SUMMARY OF 'MODULI SPACES OF FRAMED SYMPLECTIC AND ORTHOGONAL BUNDLES ON $\mathbb{P}^2$ AND THE K-THEORETIC NEKRASOV PARTITION FUNCTIONS'

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Nekrasov-Shadchin [8] proposed a partition function given by integrations of K-theory elements on the spaces of ADHM data. Our aim in this dissertation is to interpret their integral formulas in terms of instanton partition functions. More precisely in some cases when the structure group of instantons are of classical type, they are generating functions of torus characters of coordinate rings of instanton moduli spaces. Hence they become independent of choice of ADHM descriptions of instantons.

The physics origin of Nekrasov-Shadchin's formulation is 4-dimensional $\mathcal{N}=2$ supersymmetric gauge theory, especially Seiberg-Witten prepotential ([5, 6]). Seiberg-Witten prepotentials turned out to be the lowest degree coefficient of (original) Nekrasov partition functions due to Nakajima-Yoshioka [4], Nekrasov-Okounkov [7] (both for $\mathrm{SU}(N)$-instantons) and Braverman-Etingof [2] (for any gauge group).

Let K be the compact Lie group $\mathrm{USp}(N/2)$ or $\mathrm{SO}(N,\mathbb{R})$. Let $\mathcal{M}_n^K$ be the moduli space of framed K-instantons over the 4-sphere S^4 with the instanton number n. Its scheme structure is given by ADHM data due to Donaldson [3] as follows: Let W be the fundamental representation of K. Let G_k be the complex Lie group $\mathrm{SO}(k)$ (resp. $\mathrm{Sp}(k/2)$) if $K=\mathrm{USp}(N/2)$ (resp. $\mathrm{SO}(N,\mathbb{R})$). Here k is given in terms of n. Let V be the fundamental representation of G_k. Let

$$\mathbf{N}_k := \{(B_1,B_2,i,j) | B_1=B_1^*, B_2=B_2^*, j=i^*\}$$

(subspace of $\mathrm{End}(V)^{\oplus 2}\oplus\mathrm{Hom}(W,V)\oplus\mathrm{Hom}(V,W)$) where $*$ denotes the right adjoint with respect to the given nondegenerate forms on V,W. Then $\mathbf{N}_k$ is a symplectic subspace of the cotangent space of $\mathrm{End}(V)\oplus\mathrm{Hom}(W,V)$ and the natural G_k-action is Hamiltonian. Thus we have a moment map μ_k on $\mathbf{N}_k$. Now the scheme structure of $\mathcal{M}_n^K$ is endowed by the quasi-affine quotient $\mu_k^{-1}(0)^{\mathrm{reg}}/G_k$ where the subscript reg stands for the regular locus.

Nekrasov-Shadchin's integral formula is now defined as the sum over all $n\geq 0$ of equivariant integrations of the G_k-invariant part of the Koszul complex induced by μ_k on $\mathbf{N}_k/\!\!/G_k$. We compare their formula with the generating function of the coordinate rings of the instanton moduli spaces $\mu_k^{-1}(0)/G_k$. In particular these two coincide in some cases. For the purpose we need to know geometry of μ_k. Our main results are as follows: If $K=$

$\mathrm{USp}(N/2)$ then μ_k is flat and $\mu_k^{-1}(0)$ is an irreducible normal variety for any n and even N. If $K = \mathrm{SO}(N, \mathbb{R})$ the similar results are proven for low n and N. Hence the coordinate rings of two quotients $\mu_k^{-1}(0)^{\mathrm{reg}}/G_k, \mu_k^{-1}(0)/\!\!/G_k$ are naturally identified. Moreover they coincide with the above equivariant integration as elements of some completion of the representation ring of a maximal torus of the complexified group $K_{\mathbb{C}}$ of K.

Donaldson's description also gives a $K_{\mathbb{C}}$-equivariant isomorphism between the instanton moduli space and the vector bundle locus of the Gieseker moduli space. Therefore Nekrasov-Shadchin's formula also coincides with the generating function for the vector bundle locus of the Gieseker moduli spaces in the listed cases above.

References

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