J. reine angew. Math. **690** (2014), 207–216
DOI 10.1515/crelle-2012-0065

Journal für die reine und angewandte Mathematik

Property A and the operator norm localization property for discrete metric spaces

By *Hiroki Sako* at Kyoto

Abstract. We study property A defined by G. Yu and the operator norm localization property defined by X. Chen, R. Tessera, X. Wang, and G. Yu. These are coarse geometric properties for metric spaces, which have applications to operator K-theory. It is proved that the two properties are equivalent for discrete metric spaces with bounded geometry.

1. Introduction

Coarse geometry is the study of large scale uniform structure on a space, which is related to operator K-theory. In this paper we investigate the following coarse geometric properties on metric spaces: property A and the operator norm localization property. It is proved that these two properties are equivalent for metric spaces with bounded geometry.

Property A was introduced by G. Yu in [18, Definition 2.1]. A discrete metric space is said to have property A if the space satisfies a condition regarding amenability. Yu [18] proved that the property guarantees the coarse Baum–Connes conjecture for metric space. The most interesting case is when the metric space is given by a discrete group. The Novikov higher signature conjecture holds for every discrete group Γ with the property [10, Theorem 1.1]. Property A for a discrete group can be characterized by exactness of the reduced group C^*-algebra. This follows from theorems by Anantharaman-Delaroche and Renault [1], Higson and Roe [11], and Ozawa [13].

X. Chen, R. Tessera, X. Wang, and G. Yu introduced *the operator norm localization property* in [4, Definition 2.2 and Definition 2.3]. They applied this notion to prove that the coarse geometric Novikov conjecture holds for certain sequences of expanders [4, Theorem 7.1].

It has already been known that property A and the operator norm localization property (ONL) have the following common features.

- Finiteness of asymptotic dimension implies these properties. We refer to [11] (property A) and [4] (ONL).
- If two groups have property A (resp. ONL), then an amalgamated free product has property A (resp. ONL). We refer to [14] (property A) and [4] (ONL).

The author is a Research Fellow of the Japan Society for the Promotion of Science.

- If a group Γ is a hyperbolic relative to subgroups with property A (resp. ONL), then Γ has property A (resp. ONL). We refer to [14] (property A) and [5] (ONL).
- Every countable subgroup of the general linear group over a field has these properties. We refer to [8] (property A) and [9] (ONL).
- A sequence of expander graphs is an counterexample for these properties. We refer to [4] (ONL).

A C*-algebra $C_u^*(X)$, called uniform Roe algebra, is associated to a metric space X with bounded geometry. Skandalis, Tu, and Yu [16, Theorem 5.3] proved that X has property A if and only if the uniform Roe algebra is nuclear (see also Roe [15, Proposition 11.41] and Brown–Ozawa [3, Theorem 5.5.7]). Nuclearity can be characterized by a finite-dimensional approximation property (Choi–Effros [6], Kirchberg [12]). To obtain the equivalence between property A and the operator norm localization property, we manipulate approximations by completely positive maps on the uniform Roe algebra.

In the last section, we make comments on a work by Brodzki, Niblo, Špakula, Willett, and Wright. By the main theorem in this paper and their result, it turns out that two properties (MSP and ULA$_\mu$) are equivalent to property A.

2. Preliminaries

2.1. Metric space with bounded geometry and uniform Roe algebra. We fix some notations on a metric space (X, d) and recall the definition of the uniform Roe algebra $C_u^*(X)$. For $S > 0$ and $x \in X$, we denote by $N(x, S)$ the closed ball $\{y \in X \mid d(x, y) \le S\}$.

Definition 2.1. We say that (X, d) has *bounded geometry* if $\sup_{x \in X} |N(x, S)| < \infty$ for all $S > 0$.

We note that a metric space with bounded geometry is a discrete space. For a bounded linear operator a on the Hilbert space $\ell_2(X)$, write $a_{y,z}$ for the matrix coefficient $\langle a\delta_z, \delta_y \rangle \in \mathbb{C}$. The operator a has the expression $a = [a_{y,z}]_{y,z \in X}$. We define the *propagation* of a by

$$\sup\{d(y, z) \mid y, z \in X,\ a_{y,z} \neq 0\}.$$

Let E_R be the set of all the operators on $\ell_2(X)$ whose propagations are at most R. The union $\bigcup E_R$ is a $*$-subalgebra of $\mathbb{B}(\ell_2(X))$. See the book [15] for details.

Definition 2.2. The C*-algebra defined by the closure

$$C_u^*(X) = \overline{\bigcup_{R>0} E_R}^{\text{norm}}$$

is called the *uniform Roe algebra* of X.

2.2. Property A. The definition of property A was given by G. Yu.

Definition 2.3 ([18, Definition 2.1]). A discrete metric space (X, d) is said to have *property A* if for every $R > 0$ and $\epsilon > 0$, there exist $S > 0$ and finite subsets $A_x \subseteq X \times \mathbb{N}$

labeled by $x \in X$ such that

- if $d(y, z) \leq R$, then $|A_y \triangle A_z| < \epsilon |A_y \cap A_z|$, where $A_y \triangle A_z$ stands for the symmetric difference of A_y and A_z,
- $A_x \subseteq N(x, S) \times \mathbb{N}$.

Instead of the definition, we use the conditions (ii) and (iii) in the next proposition.

Proposition 2.4 ([17, Proposition 3.2]). *For a metric space X with bounded geometry, the following conditions are equivalent:*

(i) *The space X has property A.*

(ii) *For every $\epsilon > 0$ and $R > 0$, there exist $S > 0$ and unit vectors $\{\xi_x\}_{x \in X} \subseteq \ell_2(X)$ such that*

- if $d(y, z) \leq R$, then $\|\xi_y - \xi_z\| < \epsilon$,
- $\mathrm{supp}(\xi_x) \subseteq N(x, S)$ for every $x \in X$.

(iii) *For every $\epsilon > 0$ and $R > 0$, there exist $S > 0$ and a positive definite kernel $k\colon X \times X \to \mathbb{C}$ such that*

- $k(x, x) = 1$ for every $x \in X$,
- if $d(y, z) \leq R$, then $|1 - k(y, z)| < \epsilon$,
- if $d(y, z) > S$, then $k(y, z) = 0$.

A function $k\colon X \times X \to \mathbb{C}$ is said to be positive definite if for every $x(1), \ldots, x(n) \in X$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$, the inequality $\sum_{i,j=1}^{n} \overline{\lambda_i} \lambda_j k(x(i), x(j)) \geq 0$ holds true.

2.3. The operator norm localization property. X. Chen, R. Tessera, X. Wang, and G. Yu defined the operator norm localization property in [4]. We call the property "ONL" in this paper. The original definition is given for a general metric space X. Let ν be a positive locally finite Borel measure on X and let $\mathcal{H}$ be a separable infinite-dimensional Hilbert space. For an operator b on the Hilbert space $L^2((X, \nu), \mathcal{H}) = L^2(X, \nu) \otimes \mathcal{H}$, the propagation of b is also defined. See [4, Section 2] for details.

Definition 2.5 ([4, Definition 2.2]). Let ν be a positive locally finite Borel measure on a metric space X. Let f be a function $f\colon \mathbb{N} \to \mathbb{N}$. We say that (X, ν) has *ONL relative to* f with constant $0 < c \leq 1$ if for every $R \in \mathbb{N}$ and every $a \in \mathbb{B}(L^2((X, \nu), \mathcal{H}))$ whose propagation is at most R, there exists a non-zero vector $\zeta \in L^2((X, \nu), \mathcal{H})$ satisfying $\mathrm{diam}(\mathrm{supp}(\zeta)) \leq f(R)$ and $c\|a\|\|\zeta\| \leq \|a\zeta\|$.

Definition 2.6 ([4, Definition 2.3]). A metric space X is said to have *ONL* if there exists a constant $0 < c \leq 1$ and a function $f\colon \mathbb{N} \to \mathbb{N}$ such that for every positive locally finite Borel measure ν on X, (X, ν) has ONL relative to f with constant c.

For the rest of this paper, we will concentrate on a metric space X with bounded geometry. By Proposition 3.1, we have only to consider the case that ν is the counting measure on X. An operator a on the Hilbert space $\ell_2(X, \mathcal{H}) = \ell_2(X) \otimes \mathcal{H}$ has the expression

$a = [a_{y,z}]_{y,z \in X}$, where $a_{y,z}$ is an operator on $\mathcal{H}$. The propagation of b is equal to the supremum $\sup\{d(y,z) \mid y, z \in X,\ a_{y,z} \neq 0\}$. We denote by $E_R(\mathcal{H})$ the set of the operators on $\ell_2(X, \mathcal{H})$ whose propagations are at most R.

3. Characterizations of ONL

We rephrase the definition of ONL for metric spaces with bounded geometry. We need a few more notations to state the next proposition. Denote by B_S the C*-algebra

$$\prod_{x \in X} \mathbb{B}(\ell_2(N(x,S))),$$

which is isomorphic to a product of matrix algebras. An element $b \in B_S$ is a family of matrices $([b^{(x)}_{y,z}]_{y,z \in N(x,S)})_{x \in X}$ labeled by X. For $S > 0$, define a linear map $\psi_S \colon C^*_u(X) \to B_S$ by $\psi_S(a) = ([a_{y,z}]_{y,z \in N(x,S)})_{x \in X}$.

Proposition 3.1. *Let X be a metric space with bounded geometry. The following properties on X are equivalent:*

(i) *The space X has ONL.*

(ii) *There exists $0 < c \leq 1$ such that for every $R > 0$, there exists $S > 0$ satisfying condition* (α):

  (α) *for every operator $a \in E_R(\mathcal{H})$, there exists a non-zero vector $\zeta \in \ell_2(X, \mathcal{H})$ such that* $\operatorname{diam}(\operatorname{supp}(\zeta)) \leq S$ *and* $c\|a\|\|\zeta\| \leq \|a\zeta\|$.

(iii) *For every* $0 < c < 1$ *and* $R > 0$, *there exists* $S > 0$ *satisfying* (α).

(iv) *For every* $0 < c < 1$ *and* $R > 0$, *there exists* $S > 0$ *satisfying condition* (β):

  (β) *for every operator $a \in E_R$, there exists a non-zero vector $\xi \in \ell_2(X)$ such that* $\operatorname{diam}(\operatorname{supp}(\xi)) \leq S$ *and* $c\|a\|\|\xi\| \leq \|a\xi\|$.

(v) *For every $\epsilon > 0$ and $R > 0$, there exists $S > R$ satisfying*

$$\left\|(\psi_S|_{E_R})^{-1} \colon \psi_S(E_R) \to E_R\right\| < 1 + \epsilon.$$

In the next section, we will make use of property (v).

Proof of (i) $\Rightarrow$ (ii). If X has ONL, then X satisfies ONL for the counting measure. This is nothing but property (ii). □

Proof of (ii) $\Rightarrow$ (i). Suppose that X has the property (ii) for a constant c. For an arbitrary $R \in \mathbb{N}$, there exists $S = f(R)$ which satisfies (α). We may choose S from $\mathbb{N}$. Observe that for every positive measure ν, the Hilbert space $L^2((X,\nu),\mathcal{H})$ can be identified with a closed subspace of $\ell_2(X,\mathcal{H}) = \ell_2(X) \otimes \mathcal{H}$. The inclusion map is

$$V \colon L^2((X,\nu),\mathcal{H}) \ni \eta \mapsto \Sigma_{x \in X}\, \nu(\{x\})^{1/2} \delta_x \otimes \eta(x) \in \ell_2(X) \otimes \mathcal{H}.$$

Let a be an arbitrary operator on $L^2((X,\nu),\mathcal{H})$ whose propagation is at most R. Then the propagation of VaV^* is at most R. By condition (α), there exists a non-zero vector $\zeta \in \ell_2(X,\mathcal{H})$

such that

$$\operatorname{diam}(\operatorname{supp}(\zeta)) \le S, \quad c\|VaV^*\|\|\zeta\| \le \|VaV^*\zeta\|.$$

These inequalities imply $\operatorname{diam}(\operatorname{supp}(V^*\zeta)) \le S$ and $c\|a\|\|V^*\zeta\| \le \|aV^*\zeta\|$. We conclude that X has ONL. □

The part "there exists $0 < c \le 1$" in (ii) can be replaced by "for every $0 < c < 1$" in (iii). Inspired by an idea of [4, Proposition 2.4], we give a complete proof.

Proof of (ii) ⇒ (iii). Assume that X has the property (ii) with respect to a constant $0 < c < 1$. For every R and $n \in \mathbb{N}$, there exists S satisfying condition (α) for propagation $2nR$. Let $a \in E_R(\mathcal{H})$ be an arbitrary operator of norm 1. Since the propagation of $(aa^*)^n$ is at most $2nR$, there exists a non-zero vector $\zeta \in \ell_2(X, \mathcal{H})$ such that

$$\operatorname{diam}(\operatorname{supp}(\zeta)) \le S \quad \text{and} \quad c\|(aa^*)^n\|\|\zeta\| \le \|(aa^*)^n\zeta\|.$$

Since the norm of $(aa^*)^n$ is 1, we have

$$c \le \prod_{j=0}^{n-1} \|(aa^*)^{j+1}\zeta\| / \|(aa^*)^j\zeta\|.$$

It follows that there exists $j = 0, 1, \dots, n-1$ such that $c^{1/n} \le \|(aa^*)^{j+1}\zeta\| / \|(aa^*)^j\zeta\|$. We have the inequality

$$c^{1/n}\|a\|\|a^*(aa^*)^j\zeta\| \le c^{1/n}\|(aa^*)^j\zeta\| \le \|a(a^*(aa^*)^j\zeta)\|.$$

The diameter of $\operatorname{supp}(a^*(aa^*)^j\zeta)$ is at most $(2n-1)R + S$. It follows that condition (α) holds for $c^{1/n}$, $R > 0$, and $(2n-1)R + S$. We can make the constant $0 < c^{1/n} < 1$ arbitrarily close to 1, by enlarging n. Hence property (iii) holds. □

The implication from (iii) to (ii) is trivial. We further rephrase the property. The equivalence between (iii) and (iv) means that the amplification by $\mathcal{H}$ is not necessary.

Proof of (iii) ⇒ (iv). Assume that $0 < c < 1$, $R > 0$, and $S > 0$ satisfy condition (α). Fix a unit vector $\eta \in \mathcal{H}$. Denote by e the rank one projection onto $\mathbb{C}\eta \subseteq \mathcal{H}$. Let a be an arbitrary operator on $\ell_2(X)$ whose propagation is at most R. Since the propagation of $a \otimes e$ is at most R, there exists a non-zero vector $\zeta \in \ell_2(X) \otimes \mathcal{H}$ such that $\operatorname{diam}(\operatorname{supp}(\zeta)) \le S$ and $c\|a \otimes e\|\|\zeta\| \le \|(a \otimes e)\zeta\|$. The vector $(1 \otimes e)\zeta$ is of the form $\xi \otimes \eta \in \ell_2(X) \otimes \mathcal{H}$. We have

$$\operatorname{diam}(\operatorname{supp}(\xi)) \le \operatorname{diam}(\operatorname{supp}(\zeta)) \le S$$

and

$$c\|a\|\|\xi\| \le c\|a \otimes e\|\|\zeta\| \le \|(a \otimes e)\zeta\| = \|a\xi\|.$$

It follows that $0 < c < 1$, $R > 0$, and $S > 0$ satisfy condition (β). We conclude that X satisfies the property (iv). □

Proof of (iv) ⇒ (iii). Assume that condition (β) holds for $0 < 1 - \epsilon/2 < 1$, R, and S. Let a be an operator on $\ell_2(X) \otimes \mathcal{H}$ whose propagation is at most R.

We claim that there exist isometries $V, W\colon \ell_2(X) \to \ell_2(X) \otimes \mathcal{H}$ satisfying

$$V\delta_x, W\delta_x \in \mathbb{C}\delta_x \otimes \mathcal{H} \quad \text{and} \quad (1-\epsilon/2)\|a\| \le \|W^*aV\| \le \|a\|.$$

Take unit vectors $\zeta_1, \zeta_2 \in \ell_2(X) \otimes \mathcal{H}$ such that $(1-\epsilon/2)\|a\| \le \langle a\zeta_1, \zeta_2\rangle$. The vectors ζ_1, ζ_2 can be written in the following forms:

$$\zeta_1 = \sum_{x\in X} f(x)\delta_x \otimes \eta_1(x), \quad \zeta_2 = \sum_{x\in X} g(x)\delta_x \otimes \eta_2(x),$$

where $\eta_1(x)$, $\eta_2(x)$ are unit vectors and $f(x), g(x) \in \mathbb{C}$. We define two isometries

$$V, W\colon \ell_2(X) \to \ell_2(X) \otimes \mathcal{H}$$

by $V\delta_x = \delta_x \otimes \eta_1(x)$ and $W\delta_x = \delta_x \otimes \eta_2(x)$. Since the vectors ζ_1, ζ_2 are respectively in the images of V, W, the operator norm of W^*aV satisfies $(1-\epsilon/2)\|a\| \le \|W^*aV\|$. Here we get the claim.

By condition (β), there exists a unit vector $\xi \in \ell_2(X)$ satisfying

$$\operatorname{diam}(\operatorname{supp}(\xi)) \le S, \quad (1-\epsilon/2)\|W^*aV\| \le \|W^*aV\xi\|.$$

Since the support of $V\xi$ is equal to that of ξ, we have $\operatorname{diam}(\operatorname{supp}(V\xi)) \le S$. We also obtain the inequality

$$\begin{aligned}(1-\epsilon)\|a\|\|V\xi\| = (1-\epsilon)\|a\| &\le (1-\epsilon/2)^2\|a\| \\ &\le (1-\epsilon/2)\|W^*aV\| \le \|W^*aV\xi\| \\ &\le \|aV\xi\|.\end{aligned}$$

We get condition (β) for $0 < 1-\epsilon < 1$, $R > 0$, and $S > 0$. It follows that X has the property (iii). □

Proof of (iv) $\Rightarrow$ (v). Assume that X has the property (iv). For arbitrary $\epsilon > 0$ and $R > 0$, there exists S which satisfies condition (β) for $c = (1+\epsilon)^{-1}$ and $R > 0$.

It follows that for every non-zero operator $a \in E_R$, there exists a unit vector $\xi \in \ell_2(X)$ satisfying $\operatorname{diam}(\operatorname{supp}(\xi)) \le S$ and $\|a\| \le (1+\epsilon)\|a\xi\|$. Since the propagation of a is at most R, the diameter of $\operatorname{supp}(a\xi)$ is included in the R-neighborhood of $\operatorname{supp}(\xi)$. Hence there exists a unit vector η such that $\|a\xi\| = \langle a\xi, \eta\rangle$ and that supports of ξ, η are included in a common closed ball $N(x, S+R)$. By the inequality

$$\|a\| \le (1+\epsilon)\langle a\xi, \eta\rangle \le (1+\epsilon)\|[a_{y,z}]_{y,z\in N(x,S+R)}\| \le (1+\epsilon)\|\psi_{S+R}(a)\|,$$

we get $\|(\psi_{S+R}|_{E_R})^{-1}\| \le 1+\epsilon$. □

Proof of (v) $\Rightarrow$ (iv). Suppose that the property (v) holds true. For every $0 < c < 1$ and $R > 0$, take S satisfying the inequality $\|(\psi_S|_{E_R})^{-1}\colon \psi_S(E_R) \to E_R\| < c^{-1}$. Then for every operator $a \in E_R$, there exists a closed ball $N(x, S)$ with radius S satisfying

$$c\|a\| \le \|[a_{y,z}]_{y,z\in N(x,S)}\|.$$

Take a unit vector $\xi \in \ell_2(N(x,S))$ such that $\|[a_{y,z}]_{y,z\in N(x,S)}\| = \|[a_{y,z}]_{y,z\in N(x,S)}\xi\|$. The vector ξ satisfies $\operatorname{diam}(\operatorname{supp}(\xi)) \le 2S$ and $c\|a\| \le \|a\xi\|$. It follows that condition (β) holds true for $0 < c < 1$, R, and $2S$. We conclude that X has the property (v). □

For the proof of Theorem 4.1, we recall the notions of completely positive map and completely bounded map.

- A self-adjoint closed subspace F of a unital C*-algebra B such that $1_B \in F$ is called an *operator system.*
- A linear map ϕ from F to a C*-algebra C is said to be *completely positive* if the map $\phi^{(n)} = \phi \otimes \mathrm{id}\colon F \otimes \mathbb{M}_n(\mathbb{C}) \to C \otimes \mathbb{M}_n(\mathbb{C})$ is positive for every n.

The subspaces $E_R \subseteq C_u^*(X)$ and $\psi_S(E_R) \subseteq B_S$ are examples of operator systems. The map $\psi_S\colon E_R \to B_S$ is an example of a completely positive map.

- A linear map $\theta\colon F \to C$ is said to be *completely bounded* if the increasing sequence $\{\|\theta^{(n)}\colon F \otimes \mathbb{M}_n(\mathbb{C}) \to C \otimes \mathbb{M}_n(\mathbb{C})\|\}$ is bounded. The number $\|\theta\|_{\mathrm{cb}} = \sup_{n\in\mathbb{N}} \|\theta^{(n)}\|$ is called the *completely bounded norm* of θ.

The norms $\|\theta\|$ and $\|\theta\|_{\mathrm{cb}}$ are not identical in general, but we have the following proposition.

Proposition 3.2. *For every S, R such that $0 < R < S$, the completely bounded norm $\|(\psi_S|_{E_R})^{-1}\colon \psi_S(E_R) \to E_R\|_{\mathrm{cb}}$ is equal to $\|(\psi_S|_{E_R})^{-1}\|$.*

Proof. It suffices to show that

$$\|((\psi_S|_{E_R})^{-1})^{(n)}\| \leq \|(\psi_S|_{E_R})^{-1}\| \quad \text{for every } n \in \mathbb{N}.$$

Take an arbitrary positive number K satisfying $K < \|((\psi_S|_{E_R})^{-1})^{(n)}\|$. There exists an operator $a \in E_R \otimes \mathbb{M}_n(\mathbb{C})$ satisfying

$$K < \|a\| \quad \text{and} \quad \|\psi_S^{(n)}(a)\| = 1.$$

We claim that there exist isometries $V, W\colon \ell_2(X) \to \ell_2(X) \otimes \mathbb{C}^n$ satisfying

$$V\delta_x, W\delta_x \in \mathbb{C}\delta_x \otimes \mathbb{C}^n \quad \text{and} \quad K < \|W^*aV\| \leq \|a\|.$$

Indeed, the proof of (iv) $\Rightarrow$ (iii) in Proposition 3.1 also works, $\mathcal{H}$ being replaced by $\mathbb{C}^n$. Observe that the propagation of W^*aV is at most R and that $\|\psi_S(W^*aV)\| \leq \|\psi_S^{(n)}(a)\| = 1$. It follows that $K < \|W^*aV\|/\|\psi_S(W^*aV)\| \leq \|(\psi_S|_{E_R})^{-1}\|$. We conclude that

$$\|((\psi_S|_{E_R})^{-1})^{(n)}\| \leq \|(\psi_S|_{E_R})^{-1}\|. \qquad \square$$

4. Main theorem

Theorem 4.1. *Let X be a metric space with bounded geometry. The space X has property A if and only if X has ONL.*

Before proving Theorem 4.1, we recall a lemma from the book [15]. This lemma allows us to bound operator norms by matrix coefficients.

Lemma 4.2 ([15, Lemma 4.27]). *Let X be a metric space with bounded geometry. For each $R > 0$, there exists a constant $\kappa = \kappa(X, R)$ such that $\|a\| \leq \kappa \sup_{y,z\in X} |a_{y,z}|$ holds for any $a \in E_R$.*

We need a completely positive perturbation of the completely bounded map

$$(\psi_S|_{E_R})^{-1}\colon \psi_S(E_R) \to E_R \hookrightarrow \mathbb{B}(\ell_2(X)).$$

The following is Corollary B.8 in [3]:

Let $F \subseteq B$ be an operator system of a C*-algebra B and let $\theta\colon F \to \mathbb{B}(\mathcal{H})$ be a unital self-adjoint map. Then, there exists a unital completely positive map $\phi\colon B \to \mathbb{B}(\mathcal{H})$ satisfying $\|\theta - \phi|_F\|_{\mathrm{cb}} \leq 2(\|\theta\|_{\mathrm{cb}} - 1)$.

The statement is slightly modified, but this is what the proof actually shows.

Proof of Theorem 4.1. We first assume that X has property A. Take arbitrary $R > 0$ and $\epsilon > 0$. Let $\kappa = \kappa(X, R)$ is the constant given in Lemma 4.2. By condition (ii) of Proposition 2.4, there exist unit vectors $\{\xi_x\}_{x\in X} \subseteq \ell_2(X)$ and a positive number S satisfying the following:

$$\mathrm{supp}(\xi_x) \subseteq N(x, S), \quad |1 - \langle \xi_y, \xi_z\rangle| < \epsilon/\kappa \quad (d(y,z) \leq R).$$

Define a linear map $\phi\colon B_S \to C_u^*(X)$ by

$$\phi\Big(\big([b^{(x)}_{y,z}]_{y,z\in N(x,S)}\big)_{x\in X}\Big) = \Big[\sum_{x\in X} \xi_y(x)\overline{\xi_z(x)} b^{(x)}_{y,z}\Big]_{y,z\in X}.$$

We note that ϕ is unital and completely positive.

For $a \in C_u^*(X)$, we have

$$\begin{aligned}\phi\circ\psi_S(a) &= \phi\Big(\big([a_{y,z}]_{y,z\in N(x,S)}\big)_{x\in X}\Big)\\ &= \Big[\sum_{x\in X} \xi_y(x)\overline{\xi_z(x)} a_{y,z}\Big]_{y,z\in X}\\ &= \big[\langle \xi_y, \xi_z\rangle a_{y,z}\big]_{y,z\in X}.\end{aligned}$$

For $a \in E_R$, we get

$$\begin{aligned}\|a - \phi\circ\psi_S(a)\| &\leq \kappa\big(\sup_{y,z\in X} |(1 - \langle\xi_y, \xi_z\rangle) a_{y,z}|\big)\\ &\leq \epsilon \sup_{y,z\in X} |a_{y,z}|\\ &\leq \epsilon\,\|a\|.\end{aligned}$$

This implies the following inequalities:

$$\|a\| \leq \|\phi\circ\psi_S(a)\| + \|a - \phi\circ\psi_S(a)\| \leq \|\psi_S(a)\| + \epsilon\|a\|$$

and

$$(1-\epsilon)\|a\| \leq \|\psi_S(a)\|.$$

It follows that the property (v) in Proposition 3.1 holds true. Hence X has ONL.

Now assume that X has ONL. By Proposition 3.1 (v) and Proposition 3.2, for any $R > 0$ and $\epsilon > 0$, there exists $S > 0$ such that $\|(\psi_S|_{E_R})^{-1}\colon \psi_S(E_R) \to E_R\|_{\mathrm{cb}} < 1 + \epsilon/2$. It is easy to check that $(\psi_S|_{E_R})^{-1}$ is unital and self-adjoint. By [3, Corollary B.9], there exists a unital completely positive map $\phi\colon B_S \to \mathbb{B}(\ell_2(X))$ which satisfies $\|(\psi_S|_{E_R})^{-1} - \phi|_{\psi_S(E_R)}\|_{\mathrm{cb}} < \epsilon$.

Define a function k on the set $X \times X$ by $k(y,z) = \langle \phi \circ \psi_S(e_{y,z})\delta_z, \delta_y \rangle$, where $e_{y,z}$ is the rank one partial isometry which maps δ_z to δ_y. Since $\phi \circ \psi_S$ is completely positive, for every $x(1), x(2), \dots, x(n) \in X$ and $\lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{C}$, we have

$$\begin{aligned} 0 &\leq \left\langle (\phi \circ \psi_S)^{(n)} \left(\begin{bmatrix} e_{x(1),x(1)} & \cdots & e_{x(1),x(n)} \\ \vdots & \ddots & \vdots \\ e_{x(n),x(1)} & \cdots & e_{x(n),x(n)} \end{bmatrix} \right) \begin{bmatrix} \lambda_1 \delta_{x(1)} \\ \vdots \\ \lambda_n \delta_{x(n)} \end{bmatrix}, \begin{bmatrix} \lambda_1 \delta_{x(1)} \\ \vdots \\ \lambda_n \delta_{x(n)} \end{bmatrix} \right\rangle \\ &= \sum_{i,j=1}^{n} \overline{\lambda_i} \lambda_j k(x(i), x(j)). \end{aligned}$$

Thus k is a positive definite kernel on X. It is supported on the set $\{(y,z) \in X^2 \mid d(y,z) \leq S\}$, because $\psi_S(e_{y,z}) = 0$ if $d(y,z) > S$. In the case of $d(y,z) \leq R$, we have

$$\begin{aligned} \left|1 - k(y,z)\right| &= \left|\langle (e_{y,z} - \phi \circ \psi_S(e_{y,z}))\delta_z, \delta_y \rangle\right| \\ &\leq \left\| ((\psi_S|_{E_R})^{-1} - \phi)(\psi_S(e_{y,z})) \right\| < \epsilon. \end{aligned}$$

It follows that X satisfies condition (iii) in Proposition 2.4. Hence X has property A. □

The unital completely positive map ϕ in the first half of the proof was constructed in [15, Proposition 11.41].

5. Conditions equivalent to property A

In this revised version, we make comments on other coarse geometric properties. By Theorem 4.1 and a recent result [2] by Brodzki, Niblo, Špakula, Willett, and Wright, we obtain the following theorem.

Theorem 5.1. *For a metric space with bounded geometry, the following properties are equivalent:*

(i) *property A,*

(ii) *uniform local amenability (ULA$_\mu$) defined in* [2, Definition 2.5],

(iii) *the metric sparsification property (MSP) defined in* [4, Definition 3.1],

(iv) *the operator norm localization property (ONL).*

The implication (i) ⇒ (ii) is proved in [2, Proposition 3.2] and (ii) ⇒ (iii) is proved in [2, Proposition 3.8]. MSP implies ONL, which is shown in [4, Section 4]. With our result (iv) ⇒ (i), we get the above theorem.

For a connected infinite graph G of bounded vertex degrees, the notion of weighted hyperfiniteness was introduced by Elek and Timár [7]. Theorem 5.2 gives an answer to Question 1 in [7].

Theorem 5.2. *The graph G is weighted hyperfinite if and only if its vertex set V equipped with the graph metric d has property A.*

Proof. Note that weighted hyperfiniteness is invariant under quasi-isometry by [7, Proposition 3.1]. Suppose that G is weighted hyperfinite. Then the graph G_R with the same vertex set V and the edge set $\{(x, y) \in V \times V \mid d(x, y) < R\}$ is also weighted hyperfinite for every $R \geq 1$. It is routine to prove that G has MSP. Conversely, MSP of G implies weighted hyperfiniteness by definition. It follows that weighted hyperfiniteness and MSP are equivalent. □

Acknowledgement. We thank Professor Ozawa for letting us know the paper [7].

References

[1] *C. Anantharaman-Delaroche* and *J. Renault*, Amenable groupoids, Monogr. Enseign. Math. **36**, Geneva 2000.
[2] *J. Brodzki*, *G. A. Niblo*, *J. Špakula*, *R. Willett* and *N. J. Wright*, Uniform local amenability, preprint 2011, `http://arxiv.org/abs/arXiv:1203.6169`.
[3] *N. P. Brown* and *N. Ozawa*, C^*-algebras and finite-dimensional approximations, Grad. Stud. Math. **88**, American Mathematical Society, Providence 2008.
[4] *X. Chen*, *R. Tessera*, *X. Wang* and *G. Yu*, Metric sparsification and operator norm localization, Adv. Math. **218** (2008), no. 5, 1496–1511.
[5] *X. Chen* and *X. Wang*, Operator norm localization property of relative hyperbolic group and graph of groups, J. Funct. Anal. **255** (2008), no. 3, 642–656.
[6] *M. D. Choi* and *E. G. Effros*, Nuclear C^*-algebras and the approximation property, Amer. J. Math. **100** (1978), no. 1, 61–79.
[7] *G. Elek* and *A. Timár*, Quasi-invariant means and Zimmer amenability, preprint 2011, `http://arxiv.org/abs/arXiv:1109.5863`.
[8] *E. Guentner*, *N. Higson* and *S. Weinberger*, The Novikov conjecture for linear groups, Publ. Math. Inst. Hautes Études Sci. **101** (2005), 243–268.
[9] *E. Guentner*, *R. Tessera* and *G. Yu*, Operator norm localization for linear groups and its application to K-theory, Adv. Math. **226** (2011), no. 4, 3495–3510.
[10] *N. Higson*, Bivariant K-theory and the Novikov conjecture, Geom. Funct. Anal. **10** (2000), no. 3, 563–581.
[11] *N. Higson* and *J. Roe*, Amenable group actions and the Novikov conjecture, J. reine angew. Math. **519** (2000), 143–153.
[12] *E. Kirchberg*, C^*-nuclearity implies CPAP, Math. Nachr. **76** (1977), 203–212.
[13] *N. Ozawa*, Amenable actions and exactness for discrete groups, C. R. Math. Acad. Sci. Paris **330** (2000), no. 8, 691–695.
[14] *N. Ozawa*, Boundary amenability of relatively hyperbolic groups, Topology Appl. **153** (2006), no. 14, 2624–2630.
[15] *J. Roe*, Lectures on coarse geometry, Univ. Lecture Ser. **31**, American Mathematical Society, Providence 2003.
[16] *G. Skandalis*, *J.-L. Tu* and *G. Yu*, The coarse Baum–Connes conjecture and groupoids, Topology **41** (2002), no. 4, 807–834.
[17] *J.-L. Tu*, Remarks on Yu's "property A" for discrete metric spaces and groups, Bull. Soc. Math. France **129** (2001), no. 1, 115–139.
[18] *G. Yu*, The coarse Baum–Connes conjecture for spaces which admit a uniform embedding into Hilbert space, Invent. Math. **139** (2000), no. 1, 201–240.

Hiroki Sako, Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502, Japan
e-mail: sako@kurims.kyoto-u.ac.jp

Eingegangen 26. März 2012