NUMERICALLY TRIVIAL INVOLUTIONS OF KUMMER TYPE OF AN ENRIQUES SURFACE

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Abstract. There are two types of numerically trivial involutions of an Enriques surface according as their period lattice. One is $U(2) \perp U(2)$-type and the other is $U \perp U(2)$-type. An Enriques surface with an involution of $U(2) \perp U(2)$-type is doubly covered by a Kummer surface of product type, and such involutions are classified again into two types according as the parity of the corresponding Göpel subgroups. Involutions of odd $U(2) \perp U(2)$-type are constructed from the standard Cremona involutions of the quadric surface and closely related with quartic del Pezzo surfaces.

It is known that a nontrivial automorphism of a K3 surface acts nontrivially on its cohomology group. But this is not true for an Enriques surface. An automorphism of an Enriques surface S is said to be *numerically trivial* (resp. *cohomologically trivial*) if it acts on the cohomology group $H^2(S, \mathbb{Q})$ (resp. $H^2(S, \mathbb{Z})$) trivially. In this paper we classify the numerically trivial involutions, correcting [3].

Let S be a (minimal) Enriques surface, that is, a compact complex surface with $H^1(\mathcal{O}_S) = H^2(\mathcal{O}_S) = 0$ and $2K_S \sim 0$, and σ a numerically trivial (holomorphic) involution of S. We denote the covering K3 surface of S by $\tilde{S}$ and the covering involution by ε. Then the period lattice N_R of (S, σ) is isomorphic to either $U(2) \perp U(2)$ or $U \perp U(2)$ as a lattice ([3, Proposition (2.5)]). σ is called $U(2) \perp U(2)$-type, or *Kummer type*, in the former case.

In this paper, except the first appendix, we assume that $N_R \simeq U(2) \perp U(2)$ and classify the numerically trivial involutions of Kummer type using their *periods*, that is, the Hodge structures on N_R (cf. Remark 21). There exist a pair of elliptic curves E' and E'' and an isomorphism φ between $\tilde{S}$ and the Kummer surface of the product abelian surface $E' \times E''$ such that the diagram

$$(1) \qquad \begin{array}{ccccc} & \tilde{S} & \xrightarrow{\varphi} & Km(E' \times E'') & \\ \sigma_R & \downarrow & & \downarrow & \mu \\ & \tilde{S} & \xrightarrow{\varphi} & Km(E' \times E'') & \end{array}$$

is commutative, where σ_R is the anti-symplectic lift of σ (Section 1) and μ is the involution induced by $(id_{E'}, -id_{E''})$ (Proposition 6).

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Example 1. *Let β_{ev} be the involution of $Km(E' \times E'')$ induced by the translation of $E' \times E''$ by a 2-torsion point a with $a \notin E' \times 0 \cup 0 \times E''$. Then $\varepsilon_{ev} = \mu\beta_{ev}$ has no fixed points and the involution σ_{ev} of the Enriques surface $Km(E' \times E'')/\varepsilon_{ev}$ induced by μ is numerically trivial (cf. Proposition 4).*

The quotient $Km(E' \times E'')/\mu$ is the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ at the 16 points (p'_i, p''_j), $1 \le i, j \le 4$, where $\{p'_1, \dots, p'_4\}$ and $\{p''_1, \dots, p''_4\}$ are the branches of the double coverings $E' \to \mathbb{P}^1 \simeq E'/(-id)$ and $E'' \to \mathbb{P}^1 \simeq E''/(-id)$, respectively. In the course of his classification of Enriques surfaces with finite (full) automorphism groups, Kondo[2] found a numerically trivial involution of an Enriques surface which had been overlooked in [3] (*cf.* Remark 12).

Proposition 2. *Assume that*

($*$) *the ordered 4-tuples $(p'_1, \dots, p'_4)$ and $(p''_1, \dots, p''_4) \in (\mathbb{P}^1)^4$ are not projectively equivalent.*

Then the standard Cremona involution of $\mathbb{P}^1 \times \mathbb{P}^1$ with center the four points (p'_i, p''_i), $1 \le i \le 4$, lifts to a fixed point free involution ε_{odd} of $Km(E' \times E'')$ (Section 2). Moreover, the involution σ_{odd} of the Enriques surface $Km(E' \times E'')/\varepsilon_{odd}$ induced by μ is numerically trivial.

The following is the main result of this paper:

Theorem 3. *Every numerically trivial involution of Kummer type of an Enriques surface is obtained in the way of Example 1 or Proposition 2.*

First we characterize the involutions of Kummer type by their periods in Section 1. In Section 2 we construct an Enriques surface using a Cremona involution of the smooth quadric, or almost equivalently, from a smooth quartic del Pezzo surface. In Section 3 the main theorem is proved by the global Torelli theorem for Enriques surfaces and by computation of periods of Enriques surfaces of Example 1 and Proposition 2. This article has two appendices. In the first, we complete the classification of numerically trivial involutions, correcting [3]. In the second, we exibit 14 smooth rational curves on Enriques surfaces of Proposition 2 and compute the dual graph of their arrangement.

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Notation. The symbol U denotes the rank 2 lattice given by the symmetric matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. The lattice obtained from a lattice L by replacing the bilinear form (.) with r(.), r being a rational number r, is denoted by $L(r)$.

1. Involutions of Kummer type

Let $Km(E' \times E'')$ and μ be as in the introduction.

Proposition 4. *Let ε be a fixed point free involution of $Km(E' \times E'')$ which commutes with the involution μ. Then the involution of the Enriques surface $Km(E' \times E'')/\varepsilon$ induced by μ is numerically trivial.*

Proof. The invariant part of the action of μ on $H^2(Km(E' \times E''), \mathbb{Z})$ is of rank 18. On the other hand, since $\varepsilon\mu$ is symplectic, the anti-invariant part of its cohomological action is of rank 8. Therefore, μ mod ε acts on $H^2(Km(E' \times E'')/\varepsilon, \mathbb{Q})$, which is of rank 10, trivially. □

Let σ be a numerically trivial involution of an Enriques surface S. There are two involutions of the K3 cover $\tilde{S}$ of S which lift σ since $\tilde{S}$ has no fixed point free automorphisms of order 4. One is symplectic and the other is anti-symplectic. These involutions of $\tilde{S}$ are denoted by σ_K and σ_R, respectively. We denote the anti-invariant parts of the actions of $\varepsilon := \sigma_K\sigma_R$, σ_K and σ_R on $H^2(\tilde{S}, \mathbb{Z})$ by N, N_K and N_R, respectively. N is isomorphic to $U \perp U(2) \perp E_8(2)$ ([1, Chap. VIII, Lemma 19.1]) and N_K is isomorphic to $E_8(2)$ ([3, Lemma (2.1)]). N_R carries a nontrivial polarized Hodge structure of weight 2, which we call the *period* of (S, σ).

In order to compute the period for an involution in Proposition 4, we recall a basic fact on the cohomology of the Kummer surface $Km(T)$ of a (2-dimensional) complex torus T. $Km(T)$ contains sixteen $(-2)\mathbb{P}^1$'s $\{E_a\}_{a \in T_2}$ parametrized by the 2-torsion subgroup $T_2 \simeq (\mathbb{Z}/2\mathbb{Z})^4$ of T. These generate a sublattice of rank 16 in the cohomology group $H^2(Km(T), \mathbb{Z})$, which we denote by Γ_{Km}. Let Λ be the orthogonal complement of Γ_{Km} in $H^2(Km(T), \mathbb{Z})$. Λ is the image of $H^2(T, \mathbb{Z})$ by the quotient morphism from the blow-up of T at T_2 onto $Km(T)$. The following is well known ([1, Chap. VIII, §5]).

Lemma 5. $\Lambda \subset H^2(Km(T))$ *is isomorphic to* $H^2(T, \mathbb{Z})$ *as a Hodge structure and to* $H^2(T, \mathbb{Z})(2) \simeq U(2) \perp U(2) \perp U(2)$ *as a lattice.*

Being of Kummer type is characterized in terms of the period as follows:

Proposition 6. *The followings are equivalent for a numerically trivial involution σ.*

(1) *σ is of Kummer type, that is, the lattice N_R is isomorphic to $U(2) \perp U(2)$.*
(2) *σ is obtained in the way of Proposition* 4.

Proof. Γ_{Km} is fixed in the cohomological action of μ. In the action of the involution $(id_{E'}, -id_{E''})$ on $H^2(E' \times E'', \mathbb{Z}) \simeq U \perp U \perp U$, one U, generated by two elliptic curves, is invariant and the other two are anti-invariant. Hence the anti-invariant part N^- of the action involution μ on Λ is isomorphic to $U(2) \perp U(2)$ as a lattice. Therefore, $N_R \simeq U(2) \perp U(2)$ if σ is obtained in the way of Proposition 4.

Conversely assume that N_R is isomorphic to $U(2) \perp U(2)$. The lattice $U \perp U$ is isomorphic to $M_2(\mathbb{Z}) = V' \otimes V''$, the group of 2×2 matrices of integral entries endowed with the bilinear form form $(A.\, A) = 2 \det A$, where V' and V'' are free $\mathbb{Z}$-modules of rank two. The period ω of $\tilde{S}$ corresponds to a complex matrix of rank one via this isomorphism since $(\omega^2) = 0$. Hence we have $\omega = \alpha' \otimes \alpha''$ for $\alpha' \in V' \otimes \mathbb{C}$ and $\alpha'' \in V'' \otimes \mathbb{C}$. These α' and α'' determine Hodge structures of weight one since $(\omega.\bar{\omega}) > 0$. Hence, there exits a pair of elliptic curves E' and E'' such that $N_R(1/2)$ is isomorphic to $H^1(E', \mathbb{Z}) \otimes H^1(E'', \mathbb{Z})$ as a polarized Hodge structure. By Theorem 7 below and the uniqueness property of 2-elementary lattices, there exists an isomorphism φ between $\tilde{S}$ and the Kummer surface of the product $E' \times E''$ such that the diagram (1) commutes. $\square$

Theorem 7. *Let (X, σ) and (X', σ') be pairs of a K3 surface and its involution. If there exists a Hodge isometry $\alpha : H^2(X', \mathbb{Z}) \to H^2(X, \mathbb{Z})$ such that the diagram*

$$\begin{array}{ccccc} & H^2(X', \mathbb{Z}) & \xrightarrow{\alpha} & H^2(X, \mathbb{Z}) & \\ \sigma^* & \downarrow & & \downarrow & \sigma'^* \\ & H^2(X', \mathbb{Z}) & \xrightarrow{\alpha} & H^2(X, \mathbb{Z}) & \end{array}$$

commutes, then there exists an isomorphism $\varphi : X \to X'$ such that $\varphi\sigma = \sigma'\varphi$.

Proof. If neither σ nor σ' has a fixed point, this is the global Torelli theorem for Enriques surfaces. The proof in [1, Chap. VIII, §21], especially its key Proposition (21.1), works in our general case too. $\square$

Assume that (S, σ) is of Kummer type. Since $(\operatorname{disc} N_K)(\operatorname{disc} N_R) = 4 \cdot \operatorname{disc} N$, the orthogonal sum $N_K \perp N_R$ is of index two in N. Therefore, there exists a pair of nonzero 2-torsion elements $\alpha_K \in A_{N_K} = (\frac{1}{2}N_K)/N_K$ and $\alpha_R \in A_{N_R} = (\frac{1}{2}N_R)/N_R$ such that $N = N_K + N_R + \mathbb{Z}(x_K, x_R)$, where $x_K \in \frac{1}{2}N_K$ and $x_R \in \frac{1}{2}N_R$ are representatives of α_K and α_R, respectively. This pair (α_K, α_R) is uniquely determined from the involution σ. We call it the *patching pair* of σ. Since N_K and N_R are orthogonal in N, we have $q_{N_K}(\alpha_K) + q_{N_R}(\alpha_R) = 0$ in $\mathbb{Z}/2\mathbb{Z}$.

Definition 8. A numerically trivial involution σ of Kummer type, or a patching pair (α_K, α_R), is *of even type* or *of odd type* according as the common quadratic value $q_{N_K}(\alpha_K) = q_{N_R}(\alpha_R) \in \mathbb{Z}/2\mathbb{Z}$ of patching elements is 0 or 1.

Since $N_R \simeq U(2) \perp U(2)$, q_{N_R} is a non-degenerate even quadratic space of dimension 4 over $\mathbb{F}_2$. Hence the numbers of patching pairs of even and odd type are 6 and 9, respectively.

2. Cremona involutions and involutions of odd type

The Enriques surface in Proposition 2 is closely related with a del Pezzo surface of degree 4 and its small[1] involution. For our purpose it is most convenient to describe it as the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$. We identify $\mathbb{P}^1 \times \mathbb{P}^1$ with a smooth quadric surface Q in $\mathbb{P}^3 = \mathbb{P}_{(x_1:x_2:x_3:x_4)}$.

[1] An automorhism of a surface is *small* if all fixed points are isolated.

Let $p_1 = (p_1', p_1''), \dots, p_4 = (p_4', p_4'')$ be four points of $\mathbb{P}^1 \times \mathbb{P}^1$ which satisfy

$(**)$ $p_1', \dots, p_4'$ are distinct and $p_1'', \dots, p_4''$ are distinct.

In terms of a smooth quadric, this is equivalent to

$(**')$ any line $\overline{p_i p_j}$, $1 \le i < j \le 4$, is not contained in Q.

We also assume the condition $(*)$ in Proposition 2, or equivalently,

$(*')$ $p_1, \dots, p_4 \in Q \subset \mathbb{P}^3$ is not contained in a plane.

We take a system of homogeneous coordinates of $\mathbb{P}^3$ such that $p_1, \dots, p_4$ are the coordinate points $(1 : 0 : 0 : 0), \dots, (0 : 0 : 0 : 1)$. Then the defining equation of Q is of the form $\sum_{1 \le i < j \le 4} a_{ij} x_i x_j = 0$. By the assumption $(**')$, all coefficients a_{ij}'s are nonzero. Hence, replacing $x_1, \dots, x_4$ by their suitable constant multiplications, we may and do assume that $Q \subset \mathbb{P}^3$ is defined by

$$a_1 x_2 x_3 + a_2 x_1 x_3 + a_3 x_1 x_2 + (x_1 + x_2 + x_3) x_4 = 0 \tag{2}$$

for some nonzero constants a_1, a_2 and $a_3 \in \mathbb{C}$. Since Q is smooth, we have

$$a_1^2 + a_2^2 + a_3^2 - 2a_1 a_2 - 2a_1 a_3 - 2a_2 a_3 \neq 0. \tag{3}$$

Now we define a birational involution τ' of Q by

$$(x_1 : x_2 : x_3 : x_4) \mapsto \left(\frac{a_1}{x_1} : \frac{a_2}{x_2} : \frac{a_3}{x_3} : \frac{a_1 a_2 a_3}{x_4}\right)$$

and call it the *standard Cremona involution* of Q (or $\mathbb{P}^1 \times \mathbb{P}^1$) with center $p_1, \dots, p_4$.

Let B be the blow-up of a smooth quadric Q at $p_1, \dots, p_4$. By the projection from p_4, B is the blow-up of the projective plane also. By (3), the line $l : x_1 + x_2 + x_3 = 0$ and the conic $C : a_1 x_2 x_3 + a_2 x_1 x_3 + a_3 x_1 x_2 = 0$ intersect transversally in the projective plane $\mathbb{P}^2 = \mathbb{P}_{(x_1 : x_2 : x_3)}$. Let q_4 and q_5 be the two intersection points. Then B is isomorphic to the blow-up of $\mathbb{P}^2$ at the three coordinate points $(1 : 0 : 0), (0 : 1 : 0), (0 : 0 : 1)$ and the two points q_4 and q_5. The standard Cremona involution τ' is induced by the quadratic Cremona transformation

$$(x_1 : x_2 : x_3) \mapsto \left(\frac{a_1}{x_1} : \frac{a_2}{x_2} : \frac{a_3}{x_3}\right), \tag{4}$$

which interchanges l and C. In particular, it induces an automorphism of B, which we denote by τ. The following is easily verified:

Lemma 9. (1) *The indeterminacy locus of $\tau' : Q \cdots \to Q$ is $\{p_1, \dots, p_4\}$.*

(2) *For each $1 \le i \le 4$, the conic $C_i' : Q \cap \{x_i = 0\}$ is contracted to the point p_i by τ'.*

(3) *For each $1 \le i \le 4$, the two lines in Q passing through p_i are interchanged by τ'.*

(4) *The fixed points of τ' are $(\varepsilon_1 \sqrt{a_1} : \varepsilon_2 \sqrt{a_2} : \varepsilon_3 \sqrt{a_3} : \sqrt{a_1 a_2 a_3})$, where all ε_i's are ± 1 and satisfy $\varepsilon_1 \varepsilon_2 \varepsilon_3 = -1$.*

For the later use we compute the cohomological action of τ. The second cohomology group $H^2(B,\mathbb{Z})$, or equivalently, the Picard group of B is the free abelian group with the standard $\mathbb{Z}$-basis $\{h_1, h_2, e_1, \dots, e_4\}$, where h_1 and h_2 are the pull-backs of the two rulings of $\mathbb{P}^1 \times \mathbb{P}^1$ and $e_1, \dots, e_4$ are the classes of the exceptional curves over $p_1, \dots, p_4$.

Lemma 10. *The action of the standard Cremona involution τ on $H^2(B,\mathbb{Z})$ is equal to the composite of the two reflections with respect to the mutually orthogonal (-2)-classes $h_1 - h_2$ and $h_1 + h_2 - e_1 - \dots - e_4$.*

Proof. We take the description of B as the blow-up of $\mathbb{P}^2$. The cohomology group $H^2(B,\mathbb{Z})$ has $\{h, e_1, e_2, e_3, f_1, f_2\}$ as a $\mathbb{Z}$-basis. Here h is the pull-back of a line and f_1 and f_2 are the classes of the exceptional curves over q_4 and q_5. The cohomological action of the transformation (4) on the blow-up of $\mathbb{P}^2$ at the three coordinate points is the reflection r with respect to $h - e_1 - e_2 - e_3$. Since the transformation (4) interchanges q_4 and q_5, the cohomological action of τ is the composite of r and the reflection with respect to $f_1 - f_2$. This proves the lemma since $f_1 = h_1 - e_4, f_2 = h_2 - e_4$ and $h = h_1 + h_2 - e_4$. □

There are 16 smooth rational curves of degree 1 with respect to the anti-canonical divisor $-K_B = 2h_1 + 2h_2 - e_1 - \dots - e_4$:

0) the exceptional divisors $e_1, \dots, e_4$ over $p_1, \dots, p_4$,
1) the strict transforms of lines in Q passing through one of $p_1, \dots, p_4$, and
2) the strict transforms C_i's of the conics C_i''s in Lemma 9.

We denote the 8 lines of 1) by $\Gamma_{\bar{1}}$ and the 8 lines of 0) and 2) by $\Gamma_{\bar{0}}$. The Kummer surface $Km(E' \times E'')$ is the minimal resolution of the double cover

$$w^2 = (a_3x_2 + a_2x_3 + x_4)(a_3x_1 + a_1x_3 + x_4)(a_2x_1 + a_1x_2 + x_4)(x_1 + x_2 + x_3)$$

of Q with branch the union of 8 lines in Q passing through one of $p_1, \dots, p_4$. Hence it is the the minimal resolution of the double cover of B with branch the union of the 8 lines in $\Gamma_{\bar{1}}$.

Lemma 11. *$Km(E' \times E'')$ is the minimal resolution of the double cover of B with branch the union of the 8 lines $\Gamma_{\bar{0}}$ also.*

Proof. Put $g_1 = -K_B - h_1 = h_1 + 2h_2 - e_1 - \dots - e_4$. The complete linear system $|g_1|$ is a base point free pencil and the morphism $(\Phi_{|h_1|}, \Phi_{|g_1|}) : B \to \mathbb{P}^1 \times \mathbb{P}^1$ is of degree 2. The covering involution acts on $H^2(B,\mathbb{Z})$ by $\alpha \mapsto (g_1.\alpha)h_1 + (h_1.\alpha)g_1 - \alpha$ and hence interchanges $\Gamma_{\bar{0}}$ and $\Gamma_{\bar{1}}$. Hence we have our assertion. □

Proof of Proposition 2. By the above lemma, the Kummer surface $Km(E' \times E'')$ is the minimal resolution of the double cover $w^2 = x_1x_2x_3x_4$ of Q. Let β_{odd} be the involution of $Km(E' \times E'')$ induced from the birational involution

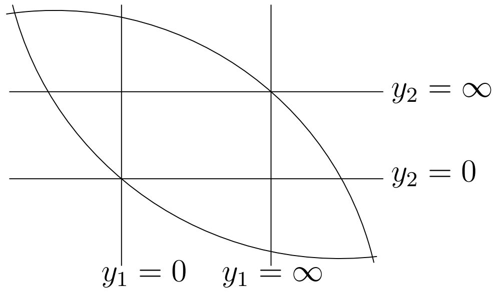
$$(w, x_1, x_2, x_3, x_4) \mapsto (a_1a_2a_3/w, a_1/x_1, a_2/x_2, a_3/x_3, a_1a_2a_3/x_4)$$

of the double cover. Then β_{odd} lifts τ and τ'. The involution $\varepsilon_{odd} := \mu\beta_{odd}$ has no fixed points by (4) of Lemma 9. σ_{odd} is numerically trivial by Proposition 4. $\square$

Horikawa expression. Let $\mathbb{P}^1_{(1)}$ and $\mathbb{P}^1_{(2)}$ be the projective lines whose inhomogenous coordinates are $y_1 = x_1/x_3$ and $y_2 = x_2/x_3$. Then the surface B is blow-up of $\mathbb{P}^1_{(1)} \times \mathbb{P}^1_{(2)}$ with center $(0,0)$, (∞,∞) and the intersection points of $y_1 + y_2 + 1 = 0$ and $a_2y_1 + a_1y_2 + a_3y_1y_2 = 0$. The involution β_{odd} is induced by the *automorphism* $(y_1, y_2) \mapsto (\dfrac{a_1}{a_3y_1}, \dfrac{a_2}{a_3y_2})$ of $\mathbb{P}^1_{(1)} \times \mathbb{P}^1_{(2)}$. By Lemma 11, $Km(E' \times E'')$ is the minimal resolution of the double cover

$$w^2 = y_1y_2(a_2y_1 + a_1y_2 + a_3y_1y_2)(y_1 + y_2 + 1)$$

whose branch locus is as follows:



(5)

Remark 12. In the special case $a_1 = a_2 = a_3 = 1$, the two elliptic curves E' and E'' are both isomorphic to $E_\omega := \mathbb{C}/(\mathbb{Z} + \mathbb{Z}e^{2\pi\sqrt{-1}/3})$. The Enriques surface $S = Km(E_\omega \times E_\omega)/\varepsilon_{odd}$ is studied in [2, (3.5)] as an Enriques surface whose automorphism group is finite. In fact, $\operatorname{Aut} S$ is the extension of $\mathbb{Z}/2\mathbb{Z}$, the group of numerically trivial automorphisms, by the symmetric group of degree 4.

3. Computation of the periods

Let $Km(T)$ and $\Lambda = (\Gamma_{Km})^\perp$ be as in Lemma 5. The discriminant group A_Λ is $(\frac{1}{2}\Lambda)/\Lambda \simeq H^2(T, \mathbb{Z}/2\mathbb{Z})$ and the discriminant form q_Λ is essentially the cup product, that is, $q_\Lambda(\bar{y}) = (y \cup y)/2 \bmod 2$ for $y \in H^2(T, \mathbb{Z})$.

Let $P = \{0, a, b, c\} \subset T_2$ be a subgroup of order 4, or equivalently, a 2-dimensional subspace of T_2. We put $E_P = E_0 + E_a + E_b + E_c \in \Gamma_{Km}$. We denote the Plücker coordinate of $P^\perp \subset T_2^\vee$ by $\pi_P \in \bigwedge^2 T_2^\vee \simeq H^2(T, \mathbb{Z}/2\mathbb{Z})$ and regard it as an element of $\Lambda/2\Lambda$. The following is easily verified ([1, Chap. VIII, §5]):

Lemma 13. *$(E_P$ mod $2) + \pi_P = 0$ holds in $H^2(Km(T), \mathbb{Z}/2\mathbb{Z})$.*

Now we specialize $Km(T)$ to $Km := Km(E' \times E'')$ of product type. Two rulings of $\mathbb{P}^1 \times \mathbb{P}^1$ give two elliptic fibrations $Km \longrightarrow \mathbb{P}^1$. We denote the classes of these fibers by $\tilde{h}_1$ and $\tilde{h}_2 \in H^2(Km, \mathbb{Z})$. These $\tilde{h}_1$ and $\tilde{h}_2$ generate a rank 2 sublattice of Λ which is isomorphic to $U(2)$. Λ is the orthogonal (direct) sum of $\langle \tilde{h}_1, \tilde{h}_2 \rangle$ and N^-, the anti-invariant part of the action of μ. As we saw in the proof of Proposition 6, N^- is isomorphic to $U(2) \perp U(2)$ as a lattice.

Observation 14. *A subgroup P of order 4 of $(E' \times E'')_2$ is naturally associated with a numerically trivial involution of Kummer type:*

(1) *Let $a = (a', a'') \in (E' \times E'')_2$ be a 2-torsion point as in Example 1 and we set $P := \{0, a, (a', 0), (0, a'')\}$. Then P is of order 4 and the Plücker coordinate π_P belongs to $N^-/2N^-$.*

(2) *Let $P \subset T_2$ be a subgroup of order 4 such that*

$$P \cap ((E')_2 \times 0) = P \cap (0 \times (E'')_2) = 0$$

and π_P the Plücker coordinate. Then $\pi_P - \tilde{h}_1 - \tilde{h}_2$ belongs to $N^-/2N^-$. Let β_P be the involution of Km induced by the standard Cremona involution τ' of $\mathbb{P}^1 \times \mathbb{P}^1$ with center the image of P. All β_{odd}'s of Proposition 2 are obtained from β_P's.

In both cases, $P \subset (E' \times E'')_2$ is a Göpel subgroup, that is, P is totally isotropic with respect to the Weil pairing.

A subgroup $P \subset T_2$ of order 4 is Göpel if and only if the Plücker coordinate π_P is parpendicular to $\tilde{h}_1 + \tilde{h}_2$. Hence either π_P or $\pi_P - \tilde{h}_1 - \tilde{h}_2$ belongs to $N^-/2N^-$. There are exactly 15 Göpel subgroups. 9 of them satisfy the above (1) and 6 satisfy (2). All 9 odd elements and 6 even non-zero elements of $N^-/2N^-$ are obtained in the way of (1) and (2), respectively.

Remark 15. The number of non-Göpel subgroups of order 4 is 20. By adding $\tilde{h}_1$ or $\tilde{h}_2$, one obtain a 2 to 1 map from the set of non-Göpel subgroups to $\{x \in N^-/2N^- \mid (x^2) = 0\}$.

Now we are ready to compute the patching pair for Examples 1 and Proposition 2.

Lemma 16. *Let $\Pi \in \Lambda$ be a representative of $\pi_P \in \Lambda/2\Lambda$.*

(1) *An Enriques involution ε_{ev} of Example 1 is of even type and the patching pair is $(\Sigma/2, \Pi/2)$ with $\Sigma := E_0 - E_a + E_{(a',0)} - E_{(0,a'')}$.*

(2) *An Enriques involution ε_{odd} of Proposition 2 is of odd type and the patching pair is $((\tilde{h}_1 + \tilde{h}_2 - E_P)/2, (\Pi - \tilde{h}_1 - \tilde{h}_2)/2)$.*

Proof. Since $\sigma_R = \mu$, N_R coincides with N^-. Hence the discriminant form of N_K is essentially the cup product on $H^2(T, \mathbb{Z}/2\mathbb{Z})$. Here we use the latter for computation.

(1) Since β_{ev} is induced by the translation of $E' \times E''$ by a, Σ belongs to N_K. By Lemma 13, $\Sigma + \Pi$ is divisible by 2. Hence the second half of (1) follows. Since π_P is the Plücker coordinate, $\frac{1}{2}(\pi_P \cup \pi_P) = 0 \in \mathbb{Z}/2\mathbb{Z}$ and σ is of even type.

(2) $\tilde{h}_1 + \tilde{h}_2 - E_P$ belongs to N_K by virtue of Lemma 10. The second half of (2) follows from this and Lemma 13. ε_{odd} is of odd type since $\frac{1}{2}(\pi_P - \tilde{h}_1 - \tilde{h}_2) \cup (\pi_P - \tilde{h}_1 - \tilde{h}_2) = \frac{1}{2}(\pi_P \cup \pi_P) + \frac{1}{2}(\tilde{h}_1 + \tilde{h}_2) \cup (\tilde{h}_1 + \tilde{h}_2) = 1 \in \mathbb{Z}/2\mathbb{Z}$. $\square$

Proof of Theorem 3. Let ε be an Enriques involution of the Kummer surface $Km = Km(E' \times E'')$ which commutes with μ. Let σ be the involution of the Enriques surface $S := Km/\varepsilon$ induced by μ. Let $(\alpha_K, \alpha_R) \in A_{N_K} \times A_{N_R}$ be the patching pair of σ. N_R coincides with N^- since $\sigma_R = \mu$ in our situation Recall that $N_R(1/2)$ is isomorphic to $U \perp U$ as a lattice and isomorphic to $H^1(E', \mathbb{Z}) \otimes H^1(E'', \mathbb{Z})$ as a polarized Hodge structure. In particular, $(\frac{1}{2}N_R)/N_R$ is isomorphic to the tensor product $(E')_2 \otimes (E'')_2$. By this isomorphism, $0 \neq \alpha_R \in (\frac{1}{2}N_R)/N_R$ corresponds to $a' \otimes a'' \in (E')_2 \otimes (E'')_2$ or to an isomorphism $\varphi : (E')_2 \xrightarrow{\sim} (E'')_2$ according as (α_K, α_R) is of even type or of odd type. ($(E')_2$ is identified with its dual since it is of dimension 2 over $\mathbb{F}_2$.) In the even case S is isomorphic to the Enriques surface Km/ε_{ev} of Example 1 with $a = (a', a'')$ by Lemma 16 and the global Torelli theorem for Enriques surfaces since the group of numerically trivial automorphisms of S is cyclic by [3, (1.1)].

Assume that (α_K, α_R) is of odd type.

Claim. There exists no isomorphism from E' to E'' whose restriction to the 2-torsion subgroups is φ.

Proof. Assume the contrary and let $\Phi \subset E' \times E''$ be the graph of such an isomorphism. Then $\Phi - E' \times 0 - 0 \times E''$ is a divisor of self-intersection -2 and its class belongs to $H^1(E', \mathbb{Z}) \otimes H^1(E'', \mathbb{Z}) \subset H^2(E' \times E'', \mathbb{Z})$. Hence $N_R \subset H^2(Km, \mathbb{Z})$ contains an algebraic cycle c' of self-intersection number -4 such that $c'/2$ represents α_R. Since $N_K \simeq E_8(2)$, α_K is represented by a (-4)-element $c \in N_K$. Then $x := (c + c')/2$ belongs to N by the definition of patching pairs and is algebraic since c is orthogonal to $H^0(\Omega^2) \subset N_R \otimes \mathbb{C}$. Since $(x^2) = -2$, either x or $-x$ is effective by the Riemann-Roch theorem. This is a contradiction since $\varepsilon(x) = -x$. $\square$

Let $P \subset T_2$ be the graph of φ and put $P = \{(p'_i, p''_i)\}_{1 \leq i \leq 4}$ as in Proposition 2. Then, by the claim, $(p'_1, \ldots, p'_4)$ and $(p''_1, \ldots, p''_4)$ are not projectively equivalent and we obtain an Enriques surface Km/ε_{odd}. Again, by Lemma 16 and the global Torelli theorem, the Enriques surface S is isomorphic to that obtained from the image of P as in (2) of Observation 14. By the same argument as the even case, we have $(S, \sigma) \simeq (Km/\varepsilon_{odd}, \sigma_{odd})$. $\square$

4. Appendix: Kummer type is not cohomologically trivial

Contrary to the erroneous Proposition (4.8) of [3], the involution of Example 1 is *not* cohomologically trivial.

Theorem 17. *A numerically trivial involution of Kummer type is not cohomologically trivial.*

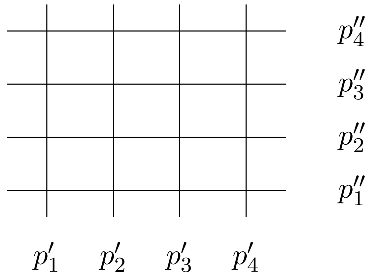
Proof. We prove our assertion by constructing an elliptic fibration.

Let $\{p'_1, \ldots, p'_4\}$ and $\{p''_1, \ldots, p''_4\}$ be the branch of the double coverings $E' \to \mathbb{P}^1 \simeq E'/(-id)$ and $E'' \to \mathbb{P}^1 \simeq E''/(-id)$, respectively. The Kummer surface $Km(E' \times E'')$ is the minimal resolution of the double cover of $\mathbb{P}^1 \times \mathbb{P}^1$

with branch

$$(p'_1 \times \mathbb{P}^1 \cup \cdots \cup p'_4 \times \mathbb{P}^1) \cup (\mathbb{P}^1 \times p''_1 \cup \cdots \cup \mathbb{P}^1 \times p''_4).$$

More precisely, it is the double cover of the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ at the 16 points (p'_i, p''_j), $i, j = 1, \ldots, 4$, with branch the strict transform of these eight rational curves.



The fixed locus of μ is the inverse images of these strict transform. We denote them by

$$(A_1 \sqcup \cdots \sqcup A_4) \sqcup (B_1 \sqcup \cdots \sqcup B_4).$$

The involution $\varepsilon := \mu\beta$ of Example 1 acts on this disjoint union. Renumbering $A_1, \ldots, A_4$ and $B_1, \ldots, B_4$, we may assume that

$$\varepsilon(A_i) = A_{i+1} \quad \text{and} \quad \varepsilon(B_i) = B_{i+1}$$

for $i = 1, 3$. Then ε interchanges two divisors $A_1 + A_3 + B_2 + B_4$ and $A_2 + A_4 + B_1 + B_3$. Let Λ be the linear pencil spanned by their images

$$H_1 := p'_1 \times \mathbb{P}^1 + p'_3 \times \mathbb{P}^1 + \mathbb{P}^1 \times p''_2 + \mathbb{P}^1 \times p''_4$$

and

$$H_2 := p'_2 \times \mathbb{P}^1 + p'_4 \times \mathbb{P}^1 + \mathbb{P}^1 \times p''_1 + \mathbb{P}^1 \times p''_3$$

on $\mathbb{P}^1 \times \mathbb{P}^1$. Then Λ induces elliptic fibrations

$$\Phi_\Lambda : Km(E' \times E'')/\mu \to \Lambda(\simeq \mathbb{P}^1)$$

of the rational surface and

$$Km(E' \times E'') \to \tilde{\Lambda}(\simeq \mathbb{P}^1)$$

of the Kummer surface. The latter is the base change of the former by the double covering $\tilde{\Lambda} \to \Lambda$ with branch $[H_1]$ and $[H_2]$, and descends to an elliptic fibration f of the Enriques surface $Km(E' \times E'')/\varepsilon$.

The action of $\langle \varepsilon, \mu \rangle \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ on $Km(E' \times E'')$ induces the action of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ on $\tilde{\Lambda} \simeq \mathbb{P}^1$. In our cases this action is effective (and hence of Heisenberg type). Let $\bar{\varepsilon}$ and $\bar{\mu}$ be the automorphisms of Λ induced by ε and μ, respectively. $\bar{\varepsilon}$ interchanges the points $[H_1]$ and $[H_2]$ underneath the singular fibers. $\bar{\mu}$ fixes exactly these two points, but the corresponding fiber of the elliptic fibration f on the Enriques surface is not multiple. Since $\bar{\mu}$ is not the identity on $\tilde{\Lambda}/\bar{\varepsilon}$, the involution μ mod ε interchanges two multiple fibers of f. Let G_1 and G_2 be the reduced part of the two multiple fibers of

f. Since the linear equivalence classes of G_1 and G_2 differ by the canonical class, μ mod ε is not cohomologically trivial.

For $\varepsilon = \varepsilon_{odd}$ in Proposition 2, we have $\varepsilon(A_i) = B_i$ for every $i = 1, \dots, 4$, since a Cremona involution interchanges $p'_i \times \mathbb{P}^1$ and $\mathbb{P}^1 \times p''_i$ for every $i = 1, \dots, 4$. The above argument works literally in this case too. Now our assertion follows from Theorem 3. □

Now we are ready to complete the classification of numerically trivial involutions, correcting [3].

At the 6th line in [3, p. 388], it is erroneously stated that the common value $q_T(\alpha) = q_{T'}(\alpha') \in \mathbb{Z}/2\mathbb{Z}$ is nonzero in the case where T', or N_R, is isomorphic to $U(2) \perp U(2)$. But the value can be both 0 and 1 mod 2. We call a primitive embedding of $T(\simeq E_8(2))$ into $N(\simeq E_8(2) \perp U(2) \perp U)$ *even* or *odd* accordingly. Then Proposition (2.6) in [3] should be replaced by

Proposition 18. *Let T_1 and T_2 be primitive sublattices of N isomorphic to $E_8(2)$. If their orthogonal complements T'_1 and T'_2 are isomorphic to each other and if in addition they have the same parity in the case $T'_1 \simeq T'_2 \simeq U(2) \perp U(2)$, then there exists an isometry of N which maps T_1 and T'_1 onto T_2 and T'_2, respectively.*

Let P be the set of periods of $E_8(2)$-polarized Enriques surfaces as defined in [3, p. 388]. Then P is the disjoint union of P_1 and P_2 for which the orthogonal complements of $E_8(2) \subset N$ are isomorphic to $U \perp U(2)$ and $U(2) \perp U(2)$, respectively. The latter decomposes into two parts, P_2^{ev} and P_2^{odd}, according to the parity. Corollary (2.7) in [3] should be replaced by

Corollary 19. *P_1/Γ, P_2^{ev}/Γ and P_2^{odd}/Γ are irreducible.*

Here Γ is the arithmetic group acting on the 10-dimensional Hermitian symmetric domain Ω^- of type IV such that the quotient Ω^-/Γ is the moduli space of Enriques surfaces. In fact, P_2^{ev}/Γ parametrizes Example 1 and an open subset of P_2^{odd}/Γ parametrizes Enriques surfaces in Proposition 2.

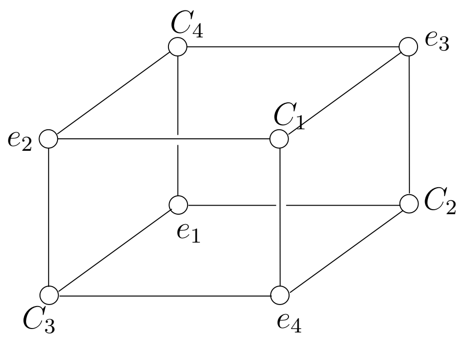
Theorem 20. *Every pair of an Enriques surface and a cohomologically trivial involution is obtained in the way of Example* 2 *of* [3]. *Moreover, they are parametrized by P_1/Γ.*

Proof. Let σ be a cohomologically trivial involution of an Enriques surface S. N_R is isomorphic to $U \perp U(2)$ by Theorem 17, and the periods of such involutions form an irreducible variety by Corollary 19. Hence (S, σ) is a deformation of Example 2 of [3]. As is shown in [3, §5], the fixed locus of the anti-symplectic involution is the disjoint union of an elliptic curve E and 8 smooth rational curves $E_1, \dots, E_8$ for Example 2. Therefore, the same holds for the anti-symplectic involution σ_R. Let $f : \tilde{S} \to \mathbb{P}^1$ be the elliptic fibration defined by the linear system $|E|$. f descends to an elliptic fibration of the quotient rational surface $\tilde{S}/\sigma_R$. We denote its minimal fibration by $f_R : R \to \mathbb{P}^1$. The rational surface R is obtained from $\tilde{S}/\sigma_R$ by blowing down an exceptional curve of the first kind 8 times. For Example 2, it is easily checked that the image of $\sum_{i=1}^{8} E_i$ is a singular fiber of type I_8 of f_R and that f_R has 4 sections. The same holds for (S, σ) as a deformation of Example 2. Hence, as is claimed in [3, §5], the configuration of the elliptic curves E and 20 rational curves is the same as Example 2, and (S, σ) is obtained in the way of Example 2. The second assertion follows from the Torelli type theorem and [3, (1.1)], the uniqueness of cohomologically trivial involution. □

Remark 21. The fixed locus of the anti-symplectic involution σ_R is the disjoint union of 8 smooth rational curves $E_1, \ldots, E_8$ for numerically trivial involutions of Kummer type. Our (main) Theorem 3 can be also proved using certain elliptic fibrations containing $E_1, \ldots, E_8$ in their fibers though the existence of such fibrations is not straightforward as above and they are not unique. Furthermore, Theorem 20 can be proved using periods also. These alternative proofs will be discussed elsewhere.

5. Appendix : Rational curves on an Enriques surface of Proposition 2

Let B, τ, $\Gamma_{\bar{0}}$ and $\Gamma_{\bar{1}}$ be as in Section 2. The dual graph of the 8 smooth rational curves in $\Gamma_{\bar{0}}$ is a cube:

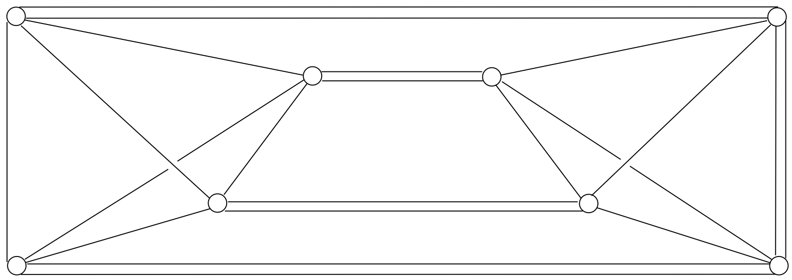


(6)

The automorphism τ sends each vertex of the cube $\Gamma_{\bar{0}}$ to its antipodal. The same holds for $\Gamma_{\bar{1}}$. The following is easily verified:

(†) for every curve m in $\Gamma_{\bar{0}}$ (resp. $\Gamma_{\bar{1}}$), there exists an antipodal pair of vertices n and n' in $\Gamma_{\bar{1}}$ (resp. $\Gamma_{\bar{0}}$) such that $(m.n) = (m.n') = 1$ and that m is disjoint from other curves in $\Gamma_{\bar{1}}$ (resp. $\Gamma_{\bar{0}}$).

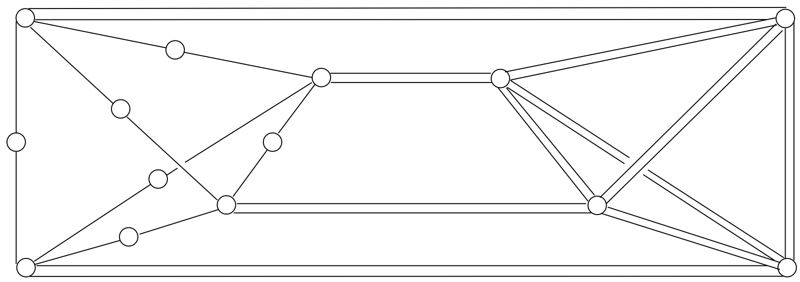
Therefore, the quotient graph $(\Gamma_{\bar{1}} \cup \Gamma_{\bar{0}})/\tau$ is as follows:



$\Gamma_{\bar{1}}/\tau$ $\Gamma_{\bar{0}}/\tau$

The Kummer surface $Km(E' \times E'')$ is the double cover of B with branch the union of the 8 curves in $\Gamma_{\bar{1}}$. The union has 12 nodes corresponding to the 12 edges of $\Gamma_{\bar{1}}$. The pull-backs of the curves in $\Gamma_{\bar{0}}$ are smooth rational curves on $Km(E' \times E'')$ by (†). Hence $Km(E' \times E'')$ has 28 smooth rational curves, 12 of which come from the nodes of the double cover and the rest from $\Gamma_{\bar{0}} \cup \Gamma_{\bar{1}}$. Since the involution τ lifts to ε_{odd} of Proposition 2, we have

Proposition 22. *On the Enriques surface $Km(E' \times E'')/\varepsilon_{odd}$ of Proposition* 2*, there are* 14 *smooth rational curves whose dual graph is as follows:*



(7)

The proposition, together with [3, (4.7)], shows the 'only if' part of [2, Theorem (1.7), (i)] in the case of Kummer type.

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