THE E_6 STATE SUM INVARIANT OF LENS SPACES

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Abstract. In this paper, we calculate the values of the E_6 state sum invariant for the lens spaces $L(p, q)$. In particular, we show that the values of the invariant are determined by $p \mod 12$ and $q \mod (p, 12)$. As a corollary, we show that the E_6 state sum is a homotopy invariant for the oriented lens spaces.

1. Introduction

In [5], Turaev and Viro constructed a state sum invariant of 3-manifolds based on their triangulations, by using the $6j$-symbols of representations of the quantum group $U_q(\mathfrak{sl}_2)$. Further, Ocneanu [2] generalized the construction to the case of other types of $6j$-symbols, say, the $6j$-symbols of subfactors. *The E_6 state sum invariant* is the state sum invariant constructed from the $6j$-symbols of the E_6 subfactor, which we denote by Z. Suzuki and Wakui [4] calculated the E_6 state sum invariant for some of the lens spaces, where they used the representation of the mapping class group of a torus $SL(2, \mathbb{Z})$.

In this paper, we calculate the E_6 state sum invariant for all of the lens spaces, as follows. For integers m, n, we denote by (m, n) the great common devisor of m and n. We put $\zeta = \exp(\pi\sqrt{-1}/12)$ and $[n] = (\zeta^n - \zeta^{-n})/(\zeta - \zeta^{-1})$ for an integer n, noting that

$$[12 - n] = [n], \quad [n + 12] = -[n],$$
$$[2] = (1 + \sqrt{3})/\sqrt{2}, \quad [3] = 1 + \sqrt{3}, \quad [4] = (3 + \sqrt{3})/\sqrt{2}.$$

Theorem 1.1. *For coprime integers p and q, the E_6 state sum invariant of the lens space $L(p,q)$ is given as*

$$Z(L(p,q)) = \begin{cases} |[p]| & \text{if } (p,12)=1, \\ [4][3]/[2] & \text{if } (p,12)=2,6, \\ \zeta^{\pm 3}[4] & \text{if } (p,12)=3 \text{ and } q\equiv \pm 1 \mod 3, \\ 2\zeta^{\pm 2}[3] & \text{if } (p,12)=4 \text{ and } q\equiv \pm 1 \mod 4, \\ 2[4][3]/[2] & \text{if } 12|p \text{ and } q\equiv \pm 1 \mod 12, \\ 0 & \text{if } 12|p \text{ and } q\equiv \pm 5 \mod 12. \end{cases} \tag{1.1}$$

In particular, the value of $Z(L(p,q))$ is determined by $p \mod 12$ and $q \mod (p,12)$.

We note that we normalize the invariant so that $Z(S^3)=1$. Thus, our Z is equal to wZ in [4], where we put $w = 2+[3]^2 = 6+2\sqrt{3}$.

Corollary 1.2. *If there exists an orientation-preserving homotopy equivalence between the two lens spaces $L(p,q)$ and $L(p',q')$, then $Z(L(p,q)) = Z(L(p',q'))$.*

We note that from Theorem 1.1 the E_6 state sum invariant distinguishes $L(p,q)$ from $L(p,-q)$ if and only if $(p,12)=3$ or 4. This is a generalization of [4, Corollary 4.3].

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2. The calculation of the E_6 state sum invariant

In this section, we briefly review the calculation of the E_6 state sum invariant for the lens spaces. Suzuki and Wakui [4] defined the representation $\rho : SL(2,\mathbb{Z}) \longrightarrow GL_{10}(\mathbb{C})$ by

$\rho(S)$

$$= \frac{1}{w}\begin{pmatrix} 1 & [3] & 1 & [2]^2 & [3] & [3] & \frac{[4][3]}{[2]} & [3] & [3] & [2]^2 \\ [3] & \frac{[4][3]}{[2]}\sqrt{-1} & -[3] & -[3] & 0 & -\frac{[4][3]}{[2]}\sqrt{-1} & 0 & [3] & -[3] & [3] \\ 1 & -[3] & 1 & [2]^2 & -[3] & -[3] & -\frac{[4][3]}{[2]} & [3] & [3] & [2]^2 \\ [2]^2 & -[3] & [2]^2 & 1 & -[3] & -[3] & \frac{[4][3]}{[2]} & -[3] & -[3] & 1 \\ [3] & 0 & -[3] & -[3] & 0 & 0 & 0 & -2[3] & 2[3] & [3] \\ [3] & -\frac{[4][3]}{[2]}\sqrt{-1} & -[3] & -[3] & 0 & \frac{[4][3]}{[2]}\sqrt{-1} & 0 & [3] & -[3] & [3] \\ \frac{[4][3]}{[2]} & 0 & -\frac{[4][3]}{[2]} & \frac{[4][3]}{[2]} & 0 & 0 & 0 & 0 & 0 & -\frac{[4][3]}{[2]} \\ [3] & [3] & [3] & -[3] & -2[3] & [3] & 0 & [3] & [3] & -[3] \\ [3] & -[3] & [3] & -[3] & 2[3] & -[3] & 0 & [3] & [3] & -[3] \\ [2]^2 & [3] & [2]^2 & 1 & [3] & [3] & -\frac{[4][3]}{[2]} & -[3] & -[3] & 1 \end{pmatrix},$$

$$\rho(T) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\zeta^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\zeta^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \zeta^8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \zeta^{-4} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix},$$

where $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ are generators of $SL(2, \mathbb{Z})$.

Let p, q be coprime integers. We choose a continued fraction expansion of p/q:

$$\frac{p}{q} = c_1 - \cfrac{1}{c_2 - \cfrac{}{\ddots - \cfrac{1}{c_n}}}.$$

From this, we have that

$$\begin{pmatrix} a & p \\ b & q \end{pmatrix} = T^{c_1} S T^{c_2} \cdots S T^{c_{n-1}} S T^{c_n} \in SL(2, \mathbb{Z})$$

for some integers a, b. From [4, Lemma 4.2], one can derive that the E_6 state sum invariant of lens spaces are given as

$$Z(L(p,q)) = w\, {}^t\mathbf{e}\rho(S T^{c_1} S T^{c_2} \cdots S T^{c_{n-1}} S T^{c_n} S)\mathbf{e} \tag{2.1}$$
$$= w\, {}^t\mathbf{e}\rho(\begin{pmatrix} -q & b \\ p & -a \end{pmatrix})\mathbf{e},$$

where we put $\mathbf{e} = {}^t(1,0,0,0,0,0,0,0,0,0)$. We note that this value does not depend on the choice of continued fraction expansions of p/q (and that of a and b).

3. Proof of the theorem

In this section, we prove Theorem 1.1 and Corollary 1.2. In order to show Theorem 1.1, we show Proposition 3.4, which says that the values of $Z(L(p,q))$ have period 12 for p and q . In order to show Proposition 3.4, we show Lemmas 3.1 and 3.3, as follows.

Lemma 3.1. *Let p, q, p', q' be integers satisfying $(p,q) = 1$, $(p',q') = 1$ and $p \equiv p', q \equiv q' \mod 12$. Then, there exist integers a, b, a', b' such that*

$$aq - bp = 1,\ a'q' - b'p' = 1 \quad \textit{and} \quad a \equiv a',\ b \equiv b' \mod 12.$$

Proof. We put integers a and b satisfying $aq - bp = 1$. Further, we put

$$\begin{pmatrix} a' & p' \\ b' & q' \end{pmatrix} = \begin{pmatrix} a & p \\ b & q \end{pmatrix} + 12 \begin{pmatrix} x & z \\ y & w \end{pmatrix},$$

noting that, by assumption, z and w are determined uniquely. It is sufficient to show that there exist integers x and y satisfying $a'q' - b'p' = 1$. The determinant of the right-hand side of the above formula is equal to

$$\begin{aligned} (a+12x)(q+12w) &- (p+12z)(b+12y) \\ &= a(q+12w) + 12xq' - (p+12z)b - 12p'y \\ &= 1 + 12(aw + xq' - zb - p'y). \end{aligned} \tag{3.1}$$

Since $(p',q') = 1$, there exists integers x and y satisfying

$$p'y - q'x = aw - zb.$$

Then, the last term of (3.1) is equal to 1. Thus, we have $a'q' - b'p' = 1$, as required. □

We denote by I_n the n-by-n identity matrix. We put

$$\Gamma = \{P \in SL(2,\mathbb{Z})\ |P \equiv I_2 \mod 12\}.$$

Lemma 3.2. *Γ is a normal closure of the set of the following 19 matrices.*

$$P_0 = \begin{pmatrix} 1 & 12 \\ 0 & 1 \end{pmatrix}, \quad P_1 = \begin{pmatrix} 1 & 0 \\ 12 & 1 \end{pmatrix}, \quad P_2 = \begin{pmatrix} -143 & 12 \\ -12 & 1 \end{pmatrix},$$

$$P_3 = \begin{pmatrix} -155 & 84 \\ -24 & 13 \end{pmatrix}, \quad P_4 = \begin{pmatrix} -191 & 156 \\ -60 & 49 \end{pmatrix}, \quad P_5 = \begin{pmatrix} -443 & 120 \\ -48 & 13 \end{pmatrix},$$

$$P_6 = \begin{pmatrix} -467 & 360 \\ -48 & 37 \end{pmatrix}, \quad P_7 = \begin{pmatrix} -299 & 108 \\ -36 & 13 \end{pmatrix}, \quad P_8 = \begin{pmatrix} -311 & 216 \\ -36 & 25 \end{pmatrix},$$

$$P_9 = \begin{pmatrix} 937 & -396 \\ 168 & -71 \end{pmatrix}, \quad P_{10} = \begin{pmatrix} 157 & -36 \\ 48 & -11 \end{pmatrix}, \quad P_{11} = \begin{pmatrix} 157 & -48 \\ 36 & -11 \end{pmatrix},$$

$$P_{12} = \begin{pmatrix} 205 & -84 \\ 144 & -59 \end{pmatrix}, \quad P_{13} = \begin{pmatrix} 157 & -72 \\ 24 & -11 \end{pmatrix}, \quad P_{14} = \begin{pmatrix} 229 & -132 \\ 144 & -83 \end{pmatrix},$$

$$P_{15} = \begin{pmatrix} 169 & -108 \\ 36 & -23 \end{pmatrix}, \quad P_{16} = \begin{pmatrix} 181 & -132 \\ 48 & -35 \end{pmatrix}, \quad P_{17} = \begin{pmatrix} 589 & -108 \\ 60 & -11 \end{pmatrix},$$

$$P_{18} = \begin{pmatrix} 649 & -384 \\ 120 & -71 \end{pmatrix}.$$

Proof. The GAP package Congruence [1] shows that Γ is generated by 97 matrices, as follows.

```
gap> LoadPackage("congruence");
true
gap> G:=PrincipalCongruenceSubgroup(12);
GeneratorsOfGroup(G); <principal congruence subgroup of level 12 in SL_2(Z)>
gap> GeneratorsOfGroup(G);
#I  Using the Congruence package for GeneratorsOfGroup ...
[ [ [ [ 1, 12 ], [ 0, 1 ] ], [ [ -143, 12 ], [ -12, 1 ] ],
  [ [ 589, -108 ], [ 60, -11 ] ], [ [ 157, -36 ], [ 48, -11 ] ],
  [ [ -443, 120 ], [ -48, 13 ] ], [ [ 157, -48 ], [ 36, -11 ] ],
  [ [ -299, 108 ], [ -36, 13 ] ], [ [ 205, -84 ], [ 144, -59 ] ],
  [ [ 937, -396 ], [ 168, -71 ] ], [ [ 157, -72 ], [ 24, -11 ] ],
  [ [ -155, 84 ], [ -24, 13 ] ], [ [ 229, -132 ], [ 144, -83 ] ],
  [ [ 649, -384 ], [ 120, -71 ] ], [ [ 169, -108 ], [ 36, -23 ] ],
  [ [ -311, 216 ], [ -36, 25 ] ], [ [ 181, -132 ], [ 48, -35 ] ],
  [ [ -467, 360 ], [ -48, 37 ] ], [ [ -191, 156 ], [ -60, 49 ] ],
  [ [ 13, -12 ], [ 12, -11 ] ], [ [ 649, -768 ], [ 60, -71 ] ],
  [ [ 205, -252 ], [ 48, -59 ] ], [ [ -491, 624 ], [ -48, 61 ] ],
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[ [ 193, -252 ], [ 36, -47 ] ], [ [ -335, 456 ], [ -36, 49 ] ],
[ [ 349, -492 ], [ 144, -203 ] ], [ [ 181, -264 ], [ 24, -35 ] ],
[ [ -179, 276 ], [ -24, 37 ] ], [ [ 373, -588 ], [ 144, -227 ] ],
[ [ 205, -336 ], [ 36, -59 ] ], [ [ -347, 588 ], [ -36, 61 ] ],
[ [ 229, -396 ], [ 48, -83 ] ], [ [ -515, 912 ], [ -48, 85 ] ],
[ [ -251, 456 ], [ -60, 109 ] ], [ [ 25, -48 ], [ 12, -23 ] ],
[ [ 709, -1548 ], [ 60, -131 ] ], [ [ 253, -564 ], [ 48, -107 ] ],
[ [ -539, 1224 ], [ -48, 109 ] ], [ [ 229, -528 ], [ 36, -83 ] ],
[ [ -371, 876 ], [ -36, 85 ] ], [ [ 493, -1188 ], [ 144, -347 ] ],
[ [ 205, -504 ], [ 24, -59 ] ], [ [ -203, 516 ], [ -24, 61 ] ],
[ [ 517, -1332 ], [ 144, -371 ] ], [ [ 241, -636 ], [ 36, -95 ] ],
[ [ -383, 1032 ], [ -36, 97 ] ], [ [ 277, -756 ], [ 48, -131 ] ],
[ [ -563, 1560 ], [ -48, 133 ] ], [ [ -311, 876 ], [ -60, 169 ] ],
[ [ 37, -108 ], [ 12, -35 ] ], [ [ 301, -972 ], [ 48, -155 ] ],
[ [ 265, -876 ], [ 36, -119 ] ], [ [ -407, 1368 ], [ -36, 121 ] ],
[ [ 637, -2172 ], [ 144, -491 ] ], [ [ 229, -792 ], [ 24, -83 ] ],
[ [ -227, 804 ], [ -24, 85 ] ], [ [ 661, -2364 ], [ 144, -515 ] ],
[ [ 277, -1008 ], [ 36, -131 ] ], [ [ -419, 1548 ], [ -36, 133 ] ],
[ [ 325, -1212 ], [ 48, -179 ] ], [ [ -371, 1416 ], [ -60, 229 ] ],
[ [ 49, -192 ], [ 12, -47 ] ], [ [ 349, -1476 ], [ 48, -203 ] ],
[ [ 301, -1296 ], [ 36, -155 ] ], [ [ 781, -3444 ], [ 144, -635 ] ],
[ [ 253, -1128 ], [ 24, -107 ] ], [ [ -251, 1140 ], [ -24, 109 ] ],
[ [ 805, -3684 ], [ 144, -659 ] ], [ [ 313, -1452 ], [ 36, -167 ] ],
[ [ 373, -1764 ], [ 48, -227 ] ], [ [ -431, 2076 ], [ -60, 289 ] ],
[ [ 61, -300 ], [ 12, -59 ] ], [ [ 397, -2076 ], [ 48, -251 ] ],
[ [ 337, -1788 ], [ 36, -191 ] ], [ [ 277, -1512 ], [ 24, -131 ] ],
[ [ -275, 1524 ], [ -24, 133 ] ], [ [ 349, -1968 ], [ 36, -203 ] ],
[ [ 421, -2412 ], [ 48, -275 ] ], [ [ -491, 2856 ], [ -60, 349 ] ],
[ [ 73, -432 ], [ 12, -71 ] ], [ [ 445, -2772 ], [ 48, -299 ] ],
[ [ 373, -2352 ], [ 36, -227 ] ], [ [ 385, -2556 ], [ 36, -239 ] ],
[ [ 469, -3156 ], [ 48, -323 ] ], [ [ -551, 3756 ], [ -60, 409 ] ],
[ [ 85, -588 ], [ 12, -83 ] ], [ [ 493, -3564 ], [ 48, -347 ] ],
[ [ 409, -2988 ], [ 36, -263 ] ], [ [ 421, -3216 ], [ 36, -275 ] ],
[ [ 517, -3996 ], [ 48, -371 ] ], [ [ -611, 4776 ], [ -60, 469 ] ],
[ [ 97, -768 ], [ 12, -95 ] ], [ [ 541, -4452 ], [ 48, -395 ] ],
[ [ 565, -4932 ], [ 48, -419 ] ], [ [ -671, 5916 ], [ -60, 529 ] ],
[ [ 109, -972 ], [ 12, -107 ] ], [ [ 121, -1200 ], [ 12, -119 ] ],
[ [ 133, -1452 ], [ 12, -131 ] ] ]
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We can verify that the set of the above 97 matrices is equal to

$$\{P_0\} \cup \{T^j P_1 T^{-j} | j = 1, 2, \ldots, 11\} \cup \bigcup_{n=2}^{18} \{T^j P_n T^{-j} | j = 0, 1, \ldots, m_n\},$$

where we put

$$m_2 = 0,\ m_3 = 5,\ m_4 = 8,\ m_5 = 2,\ m_6 = 2,\ m_7 = 3,$$
$$m_8 = 3,\ m_9 = 0,\ m_{10} = 8,\ m_{11} = 7,\ m_{12} = 4,\ m_{13} = 5,$$
$$m_{14} = 4,\ m_{15} = 7,\ m_{16} = 8,\ m_{17} = 2,\ m_{18} = 0,$$

completing the proof. □

Lemma 3.3. $\Gamma \subset \ker \rho$, *that is,* $\rho(P) = I_{10}$ *for any* $P \in \Gamma$.

Proof. From Lemma 3.2, it is sufficient to show that $\rho(P_n) = I_{10}$ for $n = 0, 1, \ldots, 18$. We can verify that these matrices is presented as the products of S and T, as follows.

$P_0 = T^{12}$, $P_1 = S^3T^{-12}S$,
$P_2 = S^2T^{12}ST^{12}S$, $P_3 = S^2T^7ST^2ST^7ST^2S$,
$P_4 = S^2T^3ST^{-5}ST^2ST^{-4}STS$, $P_5 = T^9ST^{-4}ST^3ST^4S$,
$P_6 = T^{10}ST^4ST^3ST^{-3}STS$, $P_7 = T^8ST^{-3}ST^4ST^3S$,
$P_8 = T^9ST^3ST^4ST^{-2}STS$, $P_9 = T^5ST^{-2}ST^{-4}ST^{-4}ST^{-3}ST^2S$,
$P_{10} = T^3ST^{-4}ST^{-3}ST^4S$, $P_{11} = T^4ST^{-3}ST^{-4}ST^3S$,
$P_{12} = TST^{-2}ST^3ST^4ST^{-2}ST^2S$, $P_{13} = T^6ST^{-2}ST^{-6}ST^2S$,
$P_{14} = TST^{-2}ST^{-3}ST^4ST^4ST^2S$, $P_{15} = S^2T^5ST^3ST^{-3}ST^2ST^2S$,
$P_{16} = T^4ST^4ST^{-3}ST^{-3}STS$, $P_{17} = S^2T^{10}ST^5ST^{-2}ST^5S$,
$P_{18} = T^5ST^{-2}ST^2ST^{-4}ST^3ST^2S$.

By using these formulae, we can verify that ρ takes to each of the matrices to I_{10}, completing the proof of the lemma. □

Proposition 3.4. *Let* p, q, p', q' *be integers satisfying* $(p, q) = 1$, $(p', q') = 1$ *and* $p \equiv p', q \equiv q' \mod 12$. *Then,* $Z(L(p, q)) = Z(L(p', q'))$.

Proof. From Lemma 3.1, there exist matrices

$$A = \begin{pmatrix} -q & b \\ p & -a \end{pmatrix},\ A' = \begin{pmatrix} -q' & b' \\ p' & -a' \end{pmatrix} \in SL(2, \mathbb{Z})$$

such that $A \equiv A' \mod 12$. We put $P = I_2 + A^{-1}(A' - A)$. By definition, $P \in \Gamma$ and $A' = AP$. Thus, by (2.1) and Lemma 3.3, we have

$$Z(L(p', q')) = w^t \mathbf{e} \rho(A') \mathbf{e} = w^t \mathbf{e} \rho(A) \rho(P) \mathbf{e} = w^t \mathbf{e} \rho(A) \mathbf{e} = Z(L(p, q)),$$

completing the proof. □

Proof of Theorem 1.1. By Proposition 3.4, the left-hand side of (1.1) has period 12 for p and q. On the other hand, the right-hand side of (1.1) also has period 12 for p and q. By [4, Appendix D], we can verify that (1.1) holds for coprime integers p and q with $1 \leq p \leq 12$, $q < p$. Therefore, (1.1) holds for any coprime integers p and q. □

Proof of Corollary 1.2. It is known, see [3, Remark 3], that there exists an orientation-preserving homotopy equivalence between the two lens spaces $L(p, q)$ and $L(p', q')$ if and only if $p = p'$ and $q \equiv n^2 q' \mod p$ for some integer n.

When $(p, 12) \neq 3, 4, 12$, by Theorem 1.1 the value $Z(L(p, q))$ does not depend on q. Thus, $Z(L(p, q)) = Z(L(p', q'))$.

When $(p, 12) = k$ with $k \in \{3, 4, 12\}$, we have $q \equiv n^2 q' \mod k$ for some integer n. Thus, it is enough to show that $q \equiv q' \mod k$.

(a) When $k \in \{3, 4\}$, we have $\{n^2 | n \in \mathbb{Z}/k\mathbb{Z}\} = \{0, 1\}$. Since $(p, q) = 1$, it implies that $q \equiv q' \mod k$.
(b) When $k = 12$, we have $\{n^2 | n \in \mathbb{Z}/12\mathbb{Z}\} = \{0, 1, 4, 9\}$. Again, since $(p, q) = 1$, it implies that $q \equiv q' \mod k$.

Thus, by Theorem 1.1, $Z(L(p, q)) = Z(L(p', q'))$.

□

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