepsilon_1,\varepsilon_2-\varepsilon_1,\vec{a}+\varepsilon_1\vec{l};e^{\frac{\varepsilon_1}{2r}\left\{d+l\left(-\frac{1}{2}+\frac{k}{r}\right)-\frac{r}{2}\right\}}\Lambda, e^{\frac{k\varepsilon_1}{r}}\star e^{-\frac{\varepsilon_1}{2}}(\vec{\tau}+(e^{\varepsilon_1}-1)\vec{t})) \\ & \qquad \times Z_l(\varepsilon_1-\varepsilon_2,\varepsilon_2,\vec{a}+\varepsilon_2\vec{l};e^{\frac{\varepsilon_2}{2r}\left\{d+l\left(-\frac{1}{2}+\frac{k}{r}\right)-\frac{r}{2}\right\}}\Lambda, e^{\frac{k\varepsilon_2}{r}}\star e^{-\frac{\varepsilon_2}{2}}(\vec{\tau}+(e^{\varepsilon_2}-1)\vec{t})),\end{aligned}$$

where $\vec{l}$ runs over the set $\{\vec{l}=(l_\alpha)_{\alpha=1}^r\in\mathbb{Q}^r \mid \sum l_\alpha = 0, l_\alpha\equiv -k/r \bmod \mathbb{Z}\}$, and $e^{k\varepsilon_a/r}\star\vec{\tau}$ is defined as

$$e^{k\varepsilon_a/r}\star\vec{\tau} = (\cdots, e^{-2k\varepsilon_a/r}\tau_{-2}, e^{-k\varepsilon_a/r}\tau_{-1}, e^{k\varepsilon_a/r}\tau_1, e^{2k\varepsilon_a/r}\tau_2,\cdots).$$

We shift from the previous $\vec{k}$ to $\vec{l}$ by $l_\alpha = k_\alpha - k/r$. The effect of this shift was calculated in [8, (1.36)] when $\vec{\tau}=\vec{t}=0$, and we have used it here. And $e^{k\varepsilon_a/r}\star$ comes from $\mathcal{O}(k_\alpha C)=\mathcal{O}((l_\alpha+k/r)C)$. The part l_α is absorbed into the shift $\vec{a}+\varepsilon_a\vec{l}$, but we need the remaining contribution from k/r.

Remark 3.7. We do not make precise to which ring the full partition functions belong, as a function in Λ. We just use them formally to make a formula shorter as above.

This applies all formulas below until they (more precisely their leading coefficients) are identified with one defined via Seiberg-Witten curves, which are really functions defined over an appropriate open set in Λ.

3.2. **Regularity at $\varepsilon_1 = \varepsilon_2 = 0$.** We assume $0 \le l \le r$ hereafter.

Theorem 2.11(1) means

$$\widehat{Z}_{l,0,d}(\varepsilon_1, \varepsilon_2, \vec{a}; \Lambda, \vec{\tau} = 0, \vec{t} = 0) = Z_l(\varepsilon_1, \varepsilon_2, \vec{a}; \Lambda, \vec{\tau} = 0) \tag{3.8}$$

for $0 \le d \le r$. This was conjectured in [8, (1.37)]. Combined with (3.5), we see that coefficients of $Z^{\mathrm{inst}}(\varepsilon_1, \varepsilon_2, \vec{a}; \Lambda, \vec{\tau} = 0)$ in Λ^n are determined recursively if the above holds two different values of d, as explained in [19, §5.2]. The equation (3.8), the left hand side replaced by (3.6), is called the *blow-up equation*. It gives a strong constraint on the partition function $Z(\varepsilon_1, \varepsilon_2, \vec{a}; \Lambda, \vec{\tau} = 0)$.

As an application, in [loc. cit., Prop. 1.38] we proved the followings under [loc. cit., (1.37)]:

$$Z_l^{\mathrm{inst}}(\varepsilon_1, -2\varepsilon_1, \vec{a}; \Lambda, \vec{\tau} = 0) = Z_l^{\mathrm{inst}}(2\varepsilon_1, -\varepsilon_1, \vec{a}; \Lambda, \vec{\tau} = 0) \quad \text{if } l \neq r, \tag{3.9}$$

$$\varepsilon_1\varepsilon_2 \log Z_l(\varepsilon_1, \varepsilon_2, \vec{a}; \Lambda, \vec{\tau} = 0) \text{ is regular at } \varepsilon_1 = \varepsilon_2 = 0. \tag{3.10}$$

(More precisely only the proof of (3.9) was given in [loc. cit.,(1.37)]. The proof of (3.10) was omitted since it is the same as [20, Th. 4.4].)

Remark 3.11. Though it was not stated explicitly in [loc. cit.], the second assertion holds even if $l = r$. On the other hand, the first one follows from (3.8) for d and $r + l - d$, which must be *different*. Therefore $l \neq r$ is required.

Let us apply the same argument to Proposition 2.13. We expand Z_l^{inst} as before:

$$Z_l^{\mathrm{inst}}(\varepsilon_1, \varepsilon_2, \vec{a}; \Lambda, \vec{\tau}) = \sum_{\vec{n}} \prod_{p \neq 0} \tau_p^{n_p} Z_{\vec{n}}^{\mathrm{inst}}(\varepsilon_1, \varepsilon_2, \vec{a}; \Lambda) \tag{3.12}$$

with $\vec{n} = (\cdots, n_{-1}, n_1, \cdots)$. Thus

$$Z_{\vec{n}}^{\mathrm{inst}}(\varepsilon_1, \varepsilon_2, \vec{a}; \Lambda) = \frac{1}{\vec{n}\,!} \left(\frac{\partial}{\partial \vec{\tau}}\right)^{\vec{n}} Z_l^{\mathrm{inst}}(\varepsilon_1, \varepsilon_2, \vec{a}; \Lambda, \vec{\tau} = 0).$$

By Proposition 2.13 we have

$$\left(\frac{\partial}{\partial \vec{t}}\right)^{\vec{n}} \widehat{Z}_{l,0,d}(\varepsilon_1, \varepsilon_2, \vec{a}; \Lambda, \vec{\tau} = 0, \vec{t} = 0) = 0 \tag{3.13}$$

if $\vec{n}$ is nonzero and satisfies

$$-\sum_{p<0} p n_p \le d \le r - \sum_{p>0} p n_p. \tag{3.14}$$

After substituting (3.6) to the left hand side, we also call this as the *blow-up equation*.

For simplicity we assume $\vec{n}$ is supported on either $\mathbb{Z}_{>0}$ or $\mathbb{Z}_{<0}$, i.e., $n_p = 0$ for any $p < 0$ or $n_p = 0$ for any $p > 0$. We say $\vec{n}$ is *positive* in the first case, and *negative* in the second case.

We define '$\log Z(\varepsilon_1, \varepsilon_2, \vec{a}; \Lambda, \vec{\tau})$' as follows. Since the perturbative part is already the exponential of something, we only need to define '$\log Z^{\mathrm{inst}}$'. Then observe that

$$Z^{\mathrm{inst}}(\varepsilon_1, \varepsilon_2, \vec{a}; \Lambda, \vec{\tau}) = \exp\left(\sum_{p,\alpha} \frac{\tau_p e^{pa_\alpha}}{(e^{\varepsilon_1/2} - e^{-\varepsilon_1/2})(e^{\varepsilon_2/2} - e^{-\varepsilon_2/2})}\right) \times (1 + O(\Lambda)).$$

Therefore $\log Z^{\mathrm{inst}}$ can be defined as the sum of $\sum_{p,\alpha} \tau_p e^{pa_\alpha} / (e^{\varepsilon_1/2} - e^{-\varepsilon_1/2})(e^{\varepsilon_2/2} - e^{-\varepsilon_2/2})$ and a formal power series in Λ.

Proposition 3.15. (1) *Suppose that $\vec{n}$ satisfies the followings:*

(a) *If $\sum n_p$ is odd, $-(r+l)/2 \le \sum_{p<0} p n_p$ (negative case) or $\sum_{p>0} p n_p \le (r-l)/2$ (positive case).*

(b) *If $\sum n_p$ is even, the strict inequality holds.*

Then

$$Z_{\vec{n}}(\varepsilon_1, -2\varepsilon_1, \vec{a}; \Lambda) = Z_{\vec{n}}(2\varepsilon_1, -\varepsilon_1, \vec{a}; \Lambda).$$

(2) *If $-r < \sum_{p<0} p n_p$ (negative case) or $\sum_{p>0} p n_p < r$ (positive case),*

$$\varepsilon_1 \varepsilon_2 \left(\frac{\partial}{\partial \vec{\tau}}\right)^{\vec{n}} \log Z_l(\varepsilon_1, \varepsilon_2, \vec{a}; \Lambda, \vec{\tau} = 0)$$

is regular at $\varepsilon_1 = \varepsilon_2 = 0$.

Proof. Since the proof of (2) is the same as that of (3.10), we only prove (1).

We expand Z_l^{inst} as

$$Z_l^{\mathrm{inst}}(\varepsilon_1, \varepsilon_2, \vec{a}; \Lambda, \vec{\tau}) = \sum_{N=0}^{\infty} Z_N(\varepsilon_1, \varepsilon_2, \vec{a}, \vec{\tau}) \Lambda^{2rN}.$$

Note that

$$Z_0(\varepsilon_1, \varepsilon_2, \vec{a}, \vec{\tau}) = \exp\left[\sum_{p,\alpha} \tau_p \frac{e^{pa_\alpha}}{(e^{\varepsilon_1/2} - e^{-\varepsilon_1/2})(e^{\varepsilon_2/2} - e^{-\varepsilon_2/2})}\right].$$

Therefore the assertion is true for Z_0. We prove the assertion by the induction on N.

We further expand as

$$Z_N(\varepsilon_1, \varepsilon_2, \vec{a}, \vec{\tau}) = \sum_{\vec{n}} \prod_{p \ne 0} \tau_p^{n_p} Z_{N,\vec{n}}(\varepsilon_1, \varepsilon_2, \vec{a})$$

as in (3.12).

Fix $\vec{n}$ and N and consider the coefficient of $\Lambda^{2rN} \prod \tau_p^{n_p}$ in (3.5). Setting $\varepsilon_2 = -\varepsilon_1$, we have

$$\begin{aligned} 0 = \sum_{\substack{(\vec{k},\vec{k})/2 + N_1 + N_2 = N \\ \vec{n}_1 + \vec{n}_2 = \vec{n}}} &(-1)^{\sum n_{2,p}} \frac{e^{(d-l/2)(\vec{k},\vec{a})} e^{(N_1 - N_2)(d-(r+l)/2)\varepsilon_1}}{\prod_{\alpha \in \Delta} l_\alpha^{\vec{k}}(\varepsilon_1, -\varepsilon_1, \vec{a})} \exp\left(\frac{l}{2} \sum k_\alpha^2 a_\alpha\right) \\ &\times Z_{N_1,\vec{n}_1}(\varepsilon_1, -2\varepsilon_1, \vec{a} + \varepsilon_1 \vec{k}) Z_{N_2,\vec{n}_2}(2\varepsilon_1, -\varepsilon_1, \vec{a} - \varepsilon_1 \vec{k}) \end{aligned} \tag{3.16}$$

if $\vec{n} \neq 0$ by the blow-up equation ((3.13) and (3.5)). Here the summation is over $\vec{k}$, $\vec{n}_1$, $\vec{n}_2$, N_1, N_2 and we write $\vec{n}_2 = (\cdots, n_{2,-1}, n_{2,1}, \cdots)$. We assume

$$-\sum_{p<0} pn_p \leq d \leq r - \sum_{p>0} pn_p.$$

We suppose that the same equality holds for $r+l-d$. So we assume

$$l + \sum_{p>0} pn_p \leq d \leq r + l + \sum_{p<0} pn_p.$$

By our assumption, there exists d satisfying both inequalities, e.g., we take $d = r - \sum_{p>0} pn_p$ in the positive case, $r + l + \sum_{p<0} pn_p$ in the negative case. Moreover, we may assume $d \neq (r+l)/2$ if $\sum n_p$ is even.

Substituting $r+l-d$ into d in (3.16), and replacing $\vec{k}$ by $-\vec{k}$, (N_1, N_2) by (N_2, N_1) and $(\vec{n}_1, \vec{n}_2)$ by $(\vec{n}_2, \vec{n}_1)$, we get

$$0 = \sum_{\substack{(\vec{k},\vec{k})/2+N_1+N_2=N \\ \vec{n}_1+\vec{n}_2=\vec{n}}} (-1)^{\sum n_{1,p}} \frac{e^{(d-l/2-r)(\vec{k},\vec{a})} e^{(N_1-N_2)(d-(r+l)/2)\varepsilon_1}}{\prod_{\alpha\in\Delta} l_\alpha^{-\vec{k}}(\varepsilon_1, -\varepsilon_1, \vec{a})} \exp\left(\frac{l}{2}\sum k_\alpha^2 a_\alpha\right)$$
$$\times Z_{N_1,\vec{n}_1}(2\varepsilon_1, -\varepsilon_1, \vec{a} + \varepsilon_1\vec{k}) Z_{N_2,\vec{n}_2}(\varepsilon_1, -2\varepsilon_1, \vec{a} - \varepsilon_1\vec{k}).$$

Note

$$\frac{e^{r(\vec{k},\vec{a})/2}}{\prod_{\vec{\alpha}\in\Delta} l_{\vec{\alpha}}^{\vec{k}}(\varepsilon_1, -\varepsilon_1, \vec{a})} = \frac{e^{-r(\vec{k},\vec{a})/2}}{\prod_{\vec{\alpha}\in\Delta} l_{\vec{\alpha}}^{-\vec{k}}(\varepsilon_1, -\varepsilon_1, \vec{a})}$$

by [20, Lem. 4.1 and (4.2)]. Note also that $\sum n_{1,p} = \sum n_p - \sum n_{2,p}$ and hence we can replace $(-1)^{\sum n_{1,p}}$ by $(-1)^{\sum n_{2,p}}$ in the above formula. Then the only difference of the above two equations are variables for $Z_{N_1,\vec{n}_1}$ and $Z_{N_2,\vec{n}_2}$. By the induction hypothesis those are also equal if N_1, $N_2 < N$. Therefore we have

$$\begin{aligned}
0 = \sum_{\vec{n}_1+\vec{n}_2=\vec{n}} (-1)^{\sum n_{2,p}} \Big(& e^{N(d-(r+l)/2)\varepsilon_1} Z_{N,\vec{n}_1}(\varepsilon_1, -2\varepsilon_1, \vec{a}) Z_{0,\vec{n}_2}(2\varepsilon_1, -\varepsilon_1, \vec{a}) \\
& + e^{-N(d-(r+l)/2)\varepsilon_1} Z_{0,\vec{n}_1}(\varepsilon_1, -2\varepsilon_1, \vec{a}) Z_{N,\vec{n}_2}(2\varepsilon_1, -\varepsilon_1, \vec{a}) \\
& - e^{N(d-(r+l)/2)\varepsilon_1} Z_{N,\vec{n}_1}(2\varepsilon_1, -\varepsilon_1, \vec{a}) Z_{0,\vec{n}_2}(\varepsilon_1, -2\varepsilon_1, \vec{a}) \\
& - e^{-N(d-(r+l)/2)\varepsilon_1} Z_{0,\vec{n}_1}(2\varepsilon_1, -\varepsilon_1, \vec{a}) Z_{N,\vec{n}_2}(\varepsilon_1, -2\varepsilon_1, \vec{a}) \Big), \\
= \sum_{\vec{n}_1+\vec{n}_2=\vec{n}} (-1)^{\sum n_{2,p}} \Big[& e^{N(d-(r+l)/2)\varepsilon_1} Z_{0,\vec{n}_2}(2\varepsilon_1, -\varepsilon_1, \vec{a}) \\
& \quad \times \{ Z_{N,\vec{n}_1}(\varepsilon_1, -2\varepsilon_1, \vec{a}) - Z_{N,\vec{n}_1}(2\varepsilon_1, -\varepsilon_1, \vec{a}) \} \\
& + e^{-N(d-(r+l)/2)\varepsilon_1} Z_{0,\vec{n}_1}(\varepsilon_1, -2\varepsilon_1, \vec{a}) \\
& \quad \times \{ Z_{N,\vec{n}_2}(2\varepsilon_1, -\varepsilon_1, \vec{a}) - Z_{N,\vec{n}_2}(\varepsilon_1, -2\varepsilon_1, \vec{a}) \} \Big],
\end{aligned} \tag{3.17}$$

if $\vec{n} \neq 0$.

We now prove the assertion by induction on $\vec{n}$. The case $\vec{n} = 0$ is treated already in (3.9).

Now we assume that the assertion holds for smaller $\vec{n}$. Note that the assumption on $\vec{n}$ implies that on smaller ones. Then the only remaining terms in (3.17) are either $\vec{n}_1 = 0$ or $\vec{n}_2 = 0$. Therefore we have

$$0 = \left(e^{N(d-(r+l)/2)\varepsilon_1} - (-1)^{\sum n_p} e^{-N(d-(r+l)/2)\varepsilon_1}\right)\left\{Z_{N,\vec{n}}(\varepsilon_1, -2\varepsilon_1, \vec{a}) - Z_{N,\vec{n}}(2\varepsilon_1, -\varepsilon_1, \vec{a})\right\}.$$

Hence we have the assertion for $Z_{N,\vec{n}}$. Note that we take $d \neq (r+l)/2$ when $\sum n_p$ is even, so the above is a nontrivial equality. □

Remark 3.18. We need the vanishing (3.16) for the case when $\sum n_p$ is odd, but it is enough to suppose that the right hand side of (3.16) is the same for d and $r+l-d$ when $\sum n_p$ is even. In particular, if we use (3.8) instead of (3.13), the above argument works even for $\vec{n} = 0$. This is nothing but the proof of (3.9) in [8].

3.3. **Contact term equations.** We expand as

$$\begin{aligned}&\varepsilon_1\varepsilon_2 \log Z_l(\varepsilon_1, \varepsilon_2, \vec{a}; \Lambda, \vec{\tau}) \\ &\qquad = F_0(\vec{a}; \Lambda, \vec{\tau}) + (\varepsilon_1+\varepsilon_2) H(\vec{a}; \Lambda, \vec{\tau}) + \varepsilon_1\varepsilon_2 A(\vec{a}; \Lambda, \vec{\tau}) + \frac{\varepsilon_1^2+\varepsilon_2^2}{3} B(\vec{a}, \Lambda, \vec{\tau}) + \cdots,\end{aligned}$$

where we only consider $(\partial/\partial\vec{\tau})^{\vec{n}}\big|_{\vec{\tau}=0}$ applied to this function, with $\vec{\tau}$ in the range in Proposition 3.15(2) so that singular terms do not appear.

By (3.9) $H(\vec{a}; \Lambda, \vec{\tau} = 0)$ comes only from the perturbative part. Thus we have $H(\vec{a}; \Lambda, \vec{\tau} = 0) = -\pi\sqrt{-1}\langle \vec{a}, \rho\rangle$, where ρ is the half sum of the positive roots. More generally by Proposition 3.15(1) we have

$$\left(\frac{\partial}{\partial \vec{\tau}}\right)^{\vec{n}} H(\vec{a}; \Lambda, \vec{\tau} = 0) = 0$$

if $\vec{n}$ satisfies the condition there.

We introduce a new coordinate system for $\vec{a}$ by $a^i = a_1 + a_2 + \cdots + a_i$ $(i = 1, \dots, r-1)$ as in [18, §1]. We also define k^i for $\vec{k}$ in the same way.

Let

$$\tau_{ij} = -\frac{1}{2\pi\sqrt{-1}} \frac{\partial^2 F_0(\vec{a}; \Lambda, \vec{\tau} = 0)}{\partial a^i \partial a^j}. \tag{3.19}$$

Let $\Theta_E(\vec{\xi}|\tau)$ be the Riemann theta function defined by

$$\Theta_E(\vec{\xi}|\tau) = \sum_{\vec{k}\in\mathbb{Z}^{r-1}} \exp\left(\pi\sqrt{-1}\sum_{i,j}\tau_{ij}k^i k^j + 2\pi\sqrt{-1}\sum_i k^i(\xi^i + \frac{1}{2})\right).$$

We substitute (3.6) into the left hand side of (3.8), and take the limit of $\varepsilon_1, \varepsilon_2 \to 0$. Using $H(\vec{a}; \Lambda, \vec{\tau} = 0) = -\pi\sqrt{-1}\langle\vec{a}, \rho\rangle$, we obtain

$$\begin{aligned}\exp(B-A) = \exp&\left[-\frac{1}{8r^2}\left(d - \frac{r+l}{2}\right)^2 \frac{\partial^2 F_0}{(\partial \log\Lambda)^2}\right] \times \\ &\qquad \Theta_E\left(-\frac{1}{2\pi\sqrt{-1}}\frac{1}{2r}\left(d - \frac{r+l}{2}\right)\frac{\partial^2 F_0}{\partial\log\Lambda\partial\vec{a}}\,\middle|\,\tau\right)\end{aligned}$$

for $0 \le d \le r$ as in [20, §4]. Here F_0, A, B are evaluated at $(\vec{a}; \Lambda, \vec{\tau} = 0)$.

In particular, the right hand side is independent of d. Dividing by the expression for $d = (r+l)/2$ (if $r+l$ is even), $d = (r+l-1)/2$, (if $r+l$ is odd), we have

(3.20)
$$\frac{\Theta_E\left(-\frac{1}{4\pi\sqrt{-1}r}\left(d-\frac{r+l}{2}\right)\frac{\partial^2 F_0}{\partial\log\Lambda\partial\vec{a}}\,\middle|\,\tau\right)}{\Theta_E(0|\tau)} = \exp\left[\frac{1}{8r^2}\left(d-\frac{r+l}{2}\right)^2\frac{\partial^2 F_0}{(\partial\log\Lambda)^2}\right],$$
$$\frac{\Theta_E\left(-\frac{1}{4\pi\sqrt{-1}r}\left(d-\frac{r+l}{2}\right)\frac{\partial^2 F_0}{\partial\log\Lambda\partial\vec{a}}\,\middle|\,\tau\right)}{\Theta_E\left(\frac{1}{8\pi\sqrt{-1}r}\frac{\partial^2 F_0}{\partial\log\Lambda\partial\vec{a}}\,\middle|\,\tau\right)} = \exp\left[\frac{1}{8r^2}\left\{\left(d-\frac{r+l}{2}\right)^2-\frac{1}{4}\right\}\frac{\partial^2 F_0}{(\partial\log\Lambda)^2}\right]$$

according to either $r+l$ is even or odd. This equation is called the *contact term equation*. This equation determines $F_0(\vec{a};\Lambda,\vec{\tau}=0)$ recursively in the expansion with respect to Λ, starting from the perturbation part.

Here we remind again that our full partition function, and hence τ_{ij} and $\Theta_E(\vec{\xi}|\tau)$, etc, do not have the rigorous meaning. (See Remark 3.7.) The rigorous form of the contact term equation is given by rewriting it as an equation for the instanton part. This was given in [18, (7.5)], for the homological version of Nekrasov partition function, but we do not give here since it is not enlightening.

Similarly as the limit of (3.13), we obtain

$$\begin{aligned}(3.21)\quad 0 = &-\frac{1}{2r}\left(d-\frac{r+l}{2}\right)\frac{\partial}{\partial\log\Lambda}\left(\frac{\partial}{\partial\vec{\tau}}\right)^{\vec{n}} F_0(\vec{a};\Lambda,\vec{\tau}=0)\\ &-\frac{1}{2\pi\sqrt{-1}}\sum_i \left.\frac{\partial\log\Theta_E}{\partial\xi^i}\right|_{\vec{\xi}=-\frac{1}{4\pi\sqrt{-1}r}\left(d-\frac{r+l}{2}\right)\frac{\partial^2 F_0}{\partial\log\Lambda\partial\vec{a}}}\frac{\partial}{\partial a^i}\left(\frac{\partial}{\partial\vec{\tau}}\right)^{\vec{n}} F_0(\vec{a};\Lambda,\vec{\tau}=0)\end{aligned}$$

when $\vec{n}$ is nonzero, either positive or negative and satisfies (3.14) and (3.15)(1)(a),(b). We call this as the *contact term equation* for $(\partial/\partial\vec{\tau})^{\vec{n}}F_0$. This derivation of the contact term equation from the blow-up equation can be done in the same way as in [19, §5.3].

Since $\partial\log\Theta_E/\partial\xi^i$ is divisible by Λ, this equation determines $(\partial/\partial\vec{\tau})^{\vec{n}}F_0$ recursively in the expansion with respect to Λ starting from the constant term:

$$\begin{aligned}(3.22)\quad &\left(\frac{\partial}{\partial\vec{\tau}}\right)^{\vec{n}} F_0(\vec{a};\Lambda=0,\vec{\tau}=0) = \left(\frac{\partial}{\partial\vec{\tau}}\right)^{\vec{n}}\left.\sum_{p,\alpha}\tau_p e^{pa_\alpha}\right|_{\vec{\tau},\Lambda=0}\\ &= \begin{cases}\sum_\alpha e^{pa_\alpha}|_{\Lambda=0} & \text{if } n_p = 1 \text{ and } n_q = 0 \text{ for } q\neq p,\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

In particular, we have

$$\left(\frac{\partial}{\partial\vec{\tau}}\right)^{\vec{n}} F_0(\vec{a};\Lambda,\vec{\tau}=0) = 0$$

unless $n_p = 1$ and $n_q = 0$ for $q\neq p$. The remaining $\partial F_0/\partial\tau_p$ will be determined in the next subsection.

Since higher derivatives vanish, we have

$$\left.\frac{\left((e^{\varepsilon_1/2}-e^{-\varepsilon_1/2})(e^{\varepsilon_2}-e^{-\varepsilon_2/2})\right)^{\sum_p n_p}\left(\frac{\partial}{\partial\vec{\tau}}\right)^{\vec{n}} Z_l(\varepsilon_1,\varepsilon_2,\vec{a};\Lambda,\vec{\tau}=0)}{Z_l(\varepsilon_1,\varepsilon_2,\vec{a};\Lambda,\vec{\tau}=0)}\right|_{\varepsilon_1=\varepsilon_2=0} = \prod_p \left(\frac{\partial F_0}{\partial\tau_p}(\vec{a};\Lambda,\vec{\tau}=0)\right)^{n_p}.$$

By (3.2) this is equal to

$$\begin{aligned}(3.23)\quad &\frac{1}{Z_l^{\mathrm{inst}}(\varepsilon_1,\varepsilon_2,\vec{a};\Lambda,\vec{\tau}=0)}\sum_c (\Lambda^{2r}e^{-(r+l)(\varepsilon_1+\varepsilon_2)/2})^{(\Delta(c),[\mathbb{P}^2])}\\ &\qquad\times\left.\int_{M(c)} \operatorname{ch}\left(\bigotimes_p (\psi^p(\mathcal{E})/[0])^{\otimes n_p}\right)\operatorname{td} M(c)\exp(lc_1(\mathcal{V}(\mathcal{E})))\right|_{\varepsilon_1=\varepsilon_2=0},\end{aligned}$$

where $\bullet/[0]$ is the slant product $q_{2*}(\bullet\otimes q_1^*(\mathbb{C}_0)) = (e^{\varepsilon_1/2}-e^{-\varepsilon_1/2})(e^{\varepsilon_2/2}-e^{-\varepsilon_2/2})\bullet/[\mathbb{C}^2]$ with the skyscraper sheaf $\mathbb{C}_0$ at the origin, which is nothing but the restriction to the origin.

3.4. **Seiberg-Witten prepotential.** We give a quick review of the Seiberg-Witten prepotential for the theory with 5-dimensional Chern-Simons term in this subsection. See [8, App. A] for more detail. (We change the notation slightly: m in [loc. cit.] is our l. a_i is our a_α. $\boldsymbol{\beta}$ is set to be 1.)

We consider a family of hyperelliptic curves parametrized by $\vec{U}=(U_1,\dots,U_{r-1})$:

$$\begin{aligned}C_{\vec{U},l} : Y^2 &= P(X)^2 - 4(-X)^{r+l}\Lambda^{2r}\\ P(X) &= X^r + U_1X^{r-1} + U_2X^{r-2} + \cdots + U_{r-1}X + (-1)^r\end{aligned}$$

for $-r<l<r$, $l\in\mathbb{Z}$. Note that we set $\boldsymbol{\beta}=1$ from [loc. cit.] for brevity. We call them *Seiberg-Witten curves.* We define the *Seiberg-Witten differential* by

$$dS = \frac{1}{2\pi\sqrt{-1}}\log X\,\frac{2XP'(X)-(r+l)P(X)}{2XY}dX.$$

We choose z_α ($\alpha=1,\dots,r$) so that $X_\alpha = e^{-\sqrt{-1}z_\alpha}$ are zeroes of $P(X)=0$. Then we take the cycles A_α, B_α ($\alpha=2,\dots,r$) in a way explained in [loc. cit.].

We define a_α, a_α^D by

$$a_\alpha = \int_{A_\alpha} dS,\qquad a_\alpha^D = \int_{B_\alpha} dS.$$

We then invert the role of a_α and U_p, so we consider a_α as variables and U_p are functions in a_α. Here we use $a_\alpha = -\sqrt{-1}z_\alpha + O(\Lambda)$ [loc. cit., (A.1)].

Then one can show that there exists a locally defined function $\mathcal{F}_0=\mathcal{F}_0(\vec{a};\Lambda)$ such that

$$a_\alpha^D = -\frac{1}{2\pi\sqrt{-1}}\frac{\partial\mathcal{F}_0}{\partial a_\alpha}.$$

This defines $\mathcal{F}_0$ up to a function (in Λ) independent of $\vec{a}$. This ambiguity is fixed by specifying $\partial\mathcal{F}_0/\partial\log\Lambda$ and the perturbation part of $\mathcal{F}_0$. See [loc. cit.] for detail.

It was proved in [loc. cit., (A.27), (A.33)] that $\mathcal{F}_0$ satisfies the contact term equations (3.20) where the period matrix (τ) is defined by the same formula as (3.19) by replacing F_0 by $\mathcal{F}_0$. Moreover, it was also proved that $\mathcal{F}_0$ has the same perturbation part as F_0 [loc. cit., Prop. A.6]. Therefore the recursive structure of (3.20) implies that $\mathcal{F}_0 = F_0$, i.e., the leading part of the Nekrasov partition function is equal to the Seiberg-Witten prepotential.

In [loc. cit., (A.25)], it was proved

$$0 = \frac{1}{2r}\frac{\partial U_p}{\partial \log\Lambda} + \frac{1}{2\pi\sqrt{-1}}\sum_i \left.\frac{\partial \log\Theta_E}{\partial \xi^i}\right|_{\vec{\xi}=-\frac{1}{4\pi\sqrt{-1}r}\frac{\partial^2 F_0}{\partial\log\Lambda\partial\vec{a}}} \frac{\partial U_p}{\partial a^i} \tag{3.24}$$

under the assumption $r+l$ is even. This is the same as the equation (3.21) with $d-(r+l)/2 = \pm 1$. When $r+l$ is odd, we assume that U_p satisfies (3.21) with $d = (r+l+1)/2$ for a moment, and gives a proof later.

The initial condition for U_p is

$$U_p = (-1)^p e_p(X_1,\dots,X_r) = (-1)^p e_p(e^{a_1},\dots,e^{a_r}) \quad \text{at } \Lambda = 0,$$

where e_p is the p^{th} elementary symmetric polynomial. Noticing that (3.21) holds for polynomials in $\partial F_0/\partial\tau_p$, we see that $(-1)^p U_p$ and $\partial F_0/\partial\tau_p$ are related exactly in the same way as an elementary symmetric polynomial and a power sum by (3.22).

Theorem 3.25.

$$\frac{\partial F_0}{\partial \tau_p}(\vec{a};\Lambda,\vec{\tau}=0) = X_1^p + \cdots + X_r^p$$

holds for $-(r+l)/2 \le p \le (r-l)/2$.

Note that all U_p's are written by polynomials in $\frac{\partial F_0}{\partial\tau_p}(\vec{a};\Lambda,\vec{\tau}=0)$ in the above range, as $X_1\cdots X_r = 1$. Thanks to (3.23) the polynomials can be replaced by those in Adams operators. We thus have

$$\begin{aligned} U_p = \frac{(-1)^p}{Z_l^{\text{inst}}(\varepsilon_1,\varepsilon_2,\vec{a};\Lambda,\vec{\tau}=0)} &\sum_c (\Lambda^{2r} e^{-(r+l)(\varepsilon_1+\varepsilon_2)/2})^{(\Delta(c),[\mathbb{P}^2])} \\ &\times \left.\int_{M(c)} \operatorname{ch}\left({\textstyle\bigwedge}^p(\mathcal{E}/[0])\right)\operatorname{td} M(c)\exp(lc_1(\mathcal{V}(\mathcal{E})))\right|_{\varepsilon_1=\varepsilon_2=0} \end{aligned}$$

for $0 < p \le (r-l)/2$. For $(r-l)/2 \le p < r$, the equation holds if we replace ${\textstyle\bigwedge}^p(\mathcal{E}/[0])$ by ${\textstyle\bigwedge}^{r-p}(\mathcal{E}/[0])^\vee$. This equation gives a moduli theoretic description of the coefficients U_p in the Seiberg-Witten curves.

It remains to show that U_p satisfies (3.21) with $d = (r+l+1)/2$ in the case $r+l$ is odd. Let us give a sketch of the argument. We use the notation in [loc. cit.], e.g., $E(X_1,X_2)$ is the prime form, $\omega_{\infty_+-0_-}$ is the meromorphic differential with the vanishing A-periods having poles 0_- and ∞_+ of residue -1, $+1$ respectively, etc.

By [4, Cor. 2.11] we have

$$\frac{\Theta_E^2(\frac{1}{2}\int_{0_-+\infty_+}^{2X}\vec{\omega})}{\Theta_E^2(\frac{1}{2}\int_{0_-}^{\infty_+}\vec{\omega})}\frac{E(0_-,\infty_+)}{E(X,0_-)E(X,\infty_+)} \\ = \omega_{\infty_+-0_-}(X) + 2\times\frac{1}{2\pi\sqrt{-1}}\sum_i \left.\frac{\partial\log\Theta_E}{\partial\xi^i}\right|_{\vec{\xi}=\frac{1}{2}\int_{0_-}^{\infty_+}\vec{\omega}}\omega_i(X).$$

By [loc. cit., (A.18)], we have

$$\frac{(P(X)-Y)dX}{2XY} - \frac{1}{2r}\sum_{p=1}^{r-1}\frac{\partial U_p}{\partial\log\Lambda}\frac{X^{r-p-1}dX}{Y} = \omega_{\infty_+-0_-}(X).$$

Therefore it is enough to show

$$\frac{(P(X)-Y)dX}{2XY} = \frac{\Theta_E^2(\frac{1}{2}\int_{0_-+\infty_+}^{2X}\vec{\omega})}{\Theta_E^2(\frac{1}{2}\int_{0_-}^{\infty_+}\vec{\omega})}\frac{E(0_-,\infty_+)}{E(X,0_-)E(X,\infty_+)}. \tag{3.26}$$

As in [loc. cit., §A.7] we take the branched double cover $p\colon \widehat{C}_{\vec{U},l} \to C_{\vec{U},l}$ given by $W\mapsto X = W^2$. It is given by

$$\begin{aligned} Y^2 &= P(W^2)^2 - 4(-W^2)^{r+l}\Lambda^{2r} \\ &= \left(P(W^2) - 2(\sqrt{-1}W)^{r+l}\Lambda^r\right)\left(P(W^2) + 2(\sqrt{-1}W)^{r+l}\Lambda^r\right). \end{aligned} \tag{3.27}$$

We consider the Szegö kernel for $\widehat{C}_{\vec{U},l}$ given by

$$\Psi_{\widehat{E}}(W_1,W_2) = \frac{\widehat{\Theta}_{\widehat{E}}(\int_{W_1}^{W_2}\vec{\omega}\hat{})}{\widehat{\Theta}_{\widehat{E}}(0)E(W_1,W_2)},$$

where $\widehat{\Theta}_{\widehat{E}}$ is the Riemann theta function for the curve $\widehat{C}_{\vec{U},l}$. As in [loc. cit., the first two displayed formulas in §A.6] we have

$$\Psi_{\widehat{E}}(W,\infty_+)^2 = -\frac{Y-P(W^2)}{2Y}dW\ d\left.\left(\frac{1}{W_2}\right)\right|_{W_2=\infty_+}.$$

From the defining equation (3.27) of the double cover, this has zero of order $2(r+l)$ at 0_+ and of order $2(r-l-1)$ at ∞_-. Therefore

$$\operatorname{div}\widehat{\Theta}_{\widehat{E}}(W-\infty_+) = (r+l)\cdot 0_+ + (r-l-1)\cdot\infty_-.$$

(See [4, pp.16, 17] for basic properties of $E(W_1,W_2)$.)

If we identify the half-integer characteristic $\widehat{E}$ with a vector in $\mathbb{C}^{2r-1}$ so that $\widehat{\Theta}_{\widehat{E}}(\xi) = \widehat{\Theta}(\xi-\widehat{E})$, we have

$$\widehat{E} = (r+l)\cdot 0_+ + (r-l-1)\cdot\infty_- - \infty_+ - \widehat{\Delta}, \tag{3.28}$$

where $\widehat{\Delta}$ is the Riemann's divisor class ([4, Th. 1.1]).

By [loc. cit., Lem. A.32] we have

$$\widehat{E} = p^*E - [0, c_*, 0], \tag{3.29}$$

where $p^*\colon J_0(C_{\vec{U},l}) \to J_0(\widehat{C}_{\vec{U},l})$.

On the other hand, we have

$$\widehat{\Delta} - p^*\Delta = 0_- + \infty_- + p^*\left(\frac{1}{2}\int_{0_-}^{\infty_+}\vec{\omega}\right) + [0, c_*, 0] \tag{3.30}$$

by [4, Prop. 5.3].

From (3.28,3.29,3.30) we get

$$E = \frac{r+l-1}{2}\cdot 0_+ + \frac{r-l-1}{2}\cdot\infty_- - 0_- - \frac{1}{2}\int_{0_-}^{\infty_+}\vec{\omega} - \Delta,$$

where we have used $0_+ - \infty_- = \infty_+ - 0_-$. Therefore $\Theta_E(\frac{1}{2}\int_{0_-+\infty_+}^{2X}\vec{\omega})$ has zero of order $(r+l-1)/2$ at 0_+, and of order $(r-l-1)/2$ at ∞_- again by [4, Th. 1.1].

On the other hand, the left hand side of (3.26) has zero of order $r+l-1$ at 0_+, and of order $r-l-1$ at ∞_-. Both sides of (3.26) have poles at 0_-, ∞_+ with residues -1, 1 respectively. (See [4, Cor. 2.11].) Therefore we have (3.26).

Although it is not necessary, let us also sketch how to prove (3.21) for more general d.

We first assume $r+l$ is even and generalize (3.24) as

$$0 = \frac{d}{2r}\frac{\partial U_p}{\partial\log\Lambda} + \frac{1}{2\pi\sqrt{-1}}\sum_i \left.\frac{\partial\log\Theta_E}{\partial\xi^i}\right|_{\vec{\xi}=-\frac{d}{4\pi\sqrt{-1}r}\frac{\partial^2F_0}{\partial\log\Lambda\partial\vec{a}}}\frac{\partial U_p}{\partial a^i} \tag{3.31}$$

for $|d| \le (r-l)/2$. This equation is nothing but (3.21) with $d-(r+l)/2$ replaced by d. In terms of the d in (3.21), the condition is $l \le d \le r$. This is exactly one under which we have proved (3.21) for all p from the vanishing theorem.

To show (3.31) we use [loc. cit., the first displayed formula in §A.6.2], which is [4, Cor. 2.19 (43)]. We replace d by $d+1$ and take $x_0 = X$, $y_0 = X'$, $x_1 = \cdots = x_d = 0_-$, $y_1 = \cdots = y_d = \infty_+$. We then get

$$\begin{aligned}\frac{\Theta_E\left(\int_{d0_-+X}^{d\infty_++X'}\vec{\omega}\right)}{\Theta_E(0)E(X,X')}\left(\frac{E(X,0_-)E(\infty_+,X')}{E(X,\infty_+)E(0_-,X')}\right)^d\left(E(0_-,\infty_+)\left.\sqrt{dX_1}\right|_{X_1=0}\left.\frac{\sqrt{dX_2}}{X_2}\right|_{X_2=\infty}\right)^{-d^2}\\ = \Psi_E(X,X') - \frac{\Psi_E(0_-,X')\Psi_E(X,\infty_+)}{\Psi_E(0_-,\infty_+)}\frac{1-(X/X')^d}{1-(X/X')}.\end{aligned}$$

This is proved exactly as in [loc. cit., §A.6.2], so the detail is omitted. We multiply both sides $E(X,X')$, differentiate with respect to X' and set $X'=X$. We get

$$\sum_{\alpha=2}^r\frac{\partial\log\Theta_E}{\partial\xi_\alpha}(d\int_{0_-}^{\infty_+}\vec{\omega})\omega_\alpha(X) + d\times\omega_{\infty_+-0_-}(X) = d\frac{\Psi_E(X,0_-)\Psi_E(X,\infty_+)}{\Psi_E(0_-,\infty_+)} \tag{3.32}$$

as in [15, IIIb §3, p.226]. When $d=1$, this is nothing but [loc. cit., (A.24)]. From the argument in [loc. cit., §A.6.1] we get (3.31).

We next consider the case $r+l$ is odd. We take the branched double cover $p\colon \widehat{C}_{\vec{U},l} \to C_{\vec{U},l}$ given by $W \mapsto X = W^2$ as before. Then r, l become $2r$, $2l$ for $\widehat{C}_{\vec{U},l}$, and hence we have

(3.32) for $\widehat{C}_{\vec{U},l}$, with d replaced by $2d$:

$$\sum_{\alpha=2}^{r}\frac{\partial\log\widehat{\Theta}_{\widehat{E}}}{\partial\widehat{\xi}_\alpha}(2d\int_{0_-}^{\infty_+}\vec{\omega}\widehat{\ })\widehat{\omega}_\alpha(W)+\sum_{\alpha=2}^{r}\frac{\partial\log\widehat{\Theta}_{\widehat{E}}}{\partial\widehat{\xi}'_\alpha}(2d\int_{0_-}^{\infty_+}\vec{\omega}\widehat{\ })\widehat{\omega}'_\alpha(W)$$
$$+\frac{\partial\log\widehat{\Theta}_{\widehat{E}}}{\partial\widehat{\xi}'_*}(2d\int_{0_-}^{\infty_+}\vec{\omega}\widehat{\ })\widehat{\omega}_*(W)+2d\times\widehat{\omega}_{\infty_+-0_-}(W)=2d\frac{\widehat{\Psi}_{\widehat{E}}(W,0_-)\widehat{\Psi}_{\widehat{E}}(W,\infty_+)}{\widehat{\Psi}_{\widehat{E}}(0_-,\infty_+)},$$

where we take cycles A_α, B_α, A_*, B_*, A'_α, B'_α as in [loc. cit., §A.7] and the corresponding coordinates $\widehat{\xi}_\alpha$, $\widehat{\xi}_*$, $\widehat{\xi}'_\alpha$ on $J_0(\widehat{C}_{\vec{U},l})$. This holds if $|d|\le(r-l)/2$ as above. The right hand side is

$$2d\frac{(P(W^2)-Y)dW}{2WY}=dp^*\left(\frac{(P(X)-Y)dX}{2XY}\right)$$

as [loc. cit., the second displayed equation in §A.6.1]. We also have

$$\widehat{\omega}_{\infty_+-0_-}(W)=\frac{1}{2}p^*\omega_{\infty_+-0_-}(X)$$

by definition.

On the other hand, we rewrite the theta function $\widehat{\Theta}_{\widehat{E}}$ by Θ_E by using [loc. cit., (A.29) and the second displayed formula in p.1105]. We then get

$$\frac{1}{2}\sum_{\alpha=2}^{r}\left\{\frac{\partial\log\Theta_E}{\partial\xi_\alpha}((d+\frac{1}{2})\int_{0_-}^{\infty_+}\vec{\omega})+\frac{\partial\log\Theta_E}{\partial\xi_\alpha}((d-\frac{1}{2})\int_{0_-}^{\infty_+}\vec{\omega})\right\}\omega_\alpha(X)$$
$$+d\times\omega_{\infty_+-0_-}(X)=d\frac{(P(X)-Y)dX}{2XY}.$$

From this we get

$$0=\frac{d}{2r}\frac{\partial U_p}{\partial\log\Lambda}$$
$$+\frac{1}{2\pi\sqrt{-1}}\sum_i\frac{1}{2}\left(\left.\frac{\partial\log\Theta_E}{\partial\xi^i}\right|_{\vec{\xi}=-\frac{d+1/2}{4\pi\sqrt{-1}r}\frac{\partial^2F_0}{\partial\log\Lambda\partial\vec{a}}}+\left.\frac{\partial\log\Theta_E}{\partial\xi^i}\right|_{\vec{\xi}=-\frac{d-1/2}{4\pi\sqrt{-1}r}\frac{\partial^2F_0}{\partial\log\Lambda\partial\vec{a}}}\right)\frac{\partial U_p}{\partial a^i}$$

as in [loc. cit., §A.6.1]. This is nothing but the sum of (3.21) for d replaced by $d+(r+l+1)/2$ and $d+(r+l-1)/2$. Since we have already proved (3.21) for $(r+l-1)/2$, we have (3.21) for $l\le d\le r$.

4. Quiver description

In this section we review the result of [21], rephrase that of [22] in the quiver description, and add a few things on Ext-groups.

4.1. **Moduli spaces of m-stable sheaves.**

We take vector spaces V_0, V_1, W with

$$r=\dim W,\quad (c_1,[C])=\dim V_0-\dim V_1,\quad (\mathrm{ch}_2,[\widehat{\mathbb{P}}^2])=-\frac{1}{2}(\dim V_0+\dim V_1).$$

We consider following datum $X=(B_1,B_2,d,i,j)$

- $B_1, B_2 \in \operatorname{Hom}(V_1, V_0)$, $d \in \operatorname{Hom}(V_0, V_1)$, $i \in \operatorname{Hom}(W, V_0)$, $j \in \operatorname{Hom}(V_1, W)$,

$$\begin{tikzcd} V_0 \arrow[rr, shift left, leftarrow, "B_1, B_2"] \arrow[rr, shift left=2, leftarrow] \arrow[rr, "d"'] & & V_1 \arrow[dl, "j"] \\ & W \arrow[ul, "i"] & \end{tikzcd}$$

- $\mu(B_1, B_2, d, i, j) = B_1 d B_2 - B_2 d B_1 + ij = 0.$

Let $Q := \mu^{-1}(0)$ be the subscheme of the vector space $\operatorname{Hom}(V_1, V_0)^{\oplus 2} \oplus \operatorname{Hom}(V_0, V_1) \oplus \operatorname{Hom}(W, V_0) \oplus \operatorname{Hom}(V_1, W)$ defined by the equation $\mu = 0$. It is acted by $G := \operatorname{GL}(V_0) \times \operatorname{GL}(V_1)$

$$g \cdot (B_1, B_2, d, i, j) = (g_0 B_1 g_1^{-1}, g_0 B_2 g_1^{-1}, g_1 d g_0^{-1}, g_0 i, j g_1^{-1}).$$

Let $\zeta = (\zeta_0, \zeta_1) \in \mathbb{Q}^2$.

Definition 4.1. We say $X = (B_1, B_2, d, i, j)$ is ζ-*semistable* if

(1) for subspaces $S_0 \subset V_0$, $S_1 \subset V_1$ such that $B_\alpha(S_1) \subset S_0$ $(\alpha = 1, 2)$, $d(S_0) \subset S_1$, $\operatorname{Ker} j \supset S_1$, we have $\zeta_0 \dim S_0 + \zeta_1 \dim S_1 \le 0$.
(2) for subspaces $T_0 \subset V_0$, $T_1 \subset V_1$ such that $B_\alpha(T_1) \subset T_0$ $(\alpha = 1, 2)$, $d(T_0) \subset T_1$, $\operatorname{Im} i \subset T_0$, we have $\zeta_0 \operatorname{codim} T_0 + \zeta_1 \operatorname{codim} T_1 \ge 0$.

We say X is ζ-*stable* if the inequalities are strict unless $(S_0, S_1) = (0, 0)$ and $(T_0, T_1) = (V_0, V_1)$ respectively.

We say X_1, X_2 are S-*equivalent* when the closures of orbits intersect in the ζ-semistable locus of Q.

By a standard argument, we can see that these come from quotients of Q by G in the geometric invariant theory. We only explain the result, see [10] for detail.

Let $\chi\colon G \to \mathbb{C}^*$ be the character given by $\chi(g) = \det g_0^{-\zeta_0} \det g_1^{-\zeta_1}$, where we assume $(\zeta_0, \zeta_1) \in \mathbb{Z}^2$ by multiplying a positive integer if necessary. We have a lift of G action on the trivial line bundle $Q \times \mathbb{C}$ given by $(B_1, B_2, d, i, j, z) = (g \cdot (B_1, B_2, d, i, j), \chi(g) z)$. Let L_ζ denote the corresponding G-equivariant line bundle. Then we can consider the GIT quotient

$$\widehat{M}_\zeta = \operatorname{Proj}\left(\bigoplus_{n \ge 0} A(Q)^{\chi, n}\right),$$

where $A(Q)^{\chi,n}$ is the relative invariants in the coordinate ring $A(Q)$ of Q: $\{f \in A(Q) \mid f(g \cdot X) = \chi(g)^n f(X)\}$, which is the space of invariant sections of $L_\zeta^{\otimes n}$. Then $\widehat{M}_\zeta$ is the quotient of ζ-semistable locus modulo S-equivalence relation. It contains $\widehat{M}_\zeta^{\mathrm{s}}$ of the quotient of ζ-stable locus modulo the action of G.

We have a natural projective morphism $\widehat{\pi}\colon \widehat{M}_\zeta \to \mu^{-1}(0)/\!\!/G$, where $\mu^{-1}(0)/\!\!/G$ is the affine geometric invariant theory quotient of $\mu^{-1}(0)$ by G, i.e., $\widehat{M}_0$. By [21, §1.3] $\mu^{-1}(0)/\!\!/G$ is isomorphic to M_0, the Uhlenbeck partial compactification on $\mathbb{P}^2$.

Now the main result of [21] says

Theorem 4.2. *Let $m \in \mathbb{Z}_{\geq 0}$. Suppose that $\zeta_0 < 0$, $0 > m\zeta_0 + (m+1)\zeta_1 \gg -1$. Then we have $\widehat{M}_\zeta = \widehat{M}_\zeta^{\mathrm{s}}$ and it is bijective to the set of isomorphism classes of m-stable framed sheaves on $\widehat{\mathbb{P}}^2$.*

We use this result as the definition of the moduli scheme of m-stable framed sheaves in this paper. Therefore $\widehat{M}_\zeta$ will be denoted by $\widehat{M}^m$ (or $\widehat{M}^m(c)$ when we want to write the Chern character of sheaves) hereafter. It was proved in [21, 2.4] that (a) $d\mu$ is surjective and (b) the action of G is free on the ζ-stable locus. Therefore $\widehat{M}_\zeta$ is a smooth fine moduli scheme.

The construction is given as follows: We consider the complex

$$\begin{matrix} V_0 \otimes \mathcal{O}(C - \ell_\infty) \\ \oplus \\ V_1 \otimes \mathcal{O}(-\ell_\infty) \end{matrix} \xrightarrow{\alpha} \begin{matrix} \mathbb{C}^2 \otimes V_0 \otimes \mathcal{O} \\ \oplus \\ \mathbb{C}^2 \otimes V_1 \otimes \mathcal{O} \\ \oplus \\ W \otimes \mathcal{O} \end{matrix} \xrightarrow{\beta} \begin{matrix} V_0 \otimes \mathcal{O}(\ell_\infty) \\ \oplus \\ V_1 \otimes \mathcal{O}(-C + \ell_\infty) \end{matrix}, \tag{4.3}$$

with

$$\alpha = \begin{bmatrix} z & z_0 B_1 \\ w & z_0 B_2 \\ 0 & z_1 - z_0 dB_1 \\ 0 & z_2 - z_0 dB_2 \\ 0 & z_0 j \end{bmatrix}, \qquad \beta = \begin{bmatrix} z_2 & -z_1 & B_2 z_0 & -B_1 z_0 & i z_0 \\ dw & -dz & w & -z & 0 \end{bmatrix}.$$

The equation $\mu(B_1, B_2, d, i, j) = B_1 dB_2 - B_2 dB_1 + ij = 0$ is equivalent to $\beta\alpha = 0$. The stability condition ensures the injectivity of α and the surjectivity of β. Then the sheaf corresponding to (B_1, B_2, d, i, j) is defined by $E = \operatorname{Ker}\beta / \operatorname{Im}\alpha$. By the definition it is endowed with the framing $E|_{\ell_\infty} \to W \otimes \mathcal{O}_{\ell_\infty}$. The ζ-stability is identified with the m-stability.

The inverse construction is given by

$$V_0 := H^1(E(-\ell_\infty)), \quad V_1 := H^1(E(C - \ell_\infty)),$$

and B_1, B_2, d, i, j are homomorphisms between cohomology groups induced from certain natural sections.

From this construction V_0, V_1 naturally define vector bundles over $\widehat{M}^m$, which are denoted by $\mathcal{V}_0$, $\mathcal{V}_1$ respectively. The above α, β in (4.3) are interpreted as homomorphisms between vector bundles and the universal family $\mathcal{E}$ is given by $\operatorname{Ker}\beta / \operatorname{Im}\alpha$.

When we prove that the ζ-stability corresponds to the m-stability in Definition 1.1, it is crucial to observe that the sheaf $\mathcal{O}_C(-m-1)$ corresponds to the datum $V_0 = \mathbb{C}^m$, $V_1 = \mathbb{C}^{m+1}$, $W = 0$, $d = 0$ and

$$B_1 = \begin{bmatrix} 1_m & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 & 1_m \end{bmatrix},$$

where 1_m is the identity matrix of size m. We denote this datum by C_m as above.

4.2. **Tangent complex.** From the construction the tangent space is the middle cohomology group of the complex

$$
\begin{array}{ccccc}
 & & \mathrm{Hom}(V_0,V_1) & & \\
 & & \oplus & & \\
\mathrm{Hom}(V_0,V_0) & & \mathbb{C}^2\otimes\mathrm{Hom}(V_1,V_0) & & \\
\oplus & \xrightarrow{\ \iota\ } & \oplus & \xrightarrow{\ d\mu\ } & \mathrm{Hom}(V_1,V_0), \\
\mathrm{Hom}(V_1,V_1) & & \mathrm{Hom}(W,V_0) & & \\
 & & \oplus & & \\
 & & \mathrm{Hom}(V_1,W) & &
\end{array}
\tag{4.4}
$$

with

$$
\iota\begin{bmatrix}\xi_0\\ \xi_1\end{bmatrix} = \begin{bmatrix} d\xi_0-\xi_1 d\\ B_1\xi_1-\xi_0B_1\\ B_2\xi_1-\xi_0B_2\\ \xi_0 i\\ -j\xi_1\end{bmatrix},\qquad (d\mu)\begin{bmatrix}\widetilde{d}\\ \widetilde{B}_1\\ \widetilde{B}_2\\ \widetilde{i}\\ \widetilde{j}\end{bmatrix} = \begin{array}{c} B_1d\widetilde{B}_2+B_1\widetilde{d}B_2+\widetilde{B}_1dB_2\\ -B_2d\widetilde{B}_1-B_2\widetilde{d}B_1-\widetilde{B}_2dB_1\\ +\widetilde{i}j+i\widetilde{j}, \end{array}
$$

where $d\mu$ is the differential of μ, and ι is the differential of the group action. Remark that $d\mu$ is surjective and ι is injective by the above remark if X is ζ-stable.

4.3. **A modified quiver.** We fix the vector space W with $\dim W = r \neq 0$. We define a new quiver with three vertexes 0, 1, ∞. We write two arrows from 1 to 0 corresponding to the data B_1, B_2, and one arrow from 0 to 1 corresponding to the data d. Instead of writing one arrow from ∞ to 0, we write r-arrows. Similarly we write r-arrows from 1 to ∞. And instead of putting W at ∞, we replace it the one dimensional space $\mathbb{C}$ on ∞. We denote it by V_∞. It means that instead of considering the homomorphism i from W to V_0, we take r-homomorphisms i_1,i_2,..., i_r from V_∞ to V_0 by taking a base of W. (See Figure 2.)

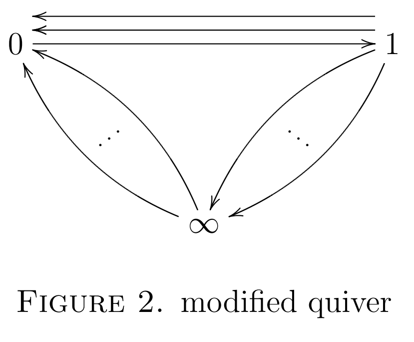

FIGURE 2. modified quiver

We consider the full subcategory of the abelian category of representations of the new quiver with the relation, consisting of representations such that $\dim V_\infty = 0$ or 1. An object can be considered as a representation of the original quiver with $\dim W = 0$, or $\dim W = r$, according to $\dim V_\infty = 0$ or 1. Note that we do not allow a representation of the original quiver with $\dim W \neq r, 0$.

It is also suitable to modify the stability condition for representations for the new quiver. Let $(\zeta_0, \zeta_1, \zeta_\infty) \in \mathbb{Q}^3$. For a representation X of the modified quiver, let us denote the underlying vector spaces by X_0, X_1, X_∞. We define the *rank*, *degree* and *slope* by

$$\begin{aligned} \operatorname{rank} X &:= \dim X_0 + \dim X_1 + \dim X_\infty, \\ \zeta \cdot \mathbf{dim}\, X &:= \zeta_0 \dim X_0 + \zeta_1 \dim X_1 + \zeta_\infty \dim X_\infty, \\ \theta(X) &:= \frac{\zeta \cdot \mathbf{dim}\, X}{\operatorname{rank} X}, \end{aligned} \tag{4.5}$$

where $\theta(X)$ is defined only when $X \neq 0$. We only consider the case $\dim X_\infty = 0$ or 1 as before. We say X is *θ-semistable* if we have

$$\theta(S) \leq \theta(X)$$

for any subrepresentation $0 \neq S$ of X. We say X is *θ-stable* if the inequality is strict unless $S = X$. If $\theta(X) = 0$, θ-(semi)stability is equivalent to ζ-(semi)stability. In fact, if a subrepresentation S has $S_\infty = 0$, then $\theta(S) \leq 0$ is equivalent to $\zeta_0 \dim S_0 + \zeta_1 \dim S_1 \leq 0$. If a subrepresentation T has $T_\infty = \mathbb{C}$, then $\theta(T) \leq 0$ is equivalent to $\zeta_0 \operatorname{codim} T_0 + \zeta_1 \operatorname{codim} T_1 \geq 0$.

The θ-stability is unchanged even if we add $c(1,1,1)$ ($c \in \mathbb{R}$) to $(\zeta_0, \zeta_1, \zeta_\infty)$. Therefore once we fix $\dim X_0$, $\dim X_1$, $\dim X_\infty$, we can always achieve the condition $\theta(X) = 0$ without the changing stable objects.

4.4. **Wall-crossing.** Let us fix a wall $\{\zeta \mid m\zeta_0 + (m+1)\zeta_1 = 0, \zeta_0 < 0\}$ and a parameter $\zeta^0 = (\zeta_0^0, \zeta_1^0)$ from the wall. We take ζ^+, ζ^- sufficiently close to ζ^0 with

$$1 \gg m\zeta_0^+ + (m+1)\zeta_1^+ > 0, \qquad -1 \ll m\zeta_0^- + (m+1)\zeta_1^- < 0.$$

(See Figure 3.) Then ζ^- is nothing but the parameter ζ appeared in Theorem 4.2 corresponding to the m-stability. On the other hand, we also know that ζ^+ corresponds to the $(m+1)$-stability by the determination of the chamber structure in [21, §2].

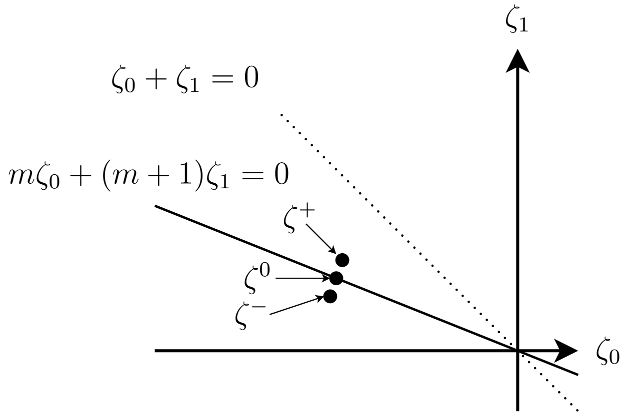

FIGURE 3. wall-crossing

Let us define the scheme $\widehat{M}^{m,m+1}$ as the GIT quotient $\widehat{M}_{\zeta^0}$ with respect to the ζ^0-semistability.

From [21] it follows that $\widehat{M}^{m,m+1}$ is the S-equivalence classes of $(m, m+1)$-semistable sheaves set-theoretically, where

Definition 4.6. **(a)** A framed sheaf (E, Φ) on $\widehat{\mathbb{P}}^2$ is called $(m, m+1)$-*semistable* if

(1) $\operatorname{Hom}(E, \mathcal{O}_C(-m-2)) = 0$,
(2) $\operatorname{Hom}(\mathcal{O}_C(-m), E) = 0$, and
(3) E is torsion free outside C.

(b) A framed sheaf (E, Φ) on $\widehat{\mathbb{P}}^2$ is called $(m, m+1)$-*stable* if it is either $\mathcal{O}_C(-m-1)$ (with the trivial framing) or both m-stable and $(m+1)$-stable, i.e., we have $\operatorname{Hom}(E, \mathcal{O}_C(-m)) = 0$ and $\operatorname{Hom}(\mathcal{O}_C(-m), E) = 0$ instead of (1),(2).

(c) A $(m, m+1)$-semistable framed sheaf (E, Φ) has a filtration $(0 = E^0 \subset E^1 \subset \cdots \subset E^N = E)$ such that E^i/E^{i-1} is $(m, m+1)$-stable with the induced framing from Φ. We say $(m, m+1)$-semistable framed sheaves (E, Φ) and (E', Φ') are S-*equivalent* if there exists an isomorphism from $\bigoplus_i E^i/E^{i-1}$ to $\bigoplus_j E'^j/E'^{j-1}$ respecting the framing in the $(m, m+1)$-stable factors.

The main result [21, Th. 1.5] was stated for m-stable framed sheaves, but it can be generalized to the case of $(m, m+1)$-stable framed sheaves. It follows from Propositions 4.1,4.2 there that X is ζ^0-semistable if and only if it satisfies the condition (S2) in [loc. cit., Th. 1.1] and the condition corresponding to (a1-2) above. Then the remaining arguments are the same.

Since $\zeta^{\pm}$-stability implies ζ^0-semistability, we have natural morphisms $\xi_m \colon \widehat{M}^m \to \widehat{M}^{m,m+1}$, $\xi_m^+ \colon \widehat{M}^{m+1} \to \widehat{M}^{m,m+1}$. Thus $\widehat{M}^m$, $\widehat{M}^{m+1}$ and $\widehat{M}^{m,m+1}$ form the diagram $(*)$.

This definition of $\widehat{M}^{m,m+1}$ is different from what are given in [22] for ordinary moduli spaces without framing, but they are the same at least set-theoretically thanks to the construction in [loc. cit., §§3.6,3.7]. It is also possible to show directly that ξ_m, ξ_m^+ have structures of stratified Grassmann bundles described there.

From the definition it is clear that ξ_m, ξ_m^+ are compatible with projective morphisms $\widehat{M}^m \to M_0$, $\widehat{M}^{m,m+1} \to M_0$, $\widehat{M}^{m+1} \to M_0$, (all denoted by $\widehat{\pi}$ before).

The change of the moduli spaces under the wall-crossing is described as follows:

Proposition 4.7 ([22, 3.15])**.** (1) *If (E^-, Φ) is m-stable, we have an exact sequence*

$$0 \to V \otimes C_m \to E^- \to E' \to 0$$

with $V = \operatorname{Hom}(C_m, E^-)$ such that

(a) *E' is $(m, m+1)$-stable, and*
(b) *the induced homomorphism $V \to \operatorname{Ext}^1(E', C_m)$ is injective.*

Conversely if (E', Φ) is $(m, m+1)$-stable and a subspace $V \subset \operatorname{Ext}^1(E', C_m)$ is given, (E^-, Φ), defined by the above exact sequence, is m-stable.

(2) *If (E^+, Φ) is $(m+1)$-stable, we have an exact sequence*

$$0 \to E' \to E^+ \to U^\vee \otimes C_m \to 0$$

with $U = \operatorname{Hom}(E^+, C_m)$ such that

(a) *E' is $(m, m+1)$-stable, and*
(b) *the induced homomorphism $U^\vee \to \operatorname{Ext}^1(C_m, E')$ is injective.*

Conversely if (E', Φ) is $(m, m+1)$-stable and a subspace $U^\vee \subset \operatorname{Ext}^1(C_m, E')$ is given, (E^+, Φ), defined by the above exact sequence, is $(m+1)$-stable.

4.5. **Computation of** Ext**-groups.** In this subsection, we continue to fix a wall $m\zeta_0 + (m+1)\zeta_1 = 0$, $\zeta_0 < 0$.

Take $X = (B_1, B_2, d, i, j)$ defined on V_0, V_1, W such that $\mu(B_1, B_2, d, i, j) = 0$. We consider the complex

$$\begin{matrix} \operatorname{Hom}(V_0, \mathbb{C}^m) \\ \oplus \\ \operatorname{Hom}(V_1, \mathbb{C}^{m+1}) \end{matrix} \xrightarrow{\sigma} \begin{matrix} \operatorname{Hom}(V_0, \mathbb{C}^{m+1}) \\ \oplus \\ \mathbb{C}^2 \otimes \operatorname{Hom}(V_1, \mathbb{C}^m) \\ \oplus \\ \operatorname{Hom}(W, \mathbb{C}^m) \end{matrix} \xrightarrow{\tau} \operatorname{Hom}(V_1, \mathbb{C}^m), \tag{4.8}$$

with

$$\sigma \begin{bmatrix} \xi_0 \\ \xi_1 \end{bmatrix} = \begin{bmatrix} \xi_1 d \\ \xi_0 B_1 - \left[\begin{smallmatrix} 1_m & 0 \end{smallmatrix}\right] \xi_1 \\ \xi_0 B_2 - \left[\begin{smallmatrix} 0 & 1_m \end{smallmatrix}\right] \xi_1 \\ \xi_0 i \end{bmatrix},$$

$$\tau \begin{bmatrix} \widetilde{d} \\ \widetilde{B}_1 \\ \widetilde{B}_2 \\ \widetilde{i} \end{bmatrix} = \left[\begin{smallmatrix} 1_m & 0 \end{smallmatrix}\right] \widetilde{d} B_2 - \left[\begin{smallmatrix} 0 & 1_m \end{smallmatrix}\right] \widetilde{d} B_1 + \widetilde{i} j + \widetilde{B}_1 d B_2 - \widetilde{B}_2 d B_1.$$

This complex is constructed as follows. We take the dual of (4.3), and replace the part $\mathbb{C}^2 \otimes V_0^* \otimes \mathcal{O} \xrightarrow{[z\ w]} V_0^* \otimes \mathcal{O}(-C + \ell_\infty)$ by $V_0^* \otimes \mathcal{O}(C - \ell_\infty)$. We then take the tensor product with C_m and apply $H^*(\widehat{\mathbb{P}}^2, \bullet)$. Therefore when X corresponds to a framed sheaf (E, Φ), the cohomology groups of the complex are $\operatorname{Ext}^\bullet(E, C_m)$.

Lemma 4.9. *Suppose that X corresponds to a framed sheaf (E, Φ).*
(1) $\operatorname{Hom}(E, C_m) \cong \operatorname{Ker}\sigma$, $\operatorname{Ext}^1(E, C_m) \cong \operatorname{Ker}\tau/\operatorname{Im}\sigma$, *and* $\operatorname{Ext}^2(E, C_m) \cong \operatorname{Coker}\tau$.
(2) *Suppose further that X is $(m, m+1)$-semistable. Then τ is surjective.*

Proof. (1) These are already explained.
(2) By the Serre duality we have $\operatorname{Ext}^2(E, C_m) = \operatorname{Hom}(C_{m-1}, E)^\vee$. But the right hand side vanishes if E is $(m, m+1)$-semistable. □

If X is ζ^--stable (i.e., (E, Φ) is m-stable), we also have $\operatorname{Ker}\sigma = 0$. But this does not hold in general if X is only $(m, m+1)$-semistable.

Next we consider the complex

$$\begin{matrix} \operatorname{Hom}(\mathbb{C}^m, V_0) \\ \oplus \\ \operatorname{Hom}(\mathbb{C}^{m+1}, V_1) \end{matrix} \xrightarrow{\sigma} \begin{matrix} \operatorname{Hom}(\mathbb{C}^m, V_1) \\ \oplus \\ \mathbb{C}^2 \otimes \operatorname{Hom}(\mathbb{C}^{m+1}, V_0) \\ \oplus \\ \operatorname{Hom}(\mathbb{C}^{m+1}, W) \end{matrix} \xrightarrow{\tau} \operatorname{Hom}(\mathbb{C}^{m+1}, V_0), \tag{4.10}$$

with

$$\sigma \begin{bmatrix} \xi_0 \\ \xi_1 \end{bmatrix} = \begin{bmatrix} d\xi_0 \\ B_1\xi_1 - \xi_0 \left[\begin{smallmatrix} 1_m & 0 \end{smallmatrix}\right] \\ B_2\xi_1 - \xi_0 \left[\begin{smallmatrix} 0 & 1_m \end{smallmatrix}\right] \\ j\xi_1 \end{bmatrix},$$

$$\tau \begin{bmatrix} \widetilde{d} \\ \widetilde{B}_1 \\ \widetilde{B}_2 \\ \widetilde{j} \end{bmatrix} = B_1\widetilde{d}\left[\begin{smallmatrix} 1_m & 0 \end{smallmatrix}\right] - B_2\widetilde{d}\left[\begin{smallmatrix} 0 & 1_m \end{smallmatrix}\right] + i\widetilde{j} + B_1 d\widetilde{B}_2 - B_2 d\widetilde{B}_1.$$

Lemma 4.11. *Suppose that X corresponds to a framed sheaf (E, Φ).*
(1) $\operatorname{Hom}(C_m, E) \cong \operatorname{Ker}\sigma$, $\operatorname{Ext}^1(C_m, E) \cong \operatorname{Ker}\tau / \operatorname{Im}\sigma$, *and* $\operatorname{Ext}^2(C_m, E) \cong \operatorname{Coker}\tau$.
(2) *Suppose further that X is $(m, m+1)$-semistable. Then τ is surjective.*

Proof. The proof of (1) is the same as in Lemma 4.9, so is omitted.
(2) We have $\operatorname{Ext}^2(C_m, E) \cong \operatorname{Hom}(E, C_{m+1})^\vee$ by the Serre duality. But the right hand side vanishes if E is $(m, m+1)$-semistable. □

If X is ζ^+-stable (i.e., (E, Φ) is $(m+1)$-stable), we also have $\operatorname{Ker}\sigma = 0$. But this does not hold in general if X is only $(m, m+1)$-semistable.

Corollary 4.12. *Let $Q^{\mathrm{ss}}(\zeta^0)$ be the open subscheme of Q consisting of ζ^0-semistable, i.e., $(m, m+1)$-semistable objects. The differential $d\mu$ is surjective on $Q^{\mathrm{ss}}(\zeta^0)$. Hence $Q^{\mathrm{ss}}(\zeta^0)$ is smooth.*

Proof. Since the surjectivity of $d\mu$ is an open condition, it is enough to check the assertion when X is a direct sum of ζ^0-stable objects. As explained in Definition 4.6, we have $X = X^0 \oplus C_m^{\oplus p}$, where X^0 is ζ^0-stable with $X_\infty \neq 0$ and $p \geq 0$. Then the tangent complex (4.4) decomposes into four parts, the tangent complex for X^0, the sum of p-copies of the complex (4.8) for X^0, the sum of p-copies of the complex (4.10) for X^0, and the sum of p^2-copies of the tangent complex for C_m.

The differential of $d\mu$ for X^0 is surjective since X^0 is ζ^0-stable by [21, 2.4]. The surjectivity of τ for (4.8) and (4.10) was proved in Lemma 4.9 and Lemma 4.11 respectively. The surjectivity of $d\mu$ for C_m follows from either [21, 2.4], Lemma 4.9 or Lemma 4.11 since C_m is $(-1/m, 1/(m+1))$-stable by [21, §2.2]. □

5. Enhanced master spaces

A naive idea to show the weak wall-crossing formula (=Theorem 1.5) is to compare the intersection products on $\widehat{M}^m(c)$ and $\widehat{M}^{m+1}(c)$ through the diagram $(*)$. Such an idea works when $\widehat{M}^m(c)$, $\widehat{M}^{m+1}(c)$ are replaced by moduli spaces of stable rank 2 torsion free sheaves over a surface with $p_g = 0$ with respect to two polarizations separated by a wall which is good [3]. (See also [7].)

However the idea does not work since the morphisms ξ_m, ξ_m^+ are much more complicated in our current situation, basically because dimensions of vector spaces V, U in Proposition 4.7 can be arbitrary. We use a refinement of the idea, due to Mochizuki [14], which was used to study wall-crossing formula for moduli spaces of higher ranks stable sheaves. It consists of two parts:

(Ma): Consider a *pair* of a sheaf and a full flag in the sheaf cohomology group.
(Mb): Use the fixed point formula for the $\mathbb{C}^*$-equivariant cohomology on Thaddeus' master space.

In this section we describe the part **(Ma)**. The results are straightforward modifications of those in [14, §4], possibly except one in §5.4.

We remark that Yamada also used moduli spaces of pairs of a sheaf and a full flag in the sheaf cohomology group to study wall-crossing of moduli spaces of higher rank sheaves [29].

We continue to fix m and choose parameters ζ^0, $\zeta^\pm$ as in the previous section. We fix a dimension vector $\mathbf{v} = (v_0, v_1, 1)$ and define ζ^0_∞ so that the normalization condition $\zeta^0_0 v_0 + \zeta^0_1 v_1 + \zeta^0_\infty = 0$ is satisfied.

5.1. Framed sheaves with flags in cohomology groups.

Let $((E,\Phi), F^\bullet)$ be a pair of a framed sheaf and a full flag $F^\bullet = (0 = F^0 \subset F^1 \subset \cdots \subset F^{N-1} \subset F^N = V_1(E))$ ($N = \dim V_1(E)$). Let ℓ be an integer between 0 and N.

Definition 5.1. An object $((E,\Phi), F^\bullet)$ is called *(m,ℓ)-stable* if the following three conditions are satisfied

(1) (E,Φ) is $(m, m+1)$-semistable.
(2) For a subsheaf $\mathfrak{S} \subset E$ isomorphic to $C_m^{\oplus p}$ with $p \in \mathbb{Z}_{>0}$, we have $V_1(\mathfrak{S}) \cap F^\ell = 0$.
(3) For a subsheaf $\mathfrak{S} \subset E$ such that the quotient $E/\mathfrak{S}$ is isomorphic to $C_m^{\oplus p}$ with $p \in \mathbb{Z}_{>0}$, we have $F^\ell \not\subset V_1(\mathfrak{S})$.

This notion makes a bridge between the m-stability and the $(m+1)$-stability by the following observation: If $\ell = 0$ (resp. $= N$), then (m,ℓ)-stability of $((E,\Phi), F^\bullet)$ is equivalent to m (resp. $(m+1)$)-stability of (E,Φ). (See Proposition 4.7.)

Proposition 5.2 ([14, 4.2.5])**.** *We have a smooth fine moduli scheme $\widetilde{M}^{m,\ell}(c)$ of (m,ℓ)-stable objects $((E,\Phi), F^\bullet)$ with $\operatorname{ch}(E) = c$. There is a projective morphism $\widetilde{M}^{m,\ell}(c) \to M_0(p_*(c))$.*

The proof will be given in the next subsection.

Let $\mathcal{E}$ be the universal sheaf over $\widehat{\mathbb{P}}^2 \times \widetilde{M}^{m,\ell}(c)$ and q_1, q_2 be the projection to the first and second factors of $\widehat{\mathbb{P}}^2 \times \widetilde{M}^{m,\ell}(c)$ respectively as before. As well as the vector bundle $\mathcal{V}_1 \equiv \mathcal{V}_1(\mathcal{E}) := R^1 q_{2*}(\mathcal{E} \otimes q_1^* \mathcal{O}(C - \ell_\infty))$, we also have the universal family $\mathcal{F}^\bullet = (0 = \mathcal{F}^0 \subset \mathcal{F}^1 \subset \cdots \subset \mathcal{F}^{N-1} \subset \mathcal{F}^N = \mathcal{V}_1)$ of flags of vector bundles over $\widetilde{M}^{m,\ell}(c)$.

For $\ell = 0$ or N, the preceding remark implies that $\widetilde{M}^{m,0}(c)$, $\widetilde{M}^{m,N}(c)$ are the full flag bundles $\operatorname{Flag}(\mathcal{V}_1, \underline{N})$ associated with vector bundles $\mathcal{V}_1$ over $\widehat{M}^m(c)$, $\widehat{M}^{m+1}(c)$ respectively. Here $\underline{N} = \{1, \dots, N\}$ and the notation $\operatorname{Flag}(\mathcal{V}_1, \underline{N})$ means that the flag is indexed by the set $\underline{N}$. This notation will become useful when we consider the fixed points in the enhanced master space.

5.2. **Proof of Proposition 5.2.** Let us rephrase the (m,ℓ)-stability in the quiver description.

We consider a pair $(X, F^\bullet) = ((B_1, B_2, d, i, j), F^\bullet)$ of $X \in \mu^{-1}(0)$ and a flag $F^\bullet = (0 = F^0 \subset F^1 \subset \cdots \subset F^{N-1} \subset F^N = V_1)$ of V_1 with $\dim(F^i/F^{i-1}) = 0$ or 1. In Definition 5.1 we have assumed that $F^\bullet$ is a full flag, i.e., $\dim(F^i/F^{i-1}) = 1$, but we slightly generalize it for a notational simplicity.

In terms of a quiver representation, Definition 5.1 is expressed as follows:

Definition 5.3. For $0 \le \ell \le N$, we say $(X, F^\bullet)$ is (m,ℓ)-*stable* if the following conditions are satisfied:

(1) X is ζ^0-semistable,
(2) for a nonzero submodule $0 \ne S \subset X$ with $\zeta^0 \cdot \mathbf{dim}\, S/\operatorname{rank} S = 0$ and $S_\infty = 0$, we have $F^\ell \cap S_1 = 0$, and
(3) for a proper submodule $S \subsetneq X$ with $\zeta^0 \cdot \mathbf{dim}\, S/\operatorname{rank} S = 0$ and $S_\infty = \mathbb{C}$, we have $F^\ell \not\subset S_1$.

The equivalence of (2),(3) and 5.1(2),(3) is an immediate consequence of [21, Th. 2.13, Prop. 5.3].

If $\ell = 0$ (resp. $= N$), then (m,ℓ)-stability is equivalent to the ζ^- (resp. ζ^+)-stability of X. (See Proposition 4.7.)

The idea of the proof of Proposition 5.2 is to relate the above condition to an usual stability condition for a linearization on the product of $\mu^{-1}(0)$ and the flag variety with respect to the group action of G. It is the tensor product of linearizations on $\mu^{-1}(0)$ and the flag variety. For $\mu^{-1}(0)$, we take one as in §4.1. On the flag variety, we take $\bigotimes(\det \mathcal{F}^i)^{\otimes(-n_i)}$ for $\mathbf{n} = (n_1, \dots, n_N) \in \mathbb{Q}^N_{>0}$, where $\mathcal{F}^i$ is the i^{th} universal bundle. (See [14, §4.2].) Then we have a natural projective morphism to $\mu^{-1}(0)/\!\!/G = M_0(p_*(c))$ as before.

The corresponding stability condition is expressed as before, but an extra term for flags is added to θ (cf. [16, Ch. 4, §4]): For a nonzero graded submodule $S \subset X$ we define

$$\mu_{\zeta,\mathbf{n}}(S) := \frac{\zeta \cdot \mathbf{dim}\, S + \sum_i n_i \dim(S \cap F^i)}{\operatorname{rank} S}.$$

We say $(X, F^\bullet)$ is $(\zeta,\mathbf{n})$-*(semi)stable* if

$$\mu_{\zeta,\mathbf{n}}(S) (\le) \mu_{\zeta,\mathbf{n}}(X)$$

holds for any nonzero proper submodule $0 \ne S \subsetneq X$. Here $(\le)$ means that $\le$ for the semistable case, and $<$ for the stable case. We say $(X, F^\bullet)$ is *strictly* $(\zeta,\mathbf{n})$-*semistable* if it is $(\zeta,\mathbf{n})$-semistable and not $(\zeta,\mathbf{n})$-stable.

A standard argument shows the following:

Lemma 5.4. *If $(X, F^\bullet)$, $(Y, G^\bullet)$ are $(\zeta,\mathbf{n})$-stable and have the same $\mu_{\zeta,\mathbf{n}}$-value, a nonzero homomorphism $\xi\colon X \to Y$ with $\xi(F^i) \subset G^i$ must be an isomorphism.*

We take $N = \dim V_1$ so that $F^\bullet$ is a full flag of V_1. Consider the following conditions when $\ell \ne 0$:

(5.5a) $\qquad \zeta_0 v_0 + \zeta_1 v_1 + \zeta_\infty = 0,$

$$n_k > \operatorname{rank} X \sum_{i=k+1}^{N} i n_i > 0 \quad \text{for } k = 1, \dots, N-1, \tag{5.5b}$$

$$\max_{\substack{S \subset V \\ \zeta^0 \cdot \mathbf{dim}\, S \neq 0}} \left| \frac{(\zeta^0 - \zeta) \cdot \mathbf{dim}\, S}{\operatorname{rank} S} \right| + \sum_{i=1}^{N} i n_i < \min_{\substack{S \subset V \\ \zeta^0 \cdot \mathbf{dim}\, S \neq 0}} \left| \frac{\zeta^0 \cdot \mathbf{dim}\, S}{\operatorname{rank} S} \right|, \tag{5.5c}$$

$$0 < \sum_{i=1}^{\ell} i n_i - \frac{\operatorname{rank} X}{2m+1} (m\zeta_0 + (m+1)\zeta_1) < n_\ell, \tag{5.5d}$$

$$\begin{aligned} \operatorname{rank} X \sum_{i=l+1}^{N} i n_i < \min \Bigg(& \sum_{i=1}^{\ell} i n_i - \frac{\operatorname{rank} X}{2m+1} (m\zeta_0 + (m+1)\zeta_1), \\ & n_\ell - \sum_{i=1}^{\ell} i n_i + \frac{\operatorname{rank} X}{2m+1} (m\zeta_0 + (m+1)\zeta_1) \Bigg), \end{aligned} \tag{5.5e}$$

where $S \subset V$ runs over the set of all graded subspaces (not necessarily submodules, as X is not fixed yet) with $\zeta^0 \cdot \mathbf{dim}\, S \neq 0$ in (c). For example, we can take ζ, $\mathbf{n}$ in the following way: First note that the right hand side of (c) is a fixed constant independent of ζ, $\mathbf{n}$. Therefore it holds if n_i and $\zeta^0 - \zeta$ are sufficiently small. Next note that (d) says that $\zeta^0 - \zeta$, up to a constant multiplication, is between $(\sum_{i=1}^{l} i n_i) - n_l$ and $\sum_{i=1}^{l} i n_i$. Therefore we can first take ζ and $n_1, \dots, n_\ell$ so that (b) (for $k = 1, \dots, \ell$),(d) and (c) hold if we set $n_{\ell+1} = \cdots = n_N = 0$. Then we take sufficiently small n_i ($i = \ell+1, \dots, N$) so that (e) holds. We can choose them very small so that (b) and (c) are not violated.

Lemma 5.6 ([14, 4.2.4])**.** *Assume* (5.5). *Then the followings hold:*

(1) *The* (m, ℓ)*-stability is equivalent to the* $(\zeta, \mathbf{n})$*-stability.*
(2) *The* $(\zeta, \mathbf{n})$*-semistability automatically implies the* $(\zeta, \mathbf{n})$*-stability.*

Lemma 5.4 also implies that $(\zeta, \mathbf{n})$-stable objects have the trivial stabilizer. Therefore we have Proposition 5.2 from this lemma.

From the construction the universal family $\mathcal{F}^\bullet = (0 = \mathcal{F}^0 \subset \mathcal{F}^1 \subset \cdots \subset \mathcal{F}^{N-1} \subset \mathcal{F}^N = \mathcal{V}_1)$ is the descent of the universal flag over $\operatorname{Flag}(V_1, \underline{N})$. Using descents of V_0, V_1 (which will be denoted by $\mathcal{V}_0$, $\mathcal{V}_1$) in the complex (4.3), we also get the universal family $\mathcal{E}$ over $\widehat{\mathbb{P}}^2 \times \widehat{M}^m(c)$.

Proof of Lemma 5.6. Suppose that $S \subset X$ is a nonzero submodule with $\zeta^0 \cdot \mathbf{dim}\, S = 0$, $S_\infty = 0$. Then $\mathbf{dim}\, S = p(m, m+1, 0)$ for some $p \in \mathbb{Z}_{>0}$. Then $\mu_{\zeta,\mathbf{n}}(S) \le \mu_{\zeta,\mathbf{n}}(X)$ means

$$\frac{m\zeta_0 + (m+1)\zeta_1}{2m+1} + \frac{\sum_{i=1}^{N} n_i \dim(S_1 \cap F^i)}{p(2m+1)} \le \frac{\sum_{i=1}^{N} i n_i}{\operatorname{rank} X}.$$

By (5.5d,e) this holds if and only if $S_1 \cap F^\ell = 0$. Moreover the equality never holds.

Next suppose that $S \subset X$ is a proper submodule with $\zeta^0 \cdot \mathbf{dim}\, S = 0$, $S_\infty = \mathbb{C}$. Then $\mathbf{dim}\, X/S = p(m, m+1, 0)$ for some $p \in \mathbb{Z}_{>0}$. Then $\mu_{\zeta,\mathbf{n}}(S) \le \mu_{\zeta,\mathbf{n}}(X)$ means

$$\frac{m\zeta_0 + (m+1)\zeta_1}{2m+1} + \frac{\sum_{i=1}^{N} n_i \dim(F^i / S_1 \cap F^i)}{p(2m+1)} \ge \frac{\sum_{i=1}^{N} i n_i}{\operatorname{rank} X}.$$

By (5.5d,e) this holds if and only if $F^i/S_1 \cap F^i \neq 0$ for some $i = 1, \dots, \ell$, i.e., $F^\ell \not\subset S_1$. Moreover the equality never holds.

Now let us start the proof. Suppose that $(X, F^\bullet)$ is $(\zeta, \mathbf{n})$-semistable. Then by (5.5c) $\mu_{\zeta,\mathbf{n}}(S) \leq \mu_{\zeta,\mathbf{n}}(X)$ implies $\zeta^0 \cdot \mathbf{dim}\, S \leq 0$, i.e., X is ζ^0-semistable. Then the above consideration shows that $(X, F^\bullet)$ is (m, ℓ)-stable and $(\zeta, \mathbf{n})$-stable.

Conversely suppose $(X, F^\bullet)$ is (m, ℓ)-stable. We want to show that $\mu_{\zeta,\mathbf{n}}(S) < \mu_{\zeta,\mathbf{n}}(X)$ for any nonzero proper submodule S. Thanks to (5.5c), it is enough to check this inequality for S with $\zeta^0 \cdot \mathbf{dim}\, S = 0$. We have either $S_\infty = 0$ or $S_\infty = \mathbb{C}$, and the above consideration shows that the inequality holds. □

Remark 5.7. A closer look of the argument gives that it is enough to assume (5.5c) for $S \subset V$ satisfying either of

(1) $\zeta^0 \cdot \mathbf{dim}\, S > 0$ and $\mu_{\zeta,\mathbf{n}}(S) \leq \mu_{\zeta,\mathbf{n}}(X)$,
(2) $\zeta^0 \cdot \mathbf{dim}\, S < 0$ and $\mu_{\zeta,\mathbf{n}}(S) \geq \mu_{\zeta,\mathbf{n}}(X)$.

5.3. **Oriented sheaves with flags in cohomology groups.** The following variant of objects in the previous subsection will show up during our analysis for the enhanced master space.

Definition 5.8. (1) Let $(E, F^\bullet)$ be a pair of a sheaf and a full flag $F^\bullet = (0 = F^0 \subset F^1 \subset \cdots \subset F^{N-1} \subset F^N = V_1(E))$ $(N = \dim V_1(E))$. We say $(E, F^\bullet)$ is $(m, +)$-*stable* if the following two conditions are satisfied:

(a) $E \cong C_m^{\oplus p}$ for $p \in \mathbb{Z}_{>0}$.
(b) For a proper subsheaf $\mathfrak{S} \subsetneq E$ isomorphic to $C_m^{\oplus q}$ with $q \in \mathbb{Z}_{>0}$, we have $V_1(\mathfrak{S}) \cap F^1 = 0$.

(2) For $(m_0, m_1) \in \mathbb{Z}^2 \setminus \{(0,0)\}$, an *orientation* of $(E, F^\bullet)$ is an isomorphism $\rho\colon \det H^1(E)^{\otimes m_0} \otimes \det H^1(E(C))^{\otimes m_1} \xrightarrow{\cong} \mathbb{C}$. We set $D := mm_0 + (m+1)m_1$.

(3) An oriented $(m,+)$-stable object means an $(m,+)$-stable object $(E, F^\bullet)$ together with an orientation.

We will choose (m_0, m_1) later when we define the enhanced master space. At this stage we only need that we will have $D > 0$. Since the orientation will be used frequently, we use the notation $L(E) := \det H^1(E(-\ell_\infty))^{\otimes m_0} \otimes \det H^1(E(C - \ell_\infty))^{\otimes m_1}$ for a sheaf E on $\widehat{\mathbb{P}}^2$. (In the above case, E is supported on C and the twisting by $\mathcal{O}(-\ell_\infty)$ is unnecessary.) If $\mathcal{E}$ is a universal family for some moduli stack, we denote the corresponding line bundle by $\mathcal{L}(\mathcal{E})$. Note that we deal only with those sheaves given by quiver representations, we have vanishing of H^0 and H^2.

We will show that we have a moduli stack $\widetilde{M}^{m,+}(pe_m)$ of oriented $(m,+)$-stable objects with $\mathrm{ch}(E) = pe_m$ with the universal family $(\mathcal{E}, \mathcal{F}^\bullet)$ where $\mathcal{F}^\bullet = (0 = \mathcal{F}^0 \subset \mathcal{F}^1 \subset \cdots \subset \mathcal{F}^{N-1} \subset \mathcal{F}^N = \mathcal{V}_1)$ is a flag of vector bundles over $\widetilde{M}^{m,+}(pe_m)$.

In the following proposition we identify $\widetilde{M}^{m,+}(pe_m)$ with a quotient stack related to the Grassmann variety $\mathrm{Gr}(m+1, p)$ of p-dimensional quotients of $V_1(C_m)^* = H^1(C_m(C))^* = \mathbb{C}^{m+1}$. Let us fix the notation. Let $\mathcal{Q}$ denote the universal quotient bundle over $\mathrm{Gr}(m+1, p)$, $\det \mathcal{Q}$ its determinant line bundle, and $(\det \mathcal{Q})^{\otimes D}$ its D^{th} tensor power. Let $\pi_G\colon ((\det \mathcal{Q})^{\otimes D})^\times \to \mathrm{Gr}(m+1, p)$ be the associated $\mathbb{C}^*$-bundle. Let $\mathcal{O}_{\mathrm{Gr}(m+1,p)} \to V_1(C_m) \otimes \mathcal{Q}$ be the homomorphism obtained from the universal homomorphism $V_1(C_m)^* \to \mathcal{Q}$.

Proposition 5.9. (1) *Forgetting F^i for $i \neq 1$, we identify $\widetilde{M}^{m,+}(pe_m)$ with the total space of the flag bundle* $\operatorname{Flag}(\mathcal{V}_1/\mathcal{F}^1, \underline{N} \setminus \{1\})$, *where the base is isomorphic to the quotient stack*

$$((\det \mathcal{Q})^{\otimes D})^{\times}/\mathbb{C}^*,$$

where $\mathbb{C}^*$ acts by the fiber-wise multiplication with weight $-pD$.

(2) *Let $\mathbb{C}^*_s = \operatorname{Spec}\mathbb{C}[s, s^{-1}]$ be a copy of $\mathbb{C}^*$ and let $\mathbb{C}^* \to \mathbb{C}^*_s$ be the homomorphism given by $t \mapsto t^{-pD} = s$. It induces an étale and finite morphism $h\colon ((\det \mathcal{Q})^{\otimes D})^{\times}/\mathbb{C}^* \to ((\det \mathcal{Q})^{\otimes D})^{\times}/\mathbb{C}^*_s = \operatorname{Gr}(m+1,p)$ of degree $= 1/pD$. The vector bundles $\mathcal{V}_1$, $\mathcal{F}^1$ and the universal sheaf $\mathcal{E}$ are related to objects on $\operatorname{Gr}(m+1,p)$ by*

$$\mathcal{V}_1 \otimes (\mathcal{F}^1)^* = h^*(V_1(C_m) \otimes \mathcal{Q}), \qquad (\mathcal{F}^1)^{\otimes -pD} = h^*((\det \mathcal{Q})^{\otimes D}),$$
$$\mathcal{E} \otimes (\mathcal{F}^1)^* = (\operatorname{id} \times h)^*(C_m \boxtimes \mathcal{Q}),$$

and the inclusion $(\mathcal{F}^1 \to \mathcal{V}_1)$ is $h^*(\mathcal{O}_{\operatorname{Gr}(m+1,p)} \to V_1(C_m) \otimes \mathcal{Q}) \otimes \operatorname{id}_{\mathcal{F}^1}$.

The proof will be given in the next subsection.

5.4. **Proof of Proposition 5.9.** Since F^i with $i > 1$ does not appear in the stability condition, the moduli stack has a structure of the flag bundle $\operatorname{Flag}(\mathcal{V}_1/\mathcal{F}^1, \underline{N} \setminus \{1\})$ over the moduli stack parametrizing (X, F^1). Here $\mathcal{V}_1$ is a vector bundle over the moduli stack and $\mathcal{F}^1$ is its line subbundle, coming from V_1 and F^1 respectively.

Next we determine the moduli stack parametrizing (X, F^1). Since we already know $X \cong C_m^{\oplus p}$, the remaining parameter is only a choice of F^1, which is a 1-dimensional subspace in $V_1(X) \cong V_1(C_m) \otimes \mathbb{C}^p$. We have an action of the stabilizer $\operatorname{GL}_p(\mathbb{C})$ of X. The above stability condition means that F^1, viewed as a nonzero homomorphism $V_1(C_m)^* \to \mathbb{C}^p$, is surjective. Therefore the moduli stack is

$$(\{\xi \in \mathbb{P}(\operatorname{Hom}(V_1(C_m)^*, \mathbb{C}^p)) \mid \xi \text{ is surjective}\} \times \mathbb{C}^*) \,/\, \operatorname{GL}_p(\mathbb{C}), \tag{5.10}$$

where the action of $\operatorname{GL}_p(\mathbb{C})$ on $\mathbb{C}^*$ is given by $g \cdot u = (\det g)^D u$ with $D = mm_0 + (m+1)m_1$.

We consider (5.10) as

$$(\{\xi \in \operatorname{Hom}(V_1(C_m)^*, \mathbb{C}^p) \mid \xi \text{ is surjective}\} \times \mathbb{C}^*) \,/\, \operatorname{GL}_p(\mathbb{C}) \times \mathbb{C}^*, \tag{5.11}$$

where $\mathbb{C}^* \ni t$ acts by $(\xi, u) \mapsto (t\xi, u)$. If we take the quotient by $\operatorname{GL}_p(\mathbb{C})$ first, we get $L^{\times} = L \setminus (0\text{-section})$, where $L = (\det \mathcal{Q})^{\otimes D}$, the D^{th} tensor power of the determinant line bundle $\det \mathcal{Q}$ of the universal quotient over the $\operatorname{Gr}(m+1,p)$. Since $\mathbb{C}^* \ni t$ acts by

$$(t\xi, u) = t \operatorname{id}_{\mathbb{C}^p} \cdot (\xi, t^{-pD} u), \tag{5.12}$$

it is the fiber-wise multiplication with weight $-pD$ on the quotient $L^{\times}$. Thus the moduli stack parametrizing (X, F^1) is isomorphic to the quotient stack $[L^{\times}/\mathbb{C}^*]$.

This action factors through $\rho\colon \mathbb{C}^* \to \mathbb{C}^*$ given by $t \mapsto s = t^{-pD}$, where the latter action is free on $L^{\times}$ and the quotient is $\operatorname{Gr}(m+1,p)$. Let us denote the latter $\mathbb{C}^*$ by $\mathbb{C}^*_s$. Then the stack $L^{\times}/\mathbb{C}^*_s$ is represented by $\operatorname{Gr}(m+1,p)$, as $L^{\times}$ is a principal $\mathbb{C}^*_s$-bundle. Since a $\mathbb{C}^*$-bundle induces a $\mathbb{C}^*_s$-bundle by taking the quotient by $\operatorname{Ker}\rho \cong \mathbb{Z}/pD$, we have a morphism $H\colon L^{\times}/\mathbb{C}^* \to L^{\times}/\mathbb{C}^*_s = \operatorname{Gr}(m+1,p)$ between stacks. It is étale and finite of degree $1/pD$.

Let us identify the pair $(\mathcal{F}^1 \subset \mathcal{V}_1)$ of the vector bundle $\mathcal{V}_1$ and its line subbundle $\mathcal{F}^1$ over the moduli stack in the description $[L^{\times}/\mathbb{C}^*]$. In the description (5.10), it is the descent of the restriction of $(\mathcal{O}_{\mathbb{P}}(-1) \boxtimes \mathcal{O}_{\mathbb{C}^*} \subset V_1(C_m) \otimes \mathbb{C}^p \otimes \mathcal{O}_{\mathbb{P}} \boxtimes \mathcal{O}_{\mathbb{C}^*})$ with respect to the natural $\mathrm{GL}_p(\mathbb{C})$-action, where $\mathbb{P} = \mathbb{P}(\mathrm{Hom}(V_1(C_m)^*, \mathbb{C}^p))$. Then in the description (5.11), it becomes the descent of the pair $(\mathcal{O}_V \subset V_1(C_m) \otimes \mathbb{C}^p \otimes \mathcal{O}_V)$, where $\mathrm{GL}_p(\mathbb{C})$ acts naturally and the $\mathbb{C}^*$-action is twisted by the weight -1 action on the first factor $\mathcal{O}_V$. Here $V = \{\xi \in \mathrm{Hom}(V_1(C_m)^*, \mathbb{C}^p) \mid \xi \text{ is surjective}\} \times \mathbb{C}^*$. Finally in the description $[L^\times/\mathbb{C}^*]$, it is the descent of

$$(\mathcal{O}_{L^\times} \subset \pi_G^*(V_1(C_m) \otimes \mathcal{Q}))\,,$$

where the $\mathbb{C}^*$-action is twisted by weight -1 on both $\mathcal{O}_{L^\times}$ and $\pi_G^*(V_1(C_m) \otimes \mathcal{Q}$. Here $\pi_G \colon L^\times \to \mathrm{Gr}(m+1,p)$ is the projection. The twist on the second factor $\pi_G^*(V_1(C_m) \otimes \mathcal{Q})$ comes from the term $t\,\mathrm{id}_{\mathbb{C}^p}$ in (5.12).

From the above description of $\mathcal{V}_1$, we have $\mathcal{V}_1 \otimes (\mathcal{F}^1)^* = h^*(V_1(C_m) \otimes \mathcal{Q})$. On the other hand, $(\mathcal{F}^1)^{\otimes -pD}$ is the descent of $\mathcal{O}_{L^\times}$ with the $\mathbb{C}^*$-action twisted by weight pD. The action factors through the $\mathbb{C}_s^*$-action, and it is twisted by weight -1. Therefore it descends to L on $\mathrm{Gr}(m+1,p)$.

5.5. **2-stability condition.** This subsection is devoted to preliminaries for a study of enhanced master spaces.

We consider the following condition on $\mathbf{n}$:

$$\sum_{i=1}^{N} k_i n_i \neq 0 \quad \text{for any } (k_1, \dots, k_N) \in \mathbb{Z}^N \setminus \{0\} \text{ with } |k_i| \le 2N^2. \tag{5.13}$$

Our flag $F^\bullet$ of V_1 again may have repetition, but assume $\dim(F^i/F^{i-1}) = 0$ or 1 as before.

Lemma 5.14. *Assume that ζ satisfies $(m+1)\zeta_0+(m+2)\zeta_1 < 0$, $(m-1)\zeta_0+m\zeta_1 > 0$, and $\mathbf{n}$ satisfies* (5.13). *If $(X, F^\bullet)$ is strictly $(\zeta,\mathbf{n})$-semistable, then there exists a submodule $0 \neq S \subsetneq X$ such that*

(1) $\mu_{\zeta,\mathbf{n}}(S, S_1 \cap F^\bullet) = \mu_{\zeta,\mathbf{n}}(X, F^\bullet)$,
(2) *$(S, S_1 \cap F^\bullet)$ and $(X/S, F^\bullet/S_1 \cap F^\bullet)$ are $(\zeta,\mathbf{n})$-stable.*

Moreover the submodule S is unique except when $(X, F^\bullet)$ is the direct sum $(S, S_1 \cap F^\bullet) \oplus (X/S, F^\bullet/S_1 \cap F^\bullet)$. In this case the another choice of the submodule is X/S.

Proof. Take a submodule S violating the $(\zeta,\mathbf{n})$-stability of X. Then we have (1). Moreover $(S, S_1 \cap F^\bullet)$ and $(X/S, F^\bullet/S_1 \cap F^\bullet)$ are $(\zeta,\mathbf{n})$-semistable. We have either $S_\infty = 0$ or $(X/S)_\infty = 0$.

Assume either $(S, S_1 \cap F^\bullet)$ or $(X/S, F^\bullet/S_1 \cap F^\bullet)$ is strictly $(\zeta,\mathbf{n})$-semistable. Then we have a filtration $0 = X^0 \subsetneq X^1 \subsetneq X^2 \subsetneq X^3 = X$ with $\mu_{\zeta,\mathbf{n}}(X^a/X^{a-1}, F_a^\bullet) = \mu_{\zeta,\mathbf{n}}(X, F^\bullet)$ for $a = 1,2,3$, where $F_a^\bullet$ denote the induced filtration on X^a/X^{a-1} from $F^\bullet$.

Among X^a/X^{a-1} $(a = 1,2,3)$, one of them has $\mathbb{C}$ and two of them have 0 at the ∞-component. Assume X^1 has $\mathbb{C}$ at the ∞-component for brevity, as the following argument can be applied to the remaining cases.

We have $\mathbf{dim}\, X^2/X^1 = p_2(m, m+1, 0)$, $\mathbf{dim}\, X^3/X^2 = p_3(m, m+1, 0)$ for some p_2, $p_3 \in \mathbb{Z}_{>0}$. Then $\mu_{\zeta,\mathbf{n}}(X^2/X^1, F_2^\bullet) = \mu_{\zeta,\mathbf{n}}(X^3/X^2, F_3^\bullet)$ implies

$$\sum n_i \left(\frac{\dim(F_2^i)}{p_2} - \frac{\dim(F_3^i)}{p_3} \right) = 0.$$

By the assumption (5.13) we have $p_3 \dim(F_2^i) = p_2 \dim(F_3^i)$ and hence

$$p_3 \dim(F_2^i/F_2^{i-1}) = p_2 \dim(F_3^i/F_3^{i-1}) \tag{5.15}$$

for any i. On the other hand, we have

$$\dim(F_1^i/F_1^{i-1}) + \dim(F_2^i/F_2^{i-1}) + \dim(F_3^i/F_3^{i-1}) = \dim(F^i/F^{i-1}) = 0 \text{ or } 1$$

for any i. Therefore at most one of three terms in the left hand side can be 1 and the other terms are 0. Combined with (5.15) this implies $\dim(F_2^i/F_2^{i-1}) = \dim(F_3^i/F_3^{i-1}) = 0$ for any i. Thus we get a contradiction $X^1 = X^2 = X^3$.

If we have another submodule S' of the same property, the $(\zeta, \mathbf{n})$-stability implies $S \cap S' = 0$ or $S \cap S' = S = S'$. In the former case we have $S' = X/S$. □

Lemma 5.16 ([14, 4.3.9])**.** *Let $(\zeta, \mathbf{n})$ as in Lemma 5.14. If $(X, F^\bullet)$ is $(\zeta, \mathbf{n})$-semistable, its stabilizer is either trivial or $\mathbb{C}^*$. In the latter case, $(X, F^\bullet)$ has the unique decomposition $(X_\flat, F_\flat^\bullet) \oplus (X_\sharp, F_\sharp^\bullet)$ such that both $(X_\flat, F_\flat^\bullet)$, $(X_\sharp, F_\sharp^\bullet)$ are $(\zeta, \mathbf{n})$-stable, and $\mu_{(\zeta,\mathbf{n})}(X_\flat) = \mu_{(\zeta,\mathbf{n})}(X_\sharp)$. The stabilizer comes from that of the factor $(X_\sharp, F_\sharp^\bullet)$ with $(X_\sharp)_\infty = 0$.*

Proof. Suppose g stabilizes $(X, F^\bullet)$. If g has an eigenvalue $\lambda \neq 1$, then we have the generalized eigenspace decomposition $(X, F^\bullet) = (X_\flat, F_\flat^\bullet) \oplus (X_\sharp, F_\sharp^\bullet)$ with $(X_\flat)_\infty = \mathbb{C}$, $(X_\sharp)_\infty = 0$. By Lemma 5.14 $(X_\flat, F_\flat^\bullet)$, $(X_\sharp, F_\sharp^\bullet)$ are $(\zeta, \mathbf{n})$-stable. Since they have the same $\mu_{\zeta,\mathbf{n}}$ and are not isomorphic, there are no nonzero homomorphisms between them. Therefore the stabilizer is $\mathbb{C}^*$ in this case. The uniqueness follows from that in Lemma 5.14.

Next suppose g is unipotent and let $n = g - 1$. Assume $n \neq 0$ and let j such that $n^j \neq 0$, $n^{j+1} = 0$. We consider the submodule $0 \neq \operatorname{Ker} n^j \subsetneq X$. From the $(\zeta, \mathbf{n})$-semistability of $(X, F^\bullet)$ we have $\mu_{\zeta,\mathbf{n}}(\operatorname{Ker} n^j, F^\bullet \cap (\operatorname{Ker} n^j)_1) = \mu_{\zeta,\mathbf{n}}(X, F^\bullet)$. Therefore $(\operatorname{Ker} n^j, F^\bullet \cap (\operatorname{Ker} n^j)_1)$ and $(X/\operatorname{Ker} n^j, F^\bullet/F^\bullet \cap (\operatorname{Ker} n^j)_1)$ are $(\zeta, \mathbf{n})$-stable by Lemma 5.14. They are not isomorphic since they have different ∞-components. However $n^j \colon X/\operatorname{Ker} n^j \to \operatorname{Ker} n^j$ is a nonzero homomorphism, and we have a contradiction by Lemma 5.4. □

5.6. **Enhanced master space.** We continue to fix c, $m \in \mathbb{Z}_{\geq 0}$, $\ell \in \underline{N}$. As we mentioned above, $\widehat{M}^m$ is constructed as a GIT quotient of a common space Q independent of m. Then the moduli schemes $\widetilde{M}^{m,0}$ and $\widetilde{M}^{m,\ell}$ will be also constructed as GIT quotients of $\widetilde{Q} = Q \times \operatorname{Flag}(V_1, \underline{N})$ by the action of the group G with respect to a common polarization, but with different lifts of the action. Here V_1 is a vector space of dimension N, on which G acts naturally. And $\operatorname{Flag}(V_1, \underline{N})$ be the variety of full flags in V_1. Let us denote by L_- and L_+ the corresponding equivariant line bundles over $\widetilde{Q}$ to define $\widetilde{M}^{m,0}$ and $\widetilde{M}^{m,\ell}$ respectively. Their descents will be denoted by the same notations.

We consider the projective bundle $\mathbb{P}(L_- \oplus L_+) \to \widetilde{Q}$ with the canonical polarization $\mathcal{O}_{\mathbb{P}}(1)$. Here $\mathbb{P}(L_- \oplus L_+)$ is the space of 1-dimensional *quotients* of $L_- \oplus L_+$. We have the natural lifts of the G-action to $\mathbb{P}(L_- \oplus L_+)$ and $\mathcal{O}_{\mathbb{P}}(1)$. Let

$$\mathcal{M} \equiv \mathcal{M}(c) \equiv \mathcal{M}^{m,\ell}(c) := \mathbb{P}(L_- \oplus L_+)^{\mathrm{ss}}/G \tag{5.17}$$

be the quotient stack of $\mathcal{O}_{\mathbb{P}}(1)$-semistable objects of $\mathbb{P}(L_- \oplus L_+)$ divided by G, where $\mathbb{P}(L_- \oplus L_+)^{\mathrm{ss}}$ denotes the semistable locus. This is called the *enhanced master space*. This space was introduced in [27] to study the change of GIT quotients under the change of linearizations. We have an inclusion $\mathcal{M}_a := \mathbb{P}(L_a)^{\mathrm{ss}}/G \to \mathcal{M}$ for $a = \pm$.

The tautological flag of vector bundles over $\mathrm{Flag}(V_1, \underline{N})$ descends to $\mathcal{M}$. We denote it by $\mathcal{F}^\bullet = (0 = \mathcal{F}^0 \subset \mathcal{F}^1 \subset \cdots \subset \mathcal{F}^{N-1} \subset \mathcal{F}^N = \mathcal{V}_1)$. We also have the universal sheaf $\mathcal{E}$ over $\widehat{\mathbb{P}}^2 \times \mathcal{M}$.

We have a natural $\mathbb{C}^*$-action on $\mathbb{P}(L_- \oplus L_+)$ given by $t \cdot [z_- : z_+] = [tz_- : z_+]$ where $[z_- : z_+]$ is the homogeneous coordinates system of $\mathbb{P}(L_- \oplus L_+)$ along fibers. It descends to a $\mathbb{C}^*$-action on $\mathcal{M}$. We have a natural $\mathbb{C}^*$-equivariant structure on the universal family $\mathcal{E}$, $\mathcal{F}^\bullet$.

The following summarizes properties of $\mathcal{M}$.

Theorem 5.18. (1) *$\mathcal{M}$ is a smooth Deligne-Mumford stack. There is a projective morphism $\mathcal{M} \to M_0(p_*(c))$.*

(2) *The fixed point set of the $\mathbb{C}^*$-action decomposes as*

$$\mathcal{M}^{\mathbb{C}^*} = \mathcal{M}_+ \sqcup \mathcal{M}_- \sqcup \bigsqcup_{\mathfrak{I} \in \mathcal{D}^{m,\ell}(c)} \mathcal{M}^{\mathbb{C}^*}(\mathfrak{I}),$$

and we have isomorphisms $\mathcal{M}_+ \cong \widetilde{M}^{m,\ell}(c)$, $\mathcal{M}_- \cong \widetilde{M}^{m,0}(c)$. The universal family $(\mathcal{E}, \mathcal{F}^\bullet)$ on $\mathcal{M}$ is restricted to ones on $\mathcal{M}_+ \cong \widetilde{M}^{m,\ell}(c)$ and $\mathcal{M}_- \cong \widetilde{M}^{m,0}(c)$ (which were denoted by the same notation $(\mathcal{E}, \mathcal{F}^\bullet)$). And the restriction of the $\mathbb{C}^*$-equivariant structure are trivial.

(3) *There is a diagram*

$$\begin{tikzcd}[column sep=small] & \mathcal{S}(\mathfrak{I}) \arrow[dl, "F"'] \arrow[dr, "G"] & \\ \mathcal{M}^{\mathbb{C}^*}(\mathfrak{I}) & & \widetilde{M}^{m,\min(I_\sharp)-1}(c_\flat) \times \widetilde{M}^{m,+}(c_\sharp), \end{tikzcd} \tag{5.19}$$

where $\mathcal{S}(\mathfrak{I})$ is a smooth Deligne-Mumford stack and both F, G are étale and finite of degree $1/pD$. There is a line bundle $L_{\mathcal{S}}$ over $\mathcal{S}(\mathfrak{I})$ with $L_{\mathcal{S}}^{\otimes pD} = G^*(\mathcal{L}(\mathcal{E}_\flat)^*)$ and the restriction of the universal family $(\mathcal{E}, \mathcal{F}^\bullet)$ over $\mathcal{M}$ and the universal families $(\mathcal{E}_\flat, \mathcal{F}_\flat^\bullet)$, $(\mathcal{E}_\sharp, \mathcal{F}_\sharp^\bullet)$ over $\widetilde{M}^{m,\min(I_\sharp)-1}(c_\flat)$, $\widetilde{M}^{m,+}(c_\sharp)$ are related by

$$F^*\mathcal{E} \cong G^*(\mathcal{E}_\flat) \oplus G^*(\mathcal{E}_\sharp) \otimes L_{\mathcal{S}}, \qquad F^*\mathcal{F}^\bullet \cong G^*(\mathcal{F}_\flat^\bullet) \oplus G^*(\mathcal{F}_\sharp^\bullet) \otimes L_{\mathcal{S}}.$$

Moreover the restriction of the $\mathbb{C}^*$-equivariant structure on the universal family $(\mathcal{E}, \mathcal{F})$ is trivial on the factor $(\mathcal{E}_\flat, \mathcal{F}_\flat)$ and of weight $1/pD$ on $(\mathcal{E}_\sharp, \mathcal{F}_\sharp)$ under the above identification.

We need to explain some notations:

- $\mathcal{D}^{m,\ell}(c)$ is the set of *decomposition types*:

$$\mathcal{D}^{m,\ell}(c) := \left\{ \mathfrak{I} = (I_\flat, I_\sharp) \;\middle|\; \begin{array}{l} \underline{N} = I_\flat \sqcup I_\sharp,\ I_\flat, I_\sharp \neq \emptyset, \\ |I_\sharp| = p(m+1) \text{ for } p \in \mathbb{Z}_{>0}, \min(I_\sharp) \le \ell \end{array} \right\}. \tag{5.20}$$

  For $\mathfrak{I} \in \mathcal{D}^{m,\ell}(c)$, we set $c_\sharp = pe_m$, $c_\flat = c - c_\sharp \in H^*(\widehat{\mathbb{P}}^2)$.
- integers (m_0, m_1) appeared in the definition of an orientation of a $(m,+)$-stable object (see Definition 5.8) will be determined by the choice of $L_\pm$.

The isomorphism $F^*\mathcal{F}^\bullet \cong G^*(\mathcal{F}_\flat^\bullet) \oplus G^*(\mathcal{F}_\sharp^\bullet) \otimes L_{\mathcal{S}}$ of universal flags in (3) means as follows: From the first statement $F^*\mathcal{E} \cong G^*(\mathcal{E}_\flat) \oplus G^*(\mathcal{E}_\sharp) \otimes L_{\mathcal{S}}$ we have a decomposition $F^*(\mathcal{V}_1(\mathcal{E})) = G^*(\mathcal{V}_1(\mathcal{E}_\flat)) \oplus G^*(\mathcal{V}_1(\mathcal{E}_\sharp)) \otimes L_{\mathcal{S}}$. Then we have $F^*(\mathcal{F}^i) = G^*(\mathcal{F}_\flat^i) \oplus G^*(\mathcal{F}_\sharp^i) \otimes L_{\mathcal{S}}$ where $\mathcal{F}_\flat^\bullet$, $\mathcal{F}_\sharp^\bullet$ are flags indexed by $\underline{N}$. If we forget irrelevant factors $\mathcal{F}_\flat^i$ with $\mathcal{F}_\flat^i = \mathcal{F}_\flat^{i-1}$

and $\mathcal{F}_\sharp^j$ with $\mathcal{F}_\sharp^j = \mathcal{F}_\sharp^{j-1}$, we get the universal flags over $\widetilde{M}^{m,\min(I_\sharp)-1}(c_\flat)$, $\widetilde{M}^{m,+}(c_\sharp)$. The above sets $I_\flat$, $I_\sharp$ consist of indexes of relevant factors. In particular, $(\mathcal{F}^1 \subset \mathcal{V}_1)$ appearing in Proposition 5.9 is identified with $(\mathcal{F}_\sharp^{\min(I_\sharp)} \subset \mathcal{F}_\sharp^{\max(I_\sharp)} = \mathcal{F}_\sharp^N)$. Let us denote $\mathcal{F}_\sharp^N$ by $\mathcal{V}_1^\sharp$ hereafter.

The fixed point substack $\mathcal{M}^{\mathbb{C}^*}$ is defined as the zero locus of the fundamental vector field generated by the $\mathbb{C}^*$-action. Note that this does not imply that the action of $\mathbb{C}^*$ is trivial on $\mathcal{M}^{\mathbb{C}^*}$, but becomes trivial on the finite cover $\mathbb{C}_s^* = \operatorname{Spec}\mathbb{C}[s,s^{-1}] \to \mathbb{C}^*; s \mapsto s^{pD} = t$. Therefore the restriction of a $\mathbb{C}^*$-equivariant sheaf to the fixed point locus is a sheaf tensored by a $\mathbb{C}_s^*$-module. In the statements (2),(3), we wrote the weights of $\mathbb{C}_s^*$-modules divided by pD, considered formally as weights of *rational* $\mathbb{C}^*$-modules.

The proofs of (1),(2),(3) will be given in §5.7, §5.8, §5.9 respectively.

Remark 5.21. We can define the master space connecting $\widehat{M}^m(c)$ and $\widehat{M}^{m+1}(c)$ in the same way. However it will be not necessarily a Deligne-Mumford stack as a semistable point possibly has a stabilizer of large dimension. This is the reason why we, following Mochizuki, consider pairs of framed sheaves and flags in cohomology groups.

5.7. **Smoothness of the Enhanced Master Space.** Let us write the enhanced master space in the quiver description. We first take ζ^- sufficiently close to ζ^0. For $l = 1, \dots, N$, we choose $(\zeta, \mathbf{n})$ satisfying (5.5,5.13). We take ζ so that $|\zeta - \zeta^0|$, $|\mathbf{n}|$ are sufficiently smaller than $|\zeta - \zeta^-|$. We take a large number k so that $k(\zeta^-, \mathbf{n})$ and $k(\zeta, \mathbf{n})$ are integral. Let L_- (resp. L_+) be the G-equivariant line bundle over $\mu^{-1}(0) \times \operatorname{Flag}(V_1, \underline{N})$ corresponding to the stability condition $k(\zeta^-, \mathbf{n})$ (resp. $k(\zeta, \mathbf{n})$). We consider the projective bundle $\mathbb{P}(L_- \oplus L_+)$ with the canonical polarization $\mathcal{O}_{\mathbb{P}}(1)$. We have the natural lifts of the G-action to $\mathbb{P}(L_- \oplus L_+)$ and $\mathcal{O}_{\mathbb{P}}(1)$. Let

$$\mathcal{M} \equiv \mathcal{M}(c) := \mathbb{P}(L_- \oplus L_+)^{\mathrm{ss}}/G$$

be the quotient stack of the semistable locus $\mathbb{P}(L_- \oplus L_+)^{\mathrm{ss}}$ divided by G.

The following was shown in e.g., [27, §§3,4].

Lemma 5.22. *A point x of $\mathbb{P}(L_- \oplus L_+) \setminus (\mathbb{P}(L_-) \sqcup \mathbb{P}(L_+))$ is semistable if and only if the corresponding $(X, F^\bullet)$ is semistable with respect to a $\mathbb{Q}$-line bundle $L_t = L_-^{\otimes(1-t)} \otimes L_+^{\otimes t}$ for some $t \in [0,1] \cap \mathbb{Q}$.*

Proposition 5.23. *$\mathcal{M}$ is a smooth Deligne-Mumford stack.*

Proof. Let x be a semistable point in $\mathbb{P}(L_- \oplus L_+)$. Then the corresponding point $(X, F^\bullet)$ in $\mu^{-1}(0) \times \operatorname{Flag}(V_1, \underline{N})$ is $(\zeta', \mathbf{n})$-stable for some ζ' on the segment connecting ζ and ζ^- (Lemma 5.22). We can apply Lemma 5.16 as ζ' satisfies $(m+1)\zeta_0 + (m+2)\zeta_1 < 0$, $(m-1)\zeta_0 + m\zeta_1 > 0$, and $\mathbf{n}$ satisfies (5.13). Therefore either the stabilizer of $(X, F^\bullet)$ is trivial or $(X, F^\bullet)$ decomposes as $(X_\flat, F_\flat^\bullet) \oplus (X_\sharp, F_\sharp^\bullet)$. Since $(X_\sharp)_\infty = 0$, $X_\sharp \cong C_m^{\oplus p}$ for some $p \in \mathbb{Z}_{>0}$ as explained in Definition 4.6. In this case the stabilizer is $\mathbb{C}^*$, coming from the automorphisms of $(X_\sharp, F_\sharp^\bullet)$. Its action on the fiber is given by $t \cdot u = t^{pk(m,m+1)\cdot(\zeta-\zeta^-)}u$ for $t \in \mathbb{C}^*$. Therefore x only has a finite stabilizer. It is also reduced as the base field is of characteristic 0. Therefore $\mathcal{M}$ is Deligne-Mumford. Since $\mathbb{P}(L_- \oplus L_+)$ is smooth, $\mathcal{M}$ is also smooth. $\square$

We set $(m_0, m_1) := k(\zeta - \zeta^-)$ (and hence $D = k(m, m+1) \cdot (\zeta - \zeta^-)$) which was used in Theorem 5.18. From our choices of ζ, ζ^-, we have $D > 0$.

5.8. $\mathbb{C}^*$**-action.** We have a natural $\mathbb{C}^*$-action on $\mathbb{P}(L_- \oplus L_+)$ given by $t \cdot [z_- : z_+] = [tz_- : z_+]$ where $[z_- : z_+]$ is the homogeneous coordinates system of $\mathbb{P}(L_- \oplus L_+)$ along fibers. It descends to a $\mathbb{C}^*$-action on $\mathcal{M}$, as it commutes with the G-action. Letting $\mathbb{C}^*$ act trivially on V_0, V_1 and the universal flag over $\mathrm{Flag}(V_1, \underline{N})$, we have the $\mathbb{C}^*$-equivariant structure on the universal family $\mathcal{E}$, $\mathcal{F}^\bullet = (0 = \mathcal{F}^0 \subset \mathcal{F}^1 \subset \cdots \subset \mathcal{F}^{N-1} \subset \mathcal{F}^N = \mathcal{V}_1)$.

The fixed point substack $\mathcal{M}^{\mathbb{C}^*}$ is defined as the zero locus of the fundamental vector field generated by the $\mathbb{C}^*$-action. Note that this does not imply that the action of $\mathbb{C}^*$ is trivial on $\mathcal{M}^{\mathbb{C}^*}$. The action becomes trivial after a finite cover $\mathbb{C}^* \to \mathbb{C}^*$.

We have an inclusion $\mathcal{M}_a := \mathbb{P}(L_a)^{\mathrm{ss}}/G \to \mathcal{M}$ for $a = \pm$. Then $\mathcal{M}_a$ is a component of the fixed point set $\mathcal{M}^{\mathbb{C}^*}$. From the construction $\mathcal{M}_a$ is the moduli stack of objects $(X, F^\bullet)$, which are stable with respect to L_a. From our choice of $(\zeta, \mathbf{n})$, we have $\mathcal{M}_+ \cong \widetilde{M}^{m,\ell}$ by Lemma 5.6. Since $\mathbf{n}$ is sufficiently smaller than $|\zeta - \zeta^-|$, $(X, F^\bullet)$ is stable with respect to L_- if and only if X is ζ^--stable. Thus we have $\mathcal{M}_- \cong \widetilde{M}^{m,0}$.

Next consider a fixed point in $\mathcal{M}^{\mathbb{C}^*}$ other than $\mathcal{M}_+ \sqcup \mathcal{M}_-$. Suppose that a point $x = ((X, F^\bullet), [z_- : z_+])$ in $\mathbb{P}(L_- \oplus L_+)^{\mathrm{ss}} \setminus (\mathbb{P}(L_-) \sqcup \mathbb{P}(L_+))$ is mapped to a fixed point in the quotient $\mathcal{M}$. It means that the tangent vector generated by the $\mathbb{C}^*$-action at x is contained in the subspace generated by the G-action. In view of Lemma 5.16 this is possible only if (X, F^*) has a nontrivial stabilizer, and hence decompose as $(X_\flat, F_\flat^\bullet) \oplus (X_\sharp, F_\sharp^\bullet)$ to the direct sum of two $(\zeta', \mathbf{n})$-stable objects with the equal $\mu_{\zeta',\mathbf{n}}$ for some ζ' on the segment connecting ζ and ζ^-. (See Lemma 5.22 and Lemma 5.16.) We number the summand so that $(X_\flat)_\infty = \mathbb{C}$. Therefore $(X_\sharp)_\infty = 0$ and hence $X_\sharp \cong C_m^{\oplus p}$ for some $p \in \mathbb{Z}_{>0}$. The data $u = z_+/z_-$ corresponds to an isomorphism $L(X) \cong \mathbb{C}$.

Conversely suppose we have such a decomposition $(X, F^\bullet) = (X_\flat, F_\flat^\bullet) \oplus (X_\sharp, F_\sharp^\bullet)$. Let $V = V^\flat \oplus V^\sharp$ be the corresponding decomposition of V. We lift the $\mathbb{C}^*$-action on $\mathcal{M}$ to $\mathbb{P}(L_- \oplus L_+)^{\mathrm{ss}}$ by

$$((X, F^\bullet), [z_- : z_+]) \mapsto (\mathrm{id}_{V^\flat} \oplus t^{1/pD} \,\mathrm{id}_{V^\sharp}) \cdot ((X, F^\bullet), [tz_- : z_+]), \tag{5.24}$$

which is well-defined on the covering $\mathbb{C}^* \to \mathbb{C}^*; s \mapsto s^{pD} = t$, and fixes $((X, F^\bullet), [z_- : z_+]) = ((X_\flat, F_\flat^\bullet) \oplus (X_\sharp, F_\sharp^\bullet), [z_- : z_+])$. Since this $\mathbb{C}^*$-action is equal to the original one up to the G-action, it is the same on the quotient $\mathcal{M}$. Therefore the point $x = ((X, F^\bullet), [z_- : z_+])$ is mapped to a fixed point in $\mathcal{M}$.

Let

$$I_\alpha := \{ i \in \underline{N} \mid \dim(F_\alpha^i / F_\alpha^{i-1}) = 1 \}$$

for $\alpha = \flat, \sharp$. Then we have the decomposition $\underline{N} = I_\flat \sqcup I_\sharp$. The datum $(I_\flat, I_\sharp)$ is called the *decomposition type* of the fixed point. Since $\dim(X_\sharp)_1 = p(m+1)$, we have $|I_\sharp| = p(m+1)$.

Lemma 5.25 ([14, 4.4.3])**.** *$\min(I_\sharp) \le \ell$.*

Proof. Suppose $\min(I_\sharp) > \ell$. Then (5.5) implies $\mu_{\zeta,\mathbf{n}}(X_\sharp) < \mu_{\zeta,\mathbf{n}}(X)$. (See the proof of Lemma 5.6.) On the other hand we have $\mu_{\zeta^-,\mathbf{n}}(X_\sharp) < \mu_{\zeta^-,\mathbf{n}}(X)$ since $\mathbf{n}$ is sufficiently smaller than $|\zeta - \zeta^-|$. Therefore we cannot have $\mu_{\zeta',\mathbf{n}}(X_\sharp) = \mu_{\zeta',\mathbf{n}}(X)$ for any ζ' on the segment connecting ζ and ζ^-. This contradicts with the assumption. $\square$

Conversely suppose an object $(X, F^\bullet) = (X_\flat, F_\flat^\bullet) \oplus (X_\sharp, F_\sharp^\bullet)$ with the decomposition type $(I_\flat, I_\sharp)$ with $\min(I_\sharp) \le \ell$ is given. We also suppose $X_\sharp \cong C_m^{\oplus p}$. We take a point x of $\mathbb{P}(L_- \oplus L_+) \setminus (\mathbb{P}(L_-) \sqcup \mathbb{P}(L_+))$ from the fiber over $(X, F^\bullet)$. Since we have $\mu_{\zeta,\mathbf{n}}(X_\sharp) > \mu_{\zeta,\mathbf{n}}(X)$ and $\mu_{\zeta^-,\mathbf{n}}(X_\sharp) < \mu_{\zeta^-,\mathbf{n}}(X)$ by the same argument as above, we can find ζ' with $\mu_{\zeta',\mathbf{n}}(X_\sharp) = \mu_{\zeta',\mathbf{n}}(X)$. Then x is semistable if and only if both $(X_\flat, F_\flat)$ and $(X_\sharp, F_\sharp)$ are $(\zeta', \mathbf{n})$-stable.

Lemma 5.26 ([14, 4.4.4])**.** (1) $(X_\flat, F_\flat)$ *is* $(\zeta', \mathbf{n})$*-stable if and only if* $(m, \min(I_\sharp) - 1)$*-stable.*

(2) $(X_\sharp, F_\sharp)$ *is* $(\zeta', \mathbf{n})$*-stable if and only if* $(m, +)$*-stable, i.e.,* $X_\sharp \cong C_m^{\oplus p}$*, and we have* $S_1 \cap F_\sharp^{\min(I_\sharp)} = 0$ *for any proper submodule* $S \subsetneq X_\sharp$ *of a form* $S \cong C_m^{\oplus q}$*.*

Note that $\dim F_\sharp^{\min(I_\sharp)} = 1$, $\dim F_\flat^{\min(I_\sharp)-1} = \min(I_\sharp) - 1$. So the definitions in §5.1 apply though $F_\flat$, $F_\sharp$ are flags which possibly have repetitions.

Proof. (1) Let $S \subset X_\flat$ be a submodule. We need to study the stability inequalities when $\zeta^0 \cdot \mathbf{dim}\, S = 0$. We first suppose $S_\infty = 0$. Then the inequality $\mu_{(\zeta',\mathbf{n})}(S) < \mu_{(\zeta',\mathbf{n})}(X_\flat) = \mu_{(\zeta',\mathbf{n})}(X_\sharp)$ is equivalent to

$$\frac{\sum_i n_i \dim(S_1 \cap F_\flat^i)}{\operatorname{rank} S} < \frac{\sum_i n_i \dim F_\sharp^i}{\operatorname{rank} X_\sharp}$$

since $\zeta' \cdot \mathbf{dim}\, S / \operatorname{rank} S = \zeta' \cdot \mathbf{dim}\, X_\sharp / \operatorname{rank} X_\sharp = (2m+1)^{-1} (m\zeta'_0 + (m+1)\zeta'_1)$. Since n_i $(i \ge \min(I_\sharp))$ is much smaller than $n_{\min(I_\sharp)-1}$ by (5.5b), we must have $S_1 \cap F_\flat^{\min(I_\sharp)-1} = 0$ if the inequality holds. Conversely suppose $S_1 \cap F_\flat^{\min(I_\sharp)-1} = 0$. Then $S_1 \cap F_\flat^{\min(I_\sharp)} = 0$. Thus the inequality holds again by (5.5b).

Next suppose $S_\infty = \mathbb{C}$. Then the inequality $\mu_{(\zeta',\mathbf{n})}(S) < \mu_{(\zeta',\mathbf{n})}(X_\flat) = \mu_{(\zeta',\mathbf{n})}(X_\sharp)$ is equivalent to

$$\frac{\sum_i n_i \dim(F_\flat^i / S_1 \cap F_\flat^i)}{\operatorname{rank}(X_\flat / S)} > \frac{\sum_i n_i \dim F_\sharp^i}{\operatorname{rank} X_\sharp}.$$

This is equivalent to $F_\flat^{\min(I_\sharp)-1} \not\subset S_1$ by the same argument as above. Thus $(X_\flat, F_\flat)$ is $(\zeta', \mathbf{n})$-stable if and only if $(m, \min(I_\sharp) - 1)$-stable.

(2) First note that $X_\sharp$ must be ζ^0-semistable as ζ' is close to ζ^0 and $\mathbf{n}$ is small. Then $X_\sharp \cong C_m^{\oplus p}$, as explained in Definition 4.6. To prove the remaining part, the same argument as above works. $\square$

If we rephrase what we have observed in terms of sheaves, we get

Proposition 5.27 ([14, 4.5.2])**.** $\mathcal{M}^{\mathbb{C}^*}(\mathfrak{I})$ *is the moduli stack of objects* $(((E_\flat, \Phi), F_\flat^\bullet), (E_\sharp, F_\sharp^\bullet), \rho)$ *where*

- $((E_\flat, \Phi), F_\flat^\bullet)$ *is* $(m, \min(I_\sharp) - 1)$*-stable,*
- $(E_\sharp, F_\sharp^\bullet)$ *is* $(m, +)$*-stable,*
- ρ *is an isomorphism* $L(E_\flat \oplus E_\sharp) \xrightarrow{\cong} \mathbb{C}$*.*

Moreover the restriction of the universal family $(\mathcal{E}, \mathcal{F}^\bullet)$ *on* $\mathcal{M}$ *decomposes as*

$$\mathcal{E} = {}^{\mathcal{M}}\mathcal{E}_\flat \oplus {}^{\mathcal{M}}\mathcal{E}_\sharp, \qquad \mathcal{F}^\bullet = {}^{\mathcal{M}}\mathcal{F}_\flat^\bullet \oplus {}^{\mathcal{M}}\mathcal{F}_\sharp^\bullet,$$

where ${}^{\mathcal{M}}\mathcal{F}_\flat^\bullet$, ${}^{\mathcal{M}}\mathcal{F}_\sharp^\bullet$ *are flags* $(0 = {}^{\mathcal{M}}\mathcal{F}_\flat^0 \subset \cdots \subset {}^{\mathcal{M}}\mathcal{F}_\flat^N = \mathcal{V}_1({}^{\mathcal{M}}\mathcal{E}_\flat))$, $(0 = {}^{\mathcal{M}}\mathcal{F}_\sharp^0 \subset \cdots \subset {}^{\mathcal{M}}\mathcal{F}_\sharp^N = \mathcal{V}_1({}^{\mathcal{M}}\mathcal{E}_\sharp))$.

The restriction of the $\mathbb{C}^*$*-equivariant structure on the universal family* $(\mathcal{E},\mathcal{F})$ *is trivial on the factor* $({}^{\mathcal{M}}\mathcal{E}_\flat, {}^{\mathcal{M}}\mathcal{F}_\flat)$ *and of weight* $1/pD$ *on* $({}^{\mathcal{M}}\mathcal{E}_\sharp, {}^{\mathcal{M}}\mathcal{F}_\sharp)$ *under the above identification.*

The last assertion follows from the description of the $\mathbb{C}^*$-action at (5.24).

5.9. **Decomposition into product of two moduli stacks.** Let $\bar{M}$ be the moduli stack of objects $((E_\flat,\Phi), F_\flat^\bullet, \rho_\flat)$ where

- $((E_\flat,\Phi), F_\flat^\bullet)$ is $(m, \min(I_\sharp)-1)$-stable,
- $\rho_\flat$ is an isomorphism $L(E_\flat) \xrightarrow{\cong} \mathbb{C}$.

We have a natural projection $\bar{M} \to \widetilde{M}^{m,\min(I_\sharp)-1}$ forgetting $\rho_\flat$. It is a principal $\mathbb{C}^*$-bundle. On the other hand, let $\widetilde{M}^{m,+}$ be the moduli stack of oriented $(m,+)$-stable sheaves with flags as in §5.3. In order to distinguish from ρ, we denote the orientation by $\rho_\sharp$. Then we have

$$\mathcal{M}^{\mathbb{C}^*}(\mathfrak{I}) \cong (\bar{M} \times \widetilde{M}^{m,+})/\mathbb{C}^*,$$

where $\mathbb{C}^*$ acts by $\rho_\flat \mapsto t\rho_\flat$, $\rho_\sharp \mapsto t^{-1}\rho_\sharp$. Let us take a covering $\mathbb{C}_s^* \to \mathbb{C}^*$ $s \mapsto s^{pD} = t$. Then we have an étale and finite morphism $F\colon (\bar{M} \times \widetilde{M}^{m,+})/\mathbb{C}_s^* \to \mathcal{M}^{\mathbb{C}^*}(\mathfrak{I})$ of degree $1/pD$.

The action of $\mathbb{C}_s^*$ on the second factor $\widetilde{M}^{m,+}$ is trivial, since it can be absorbed in the isomorphism $s^{-1}\,\mathrm{id}\colon E_\sharp \xrightarrow{\cong} E_\sharp$ as

$$\begin{array}{ccc} E_\sharp & L(E_\sharp) = \det H^1(E_\sharp)^{\otimes m_0} \otimes \det H^1((E_\sharp)(C))^{\otimes m_1} \xrightarrow{s^{-pD}\rho_\sharp} & \mathbb{C} \\ \Big\downarrow{\scriptstyle s^{-1}\,\mathrm{id}} & \Big\downarrow{\scriptstyle s^{-pD}\,\mathrm{id}} & \Big\|{\scriptstyle \mathrm{id}} \\ E_\sharp & L(E_\sharp) = \det H^1(E_\sharp)^{\otimes m_0} \otimes \det H^1((E_\sharp)(C))^{\otimes m_1} \xrightarrow[\rho_\sharp]{} & \mathbb{C}. \end{array} \tag{5.28}$$

Therefore we have

$$(\bar{M} \times \widetilde{M}^{m,+})/\mathbb{C}_s^* = \bar{M}/\mathbb{C}_s^* \times \widetilde{M}^{m,+}.$$

Furthermore we have an étale and finite morphism $G\colon \bar{M}/\mathbb{C}_s^* \to \bar{M}/\mathbb{C}^*$ of degree $1/pD$. But the latter is nothing but $\widetilde{M}^{m,\min(I_\sharp)-1}$. Hence we have the diagram in Theorem 5.18 with $\mathcal{S}(\mathfrak{I}) = (\bar{M} \times \widetilde{M}^{m,+})/\mathbb{C}_s^*$.

From (5.28) the universal sheaf $\mathcal{E}_\sharp$ over $\widetilde{M}^{m,+}$ is twisted by the line bundle over $\bar{M}/\mathbb{C}^*$ associated with the representation of $\mathbb{C}^*$ with weight 1. It is a line bundle $L_{\mathcal{S}}$ such that $L_{\mathcal{S}}^{\otimes pD} = G^*L(E_\flat)^*$. Therefore we have $F^*\left({}^{\mathcal{M}}\mathcal{E}_\sharp\right) = G^*(\mathcal{E}_\sharp) \otimes L_{\mathcal{S}}$. On the other hand, we have $F^*\left({}^{\mathcal{M}}\mathcal{E}_\flat\right) = G^*(\mathcal{E}_\flat)$.

5.10. **Normal bundle.** Let us describe the normal bundles $\mathfrak{N}(\mathcal{M}_\pm)$ of $\mathcal{M}_\pm$ and $\mathfrak{N}(\mathcal{M}^{\mathbb{C}^*}(\mathfrak{I}))$ of $\mathcal{M}^{\mathbb{C}^*}(\mathfrak{I})$ in $\mathcal{M}$ in this subsection. We need to prepare several notations.

Recall first that the covering $\mathbb{C}_s^* \to \mathbb{C}^*; s \mapsto s^{pD} = t$ acts trivially on $\mathcal{M}^{\mathbb{C}^*}(\mathfrak{I})$ (while the original $\mathbb{C}^*$ does not). Hence the tangent space at a fixed point has a natural $\mathbb{C}_s^*$-module structure. We formally consider it as a module structure of the original $\mathbb{C}^*$ dividing weights by pD.

Recall also that the restriction of the universal sheaf $\mathcal{E}$ decomposes as ${}^{\mathcal{M}}\mathcal{E}_\flat \oplus {}^{\mathcal{M}}\mathcal{E}_\sharp$ over $\mathcal{M}^{\mathbb{C}^*}(\mathfrak{I})$ (see Proposition 5.27). Let $\operatorname{Ext}^\bullet_{q_2}$ denotes the higher derived functor of the composite functor $q_{2*} \circ \mathcal{H}om$. Let

$$\mathfrak{N}({}^{\mathcal{M}}\mathcal{E}_\flat, {}^{\mathcal{M}}\mathcal{E}_\sharp) := -\sum_{a=0}^{2}(-1)^a \operatorname{Ext}^a_{q_2}({}^{\mathcal{M}}\mathcal{E}_\flat, {}^{\mathcal{M}}\mathcal{E}_\sharp), \tag{5.29}$$

where this is a class in the equivariant K-group of $\mathcal{M}^{\mathbb{C}^*}(\mathfrak{I})$. We use similar notation $\mathfrak{N}({}^{\mathcal{M}}\mathcal{E}_\sharp, {}^{\mathcal{M}}\mathcal{E}_\flat)$ exchanging the first and second factors. Later we will also use $\mathfrak{N}(\bullet,\bullet)$ replacing ${}^{\mathcal{M}}\mathcal{E}_\sharp$, ${}^{\mathcal{M}}\mathcal{E}_\flat$ by similar universal sheaves. We have already used this notation in Theorem 1.5.

Let

$$\operatorname{Flag}(V_1^\alpha, I_\alpha) := \{\text{a flag } F_\alpha^\bullet \text{ of } V_1^\alpha, \text{ indexed by } \underline{N},\ F_\alpha^i/F_\alpha^{i-1} = 0 \text{ if and only if } i \notin I_\alpha\}$$

for $\alpha = \flat, \sharp$. We have an embedding $\operatorname{Flag}(V_1^\flat, I_\flat) \times \operatorname{Flag}(V_1^\sharp, I_\sharp) \to \operatorname{Flag}(V_1, \underline{N})$ given by $(F_\flat^\bullet, F_\sharp^\bullet) \mapsto F_\flat^\bullet \oplus F_\sharp^\bullet$. Let N_0 denote its normal bundle. It has a natural $\mathbb{C}^*$-equivariant structure as $\operatorname{Flag}(V_1^\flat, I_\flat) \times \operatorname{Flag}(V_1^\sharp, I_\sharp)$ is a component of $\mathbb{C}^*$-fixed points in $\operatorname{Flag}(V_1, \underline{N})$ with respect to the $\mathbb{C}^*$-action induced by $\mathbb{C}^* \ni t \mapsto \mathrm{id}_{V_1^\flat} \oplus t^{1/pD}\, \mathrm{id}_{V_1^\sharp} \in \mathrm{GL}(V_1)$. More precisely, when we write

$$N_0 = \bigoplus_{i>j} \operatorname{Hom}(F_\flat^i/F_\flat^{i-1}, F_\sharp^j/F_\sharp^{j-1}) \oplus \bigoplus_{i>j} \operatorname{Hom}(F_\sharp^i/F_\sharp^{i-1}, F_\flat^j/F_\flat^{j-1}),$$

the first term has weight $1/pD$ and the second term has weight $-1/pD$. We have an associated vector bundle, denoted also by N_0, over $\mathcal{M}^{\mathbb{C}^*}(\mathfrak{I})$, induced from the flag bundle structure $\widetilde{Q}/G \to Q/G$ between quotient stacks.

Theorem 5.30. (1) *The normal bundle $\mathfrak{N}(\mathcal{M}_\pm)$ of $\mathcal{M}_\pm$ is $L_\mp^* \otimes L_\pm$ with the $\mathbb{C}^*$-action of weight ± 1.*

(2) *The normal bundle $\mathfrak{N}(\mathcal{M}^{\mathbb{C}^*}(\mathfrak{I}))$ of $\mathcal{M}^{\mathbb{C}^*}(\mathfrak{I})$ is equivariant K-theoretically given by*

$$N_0 + \mathfrak{N}({}^{\mathcal{M}}\mathcal{E}_\flat, {}^{\mathcal{M}}\mathcal{E}_\sharp) \otimes I_{1/pD} + \mathfrak{N}({}^{\mathcal{M}}\mathcal{E}_\sharp, {}^{\mathcal{M}}\mathcal{E}_\flat) \otimes I_{-1/pD},$$

where I_n denotes the trivial line bundle over $\mathcal{M}^{\mathbb{C}^*}(\mathfrak{I})$ with the $\mathbb{C}^*$-action of weight n.

Proof. First consider the case of $\mathcal{M}_\pm = \mathbb{P}(L_\pm)^{\mathrm{ss}}/G$. The normal bundle is the descent of the normal bundle $\mathbb{P}(L_\pm) \subset \mathbb{P}(L_- \oplus L_+)$. Then it is $L_\mp^* \otimes L_\pm$ with the $\mathbb{C}^*$-action of weight ± 1.

In the remaining of the proof we consider the case of $\mathcal{M}^{\mathbb{C}^*}(\mathfrak{I})$. The normal bundle is the sum of nonzero weight subspaces in the restriction of the tangent bundle of $\mathcal{M}$ to $\mathcal{M}^{\mathbb{C}^*}(\mathfrak{I})$.

Take a point $((X, F^\bullet), [z_- : z_+]) \in \mathbb{P}(L_- \oplus L_+)^{\mathrm{ss}}$ which descends to a fixed point in $\mathcal{M}^{\mathbb{C}^*}(\mathfrak{I})$. We have a decomposition $(X, F^\bullet) = (X_\flat, F_\flat^\bullet) \oplus (X_\sharp, F_\sharp^\bullet)$ as above. Let $V = V^\flat \oplus V^\sharp$ be the corresponding decomposition of V.

We lift the $\mathbb{C}^*$-action on $\mathcal{M}$ to $\mathbb{P}(L_- \oplus L_+)^{\mathrm{ss}}$ as in (5.24).

The tangent space at the point corresponding to $((X, F^\bullet), [z_- : z_+])$ is the quotient vector space

$$\operatorname{Ker} d\mu \oplus T_{F^\bullet} \operatorname{Flag}(V_1, \underline{N}) \oplus T_{[z_-:z_+]}\mathbb{P}^1 / \operatorname{Hom}(V_0, V_0) \oplus \operatorname{Hom}(V_1, V_1),$$

where $d\mu$ and $\operatorname{Hom}(V_0, V_0) \oplus \operatorname{Hom}(V_1, V_1) \to \operatorname{Ker} d\mu$ are as in (4.4), and $\operatorname{Hom}(V_0, V_0) \oplus \operatorname{Hom}(V_1, V_1) \to T_{F^\bullet} \operatorname{Flag}(V_1, \underline{N}) \oplus T_{[z_-:z_+]}\mathbb{P}^1$ is the differential of the G-action. This quotient space has the $\mathbb{C}^*$-module structure induced from the above lift of the $\mathbb{C}^*$-action.

In the equivariant K-group, we can replace this space by $\operatorname{Ker} d\mu - (\operatorname{Hom}(V_0, V_0) \oplus \operatorname{Hom}(V_1, V_1)) + T_{F^\bullet} \operatorname{Flag}(V_1, \underline{N}) + T_{[z_-:z_+]}\mathbb{P}^1$

The expression $\operatorname{Ker} d\mu - (\operatorname{Hom}(V_0, V_0) \oplus \operatorname{Hom}(V_1, V_1))$ is equal to the alternating sum of cohomology groups of the tangent complex (4.4) in the K-group. Moreover, the complex (4.4) decomposes into four parts, the tangent complex for $X_\flat$, the sum of p-copies of the complex (4.8) for $X_\flat$, the sum of p-copies of the complex (4.10) for $X_\flat$, and the sum of p^2-copies of the tangent complex for C_m. Since $\mathbb{C}^*$ acts on $V^\flat$ with weight 0 and $V^\sharp$ with weight $1/pD$, the first and fourth parts do not contribute to the normal bundle. The second and third terms give

$$\begin{aligned} &\big(-\operatorname{Ext}^0_p({}^{\mathcal{M}}\mathcal{E}_\flat, {}^{\mathcal{M}}\mathcal{E}_\sharp) + \operatorname{Ext}^1_p({}^{\mathcal{M}}\mathcal{E}_\flat, {}^{\mathcal{M}}\mathcal{E}_\sharp)\big) \otimes I_{1/pD} \\ &\quad + \big(-\operatorname{Ext}^0_p({}^{\mathcal{M}}\mathcal{E}_\sharp, {}^{\mathcal{M}}\mathcal{E}_\flat) + \operatorname{Ext}^1_p({}^{\mathcal{M}}\mathcal{E}_\sharp, {}^{\mathcal{M}}\mathcal{E}_\flat)\big) \otimes I_{-1/pD} \\ &= \mathfrak{N}({}^{\mathcal{M}}\mathcal{E}_\flat, {}^{\mathcal{M}}\mathcal{E}_\sharp) \otimes I_{1/pD} + \mathfrak{N}({}^{\mathcal{M}}\mathcal{E}_\sharp, {}^{\mathcal{M}}\mathcal{E}_\flat) \otimes I_{-1/pD}. \end{aligned}$$

The contribution from $T_{F^\bullet} \operatorname{Flag}(V_1, \underline{N})$ is given by N_0. Thus we have Theorem 5.30. We have no contribution from $T_{[z_-:z_+]}\mathbb{P}^1$. $\square$

5.11. **Relative tangent bundles of flag bundles.** Let Θ_{rel} be the bundles over various moduli stacks induced from the relative tangent bundle of the flag bundle $\widetilde{Q}/G \to Q/G$ as in §6.2

On the other hand, we have vector bundles $\Theta^\flat_{\mathrm{rel}}$, $\Theta^\sharp_{\mathrm{rel}}$ over $\widetilde{M}^{m,\min(I_\sharp)-1}(c_\flat) \times \widetilde{M}^{m,+}(c_\sharp)$ coming from the tangent bundles of $\operatorname{Flag}(V_1^\flat, I_\flat)$ and $\operatorname{Flag}(V_1^\sharp, I_\sharp)$. Recall the normal bundle N_0 of $\operatorname{Flag}(V_1^\flat, I_\flat) \times \operatorname{Flag}(V_1^\sharp, I_\sharp)$ in $\operatorname{Flag}(V_1, \underline{N})$ is considered as a vector bundle over $\widetilde{M}^{m,\min(I_\sharp)-1}(c_\flat) \times \widetilde{M}^{m,+}(c_\sharp)$ (see §5.10), so we have an exact sequence

$$0 \to G^*(\Theta^\flat_{\mathrm{rel}} \oplus \Theta^\sharp_{\mathrm{rel}}) \to F^*\Theta_{\mathrm{rel}}|_{\mathcal{M}^{\mathbb{C}^*}(\mathfrak{I})} \to G^*(N_0) \to 0 \tag{5.31}$$

of $\mathbb{C}^*$-equivariant vector bundles, where $\Theta^\flat_{\mathrm{rel}} \oplus \Theta^\sharp_{\mathrm{rel}}$ has weight 0 and N_0 has the $\mathbb{C}^*$-equivariant structure as describe in §5.10.

Recall also that the second factor $\widetilde{M}^{m,+}(c_\sharp)$ is the flag bundle $\operatorname{Flag}(\mathcal{V}_1^\sharp/\mathcal{F}_\sharp^{\min(I_\sharp)}, I_\sharp \setminus \{\min(I_\sharp)\})$ over the quotient stack $(\det \mathcal{Q}^{\otimes D})^\times/\mathbb{C}^*$ (see Proposition 5.9). Let Θ'_{rel} be the relative tangent bundle of the fiber. Then we have an exact sequence of vector bundles

$$0 \to \Theta'_{\mathrm{rel}} \to \Theta^\sharp_{\mathrm{rel}} \to \operatorname{Hom}(\mathcal{V}_1^\sharp/\mathcal{F}_\sharp^{\min(I_\sharp)}, \mathcal{F}_\sharp^{\min(I_\sharp)}) \to 0 \tag{5.32}$$

coming from the fibration $\operatorname{Flag}(\mathcal{V}_1^\sharp/\mathcal{F}_\sharp^{\min(I_\sharp)}) \to \operatorname{Flag}(\mathcal{V}_1^\sharp) \to \mathbb{P}(\mathcal{V}_1^\sharp)$.

6. Wall-crossing formula

We now turn to Mochizuki method **(Mb)**. (See [14, §7.2].) We will apply the fixed point formula to the equivariant homology groups $H_*^{\widetilde{T}\times\mathbb{C}^*}(\mathcal{M}(c))$ of the enhanced moduli space $\mathcal{M}(c)$, which is a module over $H^*_{\mathbb{C}^*}(\mathrm{pt}) \cong \mathbb{C}[\hbar]$. For the definition of the homology group of a Deligne-Mumford stack, see [28].

6.1. **Equivariant Euler class.** In the fixed point formula we put an equivariant Euler class in the denominator and we later consider an equivariant Euler class of a class in the K-group. Let us explain how we treat them in this subsection.

Let Y be a variety (or a Deligne-Mumford stack) with a trivial $\mathbb{C}^*$-action. The Grothendieck group of $\mathbb{C}^*$-equivariant vector bundles decomposes as $K_{\mathbb{C}^*}(Y) = K(Y) \otimes_{\mathbb{Z}} R(\mathbb{C}^*)$ where $R(\mathbb{C}^*)$ is the representation ring of $\mathbb{C}^*$. Let I_n denote the 1-dimensional representation of $\mathbb{C}^*$ with weight n. For a class $\alpha \in K(Y)$, we set

$$e(\alpha \otimes I_n) := \sum_{i \geq 0} c_i(\alpha)(n\hbar)^{r(\alpha)-i} \in H^*(Y)[\hbar^{-1}, \hbar] := H^*(Y) \otimes_{\mathbb{C}} \mathbb{C}[\hbar^{-1}, \hbar],$$

where $c_i(\alpha)$ is the i^{th} Chern class of α and $r(\alpha) = \operatorname{ch}_0(\alpha)$ is the (virtual) rank of α. If $n \neq 0$, this element is invertible. In general, if $\alpha \in K_{\mathbb{C}^*}(Y)$ is a sum of $\alpha_n \otimes I_n$ with $n \neq 0$, its equivariant Euler class $e(\alpha)$ is defined in $H^*(Y)[\hbar^{-1}, \hbar]$.

We also consider the case when another group $\widetilde{T}$ acts on Y. Then $e(\alpha \otimes I_n)$ can be still defined as an element in $\varprojlim_n H^*(Y \times_{\widetilde{T}} E_n)[\hbar^{-1}, \hbar]$, where $E_n \to E_n/\widetilde{T}$ is a finite dimensional approximation of the classifying space $E\widetilde{T} \to B\widetilde{T}$ for $\widetilde{T}$. Note $c_i(\alpha) \neq 0$ for possibly infinite i's.

6.2. **Wall-crossing formula (I).** The projective morphism $\widehat{\pi}\colon \mathcal{M}(c) \to M_0(p_*(c))$ induce a homomorphism $\widehat{\pi}_*\colon H_*^{\widetilde{T}\times\mathbb{C}^*}(\mathcal{M}(c)) \to H_*^{\widetilde{T}\times\mathbb{C}^*}(M_0(p_*(c)))$. Since $\mathbb{C}^*$ acts trivially on $M_0(p_*(c))$, we have $H_*^{\widetilde{T}\times\mathbb{C}^*}(M_0(p_*(c))) \cong H_*^{\widetilde{T}}(M_0(p_*(c))) \otimes \mathbb{C}[\hbar]$. For a cohomology class $\bullet$ on $\mathcal{M}(c)$, we denote the pushforward $\widehat{\pi}_*(\bullet \cap [\mathcal{M}(c)])$ by $\int_{\mathcal{M}(c)} \bullet$ as in §1. We also use similar push-forward homomorphisms from homology groups of various moduli stacks and denote them in similar ways, e.g., $\int_{\mathcal{M}_\pm(c)}$, $\int_{\mathcal{M}^{\mathbb{C}^*}(\mathfrak{J})}$, etc.

Let $e(\mathcal{F})$ denote the equivariant Euler class of the equivariant vector bundle $\mathcal{F}$, that is the top Chern class $c_{\text{top}}(\mathcal{F})$. See §6.1 for its generalization to a K-theory class $\mathcal{F}$ in our situation.

Let $\Phi(\mathcal{E})$ be as in §1.4. We denote also by $\Phi(\mathcal{E})$ the class on $\widetilde{M}^{m,\ell}(c)$ given by the same formula. On the enhanced master space $\mathcal{M}(c)$, we can consider the class defined by the same formula as $\Phi(\mathcal{E})$, which is regarded as a $\mathbb{C}^*$-equivariant class.

Let Θ_{rel} be the relative tangent bundle of the flag bundle $\widetilde{Q}/G \to Q/G$. Then we have the induced bundle over $\widetilde{M}^{m,\ell}(c)$ by the restriction. We denote it also by Θ_{rel} for brevity. We also have the pullback to the enhanced master space $\mathcal{M}(c)$, which is again denoted by Θ_{rel}. It has a natural $\mathbb{C}^*$-equivariant structure. We introduce $\widetilde{\Phi}(\mathcal{E}) := \frac{1}{v_1(c)!}\Phi(\mathcal{E}) \cup e(\Theta_{\text{rel}})$ so that we have

$$\int_{\widetilde{M}^{m,0}(c)} \widetilde{\Phi}(\mathcal{E}) = \int_{\widehat{M}^m(c)} \Phi(\mathcal{E}).$$

Here $v_1(c) = \dim V_1(E)$ for a sheaf E with $\operatorname{ch}(E) = c$.

Then the fixed point formula in the equivariant homology group gives us

$$\int_{\widehat{M}^m(c)} \Phi(\mathcal{E}) - \int_{\widetilde{M}^{m,\ell}(c)} \widetilde{\Phi}(\mathcal{E}) = \sum_{\mathfrak{J} \in \mathcal{D}^{m,\ell}(c)} \operatorname*{Res}_{\hbar=0} \int_{\mathcal{M}^{\mathbb{C}^*}(\mathfrak{J})} \frac{\widetilde{\Phi}(\mathcal{E})}{e(\mathfrak{N}(\mathcal{M}^{\mathbb{C}^*}(\mathfrak{J})))},$$

where '$\operatorname{Res}_{\hbar=0}$' means taking the coefficient of $\hbar^{-1}$.

By Proposition 5.9 and Theorem 5.18 together with computations of normal bundles §5.10, we can rewrite the right hand side to get

Theorem 6.1.

$$\int_{\widetilde{M}^{m,\ell}(c)} \widetilde{\Phi}(\mathcal{E}) - \int_{\widehat{M}^m(c)} \Phi(\mathcal{E})$$
$$= \sum_{\mathfrak{I}\in\mathcal{D}^{m,\ell}(c)} \int_{\widetilde{M}^{m,\min(I_\sharp)-1}(c_\flat)} \operatorname*{Res}_{\hbar=0}\left[\widetilde{\Phi}(\mathcal{E}_\flat\oplus(C_m\boxtimes\mathcal{Q}\otimes I_{-1}))\Phi'(\mathcal{E}_\flat)\right],$$

where $\widetilde{\Phi}(\mathcal{E}_\flat\oplus(C_m\boxtimes\mathcal{Q}\otimes I_{-1}))$ is defined exactly as $\widetilde{\Phi}(\mathcal{E})$ by replacing $\mathcal{E}$ by $\mathcal{E}_\flat\oplus(C_m\boxtimes\mathcal{Q}\otimes I_{-1})$ everywhere, and $\Phi'(\mathcal{E}_\flat)$ is another cohomology class given by

(6.2)
$$\Phi'(\mathcal{E}_\flat) = \frac{(v_1(c_\sharp)-1)!v_1(c_\flat)!}{v_1(c)!}\int_{\operatorname{Gr}(m+1,p)} \frac{e\left((V_1(C_m)\otimes\mathcal{Q}/\mathcal{O})^*\right)}{e(\mathfrak{N}(\mathcal{E}_\flat,C_m)\otimes\mathcal{Q}\otimes I_{-1})\,e(\mathfrak{N}(C_m,\mathcal{E}_\flat)\otimes\mathcal{Q}^*\otimes I_1)}.$$

Here I_n denotes the trivial line bundle with the $\mathbb{C}^*$-action of weight n, and $\mathfrak{N}(\bullet,\bullet)$ is the equivariant K-theory class given by the negative of the alternating sum of Ext*-groups (see* (5.29)*). (Note that Φ' depends on $\mathfrak{I}$.)*

The proof will be given in the next subsection.

6.3. **Fixed point formula on the enhanced master space.** Let $\iota_\pm$, $\iota_{\mathfrak{I}}$ be the inclusions of $\mathcal{M}^\pm(c)$, $\mathcal{M}^{\mathbb{C}^*}(\mathfrak{I})$ into $\mathcal{M}(c)$. Let $\iota_\pm^*$, $\iota_{\mathfrak{I}}^*$ be the pullback homomorphisms, which are defined as $\mathcal{M}(c)$ is smooth. They will be omitted from formulas eventually. Let $e(\mathfrak{N}(\mathcal{M}_+(c)))$, $e(\mathfrak{N}(\mathcal{M}^{\mathbb{C}^*}(\mathfrak{I})))$ be the equivariant Euler class of the normal bundles $\mathfrak{N}(\mathcal{M}_+(c))$, $\mathfrak{N}(\mathcal{M}^{\mathbb{C}^*}(\mathfrak{I}))$. The localization theorem in the equivariant cohomology groups says the following diagram is commutative:

$$\begin{array}{ccc}
\varprojlim_n H_*^{\mathbb{C}^*}(\mathcal{M}(c)\times_{\widetilde{T}} E_n)\otimes_{\mathbb{C}[\hbar]}\mathbb{C}[\hbar^{-1},\hbar] & \xrightarrow{\cong} & \varprojlim_n H_*(\mathcal{M}(c)^{\mathbb{C}^*}\times_{\widetilde{T}} E_n)[\hbar^{-1},\hbar] \\
{\scriptstyle\int_{\mathcal{M}(c)}}\Big\downarrow & & \Big\downarrow{\scriptstyle\int_{\mathcal{M}_+(c)}+\int_{\mathcal{M}_-(c)}+\sum_{\mathfrak{I}}\int_{\mathcal{M}^{\mathbb{C}^*}(\mathfrak{I})}} \\
\varprojlim_n H_*(M_0(p_*(c)\times_{\widetilde{T}} E_n))[\hbar^{-1},\hbar] & = & \varprojlim_n H_*(M_0(p_*(c)\times_{\widetilde{T}} E_n))[\hbar^{-1},\hbar],
\end{array}$$

where the upper horizontal arrow is given by

$$\frac{\iota_+^*}{e(\mathfrak{N}(\mathcal{M}_+(c)))} + \frac{\iota_-^*}{e(\mathfrak{N}(\mathcal{M}_-(c)))} + \sum_{\mathfrak{I}}\frac{\iota_{\mathfrak{I}}^*}{e(\mathfrak{N}(\mathcal{M}^{\mathbb{C}^*}(\mathfrak{I})))},$$

and $E_n\to E_n/\widetilde{T}$ is a finite dimensional approximation of $E\widetilde{T}\to B\widetilde{T}$ as above.

Let $\mathcal{T}(1)$ be the trivial line bundle with the $\mathbb{C}^*$-action of weight 1. We have

$$\int_{\mathcal{M}(c)}\widetilde{\Phi}(\mathcal{E})c_1(\mathcal{T}(1)) = \sum_{a=\pm}\int_{\mathcal{M}_a(c)}\frac{\widetilde{\Phi}(\mathcal{E})c_1(\mathcal{T}(1))}{e(\mathfrak{N}(\mathcal{M}_a))} + \sum_{\mathfrak{I}\in\mathcal{D}^{m,\ell}(c)}\int_{\mathcal{M}^{\mathbb{C}^*}(\mathfrak{I})}\frac{\widetilde{\Phi}(\mathcal{E})c_1(\mathcal{T}(1))}{e(\mathfrak{N}(\mathcal{M}^{\mathbb{C}^*}(\mathfrak{I})))}$$

holds in $\varprojlim H_*(M_0(p_*(c)\times_{\widetilde{T}} E_n))[\hbar^{-1},\hbar]$. Since $\mathcal{T}(1)$ is a trivial line bundle if we forget the $\mathbb{C}^*$-action, the left hand side is restricted to 0 at $\hbar=0$: $\int_{\mathcal{M}(c)}\widetilde{\Phi}c_1(\mathcal{T}(1))\Big|_{\hbar=0}=0$. On the other hand, $c_1(\mathcal{T}(1))|_{\mathcal{M}_a(c)} = \hbar$ and $c_1(\mathcal{T}(1))|_{\mathcal{M}^{\mathbb{C}^*}(\mathfrak{I})} = \hbar$. Moreover we have

$$\frac{1}{e(\mathfrak{N}(\mathcal{M}_a))} = a(\hbar - \omega)^{-1} = \frac{a}{\hbar}\sum_{i=0}^{\infty}\left(\frac{\omega}{\hbar}\right)^i$$

for $a = \pm$, where $\omega = c_1(L_+^* \otimes L_-)$. Combining with Theorem 5.18(2) we get

$$\int_{\widetilde{M}^{m,\ell}(c)} \widetilde{\Phi}(\mathcal{E}) - \int_{\widehat{M}^m(c)} \Phi(\mathcal{E}) = -\sum_{\mathfrak{I}\in\mathcal{D}^{m,\ell}(c)} \operatorname*{Res}_{\hbar=0} \int_{\mathcal{M}^{\mathbb{C}^*}(\mathfrak{I})} \frac{\widetilde{\Phi}(\mathcal{E})}{e(\mathfrak{N}(\mathcal{M}^{\mathbb{C}^*}(\mathfrak{I})))}. \tag{6.3}$$

We now use the diagram (5.19) to rewrite the integral in the right hand side of (6.3):

$$\int_{\mathcal{M}^{\mathbb{C}^*}(\mathfrak{I})} \frac{\widetilde{\Phi}(\mathcal{E})}{e(\mathfrak{N}(\mathcal{M}^{\mathbb{C}^*}(\mathfrak{I})))} = pD \int_{\mathcal{S}(\mathfrak{I})} \frac{F^*\widetilde{\Phi}(\mathcal{E})}{F^*e(\mathfrak{N}(\mathcal{M}^{\mathbb{C}^*}(\mathfrak{I})))}.$$

From Theorem 5.18(3) we have

$$F^*(\Phi(\mathcal{E})) = \Phi(G^*(\mathcal{E}_\flat) \oplus G^*(\mathcal{E}_\sharp) \otimes L_{\mathcal{S}} \otimes I_{1/pD}).$$

Since $L_{\mathcal{S}}^{\otimes pD} = G^*(\mathcal{L}(\mathcal{E}_\flat)^*)$, we have

$$c_1(L_{\mathcal{S}}) = -\frac{1}{pD} G^* c_1(\mathcal{L}(\mathcal{E}_\flat)).$$

Since $L_{\mathcal{S}}$ appears as $c_1(L_{\mathcal{S}})$ in $\Phi(G^*(\mathcal{E}_\sharp) \otimes L_{\mathcal{S}} \otimes I_{1/pD})$, we can formally write

$$\Phi(G^*(\mathcal{E}_\flat) \oplus G^*(\mathcal{E}_\sharp) \otimes L_{\mathcal{S}} \otimes I_{1/pD}) = G^*\Phi(\mathcal{E}_\flat \oplus \mathcal{E}_\sharp \otimes \mathcal{L}(\mathcal{E}_\flat)^{-1/pD} \otimes I_{1/pD}),$$

meaning that we replace $c_1(\mathcal{L}(\mathcal{E}_\flat)^{-1/pD})$ by $-c_1(\mathcal{L}(\mathcal{E}_\flat))/pD$.

Similarly from Theorem 5.30 and Theorem 5.18(3) we have

$$\begin{aligned}
&F^*e(\mathfrak{N}(\mathcal{M}^{\mathbb{C}^*}(\mathfrak{I}))) \\
&= F^*\left(e(\mathfrak{N}({}^{\mathcal{M}}\mathcal{E}_\flat, {}^{\mathcal{M}}\mathcal{E}_\sharp) \otimes I_{1/pD}) e(\mathfrak{N}({}^{\mathcal{M}}\mathcal{E}_\sharp, {}^{\mathcal{M}}\mathcal{E}_\flat) \otimes I_{-1/pD}) e(N_0)\right) \\
&= G^*\left(e(\mathfrak{N}(\mathcal{E}_\flat, \mathcal{E}_\sharp) \otimes \mathcal{L}(\mathcal{E}_\flat)^{-1/pD} \otimes I_{1/pD}) e(\mathfrak{N}(\mathcal{E}_\sharp, \mathcal{E}_\flat) \otimes \mathcal{L}(\mathcal{E}_\flat)^{1/pD} \otimes I_{-1/pD}) e(N_0)\right).
\end{aligned}$$

From (5.31,5.32) we have

$$\begin{aligned}
F^*(e(\Theta_{\mathrm{rel}})) &= G^*\left(e(\Theta_{\mathrm{rel}}^\flat) e(\Theta_{\mathrm{rel}}^\sharp) e(N_0)\right) \\
&= G^*\left(e(\Theta_{\mathrm{rel}}^\flat) e(\Theta'_{\mathrm{rel}}) e\left(\operatorname{Hom}(\mathcal{V}_1^\sharp/\mathcal{F}_\sharp^{\min(I_\sharp)}, \mathcal{F}_\sharp^{\min(I_\sharp)})\right) e(N_0)\right).
\end{aligned}$$

Therefore we get

$$\begin{aligned}
&pD\int_{\mathcal{S}(\mathfrak{I})} \frac{F^*\widetilde{\Phi}}{F^*e(\mathfrak{N}(\mathcal{M}^{\mathbb{C}^*}(\mathfrak{I})))} \\
&= \frac{1}{v_1(c)!}\int_{\widetilde{M}^{m,\min(I_\sharp)-1}(c_\flat)\times\widetilde{M}^{m,+}(c_\sharp)} \frac{\Phi(\mathcal{E}^\flat \oplus \mathcal{E}_\sharp \otimes \mathcal{L}(\mathcal{E}_\flat)^{-1/pD} \otimes I_{1/pD})}{e(\mathfrak{N}(\mathcal{E}_\flat, \mathcal{E}_\sharp) \otimes \mathcal{L}(\mathcal{E}_\flat)^{-1/pD} \otimes I_{1/pD})} \\
&\qquad\qquad \times \frac{e(\Theta_{\mathrm{rel}}^\flat) e(\Theta'_{\mathrm{rel}}) e\left(\operatorname{Hom}(\mathcal{V}_1^\sharp/\mathcal{F}_\sharp^{\min(I_\sharp)}, \mathcal{F}_\sharp^{\min(I_\sharp)})\right)}{e(\mathfrak{N}(\mathcal{E}_\sharp, \mathcal{E}_\flat) \otimes \mathcal{L}(\mathcal{E}_\flat)^{1/pD} \otimes I_{-1/pD})}.
\end{aligned}$$

We use the claim that $\widetilde{M}^{m,+}(c_\sharp)$ is a flag bundle over $(\det \mathcal{Q}^{\otimes D})^\times/\mathbb{C}^*$ (Proposition 5.9(1)) to rewrite this further as

$$\begin{aligned}\frac{(v_1(c_\sharp)-1)!}{v_1(c)!}\int_{\widetilde{M}^{m,\min(I_\sharp)-1}(c_\flat)\times(\det\mathcal{Q}^{\otimes D})^\times/\mathbb{C}^*} &\frac{\Phi(\mathcal{E}^\flat\oplus\mathcal{E}_\sharp\otimes\mathcal{L}(\mathcal{E}_\flat)^{-1/pD}\otimes I_{1/pD})\,e(\Theta^\flat_{\mathrm{rel}})}{e(\mathfrak{N}(\mathcal{E}_\flat,\mathcal{E}_\sharp)\otimes\mathcal{L}(\mathcal{E}_\flat)^{-1/pD}\otimes I_{1/pD})}\\ &\times\frac{e\left(\operatorname{Hom}(\mathcal{V}_1^\sharp/\mathcal{F}_\sharp^{\min(I_\sharp)},\mathcal{F}_\sharp^{\min(I_\sharp)})\right)}{e(\mathfrak{N}(\mathcal{E}_\sharp,\mathcal{E}_\flat)\otimes\mathcal{L}(\mathcal{E}_\flat)^{1/pD}\otimes I_{-1/pD})}.\end{aligned}$$

We set $\widetilde{\Phi}(\bullet) := \frac{1}{v_1(c_\flat)!}\Phi(\bullet)e(\Theta^\flat_{\mathrm{rel}})$, use Proposition 5.9(2) to replace $(\det\mathcal{Q}^{\otimes D})^\times/\mathbb{C}^*$ by $\mathrm{Gr}(m+1,p)$, and then plug into (6.3) to get

$$\int_{\widetilde{M}^{m,\ell}(c)}\widetilde{\Phi}(\mathcal{E})-\int_{\widehat{M}^m(c)}\Phi(\mathcal{E})=\sum_{\mathfrak{I}\in\mathcal{D}^{m,\ell}(c)}\int_{\widetilde{M}^{m,\min(I_\sharp)-1}(c_\flat)}\operatorname*{Res}_{\hbar=0}\widetilde{\Phi}(\mathcal{E}_\flat\oplus\mathcal{E}_\sharp\otimes\mathcal{L}(\mathcal{E}_\flat)^{-1/pD}\otimes I_{1/pD})\Phi'(\mathcal{E}_\flat),$$

where

$$\begin{aligned}\Phi'(\mathcal{E}_\flat)&=-\frac{(v_1(c_\sharp)-1)!v_1(c_\flat)!}{v_1(c)!\,pD}\int_{\mathrm{Gr}(m+1,p)}\heartsuit,\\ \heartsuit&=\frac{1}{e(\mathfrak{N}(\mathcal{E}_\flat,C_m)\otimes\mathcal{Q}\otimes\det\mathcal{Q}^{-1/p}\otimes\mathcal{L}(\mathcal{E}_\flat)^{-1/pD}\otimes I_{1/pD})}\\ &\qquad\times\frac{e\left((V_1(C_m)\otimes\mathcal{Q}/\mathcal{O})^*\right)}{e(\mathfrak{N}(C_m,\mathcal{E}_\flat)\otimes\mathcal{Q}^*\otimes\det\mathcal{Q}^{1/p}\otimes\mathcal{L}(\mathcal{E}_\flat)^{1/pD}\otimes I_{-1/pD})}.\end{aligned}$$

Here

- $\int_{\mathrm{Gr}(m+1,p)}$ means the pushforward with respect to the projection $\widetilde{M}^{m,\min(I_\sharp)-1}(c_\flat)\times\mathrm{Gr}(m+1,p)\to\widetilde{M}^{m,\min(I_\sharp)-1}(c_\flat)$.
- $\det\mathcal{Q}^{-1/p}$ is understood as before: we replace $c_1(\det\mathcal{Q}^{-1/p})$ by $-c_1(\det\mathcal{Q})/p$.

Let us slightly simplify the formula. First note that $I_{\pm 1/pD}$ appears with $\det\mathcal{Q}^{\mp 1/p}\otimes\mathcal{L}(\mathcal{E}_\flat)^{\mp 1/pD}$. Let $\omega=-(c_1(\mathcal{L}(\mathcal{E}_\flat))+c_1(\mathcal{Q}))/pD$. Thus $\Phi'(\mathcal{E}_\flat)$ is written as

$$\sum_{j=-\infty}^{\infty}A_j(\hbar-\omega)^j.$$

By a direct calculation we have

$$\operatorname*{Res}_{\hbar=0}(\hbar-\omega)^j=\begin{cases}1 & \text{if } j=-1,\\ 0 & \text{otherwise.}\end{cases}$$

Therefore we have

$$\operatorname*{Res}_{\hbar=0}\sum_{j=-\infty}^{\infty}A_j(\hbar-\omega)^j=\operatorname*{Res}_{\hbar=0}\sum_{j=-\infty}^{\infty}A_j\hbar^j.$$

This means that we can erase $\det\mathcal{Q}^{\mp 1/p}\otimes\mathcal{L}(\mathcal{E}_\flat)^{\mp 1/pD}$ from $\Phi'(\mathcal{E}_\flat)$. (The last simplification appeared in [14, Proof of Theorem 7.2.4])

Next note that nontrivial contributions of the $\mathbb{C}^*$-action appear as $I_{\pm 1/pD}$. If we take the covering $\mathbb{C}^*_s\to\mathbb{C}^*; s\mapsto t=s^{-pD}$, $I_{\pm 1/pD}$ is of weight ∓ 1 as a $\mathbb{C}^*_s$-module by our convention. Since that we have a natural isomorphism $H^*_{\mathbb{C}^*_s}(\mathrm{pt}) = \mathbb{C}[\hbar_s] \cong H^*_{\mathbb{C}^*}(\mathrm{pt}) = \mathbb{C}[\hbar]$ by $\hbar_s = -\hbar/pD$, we can replace $\hbar$ by $\hbar_s$ noticing $\operatorname{Res}_{\hbar=0} f(\hbar) = -pD \times \operatorname{Res}_{\hbar_s=0} f(\hbar_s pD)$. We use this replacement and then replace back $\hbar_s$ by $\hbar$ again. Therefore we get the formula (6.2). We have completed the proof of Theorem 6.1.

6.4. **Proof of Theorem 1.5.** The right hand side of the formula in Theorem 6.1 can be expressed by an integral over $\widehat{M}^m(c_\flat)$ by using the formula again. We continue this procedure recursively, we will get a wall-crossing formula comparing $\int_{\widehat{M}^m(c)}$ and $\int_{\widehat{M}^{m+1}(c)}$. We then get Theorem 1.5. The proof has combinatorial nature and is the same as one in [14, §7.6]. We reproduce it here for reader's convenience.

In fact, it is enough to consider the case $m = 0$, as a general case follows from $m = 0$ since we have an isomorphism $\widehat{M}^m(c) \xrightarrow{\cong} \widehat{M}^0(ce^{-m[C]})$; $(E, \Phi) \mapsto (E(-mC), \Phi)$. Then the formula in Theorem 6.1 is simplified as we can assume $p = 1$ as $\mathrm{Gr}(m+1, p) = \mathrm{Gr}(1, p)$ is empty otherwise. Theorem 1.5 is obtained in this way, but we give a proof for general m.

For $j \in \mathbb{Z}_{>0}$ let

$$S^m_j(c) := \left\{ \vec{c} = (c_\flat, p_1, \dots, p_j) \in H^*(\widehat{\mathbb{P}}^2) \times \mathbb{Z}^j_{>0} \;\middle|\; (c_\flat, [\ell_\infty]) = 0,\ c_\flat + \sum_{i=1}^j p_i e_m = c \right\}.$$

We denote the universal family for $\widehat{M}^m(c_\flat)$ by $\mathcal{E}_\flat$ as above. Let $S^m(c) = \bigsqcup_{j=1}^\infty S^m_j(c)$.

For $\vec{p} = (p_1, \dots, p_j) \in \mathbb{Z}^j_{>0}$ we consider the product of Grassmannian varieties $\prod_{i=1}^j \mathrm{Gr}(m+1, p_i)$. Let $\mathcal{Q}^{(i)}$ be the universal quotient of the i^{th} factor. We consider the j-dimensional torus $(\mathbb{C}^*)^j$ acting trivially on $\prod_{i=1}^j \mathrm{Gr}(m+1, p_i)$. We denote the 1-dimensional weight n representation of the i^{th} factor by $e^{n\hbar_i}$. (Denoted by I_n previously.) The equivariant cohomology $H^*_{(\mathbb{C}^*)^j}(\mathrm{pt})$ of the point is identified with $\mathbb{C}[\hbar_1, \dots, \hbar_j]$.

Theorem 6.4.

$$\int_{\widehat{M}^{m+1}(c)} \Phi(\mathcal{E}) - \int_{\widehat{M}^m(c)} \Phi(\mathcal{E}) = \sum_{\vec{c} \in S^m(c)} \int_{\widehat{M}^m(c_\flat)} \operatorname*{Res}_{\hbar_j=0} \cdots \operatorname*{Res}_{\hbar_1=0} \Phi(\mathcal{E}_\flat \oplus \bigoplus_{i=1}^j C_m \boxtimes \mathcal{Q}^{(i)} \otimes e^{-\hbar_i}) \cup \Psi^{\vec{p}}(\mathcal{E}_\flat),$$

where $\Psi^{\vec{p}}(\bullet)$ is another cohomology class given by

$$\Psi^{\vec{p}}(\bullet) := \frac{1}{(m+1)^j} \prod_{i=1}^j \frac{1}{\sum_{1 \le k \le i} p_k} \int_{\prod_{i=1}^j \mathrm{Gr}(m+1, p_i)} \heartsuit, \tag{6.5}$$

$$\heartsuit = \prod_{i=1}^j \frac{e\left((V_1(C_m) \otimes \mathcal{Q}^{(i)}/\mathcal{O})^*\right)}{e(\mathfrak{N}(\bullet, C_m) \otimes \mathcal{Q}^{(i)} \otimes e^{-\hbar_i})\, e(\mathfrak{N}(C_m, \bullet) \otimes \mathcal{Q}^{(i)*} \otimes e^{\hbar_i})} \times \prod_{1 \le i_1 \ne i_2 \le j} e(\mathcal{Q}^{(i_1)} \otimes \mathcal{Q}^{(i_2)*} \otimes e^{-\hbar_{i_1} + \hbar_{i_2}}).$$

(*Note that $\Psi^{\vec{p}}$ depends on $\vec{p} = (p_1, \dots, p_j)$, but not on $c_\flat$.*)

Let us prepare notation before starting the proof.

We defined a cohomology class $\Psi^{\vec{p}}(\mathcal{E}_\flat)$ by the formula (6.5), and it is an element in $H^*_{\widetilde{T}}(\widehat{\mathbb{P}}^2 \times M \times \prod_{i=1}^j \mathrm{Gr}(m+1, p^{(i)}))[\hbar_1^\pm, \dots, \hbar_j^\pm]$ for some moduli stack M with the universal family $\mathcal{E}$.

Let $\mathrm{Dec}^{(j)}(c)$ be the set of pairs $\mathfrak{I}^{(j)} = (I_\flat^{(j)}, \vec{I}_\sharp^{(j)})$ as follows:

- $\vec{I}_\sharp^{(j)}$ is a tuple $(I_\sharp^{(j)}, I_\sharp^{(j-1)}, \dots, I_\sharp^{(1)})$ of subsets of $\underline{v_1(c)}$ such that $|I_\sharp^{(i)}| = p_i(m+1)$ for some $p_i \in \mathbb{Z}_{>0}$ $(1 \le i \le j)$.
- $\min(I_\sharp^{(1)}) > \min(I_\sharp^{(2)}) > \cdots > \min(I_\sharp^{(j)})$.
- $I_\flat^{(j)}$ is also a subset of $\underline{v_1(c)}$ and we have $\underline{v_1(c)} = I_\flat^{(j)} \sqcup \bigsqcup_{i=1}^j I_\sharp^{(i)}$.

For $\mathfrak{I}^{(j)} \in \mathrm{Dec}^{(j)}(c)$ set

$$\mathfrak{k}(\mathfrak{I}^{(j)}) := \max\{i \in I_\flat^{(j)} \mid i < \min(I_\sharp^{(j)})\},$$

where we understand this to be 0 if there exists no $i \in I_\flat^{(j)}$ with $i < \min(I_\sharp^{(j)})$. We also put

$$c_\sharp^{(i)} := p_i e_m \ (1 \le i \le j), \quad c_\flat^{(j)} := c - \sum_{i=1}^j p_i e_m.$$

We have a map $\pi_j\colon \mathrm{Dec}^{(j+1)}(c) \to \mathrm{Dec}^{(j)}(c); (I_\flat^{(j+1)}, \vec{I}_\sharp^{(j+1)}) \mapsto (I_\flat^{(j)}, \vec{I}_\sharp^{(j)})$ given by

$$I_\flat^{(j)} := I_\flat^{(j+1)} \sqcup I_\sharp^{(j+1)}, \quad \vec{I}_\sharp^{(j)} := (I_\sharp^{(j)}, \dots, I_\sharp^{(1)}).$$

Let

$$\widetilde{M}(\mathfrak{I}^{(j)}) := \widetilde{M}^{m,\ell}(c_\flat^{(j)}), \qquad \widehat{M}(\mathfrak{I}^{(j)}) := \widehat{M}^m(c_\flat^{(j)}),$$

where in the first equality we take the unique order preserving bijection $I_\flat^{(j)} \cong \underline{v_1(c_\flat^{(j)})} = \underline{\#I_\flat^{(j)}}$ and take $\ell \in \underline{v_1(c_\flat^{(j)})}$ corresponding to $\mathfrak{k}(\mathfrak{I}^{(j)})$.

For the universal family $\mathcal{E}_\flat^{(j)}$ for $\widetilde{M}(\mathfrak{I}^{(j)})$ or $\widehat{M}(\mathfrak{I}^{(j)})$ let

$$\Psi^{\mathfrak{I}^{(j)}}(\mathcal{E}_\flat^{(j)}) := \Psi^{\vec{p}}(\mathcal{E}_\flat^{(j)}) \frac{v_1(c_\flat^{(j)})! \prod_{i=1}^j (v_1(c_\sharp^{(i)}) - 1)!}{v_1(c)!} (m+1)^j \prod_{i=1}^j \sum_{k=1}^i p_k,$$

where $\vec{p} = (p_1, \dots, p_j)$.

Lemma 6.6 ([14, 7.6.5])**.** *For each j, we have the formula*

$$\begin{aligned}
(6.7) \quad & \int_{\widehat{M}^{m+1}(c)} \Phi(\mathcal{E}) - \int_{\widehat{M}^m(c)} \Phi(\mathcal{E}) \\
&= \sum_{\substack{1 \le i < j \\ \mathfrak{I}^{(i)} \in \mathrm{Dec}^{(i)}(c)}} \int_{\widehat{M}(\mathfrak{I}^{(i)})} \operatorname*{Res}_{\hbar_i = 0} \cdots \operatorname*{Res}_{\hbar_1 = 0} \Phi(\mathcal{E}_\flat^{(i)} \oplus \bigoplus_{k=1}^i C_m \boxtimes \mathcal{Q}^{(k)} \otimes e^{-\hbar_k}) \Psi^{\mathfrak{I}^{(i)}}(\mathcal{E}_\flat^{(i)}) \\
&\quad + \sum_{\mathfrak{I}^{(j)} \in \mathrm{Dec}^{(j)}(c)} \int_{\widetilde{M}(\mathfrak{I}^{(j)})} \operatorname*{Res}_{\hbar_j = 0} \cdots \operatorname*{Res}_{\hbar_1 = 0} \widetilde{\Phi}(\mathcal{E}_\flat^{(j)} \oplus \bigoplus_{k=1}^j C_m \boxtimes \mathcal{Q}^{(k)} \otimes e^{-\hbar_k}) \Psi^{\mathfrak{I}^{(j)}}(\mathcal{E}_\flat^{(j)}).
\end{aligned}$$

Proof. We prove the assertion by an induction on j. If $j = 1$, this is nothing but Theorem 6.1 applied to $\ell = v_1(c)$.

Suppose that the formula is true for j. We apply Theorem 6.1 to get

$$\begin{aligned}
&\int_{\widehat{M}(\mathfrak{I}^{(j)})} \operatorname*{Res}_{\hbar_j=0} \cdots \operatorname*{Res}_{\hbar_1=0} \widetilde{\Phi}(\mathcal{E}_\flat^{(j)} \oplus \bigoplus_{k=1}^{j} C_m \boxtimes \mathcal{Q}^{(k)} \otimes e^{-\hbar_k}) \Psi^{\mathfrak{I}^{(j)}}(\mathcal{E}_\flat^{(j)}) \\
&\qquad - \int_{\widehat{M}(\mathfrak{I}^{(j)})} \operatorname*{Res}_{\hbar_j=0} \cdots \operatorname*{Res}_{\hbar_1=0} \Phi(\mathcal{E}_\flat^{(j)} \oplus \bigoplus_{k=1}^{j} C_m \boxtimes \mathcal{Q}^{(k)} \otimes e^{-\hbar_k}) \Psi^{\mathfrak{I}^{(j)}}(\mathcal{E}_\flat^{(j)}) \\
&= \sum_{\substack{\mathfrak{I}^{(j+1)} \in \operatorname{Dec}^{(j+1)}(c) \\ \pi_j(\mathfrak{I}^{(j+1)}) = \mathfrak{I}^{(j)}}} \int_{\widetilde{M}(\mathfrak{I}^{(j+1)})} \operatorname*{Res}_{\hbar_{j+1}=0} \cdots \operatorname*{Res}_{\hbar_1=0} \left[\widetilde{\Phi}(\mathcal{E}_\flat^{(j+1)} \oplus \bigoplus_{k=1}^{j+1} C_m \boxtimes \mathcal{Q}^{(k)} \otimes e^{-\hbar_k}) \Phi'(\mathcal{E}_\flat^{(j+1)}) \right],
\end{aligned}$$

where

$$\Phi'(\mathcal{E}_\flat^{(j+1)}) = \frac{v_1(c_\flat^{(j+1)})!(v_1(c_\sharp^{(j+1)}) - 1)!}{v_1(c_\flat^{(j)})!} \int_{\operatorname{Gr}(m+1, p_{j+1})} \heartsuit$$

$$\heartsuit = \frac{\Psi^{\mathfrak{I}^{(j)}}(\mathcal{E}_\flat^{(j+1)} \oplus C_m \boxtimes \mathcal{Q}^{(j+1)} \otimes e^{-\hbar_{j+1}}) e\left((V_1(C_m) \otimes \mathcal{Q}^{(j+1)}/\mathcal{O})^*\right)}{e(\mathfrak{N}(\mathcal{E}_\flat^{(j+1)}, C_m) \otimes \mathcal{Q}^{(j+1)} \otimes e^{-\hbar_{j+1}}) e(\mathfrak{N}(C_m, \mathcal{E}_\flat^{(j+1)}) \otimes \mathcal{Q}^{(j+1)*} \otimes e^{\hbar_{j+1}})}.$$

We have $\Phi'(\mathcal{E}_\flat^{(j+1)}) = \Psi^{\mathfrak{I}^{(j+1)}}(\mathcal{E}_\flat^{(j+1)})$ thanks to the multiplicative property of the Euler class and $\mathfrak{N}(C_m, C_m) = -\operatorname{Hom}(C_m, C_m) = -\mathbb{C}\operatorname{id}_{C_m}$. Hence the formula holds for $j + 1$. □

If j is sufficiently large, $\operatorname{Dec}^{(j)}(c) = \emptyset$. Hence we have

$$\begin{aligned}
&\int_{\widehat{M}^{m+1}(c)} \Phi(\mathcal{E}) - \int_{\widehat{M}^{m}(c)} \Phi(\mathcal{E}) \\
&\quad = \sum_{j=1}^{\infty} \sum_{\mathfrak{I}^{(j)} \in \operatorname{Dec}^{(j)}(c)} \int_{\widehat{M}(\mathfrak{I}^{(j)})} \operatorname*{Res}_{\hbar_j=0} \cdots \operatorname*{Res}_{\hbar_1=0} \Phi(\mathcal{E}_\flat^{(j)} \oplus \bigoplus_{i=1}^{j} C_m \boxtimes \mathcal{Q}^{(i)} \otimes e^{-\hbar_i}) \Psi^{\mathfrak{I}^{(j)}}(\mathcal{E}_\flat^{(j)}).
\end{aligned}$$

We have a map $\rho_j\colon \operatorname{Dec}^{(j)}(c) \to S_j(c)$ given by

$$\rho_j(\mathfrak{I}^{(j)}) = (c_\flat, p_1, \dots, p_j) = (c_\flat^{(j)}, \frac{|I_\sharp^{(1)}|}{m+1}, \dots, \frac{|I_\sharp^{(j)}|}{m+1}).$$

Therefore the right hand side is equal to

$$\begin{aligned}
&\sum_{j=1}^{\infty} \sum_{\vec{c} \in S_j(c)} \#\rho_j^{-1}(\vec{c}) \frac{v_1(c_\flat)! \prod_{i=1}^{j} (p_i(m+1) - 1)!}{v_1(c)!} (m+1)^j \prod_{i=1}^{j} \sum_{k=1}^{i} p_k \\
&\qquad \times \int_{\widehat{M}^m(c_\flat)} \operatorname*{Res}_{\hbar_j=0} \cdots \operatorname*{Res}_{\hbar_1=0} \Phi(\mathcal{E}_\flat \oplus \bigoplus_{i=1}^{j} C_m \boxtimes \mathcal{Q}^{(i)} \otimes e^{-\hbar_i}) \Psi^{\vec{p}}(\mathcal{E}_\flat).
\end{aligned}$$

Thus Theorem 6.4 follows from the following lemma.

Lemma 6.8 ([14, 7.6.7])**.**

$$\#\rho_j^{-1}(\vec{c})\frac{v_1(c_\flat)!\prod_{i=1}^j(p_i(m+1)-1)!}{v_1(c)!} = \frac{1}{(m+1)^j}\prod_{i=1}^j\frac{1}{\sum_{1\le k\le i}p_k}.$$

Proof. The set $\rho_j^{-1}(\vec{c})$ is

$$\left\{ (I_\flat^{(j)}, I_\sharp^{(j)}, \dots, I_\sharp^{(1)}) \;\middle|\; \begin{array}{l} \underline{v_1(c)} = I_\flat^{(j)} \sqcup \bigsqcup_{i=1}^j I_\sharp^{(i)}, |I_\flat^{(j)}| = v_1(c_\flat), |I_\sharp^{(i)}| = p_i(m+1), \\ \min(I_\sharp^{(1)}) > \min(I_\sharp^{(2)}) > \cdots > \min(I_\sharp^{(j)}) \end{array} \right\}.$$

Put $N := v_1(c)$, $N_0 := v_1(c_\flat)$, $N_i := p_i(m+1)$ $(1 \le i \le j)$.

We first choose $I_\flat^{(j)} \subset \underline{N}$. We have $\binom{N}{N_0}$ possibilities. Next we choose $I_\sharp^{(j)} \subset \underline{N} \setminus I_\flat^{(j)}$. From the second condition, we must have $\min(I_\sharp^{(j)}) = \min(\underline{N} \setminus I_\flat^{(j)})$. Let x be this number. Then the remaining choice is $I_\sharp^{(j)} \setminus \{x\} \subset (\underline{N} \setminus I_\flat^{(j)}) \setminus \{x\}$. We have $\binom{N-N_0-1}{N_j-1}$ possibilities. Next we choose $I_\sharp^{(j-1)} \subset \underline{N} \setminus (I_\flat^{(j)} \cup I_\sharp^{(j)})$. We have $\binom{N-N_0-N_j-1}{N_{j-1}-1}$ possibilities. We continue until we choose $I_\sharp^{(1)}$. Therefore we have

$$\begin{aligned} \#\rho_j^{-1}(\vec{c}) &= \binom{N}{N_0}\prod_{i=1}^j\binom{N-N_0-\sum_{k>i}N_k-1}{N_i-1} \\ &= \frac{N!}{N_0!\prod_{i=1}^j(N_i-1)!} \times \prod_{i=1}^j\frac{1}{\sum_{1\le k\le i}N_k}. \end{aligned}$$

Moreover

$$\prod_{i=1}^j\frac{1}{\sum_{1\le k\le i}N_k} = \prod_{i=1}^j\frac{1}{\sum_{1\le k\le i}p_k(m+1)}.$$

□

References

[1] R. Bott, *Homogeneous vector bundles*, Ann. of Math. (2) **66** (1957), 203–248.

[2] A. Braverman and P. Etingof, *Instanton counting via affine Lie algebras II: from Whittaker vectors to the Seiberg-Witten prepotential*, Studies in Lie theory, 61–78, Progr. Math., **243**, Birkhäuser, Boston, 2006; `arXiv:math/0409441`.

[3] G. Ellingsrud and L. Göttsche, *Variation of moduli spaces and Donaldson invariants under change of polarization*, J. Reine Angew. Math. **467** (1995), 1–49.

[4] J.D. Fay, *Theta functions on Riemann surfaces*, Lecture Notes in Math., 352, Springer, Berlin, 1973.

[5] W. Fulton and S. Lang, *Riemann-Roch algebra*, Grundlehren der Mathematischen Wissenschaften, **277**, Springer-Verlag, New York, 1985. x+203 pp.

[6] A. Gorsky, A. Marshakov, A. Mironov and A. Morozov, *RG equations from Whitham hierarchy*, Nucl. Phys. B **527** (1998), 690–716; `arXiv:hep-th/9802007`.

[7] L. Göttsche, H. Nakajima and K. Yoshioka, *Instanton counting and Donaldson invariants*, J. of Differential Geometry, **80** (2008), 343–390; `arXiv:math/0606180`.

[8] ———, *K-theoretic Donaldson invariants via instanton counting*, Pure and Applied Math. Quarterly, **5** (2009), 1029–1110; `arXiv:math/0611945`.

[9] R. Joshua, *Equivariant Riemann-Roch for G-quasi-projective varieties. I.*, K-Theory 17 (1999), no. 1, 1–35.

[10] A. King, *Moduli of representations of finite dimensional algebras*, Quarterly J. of Math. **45** (1994), 515–530.

[11] J. Kollár and S. Mori, *Birational geometry of algebraic varieties*, Cambridge Tracts in Mathematics, **134**. Cambridge University Press, Cambridge, 1998. viii+254 pp.

[12] A. Losev, N. Nekrasov and S. Shatashvili, *Issues in topological gauge theory*, Nuclear Phys. B **534** (1998), 549–611; `arXiv:hep-th/9711108`.

[13] A. Marshakov and N. Nekrasov, *Extended Seiberg-Witten Theory and Integrable Hierarchy*, J. High Energy Phys. 2007, no. 1, **104**, 39 pp; `arXiv:hep-th/0612019`.

[14] T. Mochizuki, *Donaldson type invariants for algebraic surfaces. Transition of moduli stacks*, Lecture Notes in Mathematics, **1972**. Springer-Verlag, Berlin, 2009. xxiv+383 pp.

[15] D. Mumford, *Tata lectures on theta. II*, Birkhäuser Boston, Boston, MA, 1984.

[16] D. Mumford, J. Fogarty and F. Kirwan, *Geometric invariant theory*, Third Edition, Ergebnisse der Mathematik und ihrer Grenzgebiete (2), **34**, Springer-Verlag, Berlin, 1994, xiv+292.

[17] H. Nakajima, *Lectures on Hilbert schemes of points on surfaces*, Univ. Lect. Ser. **18**, AMS, 1999.

[18] H. Nakajima and K. Yoshioka, *Instanton counting on blowup. I. 4-dimensional pure gauge theory*, Invent. Math **162** (2005), no. 2, 313–355; `arXiv:math/0306198`.

[19] ———, *Lectures on instanton counting*, Algebraic structures and moduli spaces, 31–101, CRM Proc. Lecture Notes, 38, Amer. Math. Soc., Providence, RI, 2004; `arXiv:math/0311058`.

[20] ———, *Instanton counting on blowup. II. K-theoretic partition function*, Transform. Groups **10** (2005), 489–519; `arXiv:math/0505553`.

[21] ———, *Perverse coherent sheaves on blow-up. I. A quiver description*, preprint, `arXiv:0802.3120`.

[22] ———, *Perverse coherent sheaves on blow-up. II. Wall-crossing and Betti numbers formula*, preprint, `arXiv:0806.0463`, J. of Algebraic Geometry, to appear.

[23] ———, *Instanton counting on blowup. III. Theories with matters*, in preparation.

[24] N. Nekrasov, *Seiberg-Witten prepotential from instanton counting*, Adv. Theor. Math. Phys. **7** (2003), no. 5, 831–864; `arXiv:hep-th/0206161`.

[25] N. Nekrasov and A. Okounkov, *Seiberg-Witten prepotential and random partitions*, The unity of mathematics, 525–596, Progr. Math., 244, Birkhäuser Boston, Boston, MA, 2006; `arXiv:hep-th/0306238`.

[26] Y. Tachikawa, *Five-dimensional Chern-Simons terms and Nekrasov's instanton counting*, J. High Energy Phys. 2004, no. 2, 050, 13 pp; `arXiv:hep-th/0401184`.

[27] M. Thaddeus, *Geometric invariant theory and flips*, J. Amer. Math. Soc. **9** (1996), no. 3, 691–723.

[28] A. Vistoli, *Intersection theory on algebraic stacks and on their moduli spaces*, Invent. Math. **97** (1989), no. 3, 613–670.

[29] K. Yamada, *Blowing-ups describing the polarization change of moduli schemes of semistable sheaves of general rank*, preprint, `arXiv:0704.2870`.

Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502, Japan

E-mail address: `nakajima@kurims.kyoto-u.ac.jp`

Department of Mathematics, Faculty of Science, Kobe University, Kobe 657-8501, Japan

Max-Planck-Institut für Mathematik, Vivatsgasse 7 53111 Bonn, Germany

E-mail address: `yoshioka@math.kobe-u.ac.jp`