Moduli spaces of framed symplectic and orthogonal bundles on $\mathbb{P}^2$ and the K-theoretic Nekrasov partition functions

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ABSTRACT

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Let K be the compact Lie group $\mathrm{USp}(N/2)$ or $\mathrm{SO}(N,\mathbb{R})$. Let $\mathcal{M}_n^K$ be the moduli space of framed K-instantons over S^4 with the instanton number n. By [**13**], $\mathcal{M}_n^K$ is endowed with a natural scheme structure. It is a Zariski open subset of a GIT quotient of $\mu^{-1}(0)$ where μ is a holomorphic moment map such that $\mu^{-1}(0)$ consists of the ADHM data.

The purpose of the dissertation is to study the geometric properties of $\mu^{-1}(0)$ and its GIT quotient, such as complete intersection, irreducibility, reducedness and normality. If $K = \mathrm{USp}(N/2)$ then μ is flat and $\mu^{-1}(0)$ is an irreducible normal variety for any n and even N. If $K = \mathrm{SO}(N,\mathbb{R})$ the similar results are proven for low n and N.

As an application one can obtain a mathematical interpretation of the K-theoretic Nekrasov partition function of [**48**].

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CHAPTER 1

Introduction

1.1. Brief introduction to Nekrasov partition functions

The Nekrasov partition function was formulated by Nekrasov [**45**] in the 4-dimensional $\mathcal{N} = 2$ supersymmetric gauge theory in physics, especially relevant to the Seiberg-Witten prepotential [**51**]. It is defined as a generating function of the equivariant integration of the trivial cohomology class 1 over Uhlenbeck partial compactification of the (framed) instanton moduli space on $\mathbb{R}^4$ for all the instanton numbers. Its logarithm turned out to contain the Seiberg-Witten prepotential as the coefficient of the lowest degree (with respect to two variables from the torus action on $\mathbb{R}^4$), which is the remarkable result proven by Nakajima-Yoshioka [**42**], Nekrasov-Okounkov [**47**] (both for $\mathrm{SU}(N)$) and Braverman-Etingof [**6**] (for any gauge group) in completely independent methods. The Nekrasov partition function is an equivariant version of Donaldson-type invariants for $\mathbb{R}^4$, where the ordinary Donaldson invariants are integrals over Uhlenbeck compactifications of instanton moduli spaces on compact 4-manifolds. More generally, a Nekrasov partition function for the theory with matters is defined via the equivariant integration of a cohomology class other than the trivial class. For instance, in the works of Göttsche-Nakajima-Yoshioka [**16**][**18**], such partition functions are used to express wallcrossing terms of Donaldson invariants and a relation between them and Seiberg-Witten invariants.

Our interest in this paper is the K-theoretic Nekrasov partition function. It arose from the 5-dimensional $\mathcal{N} = 2$ supersymmetric gauge theory proposed by Nekrasov [**46**]. Mathematically, one change in its definition is made [**45**]: the equivariant integration of elements in the equivariant K-theory of coherent sheaves instead of cohomology classes. There is a technical, but a subtle problem here. We need a scheme structure on the Uhlenbeck space for the definition, but there are several choices (see [**6**]). Therefore it is not clear what is the correct definition. For type A, one can use framed moduli spaces of coherent sheaves instead, which are smooth. The K-theoretic Nekrasov partition functions also appeared in mathematics literatures as K-theoretic Donaldson invariants studied in [**43**][**17**] in which the K-theory version of Nekrasov conjecture and blowup equations are proven for type A.

Nekrasov-Shadchin [**48**] took a different approach when the gauge group is of classical type. They defined the partition function using the K-theory class of the Koszul complex which defines the ADHM data of the instanton moduli spaces given in [**1**].

The main purpose of this paper is to show that Nekrasov-Shadchin's definition coincides with a generating function of the coordinate rings of the instanton moduli spaces for the classical gauge groups. The answer for the gauge group $\mathrm{SU}(N)$ has been already known by a general result of Crawley-Boevey on quiver varieties [**12**]. (See §1.3.2 for the precise motivating questions which we pursue). Our study for the gauge groups $\mathrm{USp}(N/2)$ and $\mathrm{SO}(N, \mathbb{R})$ informs of some geometry of the moduli spaces in algebraic geometry, as the instanton moduli space for $\mathrm{USp}(N/2)$

and $\mathrm{SO}(N,\mathbb{R})$ is isomorphic to the moduli space of (framed) vector bundles with symplectic and orthogonal structures on $\mathbb{P}^2$ respectively. (See §1.2.3 for explanation on the scheme structures of moduli spaces.)

1.2. Moduli spaces of instantons and ADHM description

1.2.1. Let us fix the notation and explain earlier results in order to our result precisely. Let K be a compact connected simple Lie group. Let P_K be a principal K-bundle over the 4-sphere S^4. Since $\pi_3(K) \cong \mathbb{Z}$, an integer n uniquely determines the topological type of P_K. Let $\mathcal{M}_n^K$ be the quotient of the space of ASD connections (instantons) by the group of the gauge transforms trivial at infinity $\infty \in S^4$ [**14**, §5.1.1]. By [**2**, Table 8.1] $\mathcal{M}_n^K$ is a $\mathcal{C}^\infty$-manifold with $\dim \mathcal{M}_n^K = 4nh^\vee$, where $h^\vee$ is is the dual Coxeter number of K. In this paper, we will also consider the case $K = \mathrm{SO}(4,\mathbb{R})$. Its universal cover $\mathrm{Spin}(4)$ is the product $\mathrm{SU}(2) \times \mathrm{SU}(2)$, and hence $\mathrm{SO}(4,\mathbb{R})$-bundles over S^4 are classified by pairs of integers (n_L, n_R). We have $\mathcal{M}_{(n_L,n_R)}^K \cong \mathcal{M}_{n_L}^{\mathrm{SU}(2)} \times \mathcal{M}_{n_R}^{\mathrm{SU}(2)}$, and hence $\dim \mathcal{M}_{(n_L,n_R)}^K = 8(n_L + n_R)$.

1.2.2. Let $K = \mathrm{SU}(N)$. Donaldson [**13**] showed that $\mathcal{M}_n^K$ is naturally isomorphic to the moduli space of framed holomorphic vector bundles E with $c_2(E) = n$ over $\mathbb{P}^2$, where a framing is a trivialization of E over the line l_∞ at infinity. This was proved by using the ADHM description [**1**] and the relation between moment maps and geometric invariant theory (GIT). Let $K = \mathrm{USp}(N/2)$ or $\mathrm{SO}(N,\mathbb{R})$ and G be its complexification. Since G-bundles can be identified with rank N vector bundles with symplectic or orthogonal structures, we have the following description of $\mathcal{M}_n^K$.

Let $G'_k := \mathrm{O}(k)$ (resp. $\mathrm{Sp}(k/2)$) if $G = \mathrm{Sp}(N/2)$ (resp. $\mathrm{SO}(N)$). Let $\mathfrak{p}(\mathbb{C}^k)$ be the space of symmetric endomorphisms of $\mathbb{C}^k$ (see (2.1) for the precise definition). The ADHM data are elements $x \in \mathbf{N}$ satisfying $\mu(x) = 0$ where $\mathbf{N} := \mathfrak{p}(\mathbb{C}^k)^{\oplus 2} \oplus \mathrm{Hom}(\mathbb{C}^N, \mathbb{C}^k)$ and μ is the holomorphic moment map defined on $\mathbf{N}$. The G'_k-action on $\mathbf{N}$ induces an G'_k-action on $\mu^{-1}(0)$. Now $\mathcal{M}_n^K$ is the image of the regular locus $\mu^{-1}(0)^{\mathrm{reg}}$ of the GIT quotient $\mu^{-1}(0) \to \mu^{-1}(0)/\!\!/G'_k$. Here the second Chern number k of associated vector bundles is determined from n by the argument in [**2**, §10] as

$$k = \begin{cases} n & \text{if } G = \mathrm{Sp}(N/2), \\ 2n & \text{if } G = \mathrm{SO}(N) \text{ where } N \geq 5, \\ 4n & \text{if } G = \mathrm{SO}(3). \end{cases} \tag{1.1}$$

See the details in Appendix A.

1.2.3. The scheme structure of $\mathcal{M}_n^K$ is given by $\mu^{-1}(0)^{\mathrm{reg}}/G$. Since $\mu^{-1}(0)^{\mathrm{reg}}/G$ is a Zariski-open subset of $\mu^{-1}(0)/\!\!/G$, $\mathcal{M}_n^K$ is a quasi-affine scheme. We notice that $\mathcal{M}_n^K$ may have two scheme structures because it is possible that $\mathrm{Lie}(K) = \mathrm{Lie}(K')$ for a different classical group K' (hence $\mathcal{M}_n^K = \mathcal{M}_n^{K'}$) but $\mathcal{M}_n^K$ and $\mathcal{M}_n^{K'}$ have the different ADHM descriptions. Such pairs of (K, K') are $(\mathrm{SU}(2), \mathrm{USp}(1))$, $(\mathrm{SU}(2), \mathrm{SO}(3,\mathbb{R}))$, $(\mathrm{USp}(2), \mathrm{SO}(5,\mathbb{R}))$ and $(\mathrm{SU}(4), \mathrm{SO}(6,\mathbb{R}))$. A scheme-theoretic isomorphism $\mathcal{M}_n^K \xrightarrow{\cong} \mathcal{M}_n^{K'}$ is induced by sending the associated vector bundles E to itself, $\mathrm{ad}E$, $(\Lambda^2 E)_0$ and $\Lambda^2 E$ respectively. The notations in the above are as follows: $\mathrm{ad}E$ is the trace-free part of $\mathrm{End}(E)$, and $(\Lambda^2 E)_0$ is the kernel of the symplectic form $\Lambda^2 E \to \mathcal{O}$.

In the above isomorphism we used the following two: By Donaldson's theorem [**13**], $\mathcal{M}_n^K$ is canonically isomorphic to the moduli space of framed rank N vector bundles E with $c_2(E) = k$ and G-structure. The latter space is endowed with a (smooth) scheme structure and the canonical isomorphism is scheme-theoretic. See §5.4.

1.3. Definition of Nekrasov partition functions

1.3.1. There is a canonical $G \times G'_k$-action on $\mathbf{N}$. The canonical G-action on $\mathbf{N}$ induces a G-action on $\mathcal{M}_n^K$. We define a $(\mathbb{C}^*)^2$-action on $\mathbf{N}$ by $(q_1, q_2).(B_1, B_2, i) = (q_1 B_1, q_2 B_2, \sqrt{q_1 q_2} i)$. More precisely this action is well-defined only on a double cover of $\mathbb{C}^* \times \mathbb{C}^*$, but we follow the convention in physics. It commutes with the $G \times G'_k$-action and induces a $(\mathbb{C}^*)^2$-action on $\mathcal{M}_n^K$. Since i always appears together with i^* for generators in $\mathbb{C}[\mathbf{N}]^{G'_k}$ (see [**53**, Thereoms 2.9.A and 6.1.A]), the $(\mathbb{C}^*)^2$-action is well-defined on $\mathcal{M}_n^K$.

Let T_G be a maximal torus of G. We identify $T_G = (\mathbb{C}^*)^l$ where $l = \mathrm{rank}(G)$. Let $T := T_G \times (\mathbb{C}^*)^2 = (\mathbb{C}^*)^{l+2}$. Then T acts on $\mu^{-1}(0)/\!\!/G'_k$ and $\mathcal{M}_n^K$. Nekrasov-Shadchin [**48**] defined the K-theoretic instanton partition function Z^K for the gauge group K by an explicit integral formula. Their formula can be interpreted as follows.

Fix the instanton number n. Let us consider μ be a section of a trivial vector bundle $\mathbf{E} = \mathrm{Lie}\, G'_k \times \mathbf{N}$ over $\mathbf{N}$. We endow $\mathbf{E}$ with a $G'_k \times T$-equivariant structure so that μ is a T-equivariant section. Then μ defines a Koszul complex

$$\Lambda^{\operatorname{rank}\mathbf{E}}\mathbf{E}^\vee \to \cdots \to \Lambda^2\mathbf{E}^\vee \to \mathbf{E}^\vee \to \Lambda^0\mathbf{E}^\vee = \mathcal{O}_{\mathbf{N}}. \tag{1.2}$$

The alternating sum $\sum_i (-1)^i \Lambda^i \mathbf{E}^\vee$ of terms gives an element in $K^{T\times G'_k}(\mathbf{N})$, the Grothendieck group of $T \times G'_k$-equivariant vector bundles over $\mathbf{N}$. We then take a pushforward of the class with respect to the obvious map $p\colon \mathbf{N} \to \mathrm{pt}$. This is not a class in the representation ring $R(T \times G'_k)$ because $\mathbf{N}$ is not proper. However, it is a well-defined class in $\operatorname{Frac}(R(T \times G'_k))$, the fractional field of $R(T \times G'_k)$ by *equivariant integration*: Check first that the origin 0 is the unique fixed point in $\mathbf{N}$ with respect to $T \times T'_k$, where T'_k is a maximal torus of G'_k. (See Appendix C.) Therefore the pushforward homomorphism p_* can be defined as the inverse of i_* thanks to the fixed point theorem of the equivariant K-theory, where $i\colon \{0\} \to \mathbf{N}$ is the inclusion. In practice, we take the Koszul resolution of the skyscraper sheaf at the origin as above, replacing $\mathbf{E}$ by $\mathbf{N}$, and then divide the class by $\sum_i (-1)^i \Lambda^i \mathbf{N}^\vee$.

In our situation, $\mathbf{N}$ has only positive weights with respect to the factor $(\mathbb{C}^*)^2$ of T. (See Appendix C.) Therefore the rational function can be expanded as an element in a *completed* ring of $T \times G'_k$-characters

$$\hat{R}(T \times G'_k) := R(T_G \times G'_k)[[q_1^{-1}, q_2^{-1}]],$$

where q_m denote the T-characters of the 1-dimensional representations for the second factor $(\mathbb{C}^*)^2$ of $T = T_G \times (\mathbb{C}^*)^2$. The expansion is considered as a formal character, as the sum of dimensions of weight spaces: Each weight space is finite dimensional and its dimension is given by the coefficient of $f \in \hat{R}(T \times G'_k)$ of the monomial corresponding to the weight.

Next we take the G'_k-invariant part. This is given by an integral over the maximal compact subgroup of T'_k by Weyl's integration formula. Finally taking the generating function for all $n \in \mathbb{Z}_{\geq 0}$ with the formal variable $\mathfrak{q}$, we define the instanton partition function

$$Z^K := \sum_{n \geq 0} \mathfrak{q}^n \sum_i (-1)^i p_* (\Lambda^i \mathbf{E}^\vee)^{G'_k} \in \hat{R}(T)[[\mathfrak{q}]],$$

where $\hat{R}(T) := R(T_G)[[q_1^{-1}, q_2^{-1}]]$.

Explicit computation of Z^K has been performed for small instanton numbers in physics literature. See [**4**][**23**] for instance.

1.3.2. Nekrasov-Shadchin's definition of Z^K is mathematically rigorous, but depends on the ADHM description, hence is not intrinsic in the instanton moduli space $\mathcal{M}_n^K$. We can remedy this problem under the following geometric assumptions:

(1) μ is a flat morphism;

(2) $\mu^{-1}(0) /\!\!/ G'_k$ is a normal variety and $\mathcal{M}_n^K$ has the complement codimension ≥ 2 in $\mu^{-1}(0) /\!\!/ G'_k$.

Under these assumptions, $\sum_i (-1)^i p_* (\Lambda^i \mathbf{E}^\vee)^{G'_k}$ is the formal character of the ring $\mathbb{C}[\mathcal{M}_n^K]$ of regular functions on $\mathcal{M}_n^K$: First, (1) asserts μ gives a regular sequence so that the Koszul complex (1.2) is a resolution of the coordinate ring $\mathbb{C}[\mu^{-1}(0)]$ of $\mu^{-1}(0)$. Secondly (2) asserts $\mathbb{C}[\mathcal{M}_n^K] = \mathbb{C}[\mu^{-1}(0) /\!\!/ G'_k] = \mathbb{C}[\mu^{-1}(0)]^{G'_k}$ by extension of regular functions on $\mathcal{M}_n^K = \mu^{-1}(0)^{\mathrm{reg}}/G'_k$ to $\mu^{-1}(0)/\!\!/G'_k$. As explained in §1.2.3, the scheme structure of $\mathcal{M}_n^K$ is independent of the choice of the ADHM description.

When $K = \mathrm{SU}(N)$, (1),(2) are proved by Crawley-Boevey [**11**] in a general context of quiver varieties. Moreover the Gieseker space $\widetilde{\mathcal{M}}_n^N$ of framed rank N torsion free sheaves $\mathcal{E}$ on $\mathbb{P}^2$ with $c_2(\mathcal{E}) = n$ gives a resolution of singularities of $\mu^{-1}(0)/\!\!/G'_k$. Since $\widetilde{\mathcal{M}}_n^N$ is simplectic, higher direct image sheaves vanish. Therefore $\mathbb{C}[\widetilde{\mathcal{M}}_n^N] \cong \mathbb{C}[\mu^{-1}(0)/\!\!/G'_k]$. The T-action lifts to $\widetilde{\mathcal{M}}_n^N$, where fixed points are parametrized by N-tuples of Young diagrams corresponding to direct sums of monomial ideal sheaves. Therefore the character of $\mathbb{C}[\widetilde{\mathcal{M}}_n^N]$ is given by a sum over N-tuples of Young diagrams, which is the original definition of the instanton partition function in [**45**]. See [**43**] for more detail of the latter half of this argument.

1.3.3. A goal of this paper is to prove (1),(2) for $K = \mathrm{USp}(N/2)$. (See Theorem 6.1.) A key of the proof is a result of Panyushev [**50**], which gives the flatness of μ for $N = 0$.

We also study $\mu^{-1}(0)$ for $K = \mathrm{SO}(N, \mathbb{R})$ for $(N, k) = (2, k)$, $(N, 2)$ or $(3, 4)$. (See Theorems 7.2, 7.1, 7.3 respectively.) The case $N = 2$ is less interesting since there are no $\mathrm{SO}(2, \mathbb{R})$-instantons except for $k = 0$. But $\mu^{-1}(0)$ *does* make sense, hence the study of its property is a mathematically meaningful question. We show that μ is not flat, i.e., (1) is not true in this case. Similarly there is no instanton for $(N, k) = (3, 2)$. We will see that μ is flat, but $\mu^{-1}(0)/\!\!/G'_k = \mathbb{C}^2$ in this case, if the left hand side is given the reduced structure. In particular, (2) is false in this case. In the case $(N, 2)$ with $N \geq 4$, we prove that $\mu^{-1}(0)/\!\!/G'_k$ is isomorphic to the product of $\mathbb{C}^2$ and the closure $\mathbf{P}$ of the minimal nilpotent $\mathrm{O}(N)$-orbit. When $(N,k) = (3,4)$, we show that μ is flat and $\mu^{-1}(0)/\!\!/G'_k$ is a union of two copies of $\mathbb{C}^2\times\mathbf{P}$ meeting along $\mathbb{C}^2\times\{0\}$. Here $\mathbf{P}$ is the minimal (= regular) nilpotent $\mathrm{O}(3)$-orbit closure, which is isomorphic to $\mathbb{C}^2/\mathbb{Z}_2$. In particular, (2) is false. The isomorphism $\mathcal{M}_n^{\mathrm{SU}(2)} \cong \mathcal{M}_n^{\mathrm{SO}(3)}$ asserts that both $\mathcal{M}_n^{\mathrm{SU}(2)}$ and $\mathcal{M}_n^{\mathrm{SO}(3)}$ are $\mathbb{C}^2\times(\mathbf{P}\setminus\{0\})$ for $n=1$. However their ADHM description is different, as it is $\mathbb{C}^2\times\mathbf{P}$ for $\mathrm{SU}(2)$. This phenomenon happens only for $\mathrm{SO}(3)$. See Theorem 4.1. These examples show that the definition of Z^K depends on the ADHM description, and hence must be studied with care.

The author plans to study the case $(N\geq 4, k\geq 4)$ in near future.

1.4. Organization of the thesis

The thesis is organized as follows.

In Chapter 2 we give the set-up for the entire part of this thesis. Specifically we set up linear algebra of vector spaces with nondegenerate forms in §2.1, and quiver representations in §2.2.

In Chapter 3 we identify the closed G'_k-orbits in $\mathbf{N}$. By this identification we see that $\mathcal{M}_n^K$ is the G'_k-orbit space of stable-costable representations in $\mu^{-1}(0)$. Hence $\mathcal{M}_n^K$ is endowed with the smooth quasi-affine subscheme structure of $\mu^{-1}(0)^{\mathrm{reg}}/G'_k$.

In Chapter 4 we stratify $\mu^{-1}(0)/\!\!/G'_k$ with $\mathcal{M}_n^K$ as a stratum. Each stratum is isomorphic to a product of $\mathcal{M}_{n'}^K$ and a symmetric product of $\mathbb{C}^2$ for some $n'\leq n$. From this stratification $\mu^{-1}(0)/\!\!/G'_k$ looks similar to the Uhlenbeck space. For $\mathrm{USp}(N/2)$, they are indeed equal while for $\mathrm{SO}(3,\mathbb{R})$, they are not.

In Chapter 5 we construct a Barth-type isomorphism from $\mathcal{M}_n^K$ to the moduli space of framed vector bundles with symplectic or orthogonal structure. We realize the latter moduli space as a locally closed smooth subscheme of the ordinary Gieseker space. Hence we get a scheme-theoretic isomorphism between the two moduli spaces which was mentioned in §1.2.3.

In Chapters 6 and 7 we have Theorem 6.1 and Theorems 7.1–7.3. The geometry of $\mu^{-1}(0)$ when $K = \mathrm{USp}(N/2)$ and $\mathrm{SO}(N, \mathbb{R})$ are described in these theorems.

Chapter 7 consists as follows. In §7.1 we explain Kraft-Procesi's classification theory of nilpotent pairs. In our study of $\mu^{-1}(0)$, their theory is quite useful to see contribution from the factor $\mathrm{Hom}(\mathbb{C}^N, \mathbb{C}^k)$ of $\mathbf{N}$. In the case $k = 2$, the geometry of $\mu^{-1}(0)$ is immediately deduced from Kraft-Procesi's theory on $\mathrm{Hom}(\mathbb{C}^N, \mathbb{C}^k)$ since $\mathfrak{p}(\mathbb{C}^2)$ consists of the scalars.

In §7.2 and §7.3 we will see how Kraft-Procesi's theory is applied to the cases $k = 2$ and $N = 2$. This will prove Theorem 7.1 and a part of Theorem 7.2.

In §7.4 we prove Theorem 7.3 (the case $(N, k) = (3, 4)$). The proof involves more than Kraft-Procesi's theory since the pairs $(B_1, B_2) \in \mathfrak{p}(\mathbb{C}^4)^{\oplus 2}$ are no more commuting pairs. Since $[B_1, B_2] \neq 0$ in general, the study of $\mu^{-1}(0)$ does not solely come from the factor $\mathrm{Hom}(\mathbb{C}^3, \mathbb{C}^4)$. So we study the commutator map $\mathfrak{p}(\mathbb{C}^4)^{\oplus 2} \to \mathrm{Lie}(\mathrm{Sp}(2))$, $(B_1, B_2) \mapsto [B_1, B_2]$.

In §7.5 we finish the proof of Theorem 7.2 (the case $(N, k) = (2, 4)$) based on the study of the commutator map in the above.

CHAPTER 2

Preliminary on linear algebra and quiver representations

We set up convention and notation and review basic materials for the entire part of the paper.

We are working over $\mathbb{C}$ unless otherwise stated. Vector spaces are all finite dimensional and schemes are of finite type. We say that a morphism between schemes is irreducible (resp. normal and Cohen-Macaulay) if all the nonempty fibres are irreducible (resp. normal and Cohen-Macaulay). If $\mathcal{M}$ is a scheme then $\mathcal{M}^{\mathrm{sm}}$ and $\mathcal{M}^{\mathrm{sing}}$ are the smooth locus and the singular locus of $\mathcal{M}$ respectively.

Let G be an algebraic group and $\mathfrak{g} := \mathrm{Lie}(G)$ the Lie algebra of G. Let $\mathcal{M}$ be a G-scheme. Let $G^x := \{g \in G |\, g.x = x\}$ the stabilizer subgroup of $x \in \mathcal{M}$. Let $\mathfrak{g}^x := \mathrm{Lie}(G^x)$.

2.1. Linear algebra on vector spaces with bilinear forms

2.1.1. the right adjoint. Let V_1 and V_2 be vector spaces with nondegenerate bilinear forms $(\ ,\ )_{V_1}$ and $(\ ,\ )_{V_2}$ respectively. Then for any $i \in \mathrm{Hom}(V_1, V_2)$, we have the right adjoint $i^* \in \mathrm{Hom}(V_2, V_1)$, i.e. $(v, i^*w)_{V_1} = (iv, w)_{V_2}$ where $v \in V_1$ and $w \in V_2$. The map

$$* : \mathrm{Hom}(V_1, V_2) \to \mathrm{Hom}(V_2, V_1),\ i \mapsto i^*$$

is a $\mathbb{C}$-linear isomorphism. Further if V_3 is a vector space with a nondegenerate bilinear form then for $i \in \mathrm{Hom}(V_1, V_2)$, $j \in \mathrm{Hom}(V_2, V_3)$, we have $(ji)^* = i^* j^*$.

2.1.2. anti-symmetric and symmetric forms. Let V be a vector space of dimension k with a nondegenerate form $(\,,\,)_V$. Let $\varepsilon \in \{-1, +1\}$. Let $(\;,\;)_\varepsilon$ be a nondegenerate bilinear form $(u, v)_\varepsilon = \varepsilon(v, u)_\varepsilon$ on V. If $\varepsilon = +1$ (resp. -1) then $(\;,\;)_\varepsilon$ is an orthogonal form (resp. symplectic form). We say V is orthogonal (resp. symplectic) if $\varepsilon = +1$ (resp. $\varepsilon = -1$).

We decompose $\mathfrak{gl}(V) = \mathfrak{k}(V) \oplus \mathfrak{p}(V)$ as a vector space where

$$
\begin{aligned}
\mathfrak{k}(V) &:= \{X \in \mathfrak{gl}(V) \,|\, (Xu, v)_\varepsilon = -(u, Xv)_\varepsilon\} = \{X \in \mathfrak{gl}(V) \,|\, X^* = -X\} \\
\mathfrak{p}(V) &:= \{X \in \mathfrak{gl}(V) \,|\, (Xu, v)_\varepsilon = (u, Xv)_\varepsilon\} = \{X \in \mathfrak{gl}(V) \,|\, X^* = X\}.
\end{aligned} \tag{2.1}
$$

Let $\mathfrak{k} := \mathfrak{k}(V)$ and $\mathfrak{p} := \mathfrak{p}(V)$ for short. The followings are immediate to check:

$$
[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, \;\; [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}, \;\; [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}. \tag{2.2}
$$

Let $G(V) := \{g \in \mathrm{GL}(V) \,|\, (gv, gv')_\varepsilon = (v, v')_\varepsilon \text{ for all } v, v' \in V\}$. Then $\mathfrak{k} = \mathfrak{g}(V)$ where $\mathfrak{g}(V) := \mathrm{Lie}\, G(V)$. We have $\dim G(V) = \dim \mathfrak{k} = \frac{1}{2}k(k - \varepsilon)$. So $\dim \mathfrak{p} = \frac{1}{2}k(k + \varepsilon)$.

Remark 2.1. If V is a symplectic vector space of dimension 2 then $\mathfrak{p}(V)$ consists of scalars.

If $\varepsilon = -1$ then $G(V)$ is denoted by $\mathrm{Sp}(V)$ (called the symplectic group) with the Lie algebra $\mathfrak{sp}(V)$. If $\varepsilon = +1$ then $G(V)$ is denoted by $\mathrm{O}(V)$ (called the orthogonal group) with the Lie algebra $\mathfrak{o}(V)$. Unless no confusion arises, we shorten the notation as $\mathrm{Sp}, \mathfrak{sp}, \mathrm{O}$ and $\mathfrak{o}$.

2.1.3. It is direct to check that if $(\, , \,)_{V_1} = (\, , \,)_\varepsilon$ and $(\, , \,)_{V_2} = (\, , \,)_{-\varepsilon}$ for some $\varepsilon \in \{-1, +1\}$ then $** = -\mathrm{Id}$. If both $(\, , \,)_{V_1}$ and $(\, , \,)_{V_2}$ are $(\, , \,)_\varepsilon$ for the same $\varepsilon \in \{-1, +1\}$ then $** = \mathrm{Id}$. In particular if V is symplectic and W is orthogonal then for $i \in \mathrm{Hom}(W, V)$, $ii^* \in \mathfrak{sp}$ by (2.1) since $(ii^*)^* = i^{**}i^* = -ii^*$. Similarly $i^*i \in \mathfrak{o}$ for $i \in \mathrm{Hom}(W, V)$.

2.2. Generality of quiver representations

We review some generality of quiver representations used in the later chapters. For a fixed representation space, the quiver representations form a vector space. We introduce a natural group action on the vector space. We will see the closed orbits correspond to semisimple quiver representations. This is important in understanding the GIT quotient of the space of quiver representations (i.e. *quiver variety*).

2.2.1. quiver representations. Let Q be any finite quiver. Let I be the vertex set and E be the arrow set. For $a \in E$, let $t(a)$ and $h(a)$ be the tail and head vertex respectively.

Let us assign to a vector space V_v each vertex $v \in I$. Let $\mathbf{V} := \bigoplus_{v \in I} V_v$. Frequently we denote $\mathbf{V}$ by (V_v) for short if there is no confusion for the vertex set I. Let

$$\mathbf{M_V} := \bigoplus_{a \in E} \mathrm{Hom}(V_{t(a)}, V_{h(a)}).$$

An element of $\mathbf{M_V}$ is called a *representation of Q with the representation space* $\mathbf{V}$. A representation $\mathbf{B}$ of Q is written as $\mathbf{B} = (B_a)_{a \in E} \in \mathbf{M_V}$.

Let $\mathbf{M}_{\mathbf{V}'}$ be another space of representations of Q where $\mathbf{V}' := (V'_v)$. Let $\mathbf{B}' = (B'_a)_{a\in E} \in \mathbf{M}_{\mathbf{V}'}$. A *homomorphism* $\sigma\colon \mathbf{B} \to \mathbf{B}'$ is a collection $(\sigma_v)_{v\in I}$ such that $\sigma_v\colon V_v \to V'_v$ are linear maps satisfying the commutativity of the following diagram:

$$\begin{tikzcd} V_{t(a)} \arrow[r, "B_a"] \arrow[d, "\sigma_{t(a)}"] & V_{h(a)} \arrow[d, "\sigma_{h(a)}"] \\ V'_{t(a)} \arrow[r, "B'_a"] & V'_{h(a)} \end{tikzcd}$$

for each $a \in E$. The category of the objects $\mathbf{B}$ is an abelian category.

We use the notation $\mathbf{B}' \subset \mathbf{B}$ as a subrepresentation if $V'_v \subset V_v$ and $B_a|_{V'_{t(a)}} = B'_a$ for all $v \in I$ and $a \in E$. So a subrepresentation of $\mathbf{B}$ always comes from a $\mathbf{B}$-invariant subspace $\mathbf{V}'$ of $\mathbf{V}$.

2.2.2. semisimple quiver representations. A nonzero representation $\mathbf{B}$ is called *simple* if any subrepresentation of $\mathbf{B}$ is either $\mathbf{B}$ itself or 0. A direct sum of simple representations is called a *semisimple* representation.

Let $\mathrm{GL}(\mathbf{V}) := \prod_{v\in I} \mathrm{GL}(V_v)$. Then, $\mathrm{GL}(\mathbf{V})$ acts on $\mathbf{M}_\mathbf{V}$ by

$$(g_v).(B_a) := (g_{h(a)} B_a g_{t(a)}^{-1}).$$

The following is well-known (cf. [**35**]).

LEMMA 2.2. *Let* $\mathbf{V} \neq 0$ *and* $\mathbf{B} \in \mathbf{M}_\mathbf{V}$. *Then* $\mathbf{B}$ *is semisimple if and only if the orbit* $\mathrm{GL}(\mathbf{V}).\mathbf{B}$ *is closed in* $\mathbf{M}_\mathbf{V}$.

We will not prove the lemma itself, but prove an essential ingredient of the proof:

LEMMA 2.3. *Let* $\mathbf{V} \neq 0$ *and* $\mathbf{B} \in \mathbf{M}_{\mathbf{V}}$ *be semisimple. Let* $\mathbf{W}$ *and* $\mathbf{W}'$ *be* $\mathbf{B}$-*invariant subspaces of* $\mathbf{V}$. *Then there exists a* $\mathbf{B}$-*invariant* $\mathbf{W}''$ *such that* $\mathbf{W} = \mathbf{W}' \oplus \mathbf{W}''$. *Hence any nonzero subquotient of* $\mathbf{B}$ *is semisimple.*

PROOF. We may assume $\mathbf{W} \neq 0$. Let $\mathbf{B}_{\mathbf{W}} := \mathbf{B}|_{\mathbf{W}}$ and $\mathbf{B}_{\mathbf{W}'} := \mathbf{B}|_{\mathbf{W}'}$. Let $\mathbf{B} = \bigoplus_{m=1}^{n} \mathbf{B}^m$ be a decomposition by simple representations. We denote by $\mathbf{V}^m$ the representation space of $\mathbf{B}^m$. Let $[n] := \{1, 2, ..., n\}$ for short.

We use the induction on n. If $n = 1$, then $\mathbf{B}$ is simple and the claim is obvious.

Let $p_S \colon \mathbf{V} \to \oplus_{m\in S}\mathbf{V}^m$ be the projection where S is any nonempty subset of $[n]$. We denote by $p_{S*} \colon \mathbf{B} \to \bigoplus_{m\in S} \mathbf{B}^m$ the induced homomorphism by the projection. Note that there exists $n_0 \in [n]$ such that $p_{\hat{n}_0 *}(\mathbf{B}_{\mathbf{W}}) \neq 0$, since otherwise $\mathbf{W} = 0$. We denote by $\hat{n}_0 := [n] \setminus n_0$. It is clear that $\mathrm{Ker}(p_{\hat{n}_0 *})$ is equal to $\mathbf{B}^{n_0}$ and thus simple. So if $p_{\hat{n}_0 *}|_{\mathbf{B}_{\mathbf{W}}}$ is non-injective for all $n_0 \in [n]$ then we have $\mathbf{B}_{\mathbf{W}} = \mathbf{B}$ and thus $\mathbf{W} = \mathbf{V}$.

We assume first that $\mathbf{W} \neq \mathbf{V}$. Then there exists $n_0 \in [n]$ such that $\mathbf{B}_{\mathbf{W}} \cong p_{\hat{n}_0 *}(\mathbf{B}_{\mathbf{W}})$ via $p_{\hat{n}_0 *}$. Now we have subrepresentations

$$p_{\hat{n}_0 *}(\mathbf{B}_{\mathbf{W}'}) \subset p_{\hat{n}_0 *}(\mathbf{B}_{\mathbf{W}}) \subset \mathrm{Im}(p_{\hat{n}_0 *}) = \bigoplus_{m\in\hat{n}_0} \mathbf{B}^m.$$

By the induction on n, there exists a $\left(\bigoplus_{m\in\hat{n}_0} \mathbf{B}^m\right)$-invariant subspace ${}''\mathbf{W}$ in $p_{\hat{n}_0}(\mathbf{W})$ complementary to $p_{\hat{n}_0}(\mathbf{W}')$. We set $\mathbf{W}''$ to be the pull-back of ${}''\mathbf{W}$ via $p_{\hat{n}_0}|_{\mathbf{W}}$. We are done in the case $\mathbf{W} \neq \mathbf{V}$.

We need to prove the case when $\mathbf{W} = \mathbf{V}$ (i.e., $\mathbf{B}_{\mathbf{W}} = \mathbf{B}$). We assume $\mathbf{B}_{\mathbf{W}'} \neq \mathbf{B}$ since otherwise we set $\mathbf{W}'' = 0$. By a similar argument as before, there exists

$n_0 \in [n]$ such that $\mathbf{B}_{\mathbf{W}'} \cong p_{\hat{n}_0}(\mathbf{B}_{\mathbf{W}'})$ via $p_{\hat{n}_0 *}$. And there exists a $\left(\bigoplus_{m \in \hat{n}_0} \mathbf{B}^m\right)$-invariant subspace $''\mathbf{W}$ in $\bigoplus_{m \in \hat{n}_0} \mathbf{V}^m$ complementary to $p_{\hat{n}_0}(\mathbf{W}')$. By setting $\mathbf{W}'' := {''\mathbf{W}} \oplus \mathbf{V}^{n_0}$, we are done. $\square$

Let $\lambda \in \mathrm{Hom}(\mathbb{C}^*, \mathrm{GL}(\mathbf{V}))$ where $\mathrm{Hom}(\mathbb{C}^*, \mathrm{GL}(\mathbf{V}))$ denotes the set of group homomorphisms. Write $\lambda(t) = (\lambda_v(t))_{v \in I}$ where $\lambda_v \in \mathrm{Hom}(\mathbb{C}^*, \mathrm{GL}(V_v))$. Let $\mathbf{V}_{\mathrm{wt}n} := \bigoplus_{v \in I} (V_v)_{\mathrm{wt}n}$ where $(V_v)_{\mathrm{wt}n} := \{a \in V_v |\, \lambda_v(t).a = t^n a,\, t \in \mathbb{C}^*\}$ (the weight n subspace). The decreasing filtration by weight defines a graded vector space

$$\mathrm{gr}^\lambda \mathbf{V} := \bigoplus_{v \in I} \left(\bigoplus_n (V_v)_{\mathrm{wt} \geq n} / (V_v)_{\mathrm{wt} \geq n+1} \right).$$

Let $\mathbf{B} \in \mathbf{M}_{\mathbf{V}}$. The following two are equivalent:

(i) $\mathbf{B}_0 := \lim_{t \to 0} \lambda(t).\mathbf{B}$ exists in $\mathbf{M}$.

(ii) $\mathbf{V}_{\mathrm{wt} \geq n}$ is $\mathbf{B}$-invariant for each $n \in \mathbb{Z}$.

Suppose $\mathbf{B}_0$ exists. We denote by $\mathbf{B}_{\mathrm{wt} \geq n} := \mathbf{B}|_{\mathbf{V}_{\mathrm{wt} \geq n}}$ which is a subrepresentation of $\mathbf{B}$. Then we have

$$\mathbf{B}_0 \cong \bigoplus_{n \in \mathbb{Z}} \mathbf{B}_{\mathrm{wt} \geq n-1} / \mathbf{B}_{\mathrm{wt} \geq n}.$$

by the obvious isomorphism $\mathbf{V} \cong \mathrm{gr}^\lambda \mathbf{V}$. Lemma 2.3 yields the following.

Lemma 2.4. *If* $\mathbf{B}$ *is semisimple, so is* $\mathbf{B}_0$ *(if it exists).* $\square$

CHAPTER 3

Moduli space of ADHM data for framed Sp-bundles and SO-bundles

In this chapter we study the ADHM data associated to Sp-bundles and SO-bundles. Such data are linear data originally given by Atiyah, Drinfeld, Hitchin and Manin to describe the self-dual equations of instantons [**1**]. These involve real constraints from a hyperkähler moment map (cf. [**34**][**37**]). By Kempf-Ness' theory [**28**], we can avoid complication from the hyperkähler moment map. See Appendix B for details.

The ADHM data are now the ADHM quiver representations satisfying the holomorphic moment map 0 equation which is complex algebraic. The coarse moduli space of such ADHM data is defined as a GIT quotient by a reductive group G in the context of complex algebraic geometry. In general a GIT quotient parametrizes the closed G-orbits. So our goal in this chapter is to find a necessary and sufficient condition for the closedness of the G-orbit of an ADHM quiver representation. The answer will be semisimplicity of the deframed quiver representation (Theorem 3.1). The proof is similar but subtler than Lemma 2.2 because our G is not a general linear group.

Our study of ADHM data will give an algebraic stratification of the GIT quotient of the ADHM data. Moreover the instanton moduli space $\mathcal{M}_n^K$ is a stratum of the GIT quotient. See Chapter 4.

Let

$$\mathbf{M} := \mathrm{Hom}(V,V)^{\oplus 2} \oplus \mathrm{Hom}(V,W) \oplus \mathrm{Hom}(V,W)$$

where V and W are vector spaces. An element of $\mathbf{M}$ is called an ADHM quiver representation. Let $W \cong \mathbb{C}^N$ be any isomorphism. Then we have an obvious ($\mathrm{GL}(V)$-equivariant) linear isomorphism

$$c\colon \mathbf{M} \to \mathrm{Hom}(V,V)^{\oplus 2} \oplus \mathrm{Hom}(V,\mathbb{C})^{\oplus N} \oplus \mathrm{Hom}(V,\mathbb{C})^{\oplus N}.$$

We identify the target space of c with the space of representations of the deframed quiver given in Fig. 1 (cf. Crawley-Boevey's trick [**12**, p.57]). The number of arrows from the bottom vertex to the top is N, and the number of arrows from the top vertex to the bottom is also N.

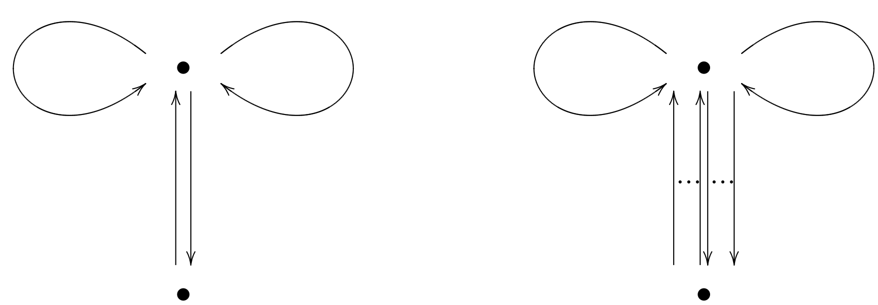

FIGURE 1. the ADHM quiver and the deframed quiver

Let $\varepsilon \in \{-1,+1\}$. In the rest of paper we fix V and W as a k-dimensional vector space with $(\ ,\ )_\varepsilon$ and an N-dimensional vector space with $(\ ,\ )_{-\varepsilon}$ respectively.

Let

$$\mathbf{N} := \{(B_1, B_2, i, j) \in \mathbf{M} |\, B_1 = B_1^*, B_2 = B_2^*, j = i^*\}\,.$$

It is clear that $\mathbf{N}$ isomorphic to the one we used in §1.2.2 by the projection.

We use the notation $\overset{\perp}{\oplus}$ for a direct sum of vector spaces orthogonal with respect to the given nondegenerate forms.

Since the $\mathrm{GL}(V) \times \mathrm{GL}(\mathbb{C})$-orbit of $c(x)$ coincides with the $\mathrm{GL}(V)$-orbit of $c(x)$, x has a closed $\mathrm{GL}(V)$-orbit if and only if $c(x)$ is semisimple by Lemma 2.2. We have a similar result for $G(V)$ as below.

Theorem 3.1. *Let $x := (B_1, B_2, i, j) \in \mathbf{N}$. Then the followings are equivalent.*

(a) *$c(x)$ is semisimple.*

(b) *there exists a decomposition*

$$V = V^s \overset{\perp}{\oplus} \overset{\perp}{\bigoplus_a} V_a \overset{\perp}{\oplus} \overset{\perp}{\bigoplus_b} (V_b \oplus V_b')$$

such that

(1) *V_b and V_b' are dual isotropic subspaces of V for each index b;*

(2) *the summands $c(B_1|_{V^s}, B_1|_{V^s}, i, j|_{V^s})$, $(B_1|_{V_a}, B_2|_{V_a}, 0, 0)$, $(B_1|_{V_b}, B_2|_{V_b}, 0, 0)$ and $(B_1|_{V_b'}, B_2|_{V_b'}, 0, 0)$ of $c(x)$ are simple quiver representations.*

(c) *$G(V).x$ is closed in $\mathbf{N}$.*

We remark that $(B_1|_{V_b}, B_2|_{V_b}, 0, 0)$ and $(B_1|_{V_b'}, B_2|_{V_b'}, 0, 0)$ in the above statement are dual to each other.

The proof will be given in §3.2.

Let us recall the two GIT stability conditions for the $\mathrm{GL}(V)$-actions on $\mathbf{M}$ (cf. [**41**, Chap. 2]).

Definition 3.2. $(B_1, B_2, i, j) \in \mathbf{M}$ is *stable* (resp. *costable*) if the following condition holds:

(1) (*stability*) there is no subspace $S \subsetneq V$ such that $B_1(S) \subset S, B_2(S) \subset S$ and $\mathrm{Im} i \subset S$,

(2) (*costability*) there is no nonzero subspace $T \subset V$ such that $B_1(T) \subset T, B_2(T) \subset T$ and $T \subset \mathrm{Ker} j$.

Note that $x \in \mathbf{M}$ is stable and costable if and only if $c(x)$ is simple when $W \neq 0$.

Let $*_{\mathbf{M}} \colon \mathbf{M} \to \mathbf{M}$, $(B_1, B_2, i, j) \mapsto (B_1^*, B_2^*, -j^*, i^*)$. Then $\mathbf{N}$ is the $*_{\mathbf{M}}$-invariant subspace of $\mathbf{M}$. If $x \in \mathbf{M}$ is stable (resp. costable) then $*_{\mathbf{M}}(x)$ is costable (resp. stable). Therefore stability and costability are equivalent on $\mathbf{N}$. See Corollary 5.13. In particular, $x \in \mathbf{N}$ is stable if and only if $c(x)$ is simple when $W \neq 0$.

Let $\mu \colon \mathbf{N} \to \mathfrak{g}(V)$ be the moment map given by $(B_1, B_2, i, j) \mapsto [B_1, B_2] + ij$. Let $\mu^{-1}(0)^{\mathrm{reg}}$ be the locus of stable-costable (abbr. regular) quiver representations in $\mu^{-1}(0)$. We have $\mu^{-1}(0)^{\mathrm{reg}} \subset \mu^{-1}(0)^{\mathrm{sm}}$ since if $x \in \mathbf{N}$ is stable then the differential $d\mu_x$ is surjective. By Theorem 3.1 the image of $\mu^{-1}(0)^{\mathrm{reg}}$ of the GIT quotient map $\mu^{-1}(0) \to \mu^{-1}(0) /\!\!/ G(V)$ is a Zariski open subset of $\mu^{-1}(0) /\!\!/ G(V)$ and it is a $G(V)$-orbit space $\mu^{-1}(0)^{\mathrm{reg}}/G(V)$. In particular $\mu^{-1}(0)^{\mathrm{reg}}/G(V)$ is a smooth quasi-affine scheme. In fact it is also irreducible by [**7**, Propositions 2.24 and 2.25].

Definition 3.3. Let $\varepsilon = -1$ (resp. $\varepsilon = +1$). We say $x = (B_1, B_2, i, j) \in \mu^{-1}(0)$ is a SO-*datum* (resp. Sp-*datum*) if $[B_1, B_2] + ij = 0$.

The rest of this chapter is devoted to prove Theorem 3.1.

3.1. Semisimple ADHM quiver representations

We study a generality for representations in $\mathbf{N}$ first. And then we give an equivalent statement to (b) of Theorem 3.1 at the end of this section.

Lemma 3.4. *Let $x \in \mathbf{N}$. Let V', W' be subspaces of V, W respectively such that (V', W') is x-invariant. Then $(V'^{\perp}, W'^{\perp})$ is also x-invariant. Hence, $y^{\perp} := x|_{(V'^{\perp}, W'^{\perp})}$ is defined.*

Proof. Let $x = (B_1, B_2, i, i^*)$. We need to show $B_n((V')^{\perp}) \subset (V')^{\perp}$, $i((W')^{\perp}) \subset (V')^{\perp}$ and $i^*((V')^{\perp}) \subset (W')^{\perp}$ $(n = 1, 2)$. For $v \in (V')^{\perp}, v' \in V'$, we have $(B_n(v), v')_V = (v, B_n(v'))_V = 0$ so that $B_n((V')^{\perp}) \subset (V')^{\perp}$.

For $v \in V'$ and $w \in (W')^{\perp}$, we have $(i(w), v)_V = (w, i^*(v))_W = 0$ so that $i((W')^{\perp}) \subset (V')^{\perp}$.

For $v \in (V')^{\perp}$ and $w \in W'$, we have $(w, i^*(v))_W = (i(w), v)_V = 0$ so that $i^*((V')^{\perp}) \subset (W')^{\perp}$. $\square$

Definition 3.5. A subspace of V is *nondegenerate* (resp. *isotropic*) if the restriction of $(\ ,\ )_V$ is nondegenerate (resp. 0). Two isotropic subspaces are *dual*, if they are complementary and their direct sum is nondegenerate.

Let $x \in \mathbf{M}$. Let y, z be any subrepresentations of x. We write $x = y \overset{\perp}{\oplus} z$, if y and z satisfies $x = y \oplus z$ and $y^\perp = z$. Let (V', W') be the representation space of y. If $W' = 0$, we say y is *of frame* 0, if $W' = 0$. When y is of frame 0, we say y is *nondegenerate* (resp. *isotropic*) if so is V'.

For $y \in \mathbf{M}_{(V',0)}$ and $z \in \mathbf{M}_{(V'',0)}$, we say they are *dual isotropic* if so are V' and V''.

LEMMA 3.6. *Let $x \in \mathbf{N}$. Let y be a simple subrepresentation of x. If y is of frame 0, it is either nondegenerate or isotropic.*

PROOF. By Lemma 3.4 and the simplicity of y, $T \cap T^\perp = 0$ or T. In the first case (resp. the second case), $(\ ,\ )_T$ is nondegenerate (resp. 0). □

If y and z are dual isotropic then $y = z^*$. To see this, let us write $y = (B'_1, B'_2, 0, 0)$, $z = (B''_1, B''_2, 0, 0)$ and $\tilde{B}_n := B'_n \oplus B''_n$ for $n = 1, 2$. Since $\tilde{B}_n = \tilde{B}^*_n$, we have $(B'_n v', v'')_V = (\tilde{B}_n v', v'')_V = (v', \tilde{B}_n v'')_V = (v', B''_n v'')_V$ where $v' \in V'$ and $v'' \in V''$. Thus $B'_n = B''^*_n$.

DEFINITION 3.7. Let $x = (B_1, B_2, i, j) \in \mathbf{N}$. We define

$$V^s := \sum_P P(B_1, B_2) i(W)$$

where P runs over all the 2-variable polynomials. Let $x^s := x|_{(V^s, W)} \in \mathbf{M}_{(V^s, W)}$.

It is clear that x^s is stable.

LEMMA 3.8. *Let $x \in \mathbf{N}$. If $c(x)$ is semisimple, V^s is nondegenerate.*

Proof. Let $Z := V^s \cap (V^s)^{\perp}$. By Lemma 3.4, Z is an x^s-invariant subspace and thus we have a subrepresentation $z := x|_{(Z,0)}$ of x^s. By Lemma 2.3, $c(x^s)$ is semisimple and there exists $z' \subset c(x^s)$ such that $c(x^s) = z \oplus z'$. Since the second factor of the representation space of z is W, we have $c(x^s) = z'$. This implies $z = 0$ and thus $Z = 0$. This proves V^s is nondegenerate. $\square$

Now we are ready to prove Theorem 3.1. In the actual proof it is convenient to replace (b) into the following equivalent statement:

(b') x is decomposed as

$$x = x^s \overset{\perp}{\bigoplus_a} y_a \overset{\perp}{\oplus} \overset{\perp}{\bigoplus_b} (z_b \oplus z_b').$$

where $c(x^s)$ and all the other summands are simple and z_b, z_b' are dual isotropic.

We also assume for convenience that the indices b are arranged as $1, 2, 3, ...$ consecutively in $\mathbb{Z}_{\geq 1}$ unless empty.

3.2. Proof of the equivalences on semisimplicity and orbit-closedness (Theorem 3.1)

We start to prove Theorem 3.1. The order of proof is (i) (a)⇔(b), (ii) (a,b)⇒(c), (iii) (c)⇒(a).

3.2.1. Proof of (a)⇔(b).

It is clear that (b) implies (a).

We prove the opposite direction (a)⇒(b').

By Lemma 3.8, we have $x = x^s \overset{\perp}{\oplus} (x^s)^\perp$. Our goal is to find $y_a, z_b, z'_b \subset (x^s)^\perp$ satisfying the statement of (b). Let $z \subset (x^s)^\perp$ be a simple subrepresentation. Since z is of frame 0, it is either nondegenerate or isotropic by Lemma 3.6.

We assume first that z is nondegenerate. Let $y_1 := z$. Since $c(z^\perp)$ is semisimple (Lemma 2.3), we are done by the induction on $\dim V$.

We assume that z is isotropic.

Lemma 3.9. *If z is given as above, there exists a simple subrepresentation z' of frame 0 in $(x^s)^\perp$ which is dual isotropic to z.*

Proof. By Lemma 3.8, V^s and thus $(V^s)^\perp$ are nondegenerate. Since $(x^s)^\perp$ is semisimple it has a simple direct summand $w := (B_1|_T, B_2|_T, 0, 0)$ such that T is not orthogonal to Z. Thus $Z \neq Z \cap T^\perp$ and $T \neq T \cap Z^\perp$. By Lemma 3.4 and simplicity of z, w, we have $Z \cap T^\perp = T \cap Z^\perp = 0$. This means $(\ ,\ )_V$ on $Z \times T$ is nondegenerate.

If T is isotropic, then we are done by $z' := x|_{(T,0)}$.

If T is not isotropic, then it is nondegenerate by Lemma 3.6. By simplicity of z, w, we have $Z \cap T = 0$. Hence $Z \oplus T$ is nondegenerate.

We will find a B_1, B_2-invariant isotropic subspace Z' in $Z \oplus T$ which is complementary to Z. Let T' be the orthogonal complement of T in $Z \oplus T$. Since $(\ ,\ )_V$ is nondegenerate on $Z \times T$, we have $Z \cap T' = 0$.

By Lemma 3.4, we have a subrepresentation $w' := x|_{(T',0)}$. Let $p_1 \colon T \oplus T' \to T$ and $p_2 \colon T \oplus T' \to T'$ be the projections. Since $Z \cap T = Z \cap T' = 0$, $p_1|_Z$ and

$p_2|_Z$ induce isomorphisms $z \cong w$ and $z \cong w'$ respectively. So we have $w \cong w'$. Let $g \in \mathrm{Isom}(T, T')$ such that $g.w' = w$. Then we have $Z = \{(v, g(v))|\, v \in T\}$. This means $gB_n(v) = B_n g(v)$ ($v \in T$, $n = 1, 2$).

Let

$$Z' := \{(v, -g(v))|v \in T\}.$$

Since $-gB_n(v) = -B_n g(v)$, we have $B_n(Z') \subset Z'$ ($n = 1, 2$). Let $z' := x|_{(Z',0)}$. Since $((v, -g(v)), (v', -g(v'))_V = ((v, g(v)), (v', g(v'))_V = 0$ for $v, v' \in V$, it is isotropic. Since $Z \cap Z' = 0$, $Z \oplus Z'$ is nondegenerate.

It remains to show z' is simple. Since $z' \cong w$ by the isomorphism induced by $p_1|_{Z'}$, we are done. $\square$

Let $z_1 := z$ and $z'_1 := z'$ where z' is given as in Lemma 3.9. By the induction on $\dim V$, we have the decomposition

$$(z_1 \oplus z'_1)^\perp = x^s \overset{\perp}{\oplus} \overset{\perp}{\bigoplus_a} y_a \overset{\perp}{\oplus} \overset{\perp}{\bigoplus_{b \neq 1}} (z_b \oplus z'_b).$$

As a result we have

$$x = x^s \overset{\perp}{\oplus} \overset{\perp}{\bigoplus_a} y_a \overset{\perp}{\oplus} \overset{\perp}{\bigoplus_{b \geq 1}} (z_b \oplus z'_b).$$

This proves (b'). $\square$

3.2.2. Proof of (a,b)⇒(c). The strategy is to show that whenever $x_0 := \lim_{t \to 0} \lambda(t).x$ exists for any given group homomorphism $\lambda \in \mathrm{Hom}(\mathbb{C}^*, G)$, it is contained in $G(V).x$. Then Iwahori's theorem [**26**] asserts $G(V).x$ is closed.

Let λ, x_0 be given as above. Let h be the highest weight. Then $h \geq 0$ and $-h$ is the lowest height, which are left as an exercise. If $h = 0$ then λ is trivial. So we may assume $h > 0$. Then $x|_{(V_{\mathrm{wt}h},0)}$ is a subrepresentation of x.

Let Z be a subspace $V_{\mathrm{wt}h}$ such that $x|_{(Z,0)}$ is simple. Let $z := x|_{(Z,0)}$. It is a subrepresentation of both x, x_0. We showed in the above proof of (a)⇒(b') that there exists a decomposition of x as in (b') with $z_1 = z$. Similarly there exists a decomposition of x_0 with $w_1 = z$ as in (b'):

$$x_0 = x_0^s \overset{\perp}{\oplus} \overset{\perp}{\bigoplus_{\alpha}} u_\alpha \overset{\perp}{\oplus} \overset{\perp}{\bigoplus_{\beta \geq 1}} (w_\beta \oplus w'_\beta).$$

Here we used that x_0 is semisimple (Lemma 2.4). We denote by $V_0^s, U_\alpha, W_\beta, W'_\beta$ (the factor of) the representation spaces of $x_0^s, u_\alpha, w_\beta, w'_\beta$ respectively.

The canonical isomorphism $V^s \oplus \bigoplus_a Y_a \oplus \bigoplus_{b \neq 1} (Z_b \oplus Z'_b) \overset{\cong}{\to} Z^\perp/Z$ is a symplectic isomorphism and induces an isomorphism

$$x^s \oplus \bigoplus_a y_a \oplus \bigoplus_{b \neq 1} (z_b \oplus z'_b) \cong z^\perp/z. \tag{3.1}$$

We need a similar isomorphism for a limit of $z^\perp/z$ with respect to λ. Let us define such a limit first. Since $\lambda(t)(Z) \subset Z$ and $\lambda(t)(Z^\perp) \subset Z^\perp$ for any $t \in \mathbb{C}^*$, we have the induced group homomorphism $\overline{\lambda}(t) \in \mathrm{Hom}(\mathbb{C}^*, G(Z^\perp/Z))$. The existence of the limit x_0 asserts $(z^\perp/z)_0 := \lim_{t \to 0} \overline{\lambda}(t).z^\perp/z$ also exists in $\mathbf{M}_{(Z^\perp/Z,W)}$. Now the canonical isomorphism $V_0^s \oplus \bigoplus_\alpha U_\alpha \oplus \bigoplus_{\beta \neq 1} (W_\beta \oplus W'_\beta) \overset{\cong}{\to} Z^\perp/Z$ is a symplectic isomorphism and induces an isomorphism

$$x_0^s \oplus \bigoplus_\alpha u_\alpha \oplus \bigoplus_{\beta \neq 1} (w_\beta \oplus w'_\beta) \cong (z^\perp/z)_0. \tag{3.2}$$

By the dimension induction we have $(z^\perp/z)_0 \in G(Z^\perp/Z).z^\perp/z$. Therefore by (3.1) and (3.2), there exists a form-preserving isomorphism

$$g\colon V^s \oplus \bigoplus_a Y_a \oplus \bigoplus_{b\neq 1}(Z_b \oplus Z'_b) \overset{\cong}{\to} V_0^s \oplus \bigoplus_\alpha U_\alpha \oplus \bigoplus_{\beta\neq 1}(W_\beta \oplus W'_\beta)$$

such that

$$x_0^s \oplus \bigoplus_\alpha u_\alpha \oplus \bigoplus_{\beta\neq 1}(w_\beta \oplus w'_\beta) = g.\left(x^s \oplus \bigoplus_a y_a \oplus \bigoplus_{b\neq 1}(z_b \oplus z'_b)\right).$$

Now it suffices to prove that there exists a form-preserving $h \in \mathrm{Isom}(Z_1 \oplus Z'_1, W_1 \oplus W'_1)$ such that $h.(z_1 \oplus z'_1) = w_1 \oplus w'_1$. For, we will have $g \oplus h \in G(V)$ and $x_0 = g.x$ so that Iwahori's theorem asserts $G(V).x$ is closed in $\mathbf{M}$. The proof comes from that z and z' are dual. More precisely it is reduced to the following lemma:

Lemma 3.10. *Let (L, L') and (M, M') be pairs of dual isotropic subspaces of V. Let $B \in \mathrm{End}(L)$ and $g \in \mathrm{Isom}(L, M)$. Then we have the following:*

(1) *There exists a unique $\tilde{B} \in \mathfrak{p}(L \oplus L')$ such that $\tilde{B}(L) \subset L$, $\tilde{B}(L') \subset L'$ and $\tilde{B}|_L = B$.*

(2) *There exists a unique form-preserving $\tilde{g} \in \mathrm{Isom}(L \oplus L', M \oplus M')$ such that $\tilde{g}(L) = M$, $\tilde{g}(L') = M'$ and $\tilde{g}|_L = g$.*

We omit the proof.

This completes the proof of (a,b')⇒(c). □

3.2.3. Proof of (c)⇒(a). We will prove (a) assuming (c): the orbit-closedness.

Let x' be a maximal direct summand of x such that the representation space of x' is either $(V', 0)$ or (V', W) for some $V' \subset V$ and $c(x')$ is 0 or semisimple. Note such x' is unique, because if x'' is another, $x'+x''$ is a direct summand with $c(x'+x'')$ being semisimple. Here we used the canonical isomorphism $x'+x'' \cong (x' \oplus x'')/x' \cap x''$ and the splitting of $x' + x'' \subset x$ by the composite $x \to x' \oplus x'' \to (x' \oplus x'')/x' \cap x''$.

We need to prove x' is equal to x. We suppose the contrary $x' \neq x$. We divide the proof into the cases: (i) V' is nondegenerate, (ii) V' is not nondegenerate.

Case (i). We have a decomposition $x = x' \oplus (x')^\perp$. Let z be a subrepresentation of x with the representation space $(Z, 0)$ or (Z, W) for some $Z \subset V$ such that $c(z)$ is simple. If z has the representation space $(Z, 0)$, it is either nondegenerate or isotropic by Lemma 3.6.

If z is nondegenerate we have decomposition $x = x' \oplus z \oplus (z^\perp \cap (x')^\perp)$. This is absurd to maximality of x'.

Suppose z is isotropic. Let $\lambda \in \mathrm{Hom}(\mathbb{C}^*, G(V))$ such that

$$V_{\mathrm{wt}1} = Z, \quad V_{\mathrm{wt}\geq 0} = z^\perp, \quad V_{\mathrm{wt}\geq -1} = V.$$

The limit $x_0 = \lim_{t\to 0} \lambda(t).x$ is isomorphic to $z \oplus z^\perp/z \oplus x/z^\perp$. Note that x' is a direct summand of $z^\perp/z$. Since x is isomorphic to x_0 by the orbit-closedness, it has a direct summand isomorphic to $z \oplus x'$. This is absurd to maximality of x'.

If z has the representation space (Z, W), Z is nondegenerate since $z \cap z^\perp = 0$. Thus we have decomposition $x = x' \oplus z \oplus (z^\perp \cap (x')^\perp)$. This is absurd to maximality of x'.

Case (ii). In the case $V' \cap (V')^\perp$ is a nonzero isotropic subspace of V. Let $z := x' \cap (x')^\perp$. Then z is a semisimple isotropic representation of frame 0. Let

$\lambda \in \mathrm{Hom}(\mathbb{C}^*, G(V))$ such that

$$V_{\mathrm{wt}1} = Z, \quad V_{\mathrm{wt}\geq 0} = z^{\perp}, \quad V_{\mathrm{wt}\geq -1} = V.$$

The limit $x_0 = \lim_{t\to 0} \lambda(t).x$ is isomorphic to $z \oplus z^{\perp}/z \oplus x/z^{\perp}$. Note that x'/z is a direct summand of $z^{\perp}/z$ and that $x/z^{\perp}$ and z are dual to each other (so that $x/z^{\perp}$ is semisimple). Since x is isomorphic to x_0 by the orbit-closedness, it has a direct summand isomorphic to $z \oplus x'/z \oplus x/z$. This is absurd to maximality of x'.

This finishes the proof of $x' = x$. □

3.3. Generalization of Theorem 3.1 to quivers with more than 2 loops

We end up this chapter with a remark. Main Theorem 3.1 is valid under a generalization of the ADHM quiver as follows. We replace the two loops into l loops where l is an arbitrary nonnegative integer. We also consider the deframed quiver. See Fig. 2 for $l = 3$.

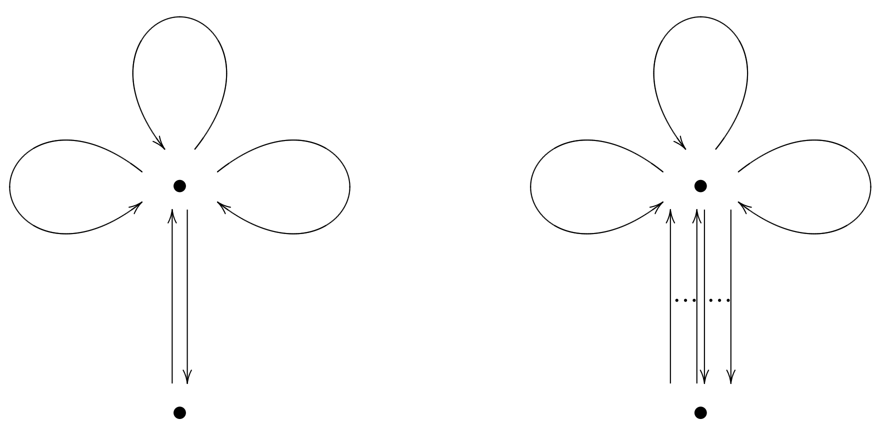

FIGURE 2. a generalized ADHM quiver and its deframed quiver ($l = 3$)

Now **N** is now modified to $\{(B_1, B_2, ..., B_l, i, j) |\, B_n = B_n^*, j = i^*, n = 1, 2, ..., l\}$. The stability notions in Definition 3.2 are obvious to generalize for **N**. The statement of Theorem 3.1 is corrected: only the corresponding statement in (b) should be changed by increasing the number of B_i in an obvious way. Similarly the statement (b') is corrected by generalizing the meaning of dual isotropic.

The proof of the generalized theorem goes similarly as above. We omit the details.

CHAPTER 4

Strafication of $\mu^{-1}(0)/\!\!/G(V)$

In this chapter we give stratification of $\mu^{-1}(0)/\!\!/G(V)$ using Theorem 3.1. In fact, it is the standard one if $\mu^{-1}(0)/\!\!/G(V)$ coincides with the Uhlenbeck space. However $\mu^{-1}(0)/\!\!/G(V)$ is not the Uhlenbeck space exactly when $G(W) = \mathrm{O}(3)$.

To emphasize $k = \dim V$, we use G_k, μ_k for $G(V), \mu$ respectively. It was denoted by G_k' in §1. Let $x := (B_1, B_2, i, j) \in \mu_k^{-1}(0)$. Assume $G_k.x$ is closed in $\mathbf{N}$. By Theorem 3.1, $(B_1|_{V^s}, B_2|_{V^s}, i, j|_{V^s})$ corresponds to a framed vector bundle, and $(B_1|_{V_a}, B_2|_{V_a}, 0, 0)$, $(B_1|_{V_b}, B_2|_{V_b}, 0, 0)$ and $(B_1|_{V_b'}, B_2|_{V_b'}, 0, 0)$ are all commuting pairs in $\mathfrak{p}(V)$. Let us focus on commuting pairs here. We simplify the situation: $B_1, B_2 \in \mathfrak{p}(V)$ with $[B_1, B_2] = 0$, where V is orthogonal or symplectic. By the semisimplicity (Theorem 3.1), B_1 and B_2 are simultaneously diagonalizable. Therefore all V_a, V_b, and V_b' of Theorem 3.1 (b) are 1-dimensional. When V is symplectic, V_a does not appear and $B_1|_{V_b \oplus V_b'}, B_2|_{V_b \oplus V_b'}$ are scalars, as $\mathfrak{p}(V) = \mathbb{C}$ if $\dim V = 2$. When V is orthogonal, the index set of b can be absorbed in the index set of a. We have the summary as follows.

Theorem 4.1. *Let $S^n\mathbb{A}^2$ be the n^{th} symmetric product of $\mathbb{A}^2$.*

(1) *Suppose V is symplectic. Then there exists a canonical set-theoretic bijection*

$$\mu_k^{-1}(0)/\!\!/G_k = \coprod_{0 \le k' \le k} \mu_{k'}^{-1}(0)^{\mathrm{reg}}/G_{k'} \times S^{\frac{k-k'}{2}}\mathbb{A}^2.$$

(2) *Suppose V is orthogonal. Then there exists a canonical set-theoretic bijection*

$$\mu_k^{-1}(0)/\!\!/G_k = \coprod_{0\le k'\le k} \mu_{k'}^{-1}(0)^{\mathrm{reg}}/G_{k'} \times S^{k-k'}\mathbb{A}^2.$$

Note that this stratification is nothing but the one of the Uhlenbeck space except the case $G = \mathrm{SO}(3)$. For $G = \mathrm{SO}(3)$, $\mu_k^{-1}(0)/\!\!/G_k$ is different from the Uhlenbeck space, where the symmetric product is $S^{\frac{k-k'}{4}}\mathbb{A}^2$ by the formula (1.1).

Since $\mu_k^{-1}(0)^{\mathrm{reg}}/G_k$ is a free quotient (unless $\mu_k^{-1}(0)^{\mathrm{reg}} = \emptyset$), we have $\dim \mu_k^{-1}(0)^{\mathrm{reg}}/G_k = \dim \mathbf{N} - 2\dim G_k$. Using $\dim \mathfrak{p} - \dim \mathfrak{t} = k$ if V is orthogonal, and $\dim \mathfrak{p} - \dim \mathfrak{t} = -k$ if V is symplectic, we have

$$\dim \mu_k^{-1}(0)^{\mathrm{reg}}/G_k = \begin{cases} k(N-2) & \text{if } V \text{ is symplectic,} \\ k(N+2) & \text{if } V \text{ is orthogonal,} \end{cases} \tag{4.1}$$

whenever $\mu_k^{-1}(0)^{\mathrm{reg}} \neq \emptyset$. Therefore we get the dimension of the strata. Hence by §1.2.2, we obtain the following.

LEMMA 4.2. (1) *Assume V is symplectic. Then $\mu_{k'}^{-1}(0)^{\mathrm{reg}}/G_{k'} \times S^{\frac{k-k'}{2}}\mathbb{A}^2$ is nonempty if and only if either $N = 3$ and $k' \in 4\mathbb{Z}_{\ge 0}$, or $N \ge 4$ and $k' \in 2\mathbb{Z}_{\ge 0}$. If it is nonempty, it is of dimension $k'(N-2) + (k-k')$*

(2) *If V is orthogonal then $\mu_{k'}^{-1}(0)^{\mathrm{reg}}/G_{k'} \times S^{k-k'}\mathbb{A}^2$ is nonempty and of dimension $k'(N+2) + 2(k-k')$.* □

REMARK 4.3. (1) Suppose V is symplectic. By $\mathrm{Spin}(3) \cong \mathrm{SU}(2)$, if $N = 3$ and $k \in 4\mathbb{Z}_{\ge 0}$ then $\mu_k^{-1}(0)^{\mathrm{reg}}/G_k$ is irreducible (§1.2.2). Hence, the strata of $\mu_4^{-1}(0)/\!\!/G_4$ are $\mu_4^{-1}(0)^{\mathrm{reg}}/G_4$ and $S^2\mathbb{A}^2$, both of which are irreducible varieties of dimension 4. See an alternative proof in Theorem 7.3 (3).

By $\mathrm{Spin}(4) \cong \mathrm{SU}(2) \times \mathrm{SU}(2)$, if $N = 4$ and $k \in 2\mathbb{Z}_{\geq 0}$ then $\mu_k^{-1}(0)^{\mathrm{reg}}/G_k$ has the $(k/2+1)$ irreducible components.

(2) We will prove in Theorem 6.1 that all the strata in Lemma 4.2 (2) are irreducible normal.

CHAPTER 5

Moduli spaces of framed vector bundles with symplectic and orthogonal structures

In this chapter we show that $\mathcal{M}_n^K = \mu^{-1}(0)^{\mathrm{reg}}/G(V)$ is scheme-theoretically isomorphic to the moduli space of framed vector bundles with symplectic or orthogonal structures. The latter space will be constructed as a closed smooth subscheme of the vector bundle locus of the ordinary Gieseker moduli space.

This chapter is organized as follows.

In the last section §5.4, we construct the above moduli scheme and the isomorphism with $\mu^{-1}(0)^{\mathrm{reg}}/G(V)$ via Barth's correspondence following Donaldson's argument [**13**]. But before §5.4, we need preliminary steps for the construction.

In §5.1, we review definitions and basic properties of framed vector bundles with symplectic and orthogonal structures.

In §5.2, we review the monad construction and the dual monad construction. More precisely, in §§5.2.1,5.2.2, we review the monad construction and Barth's correspondence for ordinary framed vector bundles. In §5.2.3, we introduce dual monads. The dual monad is necessary to study symplectic and orthogonal structures in term of maps from monads to dual monads. In §5.2.4, we modify Barth's correspondence to involve the additional $G(V)$-structure and the map between a monad and the dual monad.

In §5.3, we restrict modified Barth's correspondence to our moduli space $\mu^{-1}(0)^{\mathrm{reg}}/G(V)$. Then the image of the restriction turns out to be the set of framed vector bundles with symplectic or orthogonal structures.

The above streamline in §5.2–5.3 is included in [**13**]. See also [**9**].

5.1. Framed orthogonal bundles and framed symplectic bundles

We first define framed vector bundles with orthogonal or symplectic structure in a general context.

Let X be an algebraic variety. Let X' be a closed subvariety of X.

DEFINITION 5.1. A *framed sheaf* of rank N on X along X', is a pair of a sheaf E locally free along X' and $\Phi\colon E|_{X'} \xrightarrow{\cong} \mathcal{O}_{X'} \otimes \mathbb{C}^N$ for some N. We call Φ *frame*.

For two framed sheaves (E, Φ) and (E', Φ'), we say they are *isomorphic* if their ranks are same and there exists a sheaf isomorphism $\sigma\colon E \to E'$ such that $\Phi = \Phi'\sigma|_{X'}$.

REMARK 5.2. Huybrechts and Lehn define an isomorphism between (E, Φ) and (E', Φ') as follows ([**25**, Definition 1.4]): $(E, \Phi) \cong (E', \Phi')$ if there exists a sheaf isomorphism $\sigma\colon E \to E'$ such that $\Phi = e\Phi'\sigma|_{X'}$ for some $e \in \mathbb{C}$. Our definition looks stricter than their one, but it does not in fact because $e\sigma$ gives the isomorphism in our sense.

Let us fix a linear isomorphism $\varphi \in \mathrm{Isom}(W, W^\vee)$ where $W = \mathbb{C}^N$.

Definition 5.3. Let (E,Φ) be a framed vector bundle. A *φ-structure* on (E,Φ) is an isomorphism $\phi\colon E \xrightarrow{\cong} E^\vee$ satisfying

$$\phi|_{X'} = \Phi^\vee \phi_W \Phi,$$

where $\phi_W := \mathrm{Id}_{\mathcal{O}_{X'}} \otimes \varphi\colon \mathcal{O}_{X'} \otimes W \to \mathcal{O}_{X'} \otimes W^\vee$.

Let $(E,\Phi), (E',\Phi')$ be framed vector bundles with φ-structures ϕ, ϕ' respectively. We say they are *isomorphic* if there exists an isomorphism $\sigma\colon (E,\Phi) \to (E',\Phi')$ as framed sheaves such that $\phi = \sigma^\vee \phi' \sigma$.

Definition 5.4. A *framed orthogonal bundle* (resp. a *framed symplectic bundle*) on an algebraic variety X is a framed vector bundle (E,Φ) with a φ-structure ϕ satisfying $\phi = \phi^\vee$ (resp. $\phi = -\phi^\vee$).

In the above, we used $E = E^{\vee\vee}$ by the canonical pairing.

On each fibre E_x of E, $\phi|_x$ defines a nondegenerate form $(\ ,\ )_x$ on E_x by $(s,s')_x := \langle \phi|_x(s'), s\rangle$, where $x \in X$ and $\langle\ ,\ \rangle$ denotes the canonical pairing. So if $\phi = \phi^\vee$ (resp. $\phi = -\phi^\vee$) then $(\ ,\ )_x$ is an orthogonal form (resp. a symplectic form).

If ϕ defines an orthogonal (resp. symplectic) form then so does φ along X'. Thus we call φ-structure as orthogonal (resp. symplectic) structure or $\mathrm{SO}(N)$-structure (resp. $\mathrm{Sp}(N/2)$-structure). This clarifies the meaning of the above definition.

5.2. Construction from monad and dual monads

In this section we review relation among monads, ADHM quiver representations and framed torsion-free sheaves on $\mathbb{P}^2$. The relation is called Barth's correspondence. For the dual vector bundles we have also a similar description from dual monads. A Barth-type correspondence will be obtained for the framed vector bundles with additional φ-structures (Theorem 5.17).

Let $\mathcal{O}(m)^n := \mathcal{O}_{\mathbb{P}^2}(m)^{\oplus n}$ for short. We use the notation V, V', W, W' for vector spaces.

5.2.1. recollection of monad construction. Let M be a sequence of sheaves of the following form:

$$M : \mathcal{O}(-1) \otimes V \xrightarrow{\alpha} \begin{matrix} \mathcal{O} \otimes V^{\oplus 2} \\ \oplus \\ \mathcal{O} \otimes W \end{matrix} \xrightarrow{\beta} \mathcal{O}(1) \otimes V. \tag{5.1}$$

Definition 5.5. M is a *monad* if it is a complex, α is injective and β is surjective.

Let M' be a sequence of sheaves as above:

$$M' : \mathcal{O}(-1) \otimes V' \xrightarrow{\alpha'} \begin{matrix} \mathcal{O} \otimes V'^{\oplus 2} \\ \oplus \\ \mathcal{O} \otimes W' \end{matrix} \xrightarrow{\beta'} \mathcal{O}(1) \otimes V'.$$

We define $\mathrm{Hom}(M, M')$ as the space of triples (a, b, c) fitting in the commuting diagram:

$$
\begin{tikzcd}
M : \mathcal{O}(-1) \otimes V \arrow[r, "\alpha"] \arrow[d, "a"] & \begin{matrix} \mathcal{O} \otimes V^{\oplus 2} \\ \oplus \\ \mathcal{O} \otimes W \end{matrix} \arrow[r, "\beta"] \arrow[d, "b"] & \mathcal{O}(1) \otimes V \arrow[d, "c"] \\
M' : \mathcal{O}(-1) \otimes V' \arrow[r, "\alpha'"] & \begin{matrix} \mathcal{O} \otimes V'^{\oplus 2} \\ \oplus \\ \mathcal{O} \otimes W' \end{matrix} \arrow[r, "\beta'"] & \mathcal{O}(1) \otimes V'
\end{tikzcd}
\tag{5.2}
$$

From now on we regard a, b, c as linear maps if no confusion arises.

If M, M' are monads, their cohomology sheaves are concentrated in the middle. We denote the (middle) cohomology sheaves by $E = \mathrm{Ker}(\beta)/\mathrm{Im}(\alpha), E' = \mathrm{Ker}(\beta')/\mathrm{Im}(\alpha')$ respectively. Let $(a, b, c) \in \mathrm{Hom}(M, M')$. We denote the induced homomorphism by $\bar{b}\colon E \to E'$. By [**49**, Ch. II, Lem. 4.1.3] (see also [**44**, Lemma 5.8], [**29**, 2.2.1]) we have the following.

Lemma 5.6. *The canonical homomorphism $H\colon \mathrm{Hom}(M, M') \to \mathrm{Hom}(E, E')$, $(a, b, c) \mapsto \bar{b}$, is an isomorphism. In particular if $M = M'$, $E = E'$ and $H(a, b, c) = \mathrm{Id}_E$, we have $a = c = \mathrm{Id}_V$ and $b = \mathrm{Id}_V^{\oplus 2} \oplus \mathrm{Id}_W$.*

Let Q be the ADHM quiver and V, W be vector spaces assigned at the upper and lower vertices of Q as before (cf. Fig. 1). Let $\mathbf{M}_{(V,W)} := \mathrm{Hom}(V, V)^{\oplus 2} \oplus \mathrm{Hom}(W, V) \oplus \mathrm{Hom}(V, W)$ (the space of ADHM quiver representations). Similarly we define $\mathbf{M}_{(V',W')}$ for another pair (V',W'). Let $x := (B_1, B_2, i, j) \in \mathbf{M}_{(V,W)}$ and $x' := (B_1', B_2', i', j') \in \mathbf{M}_{(V',W')}$. Recall that $\mathrm{Hom}(x, x')$ is the space of pairs (a, τ) fitting in the following commuting diagrams:

$$\begin{tikzcd} V \arrow[r, "a"] \arrow[d, "B_1"] & V' \arrow[d, "B_1'"] \\ V \arrow[r, "a"] & V' \end{tikzcd} \quad \begin{tikzcd} V \arrow[r, "a"] \arrow[d, "B_2"] & V' \arrow[d, "B_2'"] \\ V \arrow[r, "a"] & V' \end{tikzcd} \quad \begin{tikzcd} W \arrow[r, "\tau"] \arrow[d, "i"] & W' \arrow[d, "i'"] \\ V \arrow[r, "a"] & V' \end{tikzcd} \quad \begin{tikzcd} V \arrow[r, "a"] \arrow[d, "j"] & V' \arrow[d, "j'"] \\ W \arrow[r, "\tau"] & W' \end{tikzcd} \tag{5.3}$$

Let us fix an affine 2-plane $\mathbb{A}^2$ in $\mathbb{P}^2$. Let z_1, z_2 be the coordinate functions of $\mathbb{A}^2$. A quiver representation $x = (B_1, B_2, i, j)$ defines a sequence of sheaves as in (5.1) by

$$\alpha := \begin{pmatrix} B_1 - z_1 \\ B_2 - z_2 \\ j \end{pmatrix}, \ \beta := \begin{pmatrix} -B_2 + z_2 & B_1 - z_1 & i \end{pmatrix}. \tag{5.4}$$

We denote the sequence of sheaves by $M(x)$.

Let $M' := M(x')$ where $x' = (B_1', B_2', i', j') \in \mathbf{M}_{(V',W')}$ as before.

Lemma 5.7. *If $(a, b, c) \in \mathrm{Hom}(M, M')$, we have*

$$c = a \ \textit{and} \ b = a^{\oplus 2} \oplus \tau \tag{5.5}$$

where $a \in \mathrm{Hom}(V, V')$ and $\tau \in \mathrm{Hom}(W, W')$. Hence we have a linear isomorphism

$$\mathrm{Hom}(R, R') \xrightarrow{\cong} \mathrm{Hom}(M, M'), \quad (a, \tau) \mapsto \left(a, b = a^{\oplus 2} \oplus \tau, a\right).$$

Proof. Let us write b as

$$b = \begin{pmatrix} A_{11} & A_{12} & A \\ A_{21} & A_{22} & B \\ C & D & \tau \end{pmatrix}$$

where $A_{mn} \in \mathrm{Hom}(V, V')$, $A, B \in \mathrm{Hom}(W, V')$ and $C, D \in \mathrm{Hom}(V, W')$ for $m, n = 1, 2$.

By commutativity of the left square diagram of (5.2), we have $b\alpha = \alpha' a$. Using the explicit forms of α, α', we obtain

$$b\alpha = \begin{pmatrix} A_{11}B_1 - A_{11}z_1 + A_{12}B_2 - A_{12}z_2 + Aj \\ A_{21}B_1 - A_{21}z_1 + A_{22}B_2 - A_{22}z_2 + Bj \\ CB_1 - Cz_1 + DB_2 - Dz_2 + \tau j \end{pmatrix},$$

$$\alpha' a = \begin{pmatrix} B_1'a - az_1 \\ B_2'a - az_2 \\ j'a \end{pmatrix}.$$

Comparing the coefficients of z_1 and z_2, we have $A_{11} = A_{22} = a$, $C = D = 0$, $A_{12} = A_{21} = 0$, $aB_1 = B_1'a$, $aB_2 = B_2'a$, and $j'a = \tau j$.

By commutativity of the right square diagram of (5.2), we have $c\beta = \beta' b$. Using the explicit forms of β, β', we obtain

$$c\beta = \begin{pmatrix} -cB_2 + cz_2 & cB_1 - cz_1 & ci \end{pmatrix}$$

$$\beta' b = \begin{pmatrix} -B_2'a + az_2 & B_1'a - az_1 & -B_2'A + Az_2 + B_1'B - Bz_1 + i'\tau \end{pmatrix}.$$

Comparing the coefficients of z_1, z_2, we have $A = B = 0$, $a = c$ and $ci = i'\tau$. $\square$

Note that $M(x)$ is a complex if and only if $[B_1, B_2] + ij = 0$, where $x = (B_1, B_2, i, j)$. By [**41**, Lemma 2.7], β is surjective (hence, fibre-wise surjective) if and only if x is stable. On the other hand α is always injective. Furthermore it is fibre-wise injective if and only if $\alpha^\vee$ is surjective if and only if x is costable. See [**41**, the proof of Lemma 2.7 (2)] for the latter equivalence. Thus if $[B_1, B_2] + ij = 0$, we see that x is stable if and only if $M(x)$ is monad.

Let $l_\infty := \mathbb{P}^2 \setminus \mathbb{A}^2$. If $M(x)$ is a monad, the cohomology sheaf $E = \mathrm{Ker}(\beta)/\mathrm{Im}(\alpha)$ is a torsion-free sheaf locally free along l_∞ with rank $N = \dim W$, $c_1(E) = 0$ and $c_2(E) = \dim V = k$. See [**41**, pp.18,21] for the Chern classes. The projection $p\colon \mathcal{O} \otimes V^{\oplus 2} \oplus \mathcal{O} \otimes W \to \mathcal{O} \otimes W$ induces $\widetilde{p}\colon E \to \mathcal{O} \otimes W$. By the restriction of $\widetilde{p}$ to l_∞, we have a trivialization of E.

Definition 5.8. We call this the *canonical trivialization* of E, denoted by $\overline{p}$.

We have a description of the third factor τ in (5.5) in terms of canonical trivializations as follows. For the purpose we set $x \in \mathbf{M}_{(V,\mathbb{C}^N)}$ and $x' \in \mathbf{M}_{(V',\mathbb{C}^{N'})}$ such that $M(x), M(x')$ are monads. Let $(E, \overline{p}), (E', \overline{p}')$ be the induced framed sheaves.

Lemma 5.9. *Let* $(a, b, c) \in \mathrm{Hom}(M(x), M(x'))$. *Then the third factor* τ *of* b *in* (5.5) *satisfies* $\mathrm{Id}_{\mathcal{O}_{l_\infty}} \otimes \tau = \overline{p}'\overline{b}|_{l_\infty}\overline{p}^{-1}$.

Proof. It is clear from the definitions of $\overline{b}$ and τ. $\square$

5.2.2. Barth's correspondence. So far we have reviewed the direction from quiver representations to framed sheaves (via monads). We now review the converse direction and then Barth's correspondence. An essential ingredient is Beilinson's spectral sequence which we just refer to without its explicit form. See the details in [**41**, §2.1].

Let (E, Φ) be a framed torsion-free sheaf of rank N and $c_2(E) = k$. By the Beilinson spectral sequence, there exists a stable representation $x = (B_1, B_2, i, j) \in$

$\mathbf{M}_{(\mathbb{C}^k,\mathbb{C}^N)}$ with $[B_1, B_2] + ij = 0$ such that E is isomorphic to the cohomology sheaf $\mathrm{Ker}(\beta)/\mathrm{Im}(\alpha)$ of $M(x)$. We denote this isomorphism by σ. Then we have the commuting diagram:

$$
\begin{tikzcd}
E \arrow[rr] \arrow[d, "\sigma"] & & E|_{l_\infty} \arrow[r, "\Phi"] \arrow[d, "\sigma|_{l_\infty}"] & \mathcal{O}_{l_\infty} \otimes \mathbb{C}^N \arrow[d, "\|"] \\
\mathrm{Ker}(\beta)/\mathrm{Im}(\alpha) \arrow[rr, "\widetilde{p}"] & & \mathcal{O}_{l_\infty} \otimes \mathbb{C}^N \arrow[r, "\nu"] & \mathcal{O}_{l_\infty} \otimes \mathbb{C}^N
\end{tikzcd}
\tag{5.6}
$$

where $\nu = \Phi(\sigma|_{l_\infty})^{-1}$. Hence we have $(E, \Phi) \cong (\mathrm{Ker}(\beta)/\mathrm{Im}(\alpha), \nu\overline{p})$ as framed sheaves. We regard $\nu \in \mathrm{GL}(N)$.

Let

$$x' := \nu.x, \quad M := M(x), \quad M' := M(x').$$

Let $E' := \mathrm{Ker}(\beta')/\mathrm{Im}(\alpha')$ and $\overline{p}'$ be the canonical trivialization of E'.

Lemma 5.10. *We have an isomorphism $(E, \Phi) \cong (E', \overline{p}')$ as framed sheaves.*

Proof. Let $b := \mathrm{Id}_{\mathbb{C}^k}^{\oplus 2} \oplus \nu \in \mathrm{Hom}(M, M')$. For the induced isomorphism $\overline{b}\colon E \overset{\cong}{\to} E'$, we have $\nu\overline{p} = \overline{p}'\overline{b}|_{l_\infty}$. Therefore $\overline{b}$ is also an isomorphism of framed sheaves $(\mathrm{Ker}(\beta)/\mathrm{Im}(\alpha), \nu\overline{p}) \cong (\mathrm{Ker}(\beta')/\mathrm{Im}(\alpha'), \overline{p}')$. $\square$

Now we have Barth's correspondence [**3**, Theorem 2] (see also [**41**, Proposition 2.1]):

Theorem 5.11. *The correspondence $(B_1, B_2, i, j) \mapsto (\mathrm{Ker}(\beta)/\mathrm{Im}(\alpha), \overline{p})$ gives an isomorphism between the $\mathrm{GL}(k)$-orbit space of the stable quiver representations $(B_1, B_2, i, j) \in \mathbf{M}_{(\mathbb{C}^k,\mathbb{C}^N)}$ with $[B_1, B_2] + ij = 0$ and the fine moduli space of the framed torsion-free sheaves (E, Φ) on $\mathbb{P}^2$ with rank N and $c_2(E) = k$.*

Proof. We check first that the correspondence is well-defined. Let $a \in \mathrm{GL}(k)$. Let $M := M(B_1, B_2, i, j)$ and $M' := M(a.(B_1, B_2, i, j))$. We have a morphism $(a, b, c)\colon M \to M'$ as in (5.2). It satisfies $b = a^{\oplus 2} \oplus \mathrm{Id}_{\mathbb{C}^N}$ and $c = a$ by Lemma 5.7. By Lemma 5.9, we have the isomorphism between the cohomology sheaves and its restriction to l_∞ identifies the canonical trivializations. So it is an isomorphism between the induced framed sheaves. This implies the correspondence is well-defined.

Let us check the correspondence gives a set-bijection. The surjectivity follows from Lemma 5.10.

Let us check the injectivity. Let (E, Φ) and (E', Φ') be framed torsion-free sheaves. Let $\sigma\colon (E, \Phi) \to (E', \Phi')$ be an isomorphism of framed sheaves. Then we have $\Phi\sigma|_{l_\infty} = \Phi'$. We may assume $(E, \Phi) = (\mathrm{Ker}(\beta)/\mathrm{Im}(\alpha), \overline{p})$ and $(E', \Phi') = (\mathrm{Ker}(\beta')/\mathrm{Im}(\alpha'), \overline{p}')$ by the surjectivity. Let $M := M(x)$ and $M' := M(x')$ (where x, x' are defined by $\alpha, \beta, \alpha', \beta'$). By Lemma 5.6, there exists a homomorphism $(a, b, c) \in \mathrm{Hom}(M, M')$ which induces σ. By Lemma 5.7, $c = a$ and $b = a^{\oplus 2} \oplus \mathrm{Id}_{\mathbb{C}^N}$. By Lemma 5.6 (the second statement), a is an isomorphism. By diagram-chasing in (5.2), we see $(B_1, B_2, i, j) = a.(B_1', B_2', i', j')$. This asserts the injectivity of the correspondence.

The set-bijective correspondence is, in fact, a scheme-theoretic isomorphism. See for details [**3**, §4.2] (cf. [**41**, Remark 2.2]). □

5.2.3. dual monad. In this subsection we find a monad for a dual vector bundle. The answer is the dual monad. With this we describe φ-structure in terms of linear maps. See the main Proposition 5.15 for the summary.

Let $x := (B_1, B_2, i, j) \in \mathbf{M}_{(V,W)}$. Let $M := M(x)$ as in (5.1).

We define by $M^\vee$ the dual of M:

(5.7)
$$\mathcal{O}(-1) \otimes V^\vee \xrightarrow{\beta^\vee} \begin{matrix} \mathcal{O} \otimes V^{\vee \oplus 2} \\ \oplus \\ \mathcal{O} \otimes W^\vee \end{matrix} \xrightarrow{\alpha^\vee} \mathcal{O}(1) \otimes V^\vee$$
$$\beta^\vee = \begin{pmatrix} -B_2^\vee + z_2 \\ B_1^\vee - z_1 \\ i^\vee \end{pmatrix}, \ \alpha^\vee = \begin{pmatrix} B_1^\vee - z_1 & B_2^\vee - z_2 & j^\vee \end{pmatrix}.$$

Here we used the identification $(\mathcal{O}(n)^m)^\vee = \mathcal{O}(-n)^m$ by the isomorphism $\mathcal{O}(n) \otimes \mathcal{O}(-n) \to \mathcal{O}$.

LEMMA 5.12. *Let $(a, b, c) \in \mathrm{Hom}(M, M^\vee)$. We have $c = -a$ and*

(5.8)
$$b = \begin{pmatrix} 0 & -a & 0 \\ a & 0 & 0 \\ 0 & 0 & \varphi \end{pmatrix}.$$

Furthermore $B_1^\vee a = aB_1$, $B_2^\vee a = aB_2$ and $-ai = j^\vee \varphi$.

Proof. As in the proof of Lemma 5.7, we compare the coefficients of z_1 and z_2 of the left and the right square diagrams in (5.2) respectively. Let us write

$$b = \begin{pmatrix} A_{11} & A_{12} & A \\ A_{21} & A_{22} & B \\ C & D & \varphi \end{pmatrix}.$$

By commutativity of the left square diagram we have $b\alpha = \beta^\vee a$. Using the explicit forms of $\alpha, \beta^\vee$, we obtain

$$b\alpha = \begin{pmatrix} A_{11}B_1 - A_{11}z_1 + A_{12}B_2 - A_{12}z_2 + Aj \\ A_{21}B_1 - A_{21}z_1 + A_{22}B_2 - A_{22}z_2 + Bj \\ CB_1 - Cz_1 + DB_2 - Dz_2 + \varphi j \end{pmatrix},$$

$$\beta^\vee a = \begin{pmatrix} -B_2^\vee a + az_2 \\ B_1^\vee a - az_1 \\ i^\vee a \end{pmatrix}.$$

Comparing the coefficients of z_1 and z_2, we have $A_{11} = A_{22} = C = D = 0$, $-A_{12} = A_{21} = a$, $B_1^\vee a = aB_1$, $B_2^\vee a = aB_2$ and $i^\vee a = \varphi j$.

By commutativity of the right square diagram we have $c\beta = \alpha^\vee b$. Using the explicit forms of $\beta, \alpha^\vee$, we obtain

$$c\beta = \begin{pmatrix} -cB_2 + cz_2 & cB_1 - cz_1 & ci \end{pmatrix}$$
$$\alpha^\vee b = \begin{pmatrix} B_2^\vee a - az_2 & -B_1^\vee a + az_1 & B_1^\vee A - Az_1 + B_2^\vee B - Bz_2 + j^\vee \varphi \end{pmatrix}.$$

Comparing the coefficients of z_1, z_2, we have $A = B = 0$, $a = -c$ and $ci = j^\vee\varphi$. $\square$

We have an immediate corollary of the lemma on adjoints. To state it, suppose we have $(a, b, c) \in \mathrm{Hom}(M, M^\vee)$ as above. There are two bilinear forms $(\ ,\ )_a, (\ ,\ )_\varphi$

on V, W induced by a, φ respectively where φ is the matrix coefficient of b in (5.8). To be precise, we define $(\, , \,)_a, (\, , \,)_\varphi$ by $(v, v')_a := \langle a(v'), v\rangle, (w, w')_\varphi := \langle \varphi(w'), w\rangle$ respectively where $\langle \, , \, \rangle$ denotes the canonical pairing.

COROLLARY 5.13. (1) *B_1 and B_2 are symmetric with respect to $(\, , \,)_a$.*

(2) *$(i(w), v)_a = (w, j(v))_\varphi$ and $-(v, i(w))_a = (j(v), w)_\varphi$ for $v \in V$ and $w \in W$.*

(3) *In particular if a and φ are isomorphisms, we have the identities with the right adjoints*

$$B_1 = B_1^*, \; B_2 = B_2^*, \; j = i^*, \; i = -j^*.$$

Furthermore (B_1, B_2, i, j) is stable if and only if it is costable.

PROOF. (1) follows from $B_n^\vee a = aB_n$ for $n = 1, 2$ by Lemma 5.12.

(2) By Lemma 5.12, $(i(w), v)_a = \langle a(v), i(w)\rangle = \langle i^\vee a(v), w\rangle = \langle \varphi j(v), w\rangle = (w, j(v))_\varphi$. Similarly by Lemma 5.12, $-(v, i(w))_a = -\langle v, ai(w)\rangle = \langle v, ci(w)\rangle = \langle v, j^\vee \varphi(w)\rangle = \langle j(v), \varphi(w)\rangle = (j(v), w)_\varphi$.

(3) The first statement is a direct consequence of the above.

We prove the second one. We prove that stability implies costability. We need to check $T = 0$ whenever $B_1(T), B_2(T) \subset T$ and $T \subset \mathrm{Ker} i^*$. By the symmetricity we have $B_1(T^\perp), B_2(T^\perp) \subset T^\perp$ where $T^\perp = \{v \in V \,|\, (T, v)_a = 0\}$. Since $\mathrm{Ker} i^* = (\mathrm{Im} i)^\perp$, we have $T \subset (\mathrm{Im} i)^\perp$ and thus $\mathrm{Im} i \subset T^\perp$. By stability we have $T^\perp = V$ and thus $T = 0$.

The other way implication is similarly proven. $\square$

Let us given a relation between a dual vector bundle and a dual monad. We need naturality of bundle maps:

Lemma 5.14. *Let*

$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "c"] & B \arrow[d, "b"] \\
C \arrow[r, "d"] & D
\end{tikzcd}$$

be the commutative diagram of bundle maps. Then we have the induced commutative diagram of bundle maps

$$\begin{tikzcd}
\mathrm{Coker}(a^\vee) \arrow[dr, "\cong"] & & & & & \mathrm{Ker}(a^\vee) \arrow[dl, "\cong"'] \arrow[dll] \\
& \mathrm{Ker}(a)^\vee & A^\vee \arrow[l] \arrow[ull] & B^\vee \arrow[l, "a^\vee"'] & \mathrm{Coker}(a)^\vee \arrow[l] & \\
& \mathrm{Ker}(d)^\vee \arrow[u, "\bar{c}^\vee"'] & C^\vee \arrow[l] \arrow[u, "c^\vee"'] & D^\vee \arrow[l, "d^\vee"'] \arrow[u, "b^\vee"'] & \mathrm{Coker}(d)^\vee \arrow[l] \arrow[u, "\bar{b}^\vee"'] & \\
\mathrm{Coker}(d^\vee) \arrow[ur, "\cong"'] \arrow[urr] \arrow[uuu, "\overline{c^\vee}"'] & & & & & \mathrm{Ker}(d^\vee) \arrow[ul, "\cong"] \arrow[ull] \arrow[uuu, "\overline{b^\vee}"']
\end{tikzcd}$$

where $\overline{\bullet}$ denotes the canonical induced homomorphism and all the isomorphisms are induced by the canonical pairings. □

From now on we identify two given vector bundles A and B, if there is an isomorphism $A \cong B$ by the canonical pairing.

Proposition 5.15. *Suppose M is a monad whose cohomology sheaf E is a vector bundle. Then we have the following.*

(1) $E^\vee = \mathrm{Ker}(\alpha^\vee)/\mathrm{Im}(\beta^\vee)$.

(2) *Let* $\phi\colon E \to E^\vee$. *If* $\phi = H(a,b,c)$ *then* $\phi^\vee = H(c^\vee, b^\vee, a^\vee)$. *Moreover the matrix coefficient* φ *of* b *in* (5.8) *satisfies*

$$\phi|_{l_\infty} = \overline{p}^\vee(\mathrm{Id}_{\mathcal{O}_{l_\infty}} \otimes \varphi)\overline{p}.$$

Hence $(E, \overline{p})$ *is a framed vector bundle with* φ*-structure* ϕ.

Proof. We denote the sequence of the monad M by $M_1 \xrightarrow{\alpha} M_2 \xrightarrow{\beta} M_3$ for short.

(1) Since $E^\vee = (\mathrm{Ker}(\beta)/\mathrm{Im}(\alpha))^\vee$, we need to show $\mathrm{Ker}(\alpha^\vee)/\mathrm{Im}(\beta^\vee) = (\mathrm{Ker}(\beta)/\mathrm{Im}(\alpha))^\vee$. Let $\overline{\beta}\colon \mathrm{Coker}(\alpha) \to M_3$ be the induced map by β. Then $\overline{\beta}$ is an injective bundle map and $E = \mathrm{Ker}(\overline{\beta})$. By Lemma 5.14, we have $\mathrm{Ker}\left(\overline{\beta}\right)^\vee = \mathrm{Coker}(\overline{\beta}^\vee)$ and $\mathrm{Coker}(\alpha)^\vee = \mathrm{Ker}(\alpha^\vee)$. we have $E^\vee = \mathrm{Ker}(\alpha^\vee)/\mathrm{Im}(\beta^\vee)$ from the bundle map $\overline{\beta}^\vee\colon M_3^\vee \to \mathrm{Coker}(\alpha)^\vee = \mathrm{Ker}(\alpha^\vee)$.

(2) We prove the first statement. We split the commuting diagrams (5.2) of monads M, M':

$$\begin{tikzcd}
M_1 \arrow[r, "\alpha"] \arrow[d, "a"] & \mathrm{Ker}(\beta) \arrow[r] \arrow[d, "\bar{b}"] & E \arrow[d, "\phi"] \\
M_3^\vee \arrow[r, "\overline{\beta^\vee}"] & \mathrm{Ker}(\alpha^\vee) \arrow[r] & E^\vee
\end{tikzcd}
\qquad
\begin{tikzcd}
E \arrow[r] \arrow[d, "\phi"] & \mathrm{Coker}(\alpha) \arrow[r, "\overline{\beta}"] \arrow[d, "\bar{b}"] & M_3 \arrow[d, "c"] \\
E^\vee \arrow[r] & \mathrm{Coker}(\beta^\vee) \arrow[r, "\overline{\alpha^\vee}"] & M_1^\vee
\end{tikzcd}$$

By dualizing we obtain the commutative diagrams:

$$\begin{tikzcd}
M_1^\vee & \mathrm{Coker}(\beta^\vee) \arrow[l, "\overline{\alpha^\vee}"'] & E^\vee \arrow[l] \\
M_3 \arrow[u, "a^\vee"] & \mathrm{Coker}(\alpha) \arrow[l, "\overline{\beta}"'] \arrow[u, "\overline{b^\vee}"] & E \arrow[l] \arrow[u, "\phi^\vee"]
\end{tikzcd}
\qquad
\begin{tikzcd}
E^\vee & \mathrm{Ker}(\alpha^\vee) \arrow[l] & M_3^\vee \arrow[l, "\overline{\beta^\vee}"'] \\
E \arrow[u, "\phi^\vee"] & \mathrm{Ker}(\beta) \arrow[l] \arrow[u, "\overline{b^\vee}"] & M_1 \arrow[l, "\alpha"'] \arrow[u, "c^\vee"]
\end{tikzcd}$$

Here we used Lemma 5.14. These are nothing but the splitting of $(c^\vee, b^\vee, a^\vee)$. Therefore we have $\phi^\vee = H(c^\vee, b^\vee, a^\vee)$.

The second statement follows from the obvious diagram-chasing. We omit the detail. $\square$

5.2.4. Barth's correspondence for framed vector bundles with φ-structure.

We are ready to give a version of Barth's correspondence for framed vector bundles with φ-structure (Theorem 5.17).

Let $V := \mathbb{C}^k$ and $W := \mathbb{C}^N$. We fix $\varphi \in \mathrm{Isom}(W, W^\vee)$. Let M be a monad with a cohomology vector bundle E. We denote by $\phi_a \in \mathrm{Hom}(E, E^\vee)$ the morphism corresponding to $b \in \mathrm{Hom}(M, M^\vee)$ given by

$$b = \begin{pmatrix} 0 & -a & 0 \\ a & 0 & 0 \\ 0 & 0 & \varphi \end{pmatrix}$$

where $a \in \mathrm{Isom}(V, V^\vee)$ (Lemma 5.12).

The following lemma is a direct consequence of Proposition 5.15 (2).

LEMMA 5.16. *$(E, \overline{p})$ is a framed vector bundle with φ-structure ϕ_a.* □

Note that there is a canonical $\mathrm{GL}(V)$-action on $\mathrm{Hom}(V, V^\vee)$ and hence on $\mathrm{Isom}(V, V^\vee)$.

Let $\widetilde{\mathbf{M}}_{(V,W)} := \mathbf{M}_{(V,W)} \times \mathrm{Isom}(V, V^\vee)$. It is a $\mathrm{GL}(V)$-space by the diagonal action. The following theorem is a version of Barth's correspondence for framed vector bundles with φ-structure (cf. [**9**][**27**]).

THEOREM 5.17. *The correspondence* $(B_1, B_2, i, j, a) \mapsto (\mathrm{Ker}(\beta)/\mathrm{Im}(\alpha), \overline{p}, \phi_a)$ *gives a set-theoretic bijection between*

$$\left\{ (B_1, B_2, i, j, a) \in \widetilde{\mathbf{M}}_{(V,W)} \;\middle|\; \begin{array}{l} (B_1, B_2, i, j)\colon \textit{stable} \\ [B_1, B_2] + ij = 0 \textit{ and} \\ B_1 = B_1^*,\ B_2 = B_2^* \\ j = i^*,\ i = -j^* \\ (\textit{with respect to } a, \varphi) \end{array} \right\} / \mathrm{GL}(V)$$

and

$$\left\{ (E, \Phi, \phi) \;\middle|\; \begin{array}{l} (E, \Phi)\colon \textit{framed vector bundle} \\ \textit{with rank } N,\ c_2(E) = k \textit{ and } \varphi\textit{-structure } \phi \end{array} \right\} / \textit{isomorphism}\,.$$

PROOF. We notice first that the constraints in the quiver side come from Corollary 5.13 (3).

The proof goes as in that of ordinary Barth's correspondence (Theorem 5.11). But we need to consider φ-structure additionally.

We first prove the correspondence is well-defined. Let $h \in \mathrm{GL}(V)$. Let $M := M(B_1, B_2, i, j)$ and $M' := M(h.(B_1, B_2, i, j))$. Let $(E, \overline{p}), (E', \overline{p}')$ be the framed vector bundles with the canonical trivializations from M, M' and the φ-structures $\phi_a, \phi_{h.a}$ respectively. Recall that in the proof of ordinary Barth's correspondence (Theorem 5.11) there is an induced isomorphism $\overline{b}\colon (E, \overline{p}) \xrightarrow{\cong} (E', \overline{p}')$ given by $(h, b, h) \in \mathrm{Hom}(M, M')$ where $b = h^{\oplus 2} \oplus \mathrm{Id}_W$. We need to check that it also induces an isomorphism as framed vector bundles with φ-structures. This amounts to showing $\phi = \sigma^\vee \phi' \sigma$, i.e.,

$$\begin{pmatrix} 0 & -a & 0 \\ a & 0 & 0 \\ 0 & 0 & \varphi \end{pmatrix} = \begin{pmatrix} h^\vee & 0 & 0 \\ 0 & h^\vee & 0 \\ 0 & 0 & \mathrm{Id}_W^\vee \end{pmatrix} \begin{pmatrix} 0 & -h.a & 0 \\ h.a & 0 & 0 \\ 0 & 0 & \varphi \end{pmatrix} \begin{pmatrix} h & 0 & 0 \\ 0 & h & 0 \\ 0 & 0 & \mathrm{Id}_W \end{pmatrix} \tag{5.9}$$

(Lemma 5.6). Looking at the matrix coefficients we need to check $h.a = (h^\vee)^{-1} a h^{-1}$ and $\varphi = \mathrm{Id}_W^\vee \varphi \mathrm{Id}_W$. Both identities are obvious.

Secondly we check that the correspondence is a set-theoretic bijection. We show the surjectivity first. Let (E, Φ) be a framed vector bundle with rank N, $c_2(E) = k$ and a φ-structure ϕ. Due to the surjectivity of ordinary Barth's correspondence we may assume $(E, \Phi) = (\mathrm{Ker}(\beta)/\mathrm{Im}(\alpha), \overline{p})$ where α, β are the boundary maps of the Beilinson monad. Now we need to show $\phi = \phi_a$. Since ϕ is a morphism between $E, E^\vee$, it corresponds to a morphism between the monads $M, M^\vee$ which is written as

$$\phi = H\left(a, \begin{pmatrix} 0 & -a \\ a & 0 \end{pmatrix} \oplus \varphi', -a \right)$$

for some $a \in \mathrm{Hom}(V, V^\vee)$ and $\varphi' \in \mathrm{Hom}(W, W^\vee)$ (Lemma 5.6). The matrix coefficient φ' is equal to φ. This is due to $\phi|_{l_\infty} = \overline{p}^\vee(\mathrm{Id}_{\mathcal{O}_{l_\infty}} \otimes \varphi')\overline{p}$ (Proposition 5.15 (2)) and $\phi|_{l_\infty} = \overline{p}^\vee(\mathrm{Id}_{\mathcal{O}_{l_\infty}} \otimes \varphi)\overline{p}$ from the definition of φ-structure. Thus we have $\phi = \phi_a$. This finishes the proof of the surjectivity.

Finally we show the injectivity. Let $(E, \Phi), (E', \Phi')$ be the framed vector bundles with φ-structures ϕ, ϕ' respectively. Due to the surjectivity in the above, we may assume

$$(E, \Phi, \phi) = (\mathrm{Ker}(\beta)/\mathrm{Im}(\alpha), \overline{p}, \phi_a) \text{ and } (E', \Phi', \phi') = (\mathrm{Ker}(\beta')/\mathrm{Im}(\alpha'), \overline{p}', \phi_{a'})$$

where α, β (resp. α', β') are the boundary maps of the corresponding monads as usual. Let $(B_1, B_2, i, j), (B_1', B_2', i', j')$ be quiver representations defining $(\alpha, \beta), (\alpha', \beta')$ respectively as usual. Let σ be an isomorphism between the above framed vector bundles with φ-structures. By ordinary Barth's correspondence there is a morphism between the monads corresponding to σ which is written as $\sigma = H(h, h^{\oplus 2} \oplus \mathrm{Id}_W, h)$ where $h \in \mathrm{GL}(V)$ and $(B_1, B_2, i, j) = h.(B_1', B_2', i', j')$. From $\phi_a = \sigma^\vee \phi_{a'} \sigma \in \mathrm{Hom}(E, E^\vee)$, we have

$$\begin{pmatrix} 0 & -a & 0 \\ a & 0 & 0 \\ 0 & 0 & \varphi \end{pmatrix} = \begin{pmatrix} h^\vee & 0 & 0 \\ 0 & h^\vee & 0 \\ 0 & 0 & \mathrm{Id}_W^\vee \end{pmatrix} \begin{pmatrix} 0 & -a' & 0 \\ a' & 0 & 0 \\ 0 & 0 & \varphi \end{pmatrix} \begin{pmatrix} h & 0 & 0 \\ 0 & h & 0 \\ 0 & 0 & \mathrm{Id}_W \end{pmatrix}$$

(Lemma 5.6). Comparing the matrix coefficients we obtain $a' = h.a$. This implies $(B_1, B_2, i, j, a) = h.(B_1', B_2', i', j', a')$. This completes the proof of the injectivity. $\square$

5.3. Barth's correspondence for framed symplectic and orthogonal vector bundles

In this section we will deduce another version of Barth's correspondence for framed vector bundles with symplectic or orthogonal structure (Theorem 5.21). It is obtained from restriction of the previous version.

First we need a description of symplectic and orthogonal structures in terms of morphisms of monads. It is immediate from comparison of all the matrix coefficients of $b, b^\vee$ (Proposition 5.15 (2)).

LEMMA 5.18. *A φ-structure ϕ is symplectic (resp. orthogonal) if and only if φ is symplectic (resp. orthogonal) and a is orthogonal (resp. symplectic).* □

Barth's correspondence for framed vector bundles with symplectic or orthgonal structure can be rewritten further from the previous version for framed vector bundles with φ-structure due to the above lemma (Theorem 5.17). But we can sweep out the fifth factor in the quiver side in the previous version. We need a lemma for this.

DEFINITION 5.19. We define some subsets of $\mathrm{Hom}(V, V^\vee)$ as follows.

$$\mathrm{Hom}(V, V^\vee)_{\mathrm{s}} := \{a \in \mathrm{Hom}(V, V^\vee) |\, a = a^\vee\},$$

$$\mathrm{Hom}(V, V^\vee)_{\mathrm{as}} := \{a \in \mathrm{Hom}(V, V^\vee) |\, a = -a^\vee\},$$

$$\mathrm{Isom}(V, V^\vee)_{\mathrm{s}} := \mathrm{Hom}(V, V^\vee)_{\mathrm{s}} \cap \mathrm{Isom}(V, V^\vee),$$

$$\mathrm{Isom}(V, V^\vee)_{\mathrm{as}} := \mathrm{Hom}(V, V^\vee)_{\mathrm{as}} \cap \mathrm{Isom}(V, V^\vee).$$

LEMMA 5.20. (1) *Both* $\mathrm{Hom}(V, V^\vee)_{\mathrm{s}}$ *and* $\mathrm{Hom}(V, V^\vee)_{\mathrm{as}}$ *are* $\mathrm{GL}(V)$*-invariant subsets.*

(2) *Both* $\mathrm{GL}(V)$*-actions on* $\mathrm{Isom}(V, V^\vee)_{\mathrm{s}}$ *and* $\mathrm{Isom}(V, V^\vee)_{\mathrm{as}}$ *are transitive.*

PROOF. (1) Let $h \in \mathrm{GL}(V)$. Let $a \in \mathrm{Hom}(V, V^\vee)_{\mathrm{s}}$. We need to show $h.a \in \mathrm{Hom}(V, V^\vee)_{\mathrm{s}}$. We have $\langle h.a(v), v' \rangle = \langle a(h^{-1}(v)), h^{-1}(v') \rangle = \langle h^{-1}(v), a(h^{-1}(v')) \rangle = \langle v, h.a(v') \rangle$. Therefore we have $(h.a)^\vee = h.a$.

The proof for $\mathrm{Hom}(V, V^\vee)_{\mathrm{as}}$ is similar.

(2) Let $a, a' \in \mathrm{Isom}(V, V^\vee)_{\mathrm{s}}$. We take any orthogonal bases $\mathcal{B}_a, \mathcal{B}_{a'}$ of V with respect to $(\ ,\ )_a, (\ ,\ )_{a'}$ respectively. We define h by sending $\mathcal{B}_a$ to $\mathcal{B}_{a'}$ bijectively.

Since $(v, v')_{h.a} = (h^{-1}(v), h^{-1}(v'))_a$, the two forms $(\, , \,)_{a'}, (\, , \,)_{h.a}$ coincide on $\mathcal{B}_{a'}$. Thus we have $a' = h.a$. This proves $\mathrm{Isom}(V, V^\vee) = \mathrm{GL}(V).a$.

The proof for $\mathrm{Isom}(V, V^\vee)_{\mathrm{as}}$ is similar by using symplectic bases instead. $\square$

Let us fix $\varphi \in \mathrm{Isom}(W, W^\vee)_{\mathrm{as}}$ and $a_0 \in \mathrm{Isom}(V, V^\vee)_{\mathrm{s}}$ or $\varphi \in \mathrm{Isom}(W, W^\vee)_{\mathrm{s}}$ and $a_0 \in \mathrm{Isom}(V, V^\vee)_{\mathrm{as}}$. In the first case W (resp. V) is an symplectic (resp. orthogonal) vector space. Whilst in the second case we need to exchange V and W in the above. Since we have assumed $W = \mathbb{C}^N$ and $V = \mathbb{C}^k$, we may take these forms as the standard ones. We fix the notation for the nondegenerate forms from φ, a_0 as $(\, , \,)_W := (\, , \,)_\varphi$ and $(\, , \,)_V := (\, , \,)_{a_0}$.

THEOREM 5.21. *The correspondence* $(B_1, B_2, i, j) \mapsto (\mathrm{Ker}(\beta)/\mathrm{Im}(\alpha), \overline{p})$ *gives a set-theoretic bijection between* $\mu^{-1}(0)^{\mathrm{reg}}/G(V)$ *and the set of isomorphism classes of framed vector bundles* E *with rank* N*,* $c_2(E) = k$ *and symplectic or orthogonal structure.*

PROOF. Let $\mathbf{M} := \mathbf{M}_{(V,W)}$ and $\widetilde{\mathbf{M}} := \mathbf{M} \times \mathrm{Isom}(V, V^\vee)_{\mathrm{as}}$ for short. The quiver side of the previous version of Barth's correspondence (Theorem 5.17) is a subset of $\widetilde{\mathbf{M}}/\mathrm{GL}(V)$. We use the (elementary) identification of $\widetilde{\mathbf{M}}/\mathrm{GL}(V)$ with $\mathbf{M} \times \{a_0\}/G(V)$. Here we used $G(V) = \mathrm{Stab}_G(a_0)$. Now our theorem follows from the appropriate restriction of both sides of the previous version of Barth's correspondence. $\square$

We notice that a G-structure on a framed vector bundle (E, Φ) is uniquely determined by ϕ_{a_0} up to isomorphisms of (E, Φ). So the third factor from G-structure in the image of the above correspondence is given by the isomorphism class of a framed vector bundle. Therefore the image can be viewed as a subset of the ordinary Gieseker moduli space via the projection to the framed vector bundle component. But it is not clear if it is endowed with a subscheme structure. We continue this discussion in the next section.

5.4. Scheme-structure of the moduli space of framed vector bundles with symplectic or orthogonal structure

Our goal in this section is to show that the image of the correspondence in Theorem 5.21 is a locally closed smooth subscheme of the Gieseker moduli space. Hence the correspondence will turn out to be a scheme-theoretic isomorphism.

We use the notations $(\ ,\ )_\varepsilon, (\ ,\ )_{-\varepsilon}$ for $(\ ,\ )_V, (\ ,\ )_W$ for convenience of sign convention ($\varepsilon = \pm 1$). The isomorphisms given by those pairings are denoted by $a_\varepsilon \colon V \to V^\vee$ and $\varphi_{-\varepsilon} \colon W \to W^\vee$ defined by $v \mapsto (v, \bullet)_\varepsilon$ and $w \mapsto (w, \bullet)_{-\varepsilon}$ respectively. Up to the previous section these have been denoted by a_0, φ respectively. We have $a_\varepsilon^\vee = \varepsilon a_\varepsilon$ and $\varphi_{-\varepsilon}^\vee = -\varepsilon \varphi_{-\varepsilon}$.

We use the simplified notations $\mathbf{M} = \mathbf{M}_{(V,W)}$, $* = *_\mathbf{M}$ and $x^* := *(x)$. Let $\mu_\mathbf{M} \colon \mathbf{M} \to \mathfrak{gl}(V)$ be the moment map given by $(B_1, B_2, i, j) \mapsto [B_1, B_2] + ij$. It is obvious that $\mu = \mu_\mathbf{M}|_\mathbf{N}$.

We define an involution $\overline{*}\colon \mathbf{M}^{\mathrm{reg}}/\mathrm{GL}(V) \to \mathbf{M}^{\mathrm{reg}}/\mathrm{GL}(V)$ by $\mathrm{GL}(V).x \mapsto \mathrm{GL}(V).x^*$. Then the fixed locus $\mathbf{M}^{\mathrm{reg}}/\mathrm{GL}(V)^{\overline{*}}$ (resp. $\mu_{\mathbf{M}}^{-1}(0)^{\mathrm{reg}}/\mathrm{GL}(V)^{\overline{*}}$) is a smooth subscheme of $\mathbf{M}^{\mathrm{reg}}/\mathrm{GL}(V)$ (resp. $\mu_{\mathbf{M}}^{-1}(0)^{\mathrm{reg}}/\mathrm{GL}(V)$).

Since $\mathbf{N}$ is the fixed locus of $\mathbf{M}$, we have a canonical embedding

$$\iota\colon \mu^{-1}(0)^{\mathrm{reg}}/G(V) \to \mu_{\mathbf{M}}^{-1}(0)^{\mathrm{reg}}/\mathrm{GL}(V)^{\overline{*}}.$$

We will see ι is surjective.

Let $\mathcal{G}$ be the vector bundle locus of the Gieseker moduli scheme of framed torsion-free sheaves E with rank N and $c_2(E) = k$. We denote the restriction of ordinary Barth's correspondence by $F'\colon \mu_{\mathbf{M}}^{-1}(0)^{\mathrm{reg}}/\mathrm{GL}(V) \to \mathcal{G}$. We denote by the same notation $\overline{*}$ the induced involution on $\mathcal{G}$ via F'. Thus we have the isomorphism between the fixed loci

$$F\colon \mu_{\mathbf{M}}^{-1}(0)^{\mathrm{reg}}/\mathrm{GL}(V)^{\overline{*}} \to \mathcal{G}^{\overline{*}}$$

by restriction.

According to the last version of Barth's correspondence (Theorem 5.21) the image of the composite $F\iota$ is the set of the isomorphism classes of framed vector bundles E with G-structure.

We claim that the image of F is contained in that of $F\iota$ given as above. So our claim will assert that ι is surjective. The proof itself is similar to our strategy toward Barth's correspondence (Theorem 5.21). Let $\mathrm{GL}(V).x \in \mu_{\mathbf{M}}^{-1}(0)^{\mathrm{reg}}/\mathrm{GL}(V)^{\overline{*}}$ where $x \in \mu_{\mathbf{M}}^{-1}(0)^{\mathrm{reg}}$. Then there exists $g \in \mathrm{GL}(V)$ such that $g.x = x^*$. By taking $*$ to the both sides of $g.x = x^*$, we obtain $g.x = g^*.x$. By the stability of x, we have $g = g^*$. The cohomology sheaf of the monad associated to x^* is isomorphic to

$E^\vee$ as the monad is isomorphic to the dual monad of x. Here we used Proposition 5.15 and the commuting diagram (5.10) below. The induced maps by g, g^* between the monads are explicitly written in terms of linear maps in $\mathrm{End}(V^{\oplus 2} \oplus W)$ as in (5.10). By diagram-chasing in (5.10) and passing to the cohomology sheaves, the constraint $g = g^*$ gives an isomorphism $\overline{g}\colon E \to E^\vee$ with $\overline{g}^\vee = -\varepsilon\overline{g}$ and $\overline{g}|_x = \varphi_{-\varepsilon}$ where $x \in l_\infty$. This finishes the proof of the claim.

By Zariski's main theorem we obtain the isomorphism $\mathcal{M}_n^K \cong \mathcal{G}^{\overline{*}}$, which was used in §1.2.3.

We end up with the commuting diagram used in the proof above:

$$
\begin{tikzcd}
M(x)\colon & \mathcal{O}(-1)\otimes V \arrow[r, "\alpha"] \arrow[d, "g"] & \begin{matrix}\mathcal{O}\otimes V^{\oplus 2}\\ \oplus \\ \mathcal{O}\otimes W\end{matrix} \arrow[r, "\beta"] \arrow[d, "g^{\oplus 2}\oplus \mathrm{Id}_W"] & \mathcal{O}(1)\otimes V \arrow[d, "g"] \\
M(x^*)\colon & \mathcal{O}(-1)\otimes V \arrow[r] \arrow[d, "a_\varepsilon"] & \begin{matrix}\mathcal{O}\otimes V^{\oplus 2}\\ \oplus \\ \mathcal{O}\otimes W\end{matrix} \arrow[r] \arrow[d, "A_\varepsilon"] & \mathcal{O}(1)\otimes V \arrow[d, "a_\varepsilon"] \\
M(x)^\vee\colon & \mathcal{O}(-1)\otimes V^\vee \arrow[r, "\beta^\vee"] & \begin{matrix}\mathcal{O}\otimes V^{\vee\oplus 2}\\ \oplus \\ \mathcal{O}\otimes W^\vee\end{matrix} \arrow[r, "\alpha^\vee"] & \mathcal{O}(1)\otimes V^\vee
\end{tikzcd}
\tag{5.10}
$$

where

$$A_\varepsilon := \begin{pmatrix} 0 & a_\varepsilon \\ -a_\varepsilon & 0 \end{pmatrix} \oplus \varphi_{-\varepsilon}.$$

CHAPTER 6

Geometry of the moduli spaces of Sp-data

Let V be an orthogonal vector space of dimension k, and W be a symplectic vector space of dimension $N \geq 2$. Let $G := G(V) = \mathrm{O}(V)$.

The main theorem of this chapter is the following.

Theorem 6.1. *μ is normal, irreducible, flat, reduced and surjective. Hence $\mu^{-1}(0)$ is a normal irreducible variety of dimension $k(N+2) + k(k-1)/2$. And $\mu^{-1}(0)/\!\!/G$ is a normal irreducible variety of dimension $k(N+2)$, and $(\mu^{-1}(0)/\!\!/G) \setminus (\mu^{-1}(0)^{\mathrm{reg}}/G)$ is of codimension ≥ 2 in $\mu^{-1}(0)/\!\!/G$.*

The proof will appear in §6.2.1.

Corollary 6.2. *The canonical embedding induces the equality of the ring of regular functions: $\mathbb{C}[\mu^{-1}(0)/\!\!/G] = \mathbb{C}[\mu^{-1}(0)^{\mathrm{reg}}/G]$.*

6.1. Panyushev's theorem

Let $m \colon \mathfrak{p} \times \mathfrak{p} \to \mathfrak{k}$ by $(B_1, B_2) \mapsto [B_1, B_2]$. The following is a theorem of Panyushev [**50**].

Theorem 6.3. *For any $X \in \mathfrak{k}$, $m^{-1}(X)$ is an irreducible normal variety and a complete intersection in $\mathfrak{p} \times \mathfrak{p}$ of the* $\dim \mathfrak{k}$*-equations. And the smooth locus of $m^{-1}(X)$ is the locus of $x \in m^{-1}(X)$ such that the differential dm_x is surjective.*

Proof. By [**50**, (3.5)(1)] and [**50**, Theorem 3.2], $m^{-1}(X)$ is an irreducible complete intersection. See also [**8**] for a different proof. The normality is due to [**50**, Corollary 4.4].

For $x \in m^{-1}(X)$, x is a smooth point of $m^{-1}(X)$ if and only if $\dim T_x m^{-1}(X) = \dim m^{-1}(X)\,(= 2\dim\mathfrak{p} - \dim\mathfrak{t})$ if and only if dm_x is surjective (cf. [**50**, the proof of Proposition 4.2]). $\square$

For $B \in \mathfrak{gl}$, $\mathfrak{gl}^B = \{A \in \mathfrak{gl} \mid [A, B] = 0\}$. Let $\mathfrak{p}^B := \mathfrak{gl}^B \cap \mathfrak{p}$.

Let $\mathfrak{gl}_l := \{B \in \mathfrak{gl} \mid \dim\mathfrak{gl}^B = l\}$ and $\mathfrak{p}_l := \{B \in \mathfrak{p} \mid \dim\mathfrak{p}^B = l\}$ $(l \in \mathbb{Z}_{\geq 0})$.

Lemma 6.4. ([**30**, Prop. 5]) *For any $B \in \mathfrak{p}$, $\dim\mathfrak{p}^B - \dim\mathfrak{t}^B = k$.* $\square$

Lemma 6.5. *For any $l \in \mathbb{Z}_{\geq 0}$, $\mathfrak{p}_{\geq l}$ is a closed subvariety of $\mathfrak{p}$. And $\mathfrak{p}_k$ is Zariski open dense in $\mathfrak{p}$.*

Proof. By [**40**, p.7], for the conjugation action of $\mathrm{GL}(V)$ on $\mathfrak{gl}$, the map $A \in \mathfrak{gl} \mapsto \dim\mathrm{GL}(V)^A = \dim\mathfrak{gl}^A$ is upper-semicontinuous. Thus $\mathfrak{gl}_l$ is a locally closed subvariety of $\mathfrak{gl}$. By (2.2), for any $B \in \mathfrak{p}$, $\mathfrak{gl}^B = \mathfrak{p}^B \oplus \mathfrak{t}^B$. By Lemma 6.4, for $B \in \mathfrak{p}$, $B \in \mathfrak{p}_l$ if and only if $B \in \mathfrak{gl}_{2l-k}$. Since $\mathfrak{p}$ is a closed subvariety of $\mathfrak{gl}$, $\mathfrak{p}_l = \mathfrak{p} \cap \mathfrak{gl}_{2l-k}$ is a locally closed subvariety of $\mathfrak{p}$.

The second claim comes from the upper-semicontinuity, $\mathfrak{p}_k = \mathfrak{p} \cap \mathfrak{gl}_k \neq \emptyset$ and $\mathfrak{gl}_l = \emptyset$ for $l < k$. $\square$

Let $\pi_i \colon \mathfrak{p} \times \mathfrak{p} \to \mathfrak{p}$ be the i^{th} projection $(i = 1, 2)$.

Note that for $X = 0$, the singular locus of $m^{-1}(0)$ is of codimension ≥ 3 by [**8**, p.6414, Lemma] (cf. [**50**, Theorem (4.3)]).

Let M be a smooth variety, and $\phi\colon M \to \mathfrak{t}$ be a morphism. The fibre-product $(\mathfrak{p}\times\mathfrak{p})\times_{\mathfrak{t}} M$ is cut out by the $\dim\mathfrak{t}$-equations $[B_1, B_2] - \phi(x) = 0$.

Let $S := \{x \in \mathfrak{p}\times\mathfrak{p} |\, dm_x \text{ is surjective}\}$.

Proposition 6.6. *Let M be a smooth variety and $\phi\colon M \to \mathfrak{t}$ be a morphism. Then $(\mathfrak{p}\times\mathfrak{p})\times_{\mathfrak{t}} M$ is an irreducible normal variety and a complete intersection in $\mathfrak{p}\times\mathfrak{p}\times M$ of the* $\dim\mathfrak{t}$-*equations.*

Proof. Let $\widetilde{m}\colon (\mathfrak{p}\times\mathfrak{p})\times_{\mathfrak{t}} M \to M$ be the projection. Note that $\widetilde{m}$ is surjective since so is m by Theorem 6.3. For each $x \in M$, $\widetilde{m}^{-1}(x) \cong m^{-1}(f(x))$, which means any fibre dimension of $\widetilde{m}$ is $2\dim\mathfrak{p} - \dim\mathfrak{t}$. Therefore $\dim(\mathfrak{p}\times\mathfrak{p})\times_{\mathfrak{t}} M = 2\dim\mathfrak{p} - \dim\mathfrak{t} + \dim M$ which proves $(\mathfrak{p}\times\mathfrak{p})\times_{\mathfrak{t}} M$ is a complete intersection.

Since every fibre of $\widetilde{m}$ is irreducible by Theorem 6.3 and equi-dimensional, $(\mathfrak{p}\times\mathfrak{p})\times_{\mathfrak{t}} M$ is irreducible.

By the upper-semicontinuity, S is an open subvariety of $\mathfrak{p}\times\mathfrak{p}$. By Theorem 6.3, $m^{-1}(X)\setminus S$ is of codimension ≥ 2 in $m^{-1}(X)$ for each $X \in \mathfrak{t}$. Thus $(\mathfrak{p}\times\mathfrak{p})\setminus S$ is of codimension ≥ 2 in $\mathfrak{p}\times\mathfrak{p}$. By the smooth base change ([**20**, III.10.1 (b)]), $\widetilde{m}_{S\times_{\mathfrak{t}} M}$ is a smooth morphism. By [**20**, III.10.4], $S\times_{\mathfrak{t}} M$ is smooth. By Serre's criterion ([**20**, Proposition II.8.23]), we get the normality. □

We have further description of the smooth locus of each fibre of m as follows, which is not necessary in the proof of Theorem 6.1.

Proposition 6.7. *Let $X \in \mathfrak{t}$.*

(1) $m|\colon m^{-1}(X)\cap(\pi_1^{-1}(\mathfrak{p}_k)\cup\pi_2^{-1}(\mathfrak{p}_k)) \to \mathfrak{t}$ *is a smooth morphism.*

(2) *The codimension of* $m^{-1}(X) \setminus (\pi_1^{-1}(\mathfrak{p}_k) \cup \pi_2^{-1}(\mathfrak{p}_k))$ *in* $m^{-1}(X)$ *is larger than* 1.

Proof. (1) Let $L_B \colon \mathfrak{p} \to \mathfrak{t}$ by $A \mapsto [B, A]$, where $B \in \mathfrak{p}$. If $B \in \mathfrak{p}_k$ then L_B is surjective since $\dim \mathfrak{t} + k = \dim \mathfrak{p}$. Since for $(B_1, B_2) \in \pi_1^{-1}(\mathfrak{p}_k) \cup \pi_2^{-1}(\mathfrak{p}_k)$, the differential $dm_{(B_1,B_2)} \colon \mathfrak{p} \times \mathfrak{p} \to \mathfrak{t}$ maps (A_1, A_2) to $[B_1, A_2] + [A_1, B_2]$. Since $\mathfrak{t} = \mathrm{Im} L_{B_1} + \mathrm{Im} L_{B_2} \subset \mathrm{Im} dm_{(B_1,B_2)}$, $dm_{(B_1,B_2)}$ is surjective. This proves (1).

(2) will be proven in §6.2.2. □

Remark 6.8. Let $\tilde{M} := (\mathfrak{p} \times \mathfrak{p}) \times_{\mathfrak{t}} M$, $\tilde{S} := S \times_{\mathfrak{t}} M$ and $\tilde{S}' := (\pi_1^{-1}(\mathfrak{p}_k) \cup \pi_2^{-1}(\mathfrak{p}_k)) \times_{\mathfrak{t}} M$. Then $\tilde{S}' \subset \tilde{S} \subset \tilde{M}$. By Proposition 6.6, $\tilde{S}$ is a smooth variety such that $\tilde{M} \setminus \tilde{S}$ is of codimension ≥ 2 in $\tilde{M}$. In fact, we can strengthen the result further as follows: $\tilde{S}'$ is a smooth variety such that $\tilde{M} \setminus \tilde{S}'$ is of codimension ≥ 2. By Proposition 6.7 (1) and the smooth base change, $\tilde{S}'$ is a smooth variety. By Proposition 6.7 (2), $(\mathfrak{p} \times \mathfrak{p}) \setminus (\pi_1^{-1}(\mathfrak{p}_k) \cup \pi_2^{-1}(\mathfrak{p}_k))$ is of codimension ≥ 2 in $\mathfrak{p} \times \mathfrak{p}$. Thus $\tilde{M} \setminus \tilde{S}'$ is of codimension ≥ 2 in $\tilde{M}$.

6.2. Flatness and normality of the moment map μ

The section is devoted to the proofs of Theorem 6.1 and Proposition 6.7 (2).

6.2.1. Proof of Theorem 6.1. Let $M := \mathrm{Hom}(W, V)$ and $\phi \colon M \to \mathfrak{t}$, $i \mapsto -ii^*$. By Proposition 6.6, $\mu^{-1}(0) = (\mathfrak{p} \times \mathfrak{p}) \times_{\mathfrak{t}} M$ is an irreducible normal variety and a complete intersection in $\mathfrak{p} \times \mathfrak{p} \times M$. By the method of associated cones (Theorem D.1), μ is normal, irreducible, flat and reduced. Since m is surjective, so is μ.

By [**40**, p.5], $\mu^{-1}(0)/\!\!/G$ is an irreducible normal variety. If one can show $\mu^{-1}(0)^{\mathrm{reg}} \neq \emptyset$ then by Lemma 4.2 (2), $\mu^{-1}(0)^{\mathrm{reg}}/G$ is of the complement codimension ≥ 2, since $N \geq 2$.

It remains to show $\mu^{-1}(0)^{\mathrm{reg}} \neq \emptyset$. Let $i \in \mathrm{Hom}(W,V) \setminus 0$. Let $X := -ii^*$. Let $B_1 \in \mathfrak{p}_k$. Let a_n be the eigenvalues of B_1 $(n = 1, 2, ..., k)$. Let V_{a_n} be the a_n-eigenspace. Let $p_n \colon V \to V_{a_n}$ be the projection. We may take B_1 so that $p_n(\mathrm{Im}(i)) \neq 0$ for all n. By Theorem 6.3, for any $B_1 \in \mathfrak{p}_k$, there exists $B_2 \in \mathfrak{p}$ such that $[B_1, B_2] = X$. Now, $(B_1, B_2, i, i^*) \in \mu^{-1}(0)^{\mathrm{reg}}$, since $V = \sum_P P(B_1)i(W)$ where P runs over one-variable polynomials. $\square$

6.2.2. Proof of Proposition 6.7 (2). We prove Proposition 6.7 (2) in this section: the codimension of $m^{-1}(X) \setminus (\pi_1^{-1}(\mathfrak{p}_k) \cup \pi_2^{-1}(\mathfrak{p}_k))$ in $m^{-1}(X)$ is larger than 1.

Lemma 6.9. ([**15**, Theorem XI.4]) *Let* $B \in \mathfrak{p}$. *Then* $\mathrm{O}(V).B = \mathfrak{p} \cap \mathrm{GL}(V).B$ *where* $\mathrm{O}(V).B$ *and* $\mathrm{GL}(V).B$ *are the orbits by conjugation.* $\square$

Let $E_{\mathfrak{gl}} \colon \mathfrak{gl} \to S^k\mathbb{C}$ be the morphism mapping B to the unordered set of eigenvalues of B. Here, S_k is the symmetric group of k-letters and its acts on $\mathbb{C}^k$ by permutation of coordinates so that $S^k\mathbb{C} := \mathbb{C}^k/S_k$. Let $E := E_{\mathfrak{gl}}|_{\mathfrak{p}}$ and $E_l := E|_{\mathfrak{p}_l}$. To construct $E_{\mathfrak{gl}}$ explicitly, let $P \colon S^k\mathbb{C} \to \mathbb{C}^k$ be the isomorphism by $[(a_1, ..., a_k)] \mapsto (p_1(a), ..., p_k(a))$ where $a := (a_1, ..., a_k)$ and $p_i(a) = a_1^i + ... + a_k^i$ (the i^{th} power sum). Let $E'_{\mathfrak{gl}} \colon \mathfrak{gl} \to \mathbb{C}^k$ by $B \mapsto (\mathrm{tr}B, \mathrm{tr}B^2, ..., \mathrm{tr}B^k)$. Let $E_{\mathfrak{gl}} := P^{-1} \circ E'_{\mathfrak{gl}}$.

Let $\mathfrak{p}_l^{(e)} := \{B \in \mathfrak{p}_l | \, B$ has e distinct eigenvalues$\}$. Let $\mathfrak{p}_l^{(\leq e)} := \bigsqcup_{e' \leq e} \mathfrak{p}_l^{(e')}$. Then $\mathfrak{p}_l^{(\leq e)}$ is a closed subvariety of $\mathfrak{p}$. Indeed, let $\Delta^{(e)} \subset S^k\mathbb{C}$ be the locus of all the unordered sets of e distinct points. Let $\Delta^{(\leq e)} := \bigsqcup_{e' \leq e} \Delta^{(e')}$. Then $\Delta^{(\leq e)}$ is a closed subvariety of $S^k\mathbb{C}$. Therefore $E^{-1}(\Delta^{(\leq e)})$ is a closed subvariety of $\mathfrak{p}$. In particular, $\mathfrak{p}_l^{(e)} = \mathfrak{p}_l \cap E^{-1}(\Delta^{(e)})$ is locally closed in $\mathfrak{p}$ by Lemma 6.5. It is manifest that if $e > k$ then for any l, $\mathfrak{p}_l^{(e)} = \emptyset$ and that if $e = k$ then $\mathfrak{p}_l^{(e)} \neq \emptyset$ if and only if $l = k$.

LEMMA 6.10. (1) *If* $\mathfrak{p}_l^{(e)} \neq \emptyset$ *then* $\dim \mathfrak{p}_l^{(e)} \leq \dim \mathfrak{p} - l + e$. *In particular, if* $l > k$ *then* $\dim \mathfrak{p}_l^{(e)} \leq \dim \mathfrak{p} - 2$.

(2) *If* $e = k - 1$ *then* $\mathfrak{p}_l^{(e)} \neq \emptyset$ *if and only if* $l \in \{k, k+1\}$.

(3) $\mathfrak{p}_{k+1}^{(k-1)}$ *consists of* $B \in \mathfrak{p}$ *conjugate by* $\mathrm{O}(V)$ *to* $\mathrm{diag}(a_1, a_1, a_2, ..., a_{k-1})$, *where* $a_1, a_2, ..., a_{k-1}$ *are distinct in* $\mathbb{C}$.

PROOF. (1) The image $E_l|\colon \mathfrak{p}_l^{(e)} \to S^k\mathbb{C}$ is contained in $\Delta^{(e)}$. Any nonempty fibre of E_l is a union of $\mathrm{O}(V).B$ for finitely many $B \in \mathfrak{p}_l^{(e)}$ (Lemma 6.9), so that its dimension is $\dim \mathrm{O}(V).B = \dim \mathrm{O}(V) - \dim \mathrm{O}(V)^B = \dim \mathfrak{t} - \dim \mathfrak{t}^B = \dim \mathfrak{p} - \mathfrak{p}^B = \dim \mathfrak{p} - l$ where the third identity comes from Lemma 6.4. Therefore $\dim \mathfrak{p}_l^{(e)} \leq \dim \mathfrak{p} - l + \dim \Delta^{(e)}$. Since $\dim \Delta^{(e)} = e$, we have proven (1).

(2) Let $e = k - 1$. Let $B \in \mathfrak{p}_l^{(e)}$. Let $a_1, ..., a_{k-1}$ be the (distinct) eigenvalues of B. We may assume that only the a_1-eigenspace of B is 2-dimensional while the other ones are all 1-dimensional. The Jordan normal form of B is either $\begin{pmatrix} a_1 & 1 \\ 0 & a_1 \end{pmatrix} \oplus \mathrm{diag}(a_2, ..., a_{k-1})$ or $\mathrm{diag}(a_1, a_1, ..., a_{k-1})$. Both cases actually happen,

since $\begin{pmatrix} a_1 & 1 \\ 0 & a_1 \end{pmatrix}$ is conjugate by GL(2) to a symmetric matrix $\begin{pmatrix} a_1+\sqrt{-1} & 1 \\ 1 & a_1-\sqrt{-1} \end{pmatrix}$. We have $\dim \mathfrak{gl}^B = k$ and $k+2$ respectively. Using $\mathfrak{gl}^B = \mathfrak{p}^B \oplus \mathfrak{t}^B$ (by (2.2)) and Lemma 6.4, we have $\dim \mathfrak{p}^B = k$ and $k+1$ respectively. This proves (2).

(3) follows from Lemma 6.9. $\square$

LEMMA 6.11. *Let $X \in \mathfrak{t}$. Let $i \in \{1,2\}$. Suppose $\pi_i^{-1}(\mathfrak{p}_l^{(e)}) \cap m^{-1}(X) \neq \emptyset$. Then $\dim \pi_i^{-1}(\mathfrak{p}_l^{(e)}) \cap m^{-1}(X) \leq \dim \mathfrak{p} + e$. In particular, if $e \leq k-2$ then $\dim \pi_i^{-1}(\mathfrak{p}_l^{(e)}) \cap m^{-1}(X) \leq \dim \mathfrak{p} + k - 2 = \dim m^{-1}(X) - 2$.*

PROOF. We claim that any nonempty fibre of $\pi_i|\colon \pi_i^{-1}(\mathfrak{p}_l^{(e)}) \cap m^{-1}(X) \to \mathfrak{p}_l^{(e)}$ is of dimension l. Take $B \in \mathfrak{p}^{(e)}$ and identify $\pi_i^{-1}(B)$ with $\mathfrak{p}$. Then $\pi_i^{-1}(B) \cap m^{-1}(X)$, unless empty, is an affine space isomorphic to $\mathfrak{p}^B$, since for any $B' \in \pi_i^{-1}(B) \cap m^{-1}(X)$, $B' - B \in \mathfrak{p}^B$. The base dimension $\dim \mathfrak{p}_l^{(e)}$ is estimated in Lemma 6.10. Thus the lemma is proven. $\square$

Now we are ready to estimate the codimension of $m^{-1}(X) \setminus (\pi_1^{-1}(\mathfrak{p}_k) \cup \pi_2^{-1}(\mathfrak{p}_k))$ in $m^{-1}(X)$. By Lemma 6.11 and Lemma 6.10 (2), to check the codimension ≥ 2, it suffices to check that so is the codimension of $\pi_i^{-1}(\mathfrak{p}_{k+1}^{(k-1)}) \cap \pi_j^{-1}(\mathfrak{p} \setminus \mathfrak{p}_k) \cap m^{-1}(X)$ in $m^{-1}(X)$, whenever $\{i,j\} = \{1,2\}$. By Lemma 6.11, $\pi_i^{-1}(\mathfrak{p}_{k+1}^{(k-1)}) \cap m^{-1}(X)$ is of codimension ≥ 1 in $m^{-1}(X)$. It remains to prove $\pi_i^{-1}(\mathfrak{p}_{k+1}^{(k-1)}) \cap \pi_j^{-1}(\mathfrak{p}_k) \cap m^{-1}(X)$ is Zariski (open) dense in $\pi_i^{-1}(\mathfrak{p}_{k+1}^{(k-1)}) \cap m^{-1}(X)$. Let $B_1 \in \mathfrak{p}_{k+1}^{(k-1)}$. This is reduced to check that

$$\pi_i^{-1}(B_1) \cap \pi_j^{-1}(\mathfrak{p}_k) \cap m^{-1}(X) \neq \emptyset \text{ provided } \pi_i^{-1}(B_1) \cap m^{-1}(X) \neq \emptyset \tag{6.1}$$

since $\pi_i^{-1}(B_1) \cap m^{-1}(X) \cong \mathfrak{p}^{B_1}$ irreducible (see the proof of Lemma 6.11). Let $B_0 \in \pi_i^{-1}(B_1) \cap m^{-1}(X) \subset \mathfrak{p}$ where $\pi_i^{-1}(B_1)$ are canonically identified with $\mathfrak{p}$. Let us write $B_1 = g.\mathrm{diag}(a_1, a_1, a_2, ..., a_{k-1})$ where $g \in \mathrm{O}(V)$ and $a_1, ..., a_{k-1}$ are distinct (Lemma 6.9 and Lemma 6.10 (3)). Let $B_2 := g.\mathrm{diag}(b_1, ..., b_k)$ where $b_1, ..., b_k$ are distinct so that $B_2 \in \mathfrak{p}_k$. By the Zariski openness of $\mathfrak{p}_k$ in $\mathfrak{p}$, there exists $u \in \mathbb{C} \setminus \{1\}$ such that $(1-u)B_0 + uB_2 \in \mathfrak{p}_k$ since for $u = 1$, $B_2 \in \mathfrak{p}_k$. Therefore $B_0 + \frac{u}{1-u}B_2 \in \mathfrak{p}_k$. Now we have $(B_1, B_0 + \frac{u}{1-u}B_2)$ or $(B_0 + \frac{u}{1-u}B_2, B_1) \in \pi_i^{-1}(B_1) \cap \pi_j^{-1}(\mathfrak{p}_k) \cap m^{-1}(X)$, which shows (6.1). This completes the proof of Proposition 6.7 (2).

CHAPTER 7

Geometry of the moduli spaces of SO-data

We assume that V is symplectic and W is orthogonal in the rest of the paper, except §7.1.1 where preliminaries from linear algebra will be given. Let $k = \dim V$ and $N = \dim W$.

If $k = 0$ then we have $\mathbf{N} = 0$. If $N = 1$ and $k \geq 2$, $\mu^{-1}(0)^{\mathrm{reg}} = \emptyset$ as the right hand side of (4.1) is negative. We will study the next simplest cases, $k = 2$ (Theorem 7.1), $N = 2$ (Theorem 7.2) and $(k, N) = (4, 3)$ (Theorem 7.3). μ is not flat nor irreducible in general unlike the case of the moduli spaces of Sp-data.

In the following theorems, $\mathbf{P}$ denotes the minimal nilpotent $\mathrm{O}(W)$-orbit closure in $\mathfrak{o}(W)$. If $N = 3$ or $N \geq 5$ then $\mathbf{P}$ is irreducible and normal. When $N = 4$, $\mathbf{P}$ has two irreducible components, which are isomorphic by the action of an element in $\mathrm{O}(W) \setminus \mathrm{SO}(W)$. Each irreducible component is the closure of a $\mathrm{SO}(W)$-orbit. See [**10**, Theorems 5.1.4 and 5.1.6].

Theorem 7.1. *Let $k = 2$. Then we have the followings.*

(1) *$\mu^{-1}(0)$ is isomorphic to $\mathbb{C}^2 \times \{i \in \mathrm{Hom}(W, V) \,|\, ii^* = 0\}$ by $(B_1, B_2, i, i^*) \mapsto (\mathrm{tr} B_1, \mathrm{tr} B_2, i)$.*

(2) *If $N \geq 3$, μ is flat.*

(3) *If* $N \geq 5$*,* $\mu^{-1}(0)$ *is irreducible and normal. Hence,* μ *is irreducible and normal. If* $N = 4$*, it is a reduced scheme and a union of two isomorphic irreducible components. If* $N = 3$*, it is irreducible, but non-reduced.*

(4) *If* $N \leq 3$*,* $\mu^{-1}(0)^{\mathrm{reg}} = \emptyset$*. If* $N \geq 4$*,* $\mu^{-1}(0)^{\mathrm{reg}} = \mu^{-1}(0)^{\mathrm{sm}}$*.*

(5) *If* $N \leq 3$*, the reduced scheme* $\mu^{-1}(0)/\!\!/\mathrm{Sp}(V)_{\mathrm{red}}$ *is isomorphic to* $\mathbb{C}^2$*.*

(6) *If* $N \geq 4$*,* $\mu^{-1}(0)/\!\!/\mathrm{Sp}(V)$ *is isomorphic to* $\mathbb{C}^2 \times \mathbf{P}$*. Moreover the isomorphism restricts to* $\mu^{-1}(0)^{\mathrm{reg}}/\mathrm{Sp}(V) \cong \mathbb{C}^2 \times (\mathbf{P} \setminus 0)$*.*

THEOREM 7.2. (1) *Let* $N = 2$ *and* $k \geq 2$*. Then* $\mu^{-1}(0)$ *is not a complete intersection. Hence* μ *is not flat.*

(2) *Let* $N = 2$ *and* $k = 4$*. Then* $\mu^{-1}(0)^{\mathrm{reg}} = \emptyset$*.*

It is true that $\mu^{-1}(0)^{\mathrm{reg}} = \emptyset$ for $N = 2$ and any $k \geq 2$, as there is no nontrivial SO(2)-instantons. But the author does not find its proof in terms of the ADHM description.

THEOREM 7.3. *Let* $N = 3$ *and* $k = 4$*. Then we have the followings.*

(1) $\mu^{-1}(0)$ *is a reduced complete intersection and a union of two irreducible components. Hence* μ *is flat.*

(2) *One irreducible component of* $\mu^{-1}(0)$ *is the closure of* $\mu^{-1}(0)^{\mathrm{reg}}$*.*

(3) $\mu^{-1}(0)/\!\!/\mathrm{Sp}(V)$ *is isomorphic to* $\mathbb{C}^2 \times (\mathbf{P} \sqcup_0 \mathbf{P})$*. Moreover the isomorphism restricts to* $\mu^{-1}(0)^{\mathrm{reg}}/\mathrm{Sp}(V) \cong \mathbb{C}^2 \times (\mathbf{P} \setminus 0)$*.*

Here, $\mathbf{P} \sqcup_0 \mathbf{P} := (\mathbf{P} \times \{0\}) \cup (\{0\} \times \mathbf{P})$ in $\mathbf{P} \times \mathbf{P}$. The proofs will appear in the subsequent sections.

Note that the first statement of Theorem 7.1 (4), Theorem 7.2 (2) and the second statement of Theorem 7.3 (3) are either obvious or well-known in the context of instantons on S^4 in §1.2.2 (See also Remark 4.3).

The properties of the moment map μ in Theorem 7.1 (2), (3), Theorem 7.2 (1) and Theorem 7.3 (1) follow from the corresponding properties of $\mu^{-1}(0)$ by the method of associated cones (Theorem D.1), as in the proof of Theorem 6.1.

The author is planning to study on flatness of μ for $N \geq 4$ and normality of μ for $N \geq 5$ near future. At least the following is true for small k for a fixed N.

REMARK 7.4. It is known by [**33**, Remark 11.3] that $\rho\colon \mathrm{Hom}(W, V) \to \mathfrak{sp}(V)$, $i \mapsto ii^*$, is flat if $N \geq 2k$. Moreover ρ is normal and irreducible if $N \geq 2k+1$. By the base change argument used in the proof of Proposition 6.6, the same is true for μ.

7.1. Kraft-Procesi's classification theory of nilpotent pairs

In this section we review some geometry of $\mathrm{Hom}(W, V)$ following Kraft-Procesi [**33**], which will be used in the proofs of the main theorems above.

7.1.1. generalized eigenspaces and bilinear forms. Let V be a vector space with $(\ ,\ ) := (\ ,\ )_\varepsilon$ where $\varepsilon \in \{-1, +1\}$.

DEFINITION 7.5. Let $X \in \mathfrak{gl}(V)$. Let $V_a^n := \mathrm{Ker}(X - a\mathrm{Id})^n$ for $n \geq 0$. We call $V_a := \cup_{n \geq 1} V_a^n$ the *generalized a-eigenspace of X*.

LEMMA 7.6. (1) *If $X \in \mathfrak{p}(V)$ and $a \neq b$ then $V_a \perp V_b$.*

(2) *If* $X \in \mathfrak{k}(V)$ *and* $a \neq -b$ *then* $V_a \perp V_b$.

PROOF. (1) Let $m, n \geq 0$. Let $v \in V_a^n$ and $w \in V_b^m$. We will show $(v, w) = 0$. We use the induction on m, n. If m or $n = 0$, the assertion is obvious. Suppose the assertion is true for $m - 1, n$ and $m, n - 1$. Let $v' := (X - a)v \in V_a^{m-1}$ and $w' := (X - a)w \in V_b^{n-1}$. We have $(Xv, w) = a(v, w)$ as $(v', w) = 0$ by our assumption. Similarly we also have $(v, Xw) = b(v, w)$. Thus we have $a(v, w) = b(v, w)$ which asserts $(v, w) = 0$.

(2) The proof is same if one uses $(Xv, w) = -(v, Xw)$. $\square$

Let $X \in \mathfrak{k}(V)$ or $\mathfrak{p}(V)$. Let us define a bilinear form $|\,,\,|$ on $\mathrm{Im}(X)$ by $|Xv, Xv'| := (v, Xv')$ (cf. [**33**, §4.1]).

LEMMA 7.7. (1) *If* $X \in \mathfrak{k}(V)$ *then* $|\,,\,|$ *on* $\mathrm{Im}(X)$ *is a nondegenerate bilinear form of type* $-\varepsilon$.

(2) *If* $X \in \mathfrak{p}(V)$ *then* $|\,,\,|$ *on* $\mathrm{Im}(X)$ *is a nondegenerate bilinear form of type* ε.

PROOF. (1) is obvious, as noted in [**33**, §4.1]. (2) is also clear. $\square$

PROPOSITION 7.8. *Let* V *be a* 4*-dimensional symplectic vector space. Let* $X \in \mathfrak{p}(V)$. *Then* X *has an eigenspace of dimension* ≥ 2.

PROOF. By Lemma 7.6, *either* $V = V_a \oplus V_b$ for some $a \neq b$, *or* $V = V_a$ for some a. In the first case, V_a and V_b are 2-dimensional symplectic subspaces of V and thus $X|_{V_a} = a$ and $X|_{V_b} = b$ by Remark 2.1. In the second case, by Lemma 7.7, $\mathrm{Im}(X-a)$ is a symplectic subspace of dimension 0 or 2. If $\dim \mathrm{Im}(X - a) = 0$ then $X = a$. If

$\dim \mathrm{Im}(X-a) = 2$ then $X|_{\mathrm{Im}(X-a)} = a$ by Remark 2.1 as $X \in \mathfrak{p}(\mathrm{Im}(X-a))$. We are done. $\square$

Let W be a vector space with $(,)_{-\varepsilon}$. Recall that for $i \in \mathrm{Hom}(W, V)$, we have $ii^* \in \mathfrak{k}(V)$ and $i^*i \in \mathfrak{k}(W)$ (§2.1.3). Let V_a (resp. W_a) be the generalized a-eigenspace of ii^* (resp. i^*i).

Lemma 7.9. *We have $i^*(V_a) \subset W_a$ and $i(W_a) \subset V_a$. Moreover if $a \neq 0$ then i and i^* are isomorphisms between V_a and W_a.*

Proof. For any $a \in \mathbb{C}$ and $n \in \mathbb{Z}_{\geq 0}$, we have $i^*(ii^*-a)^n = (i^*i-a)^n i^*$ and $(ii^*-a)^n i = i(i^*i-a)^n$. If $v \in V_a$ then $(i^*i-a)^n i^* v = i^*(ii^*-a)^n v = 0$ for $n \gg 0$. Similarly, if $w \in W_a$ then $(ii^*-a)^n iw = i(i^*i-a)^n w = 0$ for $n \gg 0$. Therefore $i^*(V_a) \subset W_a$ and $i(W_a) \subset V_a$. This proves the first claim.

The restriction of ii^* to V_a is $a\mathrm{Id}_{V_a}$ plus a nilpotent endomorphism of V_a. So it is an isomorphism of V_a if $a \neq 0$. Similarly i^*i gives an isomorphism of W_a. Therefore i and i^* give isomorphism between V_a and W_a. $\square$

7.1.2. Kraft-Procesi's results on nilpotent endomorphisms in $\mathfrak{k}$. Let $\mathrm{Sp} := \mathrm{Sp}(V), \mathrm{O} := \mathrm{O}(W), \mathfrak{sp} := \mathfrak{sp}(V)$ and $\mathfrak{o} := \mathfrak{o}(W)$, for short. Let X and Y be nilpotent elements of $\mathfrak{sp}(V)$ and $\mathfrak{o}(V)$ respectively.

A nilpotent orbit $\mathrm{Sp}.X$ corresponds uniquely to a partition $\eta = (\eta_1, \eta_2, ...)$ such that its transpose $(\hat{\eta}_1, \hat{\eta}_2, ...)$ is given by $\hat{\eta}_n := \dim \mathrm{Ker}(X^n)/\mathrm{Ker}(X^{n-1})$. We denote η by η_X. We represent η_X by the Young diagram with the boxes replaced by b (*b-diagram*). For instance, a partition $(5, 3, 3, 2, 2, 0, 0, ...)$ is represented by the

b-diagram:

$$\begin{array}{c} bbbbb \\ bbb \\ bbb \\ bb \\ bb \end{array}$$

For a nilpotent orbit O.Y, we use the symbol a instead of b, and the Young diagram is called an *a-diagram*.

REMARK 7.10. By [**52**, IV 2.15], for $\eta_X = (\eta_1, \eta_2, ...)$ (resp. $\eta_Y = (\eta_1, \eta_2, ...)$), $\#\{n | \, \eta_n = m\}$ is even for any odd m (resp. for any even m).

PROPOSITION 7.11. ([**33**, Proposition 2.4]) *For $\eta_X = \eta := (\eta_1, \eta_2, ...)$,*

$$\dim \mathrm{Sp}.X = \frac{1}{2}\left(|\eta|^2 + |\eta| - \sum \hat{\eta}_n^2 - \#\{n | \, \eta_n \text{ is odd}\}\right).$$

For $\eta_Y = \eta := (\eta_1, \eta_2, ...)$,

$$\dim \mathrm{O}.Y = \frac{1}{2}\left(|\eta|^2 - |\eta| - \sum \hat{\eta}_n^2 + \#\{n | \, \eta_n \text{ is odd}\}\right).$$

We say $\sigma \geq \eta$ for two partitions $\sigma = (\sigma_1, \sigma_2, ...)$ and $\eta = (\eta_1, \eta_2, ...)$ with $|\sigma| = |\eta|$ if $\sum_{1 \leq i \leq j} \sigma_i \geq \sum_{1 \leq i \leq j} \eta_i$ for any $j \geq 1$ (*dominance order* [**38**, §I.1])

PROPOSITION 7.12. ([**21**, Theorem 3.10]) *Let X and X' be nilpotent elements in $\mathfrak{sp}$. Then $\eta_X \geq \eta_{X'}$ if and only if $\mathrm{Sp}.X' \subset \overline{\mathrm{Sp}.X}$. The similar holds for $\mathfrak{o}$.*

COROLLARY 7.13. (1) *If $\dim W \leq 2$, there is no nonzero nilpotent orbit in $\mathfrak{o}$.*

(2) *If $\dim W = 3$, there exists a unique nonzero nilpotent orbit is given by the a-diagram aaa.*

(3) *If* $\dim W \geq 4$, *the minimal nilpotent orbit is given by the a-diagram*

$$\begin{array}{c} aa \\ aa \\ a \\ \vdots \\ a \end{array} \tag{7.1}$$

PROOF. These come from Remark 7.10 and Proposition 7.12. $\square$

Let us define two maps from $\mathrm{Hom}(W, V)$

$$\pi\colon \mathrm{Hom}(W,V) \to \mathfrak{o},\ i \mapsto i^*i,$$

$$\rho\colon \mathrm{Hom}(W,V) \to \mathfrak{sp},\ i \mapsto ii^*.$$

THEOREM 7.14. ([**33**, Theorem 1.2]) π *and* ρ *are the GIT quotient maps onto the images, by* Sp *and* O *respectively.*

We review Kraft-procesi's classification theory of nilpotent pairs in [**32**] and [**33**]. A pair $(i, j) \in \mathrm{Hom}(W, V) \times \mathrm{Hom}(V, W)$ is a nilpotent pair if ij is a nilpotent endomorphism. As in the case of nilpotent endomorphisms, a Young diagram plays an important role in the classification of the $\mathrm{GL}(V) \times \mathrm{GL}(W)$-orbits of nilpotent pairs.

DEFINITION 7.15. ([**32**, §§4.2–4.3]) By an *ab-diagram*, we mean a Young diagram whose rows consists of alternating a and b. E.g.,

$$\begin{array}{c} ababa \\ aba \\ aba \\ ab \\ ba \end{array} \tag{7.2}$$

An ab-diagram A gives a nilpotent pair as follows. Suppose the number of a (resp. b) in A is k (resp. N). Let us take any basis $\{b_1, b_2, ..., b_k\}$ of V (resp. $\{a_1, a_2, ..., a_N\}$ of W). We replace all the a and b in A by a_i and b_i. We define a nilpotent pair (i,j) such that i maps a_m to b_n in the right adjacent position or 0 if there is no such b_n and j maps b_m to a_n or 0 similarly. In the above example of ab-diagram, we have a nilpotent pair as follows:

$$\begin{array}{c} a_1 \mapsto b_1 \mapsto a_2 \mapsto b_2 \mapsto a_3 \mapsto 0 \\ a_4 \mapsto b_3 \mapsto a_5 \mapsto 0 \\ a_6 \mapsto b_4 \mapsto a_7 \mapsto 0 \\ a_8 \mapsto b_5 \mapsto 0 \\ b_6 \mapsto a_9 \mapsto 0 \end{array} \tag{7.3}$$

The above correspondence from an ab-diagram does not determine uniquely a nilpotent pair as the bases of V and W can be changed. Therefore up to the change of the bases, we have the bijective correspondence ([**32**, §4.3])

$$\{ab\text{-diagram with } \#a = \dim V \text{ and } \#b = \dim W\}$$

$$\to \{(\mathrm{GL}(V) \times \mathrm{GL}(W)).(i,j) \,|\, (i,j) \text{ is a nilpotent pair}\}.$$

Suppose (i, i^*) is a nilpotent pair. From the ab-diagram of (i,i^*), the a-diagram of i^*i and the b-diagram of ii^* are obtained from the ab-diagram of i by removing b and a respectively. For example,

$$\begin{array}{c} bb \\ b \\ b \\ b \\ b \end{array} \xleftarrow{\ \rho\ } \begin{array}{c} ababa \\ aba \\ aba \\ ab \\ ba \end{array} \xrightarrow{\ \pi\ } \begin{array}{c} aaa \\ aa \\ aa \\ a \\ a \end{array}$$

For an ab-diagram we define

(7.4)

$$\Delta_{ab} := \sum_{n:\,\mathrm{odd}} (\#\text{rows of length } n \text{ starting with } a)\cdot(\#\text{rows of length } n \text{ starting with } b).$$

Theorem 7.16. ([**33**, Theorem 6.5 and Proposition 7.1]) *Let* $i \in \mathrm{Hom}(W, V)$.

(1) *The orbits* $(\mathrm{Sp} \times \mathrm{O}).i$ *such that* $X := ii^*$ *and* $Y := i^*i$ *are nilpotent, are in 1-1 correspondence with the ab-diagrams whose rows are one of the types* $\alpha, \beta, \gamma, \delta, \epsilon$ *of Table 1 with* $\#a = \dim W,\ \#b = \dim V$.

(2) *For the associated ab-diagram to i,*

$$\dim(\mathrm{Sp}\times\mathrm{O}).i = \frac{1}{2}(\dim\mathrm{Sp}.X + \dim\mathrm{O}.Y + \dim V.\dim W - \Delta_{ab}). \tag{7.5}$$

TABLE 1. Rows of ab-diagrams

Type	α_n	β_n	γ_n	δ_n	ϵ_n
ab-diagram	$aba\cdots ba$	$bab\cdots ab$	$aba\cdots ba$ $aba\cdots ba$	$bab\cdots ab$ $bab\cdots ab$	$aba\cdots ab$ $bab\cdots ba$
n			odd	even	
$\#a$	$2n+1$	$2n-1$	$2(n+1)$	$2n$	$2n$
$\#b$	$2n$	$2n$	$2n$	$2(n+1)$	$2n$

7.2. Moduli spaces of $\mathrm{SO}(N)$-data with $N \geq 2$ and $k = 2$

This section contains the proof of Theorem 7.1.

Let $\dim V = k = 2$ and $\dim W = N \geq 2$. Then $\mathfrak{p} := \mathfrak{p}(V)$ consists of scalars by Remark 2.1. Thus $(B_1, B_2, i, i^*) \in \mu^{-1}(0)$ implies $[B_1, B_2] = ii^* = 0$, and $\mu^{-1}(0) \cong \mathbb{C}^2 \times \rho^{-1}(0)$ by $(B_1, B_2, i, i^*) \mapsto (\mathrm{tr}(B_1), \mathrm{tr}(B_2), i)$. This proves (1).

Let $i \in \rho^{-1}(0)$. Since $ii^* = 0$, b cannot appear twice in the same row in the ab-diagram of i. Looking at Table 1, we find that the ab-diagram of i is one of the followings:

$$\begin{matrix} aba \\ aba \\ a \\ \vdots \\ a \end{matrix} \qquad\qquad \begin{matrix} ab \\ ba \\ a \\ \vdots \\ a \end{matrix} \qquad\qquad \begin{matrix} b \\ b \\ a \\ \vdots \\ a \end{matrix} \tag{7.6}$$

where the left-most actually happens only when $N \geq 4$. We denote elements of $\mathrm{Hom}(W, V)$ corresponding to the above ab-diagrams by i_1, i_2 and $i_3(= 0)$ respectively where i_1 does not exist unless $N \geq 4$. By Theorem 7.16, $\dim(\mathrm{Sp} \times \mathrm{O}).i_1 = 2N - 3$, $\dim(\mathrm{Sp} \times \mathrm{O}).i_2 = N$ and $\dim(\mathrm{Sp} \times \mathrm{O}).i_3 = 0$. Therefore $\rho^{-1}(0)$ is of the expected dimension. By the method of associated cones (Theorem D.1), ρ is flat. Thus μ is flat. This proves (2). See Remark 7.4 for the flatness when $N \geq 4$.

Suppose $N = 3$. Then $\mathrm{O} = \mathrm{SO} \sqcup -\mathrm{SO}$. Since $-\mathrm{Id}_V \in \mathrm{Sp}$ and $-\mathrm{Id}_W.i = -\mathrm{Id}_V.i$ for any $i \in \mathrm{Hom}(W, V)$, $(\mathrm{Sp} \times \mathrm{O}).i_2\,(= (\mathrm{Sp} \times \mathrm{SO}).i_2)$ is irreducible and its Zariski closure is $\rho^{-1}(0)$. Let us check $\rho^{-1}(0)$ is not reduced. By [**33**, Remark 11.4], $\rho^{-1}(0)^{\mathrm{sm}} = \{i \in \rho^{-1}(0) |\, i \text{ is surjective}\}$. By (7.6) any $i \in \rho^{-1}(0)$ is not surjective, which implies $\rho^{-1}(0)$ is not reduced.

If $N = 4$ then by [**33**, §11.3], $\rho^{-1}(0)$ is reduced and consists of two irreducible components which are isomorphic by the action of an element of $\mathrm{O} \setminus \mathrm{SO}$.

If $N \geq 5$, [**33**, §11.3] asserts $\rho^{-1}(0)$ is normal and irreducible. This proves (3).

Let $x := (B_1, B_2, i, i^*) \in \mu^{-1}(0)$. By the ab-diagrams in (7.6), $\mathrm{Ker}(i_2^*)$ and $\mathrm{Ker}(i_3^*)$ are nonzero. Any nonzero vector in $\mathrm{Ker}(i_2^*)$ or $\mathrm{Ker}(i_3^*)$ is a common eigenvector of scalars B_1 and B_2. Thus, if $i = i_2$ or i_3 then x is not costable. Since $\mathrm{Ker}(i_1^*) = 0$, if $i = i_1$ then x is costable. So $\mu^{-1}(0)^{\mathrm{reg}} = \emptyset$ if $N \leq 3$. If $N \geq 4$, $\mu^{-1}(0)^{\mathrm{reg}} = (\mathrm{Sp} \times \mathrm{O}).i_1$.

We describe the smooth locus of $\mu^{-1}(0) = \mathbb{C}^2 \times \rho^{-1}(0)$ when $N \geq 4$. Since $T_x\rho^{-1}(0) = \mathrm{Ker} d\rho_x$ and $\dim \rho^{-1}(0) = \dim \mathrm{Hom}(W, V) - \dim \mathrm{Sp}$, $\rho^{-1}(0)^{\mathrm{sm}}$ is the locus of x such that $d\rho_x$ is surjective. Thus it is the locus of x such that ρ is smooth at x. By [**33**, Remark 11.4], $\rho^{-1}(0)^{\mathrm{sm}} = \{i \in \rho^{-1}(0) | \, i \text{ is surjective}\}$. Among i_1, i_2 and i_3, only i_1 is surjective. This proves (4).

Since Sp acts trivially on $\mathfrak{p}$, we have $\mu^{-1}(0)/\!\!/\mathrm{Sp} \cong \mathbb{C}^2 \times (\rho^{-1}(0)/\!\!/\mathrm{Sp})$. By Theorem 7.14 and the ab-diagrams in (7.6), the reduced scheme

$$\rho^{-1}(0)/\!\!/\mathrm{Sp}_{\mathrm{red}} \cong \begin{cases} 0 & \text{if } N \leq 3 \\ \mathbf{P} & \text{if } N \geq 4. \end{cases}$$

This proves the statement on $\mu^{-1}(0)/\!\!/\mathrm{Sp}$ in (5) and (6). By Theorem 7.14 and the a-diagrams coming from the ab-diagrams in (7.6), we have $\overline{(\mathrm{Sp} \times \mathrm{O}).i_2}/\!\!/\mathrm{Sp} = 0$. This proves the second claim of (6). $\square$

7.3. Moduli spaces of $\mathrm{SO}(2)$-data

This section contains the proof of Theorem 7.2 (1). The proof of Theorem 7.2 (2) will appear in §7.5.

Let $\dim V = k \geq 2$ and $\dim W = N = 2$. Let $m \colon \mathfrak{p} \times \mathfrak{p} \to \mathfrak{t}$, $(B, B') \mapsto [B, B']$.

We will show there exists a subvariety in $\mu^{-1}(0)$ of dimension $> 2 \dim \mathfrak{p} - \dim \mathfrak{t} + \dim \mathrm{Hom}(W, V)$. The idea is to show that $S^{k/2}_{(1^{k/2})}\mathbb{C}^2$ in $\mu^{-1}(0)/\!\!/\mathrm{Sp}$ gives such a subvariety. Let $X := \{(B, B', i, i^*) | \, [B, B'] = 0 = ii^*\} = m^{-1}(0) \times \rho^{-1}(0) \subset \mu^{-1}(0)$. We will show $\dim X > 2 \dim \mathfrak{p} - \dim \mathfrak{t} + \dim \mathrm{Hom}(W, V)$.

Let us estimate $\dim m^{-1}(0)$. Let $\mathfrak{p}^{(e)} := \{B \in \mathfrak{p} | \, B \text{ has distinct } e \text{ eigenvalues}\}$. As shown in Appendix 6.2.2, $\mathfrak{p}^{(\leq e)}$ is a closed subvariety of $\mathfrak{p}$. By Lemma 7.6 (1), if $e > k/2$, $\mathfrak{p}^{(e)} = \emptyset$. With respect to a symplectic basis of V, $\mathrm{diag}(a_1, a_2, ..., a_{k/2})^{\oplus 2} \in \mathfrak{p}^{(k/2)}$ where $a_1, a_2, ..., a_{k/2}$ are all distinct. Therefore $\mathfrak{p}^{(k/2)}$ is a Zariski dense open subset of $\mathfrak{p}$. Let $p \colon m^{-1}(0) \to \mathfrak{p}$ be the first projection. It is clear that p is surjective.

And $p^{-1}(B_0) \cong \mathfrak{p}^{B_0}$ for any $B_0 \in \mathfrak{p}$ (see §6.1 for the notation). If $B_0 \in \mathfrak{p}^{(k/2)}$ then we claim $\dim \mathfrak{p}^{B_0} = k/2$. We have the (2-dimensional) eigenspace decomposition $V = \bigoplus_{i=1}^{k/2} V_{a_i}$ of B_0. Since $\mathfrak{gl}^{B_0} = \bigoplus_{i=1}^{k/2} \mathfrak{gl}(V_{a_i})$ we have $\mathfrak{p}^{B_0} = \bigoplus_{i=1}^{k/2} \mathfrak{p}(V_{a_i}) = \bigoplus_{i=1}^{k/2} \mathbb{C}$.

By the claim we obtain an estimate:

$$\dim m^{-1}(0) \geq \frac{k}{2} + \dim \mathfrak{p}.$$

Let us compute $\dim \rho^{-1}(0)$. Let $i \in \rho^{-1}(0)$. Since $\dim W = 2$ and i^*i is nilpotent, $i^*i = 0$ (Corollary 7.13 (1)). By a similar argument in §7.2, the ab-diagram of any nonzero i is

$$\begin{array}{c} ab \\ ba \\ b \\ \vdots \\ b \end{array}$$

Thus $\rho^{-1}(0)$ is the Zariski closure of $(\mathrm{Sp} \times \mathrm{O}).i$ and $\dim \rho^{-1}(0) = k$ by Theorem 7.16 (2).

To sum up, $\dim X \geq \frac{3}{2}k + \dim \mathfrak{p} > 2\dim \mathfrak{p} - \dim \mathfrak{k} + 2k$ because $\dim \mathfrak{k} - \dim \mathfrak{p} = k$, which means $\mu^{-1}(0)$ is not a complete intersection in $\mathbf{N}$. $\square$

7.4. Moduli spaces of $\mathrm{SO}(3)$-data with $k = 4$

This section contains the proof of Theorem 7.3.

Let $\dim V = k = 4$ and $\dim W = N = 3$. Let $\mathrm{SO} := \mathrm{SO}(W)$ for short.

7.4.1. description of $\mu^{-1}(0)$. Let $\mathfrak{p}' := \{D \in \mathfrak{p} \mid \mathrm{tr} D = 0\}$. Let $\widetilde{X} := \{(B_1, B_2, i, i^*) \in \mu^{-1}(0) \mid B_1, B_2 \in \mathfrak{p}'\}$. Then

$$\mu^{-1}(0) \cong \mathbb{C}^2 \times \widetilde{X},$$

$$(B_1, B_2, i, i^*) \mapsto \left((\mathrm{tr}(B_1), \mathrm{tr}(B_2)), (B_1 - \frac{1}{4}\mathrm{tr}(B_1)\mathrm{Id}_V, B_2 - \frac{1}{4}\mathrm{tr}(B_2)\mathrm{Id}_V, i, i^*) \right).$$

Let $\mathbf{N}' := \mathfrak{p}'^{\oplus 2} \oplus \{(i, i^*) |\, i \in \mathrm{Hom}(W, V)\}$. Then $\widetilde{X}$ is defined by $\mu = 0$ in $\mathbf{N}'$. Therefore it is enough to show the corresponding statements for $\widetilde{X}$ to prove Theorem 7.3.

Let $\widetilde{X}_1 := \{(B_1, B_2, i, i^*) \in \widetilde{X} |\, [B_1, B_2] = 0\}$. Then $\widetilde{X}_1 = \{(B_1, B_2) \in \mathfrak{p}' \times \mathfrak{p}' |\, [B_1, B_2] = 0\} \times \rho^{-1}(0)$.

LEMMA 7.17. $\rho^{-1}(0)$ *consists of two irreducible* $(\mathrm{Sp} \times \mathrm{O})$*-orbits of dimension 5 and 0 respectively.*

PROOF. By a similar argument as in §7.2, the possible ab-diagrams of i of $\rho^{-1}(0)$ are

$$\begin{array}{cc} & b \\ ab & b \\ ba & b \\ b & b \\ b & a \\ a & a \\ & a \end{array} \tag{7.7}$$

By Theorem 7.16, $\rho^{-1}(0)$ consists of two $(\mathrm{Sp} \times \mathrm{O})$-orbits associated to the above ab-diagrams of dimension 5 and 0 respectively.

Since $-\mathrm{Id}_W \in \mathrm{O} \setminus \mathrm{SO}$ and $-\mathrm{Id}_V \in \mathrm{Sp}$, $(\mathrm{Sp} \times \mathrm{O}).i = (\mathrm{Sp} \times \mathrm{SO}).i$ for any $i \in \mathrm{Hom}(W, V)$. So we have irreducibility. $\square$

Let e_1, e_2, e_3, e_4 be a basis of V such that $(e_1, e_2)_V = (e_3, e_4)_V = 1$ and $(e_l, e_m)_V = 0$ for other l, m with $l \leq m$. Let f_1, f_2, f_3 be an orthogonal basis of W so that $(f_i, f_j) = \delta_{ij}$.

Let I be the 2×2 identity matrix. Let

$$J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \; H := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \; X := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \text{ and } Y := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{7.8}$$

We identify $\mathfrak{gl}(V) = \mathrm{Mat}_4$ with respect to $e_1, ..., e_4$. Then we can write the elements of $\mathfrak{k}$ and $\mathfrak{p}$ in matrix forms in 2×2 subminors:

$$\begin{aligned} \mathfrak{k} &:= \left\{ \begin{pmatrix} P & JQ \\ JQ^t & S \end{pmatrix} \in \mathfrak{gl}(V) \,\middle|\, \mathrm{tr}(P) = \mathrm{tr}(S) = 0 \right\}, \\ \mathfrak{p} &:= \left\{ \begin{pmatrix} aI & JR \\ -JR^t & bI \end{pmatrix} \in \mathfrak{gl}(V) \,\middle|\, a, b \in \mathbb{C} \right\}. \end{aligned} \tag{7.9}$$

where $P, Q, R, S \in \mathrm{Mat}_2$. Note that

$$\begin{pmatrix} 0 & H \\ -H & 0 \end{pmatrix}, \begin{pmatrix} 0 & X \\ -X & 0 \end{pmatrix}, \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & Y \\ -Y & 0 \end{pmatrix}$$

are elements of $\mathfrak{p}'$ by letting $R = -JH$, $R = -JX$, $R = -JY$ and $R = -J$ respectively. Therefore we obtain a basis of $\mathfrak{p}'$ as

$$\begin{aligned} v_1 &:= \frac{1}{2} \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, v_2 := \begin{pmatrix} 0 & H \\ -H & 0 \end{pmatrix}, v_3 := \begin{pmatrix} 0 & X \\ -X & 0 \end{pmatrix}, \\ v_4 &:= \begin{pmatrix} 0 & Y \\ -Y & 0 \end{pmatrix}, v_5 := \frac{1}{2} \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}. \end{aligned} \tag{7.10}$$

By direct computation, the Lie brackets of pairs of basis elements of $\mathfrak{p}'$ are

(7.11)

$$[v_1, v_2] = \begin{pmatrix} 0 & H \\ H & 0 \end{pmatrix}, \ [v_1, v_3] = \begin{pmatrix} 0 & X \\ X & 0 \end{pmatrix}, \ [v_1, v_4] = \begin{pmatrix} 0 & Y \\ Y & 0 \end{pmatrix}, \ [v_1, v_5] = \frac{1}{2}\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},$$

$$[v_2, v_3] = -2\begin{pmatrix} X & 0 \\ 0 & X \end{pmatrix}, \ [v_2, v_4] = 2\begin{pmatrix} Y & 0 \\ 0 & Y \end{pmatrix}, \ [v_2, v_5] = \begin{pmatrix} H & 0 \\ 0 & -H \end{pmatrix},$$

$$[v_3, v_4] = -\begin{pmatrix} H & 0 \\ 0 & H \end{pmatrix}, \ [v_3, v_5] = \begin{pmatrix} X & 0 \\ 0 & -X \end{pmatrix}, \ [v_4, v_5] = \begin{pmatrix} Y & 0 \\ 0 & -Y \end{pmatrix}.$$

These form a basis of $\mathfrak{t}$.

Let us define a linear map $F\colon \wedge^2 \mathfrak{p}' \to \mathfrak{t}$ by setting $F(v_i \wedge v_j) := [v_i, v_j]$ where $1 \le i < j \le 5$. Then we have a commuting diagram

(7.12)
$$\begin{tikzcd} \mathfrak{p}' \times \mathfrak{p}' \arrow[r, "\omega"] \arrow[dr, "{[\ ,\ ]}"'] & \wedge^2\mathfrak{p}' \arrow[d, "F"] \\ & \mathfrak{t}. \end{tikzcd}$$

By (7.11), $\mathrm{Im}(F)$ contains the basis of $\mathfrak{t}$ and $\dim \mathfrak{t} = \dim \wedge^2\mathfrak{p}' = 10$. Thus F is an isomorphism. In particular, for $B_1, B_2 \in \mathfrak{p}'$, $[B_1, B_2] = 0$ if and only if $B_1 \wedge B_2 = 0$. This proves the following lemma.

LEMMA 7.18. $\{(B_1, B_2) \in \mathfrak{p}' \times \mathfrak{p}' \,|\, [B_1, B_2] = 0\} = \{(aB, bB) \,|\, a, b \in \mathbb{C},\ B \in \mathfrak{p}'\}$ *and it is irreducible.* □

COROLLARY 7.19. (1) $\widetilde{X}_1$ *is irreducible of dimension 11.*

(2) *Any element of* $\widetilde{X}_1$ *is an unstable quiver representation.*

PROOF. (1) follows from Lemmas 7.17 and 7.18.

(2) The non-costability amounts to the existence of a common eigenvector v of B_1 and B_2 such that $i^*(v) = 0$. From (7.7), we have $\dim \operatorname{Ker}(i^*) \geq 3$. Let $B_1 = a_1 B$ and $B_2 = a_2 B$ for some $a_1, a_2 \in \mathbb{C}$ and $B \in \mathfrak{p}'$ (Lemma 7.18). By Proposition 7.8, we have an eigenspace of B of dimension ≥ 2. By the dimension reason it has a nonzero vector contained in the eigenspace and $\operatorname{Ker}(i^*)$. $\square$

On the other hand, if $\sigma \in \operatorname{Im} \omega$ is nonzero then we have

$$\omega^{-1}(\sigma) \cong \mathrm{SL}(2). \tag{7.13}$$

since $\operatorname{Im} \omega \setminus 0$ is the set of 2-dimensional subspaces S of $\mathfrak{p}'$ with a volume form of S. This isomorphism can be described in more detail as follows. Define an $\mathrm{SL}(2)$-action on $\mathfrak{p}' \times \mathfrak{p}'$ as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.(B_1, B_2) := (aB_1 + bB_2, cB_1 + dB_2). \tag{7.14}$$

The $\mathrm{SL}(2)$-action on $\omega^{-1}(\wedge^2 \mathfrak{p}' \setminus 0)$ is free. The $\mathrm{SL}(2)$-action on $\wedge^2 \mathfrak{p}'$ is trivial and $\omega|_{\omega^{-1}(\wedge^2 \mathfrak{p}' \setminus 0)}$ is $\mathrm{SL}(2)$-equivariant. Now (7.13) is nothing but the identification of a free $\mathrm{SL}(2)$-orbit.

Definition 7.20. Let $G := \mathrm{SL}(2) \times \mathrm{Sp} \times \mathrm{O}$. Then we have a G-action on $\widetilde{X}$ by

$$(g_1, g_2, g_3).((B_1, B_2), i, i^*) := (g_1.(g_2.B_1, g_2.B_2), g_2 i g_3^{-1}, (g_2 i g_3^{-1})^*).$$

Lemma 7.21. *Let $B_1, B_2 \in \mathfrak{p}'$. Then $([B_1, B_2])^2$ is a scalar endomorphism.*

Proof. This follows from a tedious, but direct computation. $\square$

Let $\widetilde{X}_2 := \widetilde{X} \setminus \widetilde{X}_1 = \{(B_1, B_2, i, i^*) \in \widetilde{X} \,|\, [B_1, B_2] \neq 0\}$. Let $p\colon \widetilde{X}_2 \to \mathrm{Hom}(W, V)$ be the projection.

COROLLARY 7.22. *Let $i \in p(\widetilde{X}_2)$. Then $(ii^*)^2 = 0$.*

PROOF. Write $i = p(B_1, B_2, i, i^*)$ for some $B_1, B_2 \in \mathfrak{p}'$. By Lemma 7.21, $[B_1, B_2]^2 = (ii^*)^2$ is a scalar endomorphism. Since $\operatorname{rank} i \leq 3$, the scalar is 0. $\square$

COROLLARY 7.23. *Let $i \in p(\widetilde{X}_2)$. Then the ab-diagram of i is one of the followings:*

$$\text{(7.15)} \qquad \text{(I)}\ \begin{matrix} bab \\ bab \\ a \end{matrix} \qquad \text{(II)}\ \begin{matrix} ababa \\ b \\ b \end{matrix} \qquad \text{(III)}\ \begin{matrix} bab \\ ba \\ ab \end{matrix} \qquad \text{(IV)}\ \begin{matrix} bab \\ b \\ b \\ a \\ a \end{matrix}$$

PROOF. Since $(ii^*)^2 = 0$ by Corollary 7.22 and $ii^* \neq 0$, the maximal length of rows of the b-diagram of ii^* is 2. Thus the b-diagram of ii^* is one of the followings:

$$\text{(7.16)} \qquad \begin{matrix} bb \\ bb \end{matrix} \qquad \begin{matrix} bb \\ b \\ b \end{matrix}$$

From Table 1, we get the list of ab-diagrams as above. $\square$

LEMMA 7.24. (1) *Let $B_1 := v_5$ and $B_2 := v_1 - \frac{1}{2}v_2$. Then $[B_1, B_2]$ is a nilpotent matrix whose b-diagram is*

$$\text{(7.17)} \qquad \begin{matrix} bb \\ bb \end{matrix}$$

(2) *Let $B_1 := v_3$ and $B_2 := \sqrt{-1}v_1 - v_5$. Then $[B_1, B_2]$ is a nilpotent matrix whose b-diagram is*

$$\text{(7.18)} \qquad \begin{matrix} bb \\ b \\ b \end{matrix}$$

PROOF. (1) By direct calculation we have

$$[B_1, B_2] = \frac{1}{2}\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

which maps

$$e_1 \mapsto \frac{1}{2}(e_1 + e_3),\ \frac{1}{2}(e_1 + e_3) \mapsto 0,\ e_2 \mapsto -\frac{1}{2}(e_2 - e_4),\ \frac{1}{2}(e_2 - e_4) \mapsto 0. \tag{7.19}$$

(2) By direct calculation we have

$$[B_1, B_2] = \begin{pmatrix} 0 & -1 & 0 & -\sqrt{-1} \\ 0 & 0 & 0 & 0 \\ 0 & -\sqrt{-1} & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

which maps

$$e_2 \mapsto -(e_1 + \sqrt{-1}e_3),\ e_1 + \sqrt{-1}e_3 \mapsto 0,\ e_1 \mapsto 0,\ e_2 + \sqrt{-1}e_4 \to 0. \tag{7.20}$$

□

We denote B_1, B_2 in Lemma 7.24 (1) by $B_1^{(\mathrm{I})}, B_2^{(\mathrm{I})}$ respectively. We denote B_1, B_2 in Lemma 7.24 (2) by $B_1^{(\mathrm{II})}, B_2^{(\mathrm{II})}$ respectively. Let $B_n^{(\mathrm{III})}$ and $B_n^{(\mathrm{IV})}$ be $B_n^{(\mathrm{II})}$ $(n = 1, 2)$. For each $A = \mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{IV}$ there exists $i^{(A)} \in \mathrm{Hom}(W, V)$ whose ab-diagram is of type (A) in (7.15) and $[B_1^{(A)}, B_2^{(A)}] + i^{(A)} i^{(A)*} = 0$ by Theorem 7.16 (1).

Let us define $i^{(\mathrm{I})}$ in a matrix form. We take a basis $\{\widetilde{e}_1, \widetilde{e}_2, \widetilde{e}_3, \widetilde{e}_4\}$ of V by setting $\widetilde{e}_1 := e_1$, $\widetilde{e}_2 := e_2 - e_4$, $\widetilde{e}_3 := e_1 + e_3$ and $\widetilde{e}_4 := e_4$. We take a basis $\{\widetilde{f}_1, \widetilde{f}_2, \widetilde{f}_3\}$ of W

by setting $\widetilde{f}_1 := \frac{1}{\sqrt{2}}(f_1 + \sqrt{-1}f_2)$, $\widetilde{f}_2 := \frac{1}{\sqrt{2}}(f_1 - \sqrt{-1}f_2)$ and $\widetilde{f}_3 := f_3$. Let

$$i^{(\mathrm{I})} := \frac{1}{\sqrt{-2}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

with respect to $\{\widetilde{e}_1, \widetilde{e}_2, \widetilde{e}_3, \widetilde{e}_4\}$ and $\{\widetilde{f}_1, \widetilde{f}_2, \widetilde{f}_3\}$.

Let us define $i^{(\mathrm{II})}$, $i^{(\mathrm{III})}$ and $i^{(\mathrm{IV})}$ in matrix forms respectively. We take a basis $\{\widetilde{e}'_1, \widetilde{e}'_2, \widetilde{e}'_3, \widetilde{e}'_4\}$ of V by setting $\widetilde{e}'_1 := e_2$, $\widetilde{e}'_2 := -(e_1 + \sqrt{-1}e_3)$, $\widetilde{e}'_3 := e_3$ and $\widetilde{e}'_4 := -\sqrt{-1}e_2 + e_4$. We take a basis $\{\widetilde{f}'_1, \widetilde{f}'_2, \widetilde{f}'_3\}$ of W by setting $\widetilde{f}'_1 := \frac{1}{\sqrt{2}}(f_1 + \sqrt{-1}f_3)$, $\widetilde{f}'_2 := f_2$ and $\widetilde{f}'_3 := \frac{1}{\sqrt{2}}(f_1 - \sqrt{-1}f_3)$. Let

$$i^{(\mathrm{II})} := \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad i^{(\mathrm{III})} := \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad i^{(\mathrm{IV})} := \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

with respect to $\{\widetilde{e}'_1, \widetilde{e}'_2, \widetilde{e}'_3, \widetilde{e}'_4\}$ and $\{\widetilde{f}'_1, \widetilde{f}'_2, \widetilde{f}'_3\}$.

Corollary 7.25. *$\widetilde{X}_2$ is the union of four G-orbits through $(B_1^{(A)}, B_2^{(A)}, i^{(A)}, i^{(A)*})$ for $A = \mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{IV}$. And each G-orbit is irreducible of dimension 12, 12, 11 and 9 respectively.*

Proof. Let $\widetilde{X}'_2$ be the union of four G-orbits. Since $\widetilde{X}'_2 \subset \widetilde{X}_2$, we need to show the opposite inclusion. By Corollary 7.23, we have $p(\widetilde{X}_2) \subset \bigcup_{A=\mathrm{I},\mathrm{II}} \mathrm{Sp}.i^{(A)} = p(\widetilde{X}'_2)$.

Thus $p(\widetilde{X}_2) = \bigcup_{A=\mathrm{I},\mathrm{II}} \mathrm{Sp}.i^{(A)} = p(\widetilde{X}'_2)$. On the other hand, $p^{-1}(p(x)) \cong \mathrm{SL}(2)$ for $x \in \widetilde{X}_2$ by (7.13). Therefore $\widetilde{X}_2 \subset p^{-1}(p(\widetilde{X}_2)) = p^{-1}(p(\widetilde{X}'_2)) = \widetilde{X}'_2$.

By Theorem 7.16 (2), the explicit value of $\dim G.(B_1^{(A)}, B_2^{(A)}, i^{(A)}, i^{(A)*}) = 3 + \dim(\mathrm{Sp} \times \mathrm{O}).i^{(A)}$ for each A, is computed as above.

The irreducibility comes from the above G-orbits are the $(\mathrm{SL}(2) \times \mathrm{Sp} \times \mathrm{SO})$-orbits. To see this we observe that $-\mathrm{Id}_W \in \mathrm{O}$ acts on $\mathbf{N}$ in the same way as $-\mathrm{Id}_V \in \mathrm{Sp}$. $\square$

Lemma 7.26. *Let $x \in G.(B_1^{(A)}, B_2^{(A)}, i^{(A)}, i^{(A)*})$ for $A = \mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{IV}$. Then we have the followings.*

(1) *If $A = \mathrm{I}$, x is unstable and $\mathfrak{sp}^x$ is trivial.*

(2) *If $A = \mathrm{II}$, x is stable (hence Sp^x is trivial).*

(3) *If $A = \mathrm{III}, \mathrm{IV}$, x is unstable.*

(4) *$\widetilde{X}$ is a reduced complete intersection.*

Proof. (1) It is direct to check $e_1 + e_3$ is a common eigenvector of $B_1^{(\mathrm{I})}$ and $B_2^{(\mathrm{I})}$. We have $i^{(\mathrm{I})*}(e_1 + e_3) = i^{(\mathrm{I})*}(\widetilde{e}_3) = 0$. So $e_1 + e_3$ violates the costability of x.

We check $\mathfrak{sp}^x$ is trivial. Use (7.9). Then we have

$$\mathfrak{sp}^{B_1^{(\mathrm{I})}} = \left\{ \begin{pmatrix} P & JQ^t \\ JQ & S \end{pmatrix} \in \mathfrak{sp} \,\middle|\, [\begin{pmatrix} P & JQ^t \\ JQ & S \end{pmatrix}, \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}] = 0 \right\}$$

$$= \left\{ \begin{pmatrix} P & R \\ R & P \end{pmatrix} \,\middle|\, \mathrm{tr}P = 0,\ R = JQ,\ Q = Q^t \right\},$$

$$\mathfrak{sp}^{B_1^{(\mathrm{I})}} \cap \mathfrak{sp}^{B_2^{(\mathrm{I})}} = \left\{ \begin{pmatrix} P & R \\ R & P \end{pmatrix} \,\middle|\, [\begin{pmatrix} P & R \\ R & P \end{pmatrix}, \begin{pmatrix} I & H \\ -H & -I \end{pmatrix}] = 0,\ \mathrm{tr}P = 0 \right\}$$

$$= \mathbb{C}\left\langle \begin{pmatrix} X & -X \\ -X & X \end{pmatrix}, \begin{pmatrix} Y & Y \\ Y & Y \end{pmatrix}, \begin{pmatrix} H & 0 \\ 0 & H \end{pmatrix} \right\rangle.$$

On the other hand, $\mathrm{Im}(i^{(\mathrm{I})}) = \mathbb{C}\langle \widetilde{e}_2, \widetilde{e}_3 \rangle = \mathbb{C}\langle e_1 + e_3, e_2 - e_4 \rangle$. So

$$\mathfrak{sp}^{i^{(\mathrm{I})}} = \{ g \in \mathfrak{sp} |\, g(e_1 + e_3) = g(e_2 - e_4) = 0 \}.$$

We claim $\mathfrak{sp}^{B_1^{(\mathrm{I})}} \cap \mathfrak{sp}^{B_2^{(\mathrm{I})}} \cap \mathfrak{sp}^{i^{(\mathrm{I})}} = 0$. The solution $a, b, c \in \mathbb{C}$ of the following equations

$$\left(a \begin{pmatrix} X & -X \\ -X & X \end{pmatrix} + b \begin{pmatrix} Y & Y \\ Y & Y \end{pmatrix} + c \begin{pmatrix} H & 0 \\ 0 & H \end{pmatrix} \right) (e_1 + e_3) = 0$$

$$\left(a \begin{pmatrix} X & -X \\ -X & X \end{pmatrix} + b \begin{pmatrix} Y & Y \\ Y & Y \end{pmatrix} + c \begin{pmatrix} H & 0 \\ 0 & H \end{pmatrix} \right) (e_2 - e_4) = 0$$

is trivial. So the claim is proven.

(2) We show there does not exist a common eigenvector v of $B_1^{(\mathrm{II})}$ and $B_2^{(\mathrm{II})}$ such that ${i^{(\mathrm{II})}}^*(v) = 0$. By direct computation we have $\mathrm{Ker}(B_1^{(\mathrm{II})}) = \mathbb{C}\langle e_1, e_3 \rangle$ and

$\mathrm{Ker}(B_2^{(\mathrm{II})}) = \mathbb{C}\langle e_1 + \sqrt{-1}e_3, e_2 + \sqrt{-1}e_4\rangle$. So we have $\mathrm{Ker}(B_1^{(\mathrm{II})}) \cap \mathrm{Ker}(B_2^{(\mathrm{II})}) = \mathbb{C}\langle e_1 + \sqrt{-1}e_3\rangle$.

On the other hand $i^{(\mathrm{II})^*}(e_1 + \sqrt{-1}e_3) = -i^{(\mathrm{II})^*}(\widetilde{e}_2') \neq 0$. This proves (2).

(3) In (2) we checked that $e_1 + \sqrt{-1}e_3$ is a unique common eigenvector of $B_1^{(A)}$ and $B_2^{(A)}$ for $A = \mathrm{III}, \mathrm{IV}$ up to constant.

On the other hand $i^{(A)^*}(e_1 + \sqrt{-1}e_3) = -i^{(A)^*}(\widetilde{e}_2') = 0$ for $A = \mathrm{III}, \mathrm{IV}$. So $e_1 + \sqrt{-1}e_3$ violates the costability of x.

(4) By Corollaries 7.19, 7.25 and the dimension reason, $\widetilde{X}$ is a complete intersection of dimension 12. By (1) and (2), $\widetilde{X}$ is smooth along the two 12-dimensional G-orbits since $\mathfrak{sp}^x = 0$ implies that $d\mu_x \colon T_x\mathbf{N}' \to \mathfrak{sp}$ is surjective. Therefore $\widetilde{X}$ is reduced along the two 12-dimensional G-orbits. Hence $\widetilde{X}$ itself is reduced by [**19**, Prop. 5.8.5]. $\square$

7.4.2. description of $\mu^{-1}(0)/\!\!/\mathrm{Sp}$. In the previous subsection we proved

(1) $\widetilde{X}$ is a reduced complete intersection with two irreducible components $\overline{\mathbf{S}}$ and $\overline{\mathbf{U}}$, where $\mathbf{S}$ is the stable locus and $\mathbf{U}$ is a G-orbit $G.(B_1^{(\mathrm{I})}, B_2^{(\mathrm{I})}, i^{(\mathrm{I})}, i^{(\mathrm{I})^*})$.

(2) $\mathbf{S}$ is a G-orbit $G.(B_1^{(\mathrm{II})}, B_2^{(\mathrm{II})}, i_4, i_4^*)$.

(3) The Sp-action on $\mathbf{U}$ is locally free (i.e., $\mathfrak{sp}^x = 0$ for any $x \in \mathbf{U}$).

(4) $\widetilde{X}/\!\!/\mathrm{Sp} = \overline{\mathbf{S}}/\!\!/\mathrm{Sp} \cup \overline{\mathbf{U}}/\!\!/\mathrm{Sp}$ as varieties (as $\widetilde{X}$ is reduced).

From now on we fix an orthogonal basis of W. Then $\mathfrak{o}$ is the set of anti-symmetric matrices. Let us identify

$$\mathfrak{o} \cong \mathbb{C}^3, \quad \begin{pmatrix} 0 & e & f \\ -e & 0 & g \\ -f & -g & 0 \end{pmatrix} \mapsto (e, f, g). \tag{7.21}$$

The characteristic polynomial of $A := \begin{pmatrix} 0 & e & f \\ -e & 0 & g \\ -f & -g & 0 \end{pmatrix}$ in t is $t^3 + (e^2 + f^2 + g^2)t$. Therefore A is nilpotent if and only if $e^2 + f^2 + g^2 = 0$. Since any nonzero nilpotent element x in $\mathfrak{o}$ has the a-diagram aaa, the minimal nilpotent orbit is $\mathrm{O}.x$. Hence, $\mathbf{P}$ is the quadric surface in $\mathbb{C}^3$ defined by $e^2 + f^2 + g^2 = 0$, which also equals the nilpotent variety.

LEMMA 7.27. *The map* $[(x, y)] \mapsto (x^2, \sqrt{-1}xy, y^2)$ *gives an isomorphism* $\mathbb{C}^2/\mathbb{Z}_2 \cong \mathbf{P}$, *and hence* $\mathbf{P}$ *is an irreducible normal variety. Moreover,* $\mathbf{P}^{\mathrm{rank}2} := \{A \in \mathbf{P} | \operatorname{rank} A = 2\} = \mathbf{P} \setminus 0 \cong (\mathbb{C}^2 \setminus 0)/\mathbb{Z}_2$.

PROOF. The first isomorphism is well-known in invariant theory. Since $\mathbb{C}^2$ is irreducible and normal, so is $\mathbb{C}^2/\mathbb{Z}_2$.

We prove the second assertion. The a-diagram of a nilpotent matrix $A \in \mathfrak{o}$ is either

$$\begin{matrix} a \\ a \\ a \end{matrix} \quad \text{or} \quad aaa \tag{7.22}$$

Therefore $A \neq 0$ means $\operatorname{rank} A = 2$. $\square$

Lemma 7.28. *$\Phi_{\widetilde{X}_1}\colon \widetilde{X}_1 \to \mathbf{P}$, $(B_1, B_2, i, i^*) \mapsto (\mathrm{tr}(B_1^2), \sqrt{-1}\mathrm{tr}(B_1B_2), \mathrm{tr}(B_2^2))$, is the GIT quotient by* Sp.

Proof. Let $x := (B_1, B_2, i, i^*) \in \widetilde{X}_1$. By Corollary 7.19 (2), x is unstable. Suppose Sp.x is closed in $\mathbf{N}$. Since $x \notin \mu^{-1}(0)^{\mathrm{reg}}$, we have $x^s = 0$ by Theorem 3.1 and Theorem 7.1 (4). Thus $i = 0$. Let $T := \{(aB, bB, 0, 0) |\, B \in \mathfrak{p}',\ a, b \in \mathbb{C}\}$. Let $\phi := \Phi_{\widetilde{X}_1}|_T$. It is enough to show that ϕ is the GIT quotient by Sp by Lemma 7.18.

Since the Sp-action on $\mathbf{P}$ is trivial and $\mathbf{P}$ is normal, we need to show that $\phi /\!\!/ \mathrm{Sp}\colon T /\!\!/ \mathrm{Sp} \to \mathbf{P}$ is bijective by Zariski's main theorem.

Since $\mathrm{tr}(v_1^2) \neq 0$, ϕ is surjective and thus so is $\phi /\!\!/ \mathrm{Sp}$.

To show injectivity, it is enough to show that $\phi^{-1}(c)$ is an Sp-orbit for any $c \in \mathbf{P} \setminus 0$ and that 0 is the unique closed Sp-orbit in $\phi^{-1}(0)$.

Let $c \in \mathbf{P} \setminus 0$. Then $c = (a^2, \sqrt{-1}ab, b^2)$ for some $a, b \in \mathbb{C}$. Then we have $\phi^{-1}(1, 0, 0) \cong \phi^{-1}(c)$, $(B, 0, 0, 0) \mapsto (aB, bB, 0, 0)$. Thus $\phi^{-1}(c)$ is an irreducible variety of dimension 4 since $\dim T = 6$ and $\dim \mathbf{P} = 2$.

On the other hand, we have ${\mathfrak{p}'}^B = \mathbb{C}\langle B\rangle$ for any $B \in \mathfrak{p}' \setminus 0$ by Lemma 7.18. So $\dim \mathfrak{p}^B = 2$. By [**30**, Prop. 5] we have $\dim \mathfrak{k}^B = 6$ and $\dim \mathrm{Sp}.B = 4$. This means $\phi^{-1}(c)$ is a Sp-orbit by irreducibility and the dimension reason. This proves the first item.

By a similar argument we have $\varphi /\!\!/ \mathrm{Sp}\colon \mathfrak{p}' /\!\!/ \mathrm{Sp} \to \mathbb{C}$ is a birational surjective morphism where $\varphi\colon \mathfrak{p}' \to \mathbb{C}$ is given by $B \mapsto \mathrm{tr}B^2$. Since both $\mathfrak{p}' /\!\!/ \mathrm{Sp}$ and $\mathbb{C}$ are irreducible normal varieties of dimension 1, $\varphi /\!\!/ \mathrm{Sp}$ is an isomorphism by Zariski's main theorem. Therefore 0 is the unique closed Sp-orbit in $\varphi^{-1}(0)$. This proves the second item. □

LEMMA 7.29. *$\Phi_{\overline{\mathbf{S}}}\colon \overline{\mathbf{S}} \to \mathbf{P}$, $(B_1, B_2, i, i^*) \mapsto i^*i$, is the GIT quotient by* Sp.

PROOF. Let $x := (B_1, B_2, i, i^*) \in \mathbf{S}$. By the ab-diagram (7.15) of i, we have $i^*i \in \mathbf{P}^{\text{rank2}}$. So $\Phi_{\overline{\mathbf{S}}}(\mathbf{S}) = \mathbf{P}^{\text{rank2}}$ and $\Phi_{\overline{\mathbf{S}}}$ is well-defined.

We claim that $\Phi_{\overline{\mathbf{S}}}^{-1}(a)$ is an Sp-orbit for any $a \in \mathbf{P} \setminus 0 = \mathbf{P}^{\text{rank2}}$. Note that since $\mathbf{S}$ is irreducible, so is $\Phi_{\overline{\mathbf{S}}}^{-1}(a)$. Since x is stable, Sp^x is trivial. Thus $\dim \mathrm{Sp}.x = 10$. On the other hand, $\Phi_{\overline{\mathbf{S}}}^{-1}(i^*i) = (\mathrm{SL}(2) \times \mathrm{Sp}).x$ by Theorem 7.14 and (7.13). Thus $\Phi_{\overline{\mathbf{S}}}^{-1}(i^*i)$ is an irreducible 10-dimensional variety. If $\mathrm{Sp}.x \subsetneq (\mathrm{SL}(2) \times \mathrm{Sp}).x$ then for $y \in (\mathrm{SL}(2) \times \mathrm{Sp}).x \setminus \mathrm{Sp}.x$, we have $\dim \mathrm{Sp}.y = 10$. But then $\Phi_{\overline{\mathbf{S}}}^{-1}(i^*i)$ contains two disjoint locally closed subvarieties $\mathrm{Sp}.x$ and $\mathrm{Sp}.y$ of dimension 10. This is absurd to the irreducibility of $\Phi_{\overline{\mathbf{S}}}^{-1}(i^*i)$. Therefore $\mathrm{Sp}.x = \Phi_{\overline{\mathbf{S}}}^{-1}(i^*i)$ as desired.

Since $\mathbf{S}$ consists of Sp-closed orbits (Theorem 3.1), $\mathbf{S}/\mathrm{Sp}$ is Zariski open in $\overline{\mathbf{S}}/\!\!/\mathrm{Sp}$. By Luna's slice theorem [**36**], $\mathbf{S}/\mathrm{Sp}$ is a smooth variety. By the above claim and Zariski's main theorem, $\overline{\Phi}_{\overline{\mathbf{S}}}|_{\mathbf{S}/\mathrm{Sp}}\colon \mathbf{S}/\mathrm{Sp} \to \mathbf{P}^{\text{rank2}}$ is an isomorphism where $\overline{\Phi}_{\overline{\mathbf{S}}}\colon \overline{\mathbf{S}}/\!\!/\mathrm{Sp} \to \mathbf{P}$ is the induced morphism. Let us finish the proof of the lemma. Let $f \in \mathbb{C}[\overline{\mathbf{S}}]^{\mathrm{Sp}}$. Then $f|_S \in \Phi^*\Gamma(\mathcal{O}_{\mathbf{P}\setminus 0})$. By the normality of $\mathbf{P}$, $\Gamma(\mathcal{O}_{\mathbf{P}\setminus 0}) = \mathbb{C}[\mathbf{P}]$. Thus $f \in \Phi_{\overline{\mathbf{S}}}^*\mathbb{C}[\mathbf{P}]$. This means $\mathbb{C}[\overline{\mathbf{S}}]^{\mathrm{Sp}} = \Phi_{\overline{\mathbf{S}}}^*\mathbb{C}[\mathbf{P}]$, equivalently Φ is the GIT quotient by Sp as desired. □

LEMMA 7.30. *$\widetilde{X}_1 \subset \overline{\mathbf{U}}$ and $\widetilde{X}_1/\!\!/\mathrm{Sp} = \overline{\mathbf{U}}/\!\!/\mathrm{Sp}$.*

Proof. Since $\mathbf{U}$ is an irreducible reduced locally free G-orbit of dimension 12, we have $\overline{\mathbf{U}}/\!\!/\mathrm{Sp}$ is an irreducible reduced variety of dimension ≤ 2. Now the second assertion follows from the first by Lemma 7.28.

Let us prove the first assertion. Suppose $\widetilde{X}_1 \not\subseteq \overline{\mathbf{U}}$. Then $\widetilde{X}_1 \subset \overline{\mathbf{S}}$ since $\widetilde{X}_1$ is irreducible by Corollary 7.19 (1). Let $T := \{(aB, bB) \in \mathfrak{p}' \times \mathfrak{p}' \,|\, a, b \in \mathbb{C},\ B \in \mathfrak{p}'\}$. By Lemma 7.17 and (7.7), $\widetilde{X}_1$ is the closure of $T \times (\mathrm{Sp} \times \mathrm{O}).i$ for a nonzero $i \in \mathrm{Hom}(W, V)$ with $i^*i = 0$. By Lemma 7.29, $\Phi_{\overline{\mathbf{S}}}(\widetilde{X}_1) = 0$. This contradicts Lemma 7.28. $\square$

Definition 7.31. Define $\Phi\colon \widetilde{X} \to (\mathbf{P}\times 0)\cup(0\times\mathbf{P})$ $(\subset \mathbf{P}\times\mathbf{P})$, $(B_1, B_2, i, i^*) \mapsto ((\mathrm{tr}(B_1^2), \sqrt{-1}\mathrm{tr}(B_1B_2), \mathrm{tr}(B_2^2)), i^*i)$.

To see Φ is well-defined morphism one notices that $i^{(\mathrm{I})*}i^{(\mathrm{I})} = i^{(\mathrm{II})*}i^{(\mathrm{II})} = i^{(\mathrm{III})*}i^{(\mathrm{III})} = 0$ and $\mathrm{tr}({B_1^{(\mathrm{II})}}^2) = \sqrt{-1}\mathrm{tr}(B_1^{(\mathrm{II})}B_2^{(\mathrm{II})}) = \mathrm{tr}({B_2^{(\mathrm{II})}}^2) = 0$, which come from Corollary 7.23 and the direct computation respectively.

Theorem 7.32. *Φ is the GIT quotient by* Sp *onto* $(\mathbf{P}\times 0)\cup(0\times\mathbf{P})$. *Hence,* $\mu^{-1}(0)/\!\!/\mathrm{Sp} \cong \mathbb{C}^2 \times ((\mathbf{P}\times 0)\cup(0\times\mathbf{P}))$.

Proof. By Lemmas 7.28, 7.29 and 7.30, Φ factors through $\widetilde{X}/\!\!/\mathrm{Sp}$. The canonical morphism given by the composite

$$(\mathbf{P}\times 0)\cup(0\times\mathbf{P}) \xrightarrow{f} \widetilde{X}_1/\!\!/\mathrm{Sp} \coprod_{[0]} \overline{\mathbf{S}}/\!\!/\mathrm{Sp} \longrightarrow \widetilde{X}/\!\!/\mathrm{Sp}$$

is the inverse of $\Phi/\!\!/\mathrm{Sp}$ where $f := (\Phi|_{\widetilde{X}_1}/\!\!/\mathrm{Sp})^{-1} \coprod_{[0]} (\Phi|_{\overline{\mathbf{S}}}/\!\!/\mathrm{Sp})^{-1}$ and $\Phi/\!\!/\mathrm{Sp}\colon \widetilde{X}/\!\!/\mathrm{Sp} \to (\mathbf{P}\times 0)\times(0\times\mathbf{P})$ is the induced morphism. $\square$

7.5. Moduli spaces of $\mathrm{SO}(2)$-data with $k = 4$

This section will be devoted to the proof of Theorem 7.2 (2).

Let $\dim V = k = 4$ and $\dim W = N = 2$. Let $\mathfrak{p}' := \{B \in \mathfrak{p} \,|\, \mathrm{tr}(B) = 0\}$. Let $\widetilde{X} := \{(B_1, B_2, i, i^*) \in \mu^{-1}(0) \,|\, B_1, B_2 \in \mathfrak{p}'\}$. Then as in §7.4.1, $\mu^{-1}(0) \cong \mathbb{C}^2 \times \widetilde{X}$. Let $p\colon \widetilde{X} \to \mathrm{Hom}(W, V)$ be the projection.

LEMMA 7.33. *Let $i \in \mathrm{Im}(p)$. Then ii^* is nilpotent and the ab-diagram of i is one of the followings:*

$$\begin{array}{cccc} & b & & \\ ab & b & & bab \\ ba & b & bab & b \\ b & b & bab & b \\ b & a & & a \\ & a & & \end{array} \tag{7.23}$$

PROOF. Let $(B_1, B_2, i, i^*) \in \widetilde{X}$. By Lemma 7.21, $([B_1, B_2])^2 = (ii^*)^2$ is a scalar. Since $\mathrm{rank}\, i \leq 2$ and $\dim V = 4$, the scalar is 0. Since there is no nontrivial nilpotent element in $\mathfrak{o}$, we have $i^*i = 0$. By a similar argument as in §7.2, the ab-diagram of i is one of (7.23) from Table 1. $\square$

Let $\widetilde{X}_1 := \{(B_1, B_2) \in \mathfrak{p}' \times \mathfrak{p}' \,|\, [B_1, B_2] = 0\} \times \{i \in \mathrm{Hom}(W, V) \,|\, ii^* = 0\}$. Let $\widetilde{X}_2 := \widetilde{X} \setminus \widetilde{X}_1$.

THEOREM 7.34. $\mu^{-1}(0)^{\mathrm{reg}} = \emptyset$.

PROOF. Let $x := (B_1, B_2, i, i^*) \in \widetilde{X}$. We will prove that x is not costable.

Suppose $x \in \widetilde{X}_1$. The ab-diagram of i is either the first or the second in (7.23). Therefore $\dim \mathrm{Ker}(i^*) \geq 3$. As in the proof of Corollary 7.19 (2), we see that x is not costable.

Suppose $x \in \widetilde{X}_2$. As in the proofs of Lemma 7.26 (1) and (3), we deduce that x is not costable. $\square$

APPENDIX A

Instanton numbers and the second Chern numbers

We find a relation between $c_2(E)$ and the instanton number where E is an associated vector bundle of a principal K-bundle over S^4. As usual we use the notation $c_2(E)$ both for a 2-form (the second Chern class) and an integer via integration over S^4 (the second Chern number).

A.1. Basic case $K = \mathrm{SU}(2)$

In this section we assume $K = \mathrm{SU}(2)$. We identity $S^4 = \mathbb{R}^4 \sqcup \{\infty\}$ so that $\mathbb{R}^4_{\le 1}$ (resp. $\mathbb{R}^4_{\ge 1} \sqcup \{\infty\}$) is the lower (resp. upper) hemisphere. We identify $\mathbb{R}^4 = \mathbb{H}$ (the space of quaternions). We also identify $\mathrm{SU}(2) = \{x \in \mathbb{H} | \, |x| = 1\}$. Let $\sigma(g) := g^{-1}dg$ (the Maurer-Cartan form on SU(2)). We can view σ as a 1-form on $\mathbb{R}^4 \setminus 0$ using the quaternion coordinates $x = x_1 + x_2 i + x_3 j + x_4 k$. Let

$$\omega := -\frac{1}{12}\mathrm{tr}(\sigma \wedge \sigma \wedge \sigma) \, \left(= \frac{\sum_{i=1}^{4}(-1)^{i-1}x_i dx_1 \wedge dx_2 \wedge ... \wedge \widehat{dx_i} \wedge ... \wedge dx_4}{r^4} \right).$$

Thus ω restricts to the volume form of SU(2) and $dr \wedge \omega = r^{-3}\omega_{\mathbb{R}^4}$ where $\omega_{\mathbb{R}^4}$ is the volume form of $\mathbb{R}^4$.

Let us define a complex vector bundle E of rank 2 over S^4 as follows. Let $r := |x|$ where $x \in \mathbb{R}^4$. Let $\phi\colon \mathbb{R}^4 \setminus 0 \to \mathrm{SU}(2)$ be any smooth map independent of r (e.g. $\phi(x) = x/|x|$). We define E using ϕ as a transition matrix: $\mathbb{C}^2 \times S^3 \to \mathbb{C}^2 \times S^3$,

$(v, x) \mapsto (\phi(x).v, x)$, where the left hand side S^3 (resp. the right hand side S^3) denotes the equator of the lower (resp. upper) hemisphere.

Let us define a connection A on E. To define a connection matrix we need an auxiliary smooth function $s\colon \mathbb{R}^4 \to [0, 1]$ satisfying

(1) $s(x)$ depends only on $r = |x|$,

(2) s is monotonically increasing with respect to r,

(3) $s(0) = 0$, $s|_{r=1} = 1$ and

(4) $s\phi^*\sigma$ extends to a smooth 1-form on $\mathbb{R}^4$.

Let $A := d + s\phi^*\sigma$. To see that A extends to a connection of E, we note that $\phi d\phi^{-1} + \phi\phi^*\sigma\phi^{-1}$ (the connection matrix on the upper hemisphere) is 0.

The curvature form of A is

$$F_A = ds \wedge \tau + (s^2 - s)\tau \wedge \tau.$$

where $\tau := \phi^*\sigma$. We have the second Chern class $c_2(E) = \frac{1}{8\pi^2}\mathrm{tr}(F_A \wedge F_A) = \frac{1}{4\pi^2}(s^2 - s)\mathrm{tr}(ds \wedge \tau \wedge \tau \wedge \tau)$. By direct calculation we have

$$\mathrm{tr}(ds \wedge \tau \wedge \tau \wedge \tau) = -12\frac{ds}{dr}dr \wedge \phi^*(\omega) \equiv -12n\frac{ds}{dr}dr \wedge \omega = -12n\frac{ds}{dr}r^{-3}\omega_{\mathbb{R}^4}$$

where $\equiv$ denotes the cohomology equivalence as classes in the De Rham cohomology group $H^4_{\mathrm{DR}}(S^4)$ and n denotes $\deg \phi|_{r=1}$. Therefore we have

$$\int_{S^4} c_2(E) = -\frac{3n}{\pi^2}\int_{\mathbb{R}^4}(s^2 - s)r^{-3}\frac{ds}{dr}\omega_{\mathbb{R}^4} = n. \tag{A.1}$$

Since ϕ is SU(2)-valued, it also defines a principal SU(2)-bundle. We denote this bundle by $P_{\mathrm{SU}(2)}$. It is clear that $E = P_{\mathrm{SU}(2)} \times_{\mathrm{SU}(2)} \mathbb{C}^2$. The instanton number of $P_{\mathrm{SU}(2)}$ is n since in general the instanton number of a principal SU(2)-bundle is defined as the degree of the transition from the lower to upper hemisphere. Hence (A.1) asserts that the instanton number of a principal SU(2)-bundle and the second Chern number of the vector bundle associated to the fundamental representation are equal.

In the next section we generalize this result to K-instantons for any compact simple Lie group K.

A.2. Cases of simple groups

Let K be any compact simple Lie group. Here 'simple' means that $\mathfrak{k} := \mathrm{Lie}(K)$ is a simple Lie algebra.

We follow the argument of Atiyah-Hitchin-Singer [**2**, p.453] for generalization of the previous result to K-instantons P_K. Our steps toward the generalization are as follows:

(1) find first a principal SU(2)-subbundle $P_{\mathrm{SU}(2)}$ of P_K with the same instanton number;

(2) compare the second Chern numbers of the two associated vector bundles from $P_{\mathrm{SU}(2)}$ and P_K.

We now start (1) using Bott's periodicity. We need a set-up for this.

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{k}$. Let $\alpha \in \mathfrak{h}_{\mathbb{C}}^{\vee}$ be any root of $\mathfrak{k}_{\mathbb{C}}$. Let $L_\alpha \subset \mathfrak{k}_{\mathbb{C}}$ be the weight α subspace (root space). Let $h_\alpha \in \mathfrak{h}_{\mathbb{C}}$ such that $\langle h_\alpha, \beta \rangle = \frac{2(\alpha,\beta)}{(\alpha,\alpha)}$, where $\langle\, , \rangle$ and $(\, ,)$ denote the canonical pairing and the dual invariant form respectively (cf. [**24**, §8.3]). Then h_α spans $[L_\alpha, L_{-\alpha}]$. Since the invariant form on $\mathfrak{h}$ is negative definite, $\mathfrak{h}$ is spanned by $\sqrt{-1}h_\alpha$ over $\mathbb{R}$. We define a lattice in $\sqrt{-1}\mathfrak{h}$

$$L := \mathbb{Z}\langle h_\alpha |\, \alpha : \mathrm{root}\rangle .$$

It is isometric to the coroot lattice.

Let $\exp\colon \mathfrak{k} \to \widetilde{K}$ be the exponential map where $\widetilde{K}$ is the universal covering of K. Let $\mathrm{SU}(2)_\alpha$ be the subgroup of $\widetilde{K}$ generated by $\exp((L_\alpha + L_{-\alpha})_\mathbb{R})$ where $(L_\alpha + L_{-\alpha})_\mathbb{R}$ denotes the space of real forms in $L_\alpha + L_{-\alpha}$. We have $\mathrm{SU}(2)_\alpha \cong \mathrm{SU}(2)$.

Let $p, q \in \mathrm{SU}(2)_\alpha$ be the points corresponding to $I, -I \in \mathrm{SU}(2)$ respectively. The geodesics in $\widetilde{K}$ from p to q are written as

$$C_X(t) = g.\exp(2\pi t X) \quad (t \in [0,1])$$

where $X \in \mathfrak{h}$, $C_X(1) = q$ and $g \in Z_{\widetilde{K}}(q)$ ([**39**, Lemma 21.2]). Here $Z_{\widetilde{K}}(q)$ denotes the centralizer subgroup of q in $\widetilde{K}$. We notice from $C_X(1) = q$ that $X \in \sqrt{-1}(\frac{1}{2}h_\alpha + L)$ since $2\pi\sqrt{-1}L = \exp^{-1}(\mathrm{id}_{\widetilde{K}}) \cap \mathfrak{h}$.

We use the notation $(\ ,\ )$ also for the invariant form if no confusion arises. By normalizing the length of a 1-dimensional torus in $\mathrm{SU}(2)$, the length of C_X is $-(X, X)$. We have a criterion for the minimal length of C_X where $X \in \sqrt{-1}(\frac{1}{2}h_\alpha + L)$:

Lemma A.1. *Suppose α is a long root. Then $-(X, X)$ attains the minimal value $\frac{1}{(\alpha,\alpha)}$ if and only if $X = \pm\frac{1}{2}\sqrt{-1}h_\alpha$.*

We say that $C_X(t)$ has *index* l if it has l conjugate points. We will not give the precise definition of conjugate points, but in our situation the index l coincides with the number of pairs of a root γ and a positive integer n such that $|\gamma(X)| > n$. See

[**39**, Theorem 20.5 and the proof of 21.7 on pp.116–117]. We need a lower bound of the index l to use Bott's periodicity later.

LEMMA A.2. *Suppose α is a long root. If $X \neq \pm\frac{1}{2}\sqrt{-1}h_\alpha$ then $l \geq 4$.*

So the two lemmas assert that any non-minimal geodesic of the form $C_X(t)$ has index ≥ 4 provided α is a long root.

THEOREM A.3. *If α is a long root, the canonical homomorphism $\pi_3(\mathrm{SU}(2)_\alpha) \to \pi_3(\widetilde{K})$ is an isomorphism. Hence the canonical homomorphism $\pi_3(\mathrm{SU}(2)_\alpha) \to \pi_3(K)$ is an isomorphism.*

PROOF. Let $\Omega_{\mathrm{SU}(2)_\alpha}$ (resp. $\Omega_{\widetilde{K}}$) be the loop space of $\mathrm{SU}(2)_\alpha$ (resp. $\widetilde{K}$). Let $\Omega^d_{\mathrm{SU}(2)_\alpha}$ (resp. $\Omega^d_{\widetilde{K}}$) be the space of minimal geodesics from p to q in $\mathrm{SU}(2)_\alpha$ (resp. $\widetilde{K}$). By fixing any path C from q to p in $\mathrm{SU}(2)_\alpha$, we have an embedding $\Omega^d_{\mathrm{SU}(2)_\alpha} \to \Omega_{\mathrm{SU}(2)_\alpha}$ by the amalgamation product $\tau \mapsto C * \tau$. Similarly C induces an embedding $\Omega^d_{\widetilde{K}} \to \Omega_{\widetilde{K}}$. These embeddings induce homomorphisms between homotopy groups and thus fit in the following commuting diagram of homotopy exact sequences:

$$\begin{array}{ccccccc}
\pi_1(\Omega_{\mathrm{SU}(2)_\alpha}, \Omega^d_{\mathrm{SU}(2)_\alpha}) & \longrightarrow & \pi_2(\Omega^d_{\mathrm{SU}(2)_\alpha}) & \longrightarrow & \pi_2(\Omega_{\mathrm{SU}(2)_\alpha}) & \longrightarrow & \pi_2(\Omega_{\mathrm{SU}(2)_\alpha}, \Omega^d_{\mathrm{SU}(2)_\alpha}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\pi_1(\Omega_{\widetilde{K}}, \Omega^d_{\widetilde{K}}) & \longrightarrow & \pi_2(\Omega^d_{\widetilde{K}}) & \longrightarrow & \pi_2(\Omega_{\widetilde{K}}) & \longrightarrow & \pi_2(\Omega_{\widetilde{K}}, \Omega^d_{\widetilde{K}})
\end{array}$$

We claim that the relative homotopy groups in the above diagram are all trivial. This will follow from Bott's periodicity [**39**, Theorem 22.1], once we check that the assumptions of the theorem are fulfilled: Both $\Omega^d_{\mathrm{SU}(2)_\alpha}$ and $\Omega^d_{\widetilde{K}}$ are $\mathcal{C}^\infty$-manifolds and the indices of non-minimal geodesics in both $\mathrm{SU}(2)_\alpha$ and $\widetilde{K}$ are not less than

4. The second assumption was readily checked in Lemma A.2. The first assumption was also checked since the action of $Z_{\widetilde{K}}(q)$ (resp. $Z_{\mathrm{SU}(2)_\alpha}(q)$) on $\Omega^d(\widetilde{K})$ (resp. $\Omega^d(\mathrm{SU}(2)_\alpha)$) is transitive.

Now the two horizontal maps in the middle in the above diagram are isomorphisms. To conclude that $\pi_2(\Omega_{\mathrm{SU}(2)_\alpha}) \cong \pi_2(\Omega_{\widetilde{K}})$, we need to prove

$$\pi_2(\Omega^d_{\mathrm{SU}(2)_\alpha}) \cong \pi_2(\Omega^d_{\widetilde{K}}). \tag{A.2}$$

Let $X' := \pi\sqrt{-1}h_\alpha$ (so that $q = \exp(X')$). We have $\Omega^d_{\widetilde{K}} = Z_{\widetilde{K}}(q)/Z_{\widetilde{K}}(\exp(\mathbb{R}X'))$ and $\Omega^d_{\mathrm{SU}(2)_\alpha} = Z_{\mathrm{SU}(2)_\alpha}(q)/Z_{\mathrm{SU}(2)_\alpha}(\exp(\mathbb{R}X'))$. Thus $\Omega^d_{\mathrm{SU}(2)_\alpha}$ is a submanifold of $\Omega^d_{\widetilde{K}}$. We claim that $(\Omega^d_{\widetilde{K}})_0 = (\Omega^d_{\mathrm{SU}(2)_\alpha})_0$ where $(\bullet)_0$ denotes the identity component. The claim will assert (A.2). It amounts to showing

$$\mathrm{Lie}(Z_{\widetilde{K}}(q)) = \mathfrak{h} + \mathfrak{su}(2)_\alpha, \quad \mathrm{Lie}(Z_{\widetilde{K}}(\exp(\mathbb{R}X'))) = \mathfrak{h}, \tag{A.3}$$

where $\mathfrak{su}(2)_\alpha := \mathrm{Lie}(\mathrm{SU}(2)_\alpha)$. For, these induce a canonical isomorphism

$$T_{[I]}Z_{\mathrm{SU}(2)_\alpha}(q)/Z_{\mathrm{SU}(2)_\alpha}(\exp(\mathbb{R}X')) \cong T_{[I]}Z_{\widetilde{K}}(q)/Z_{\widetilde{K}}(\exp(\mathbb{R}X'))$$

as both sides are isomorphic to $\mathfrak{su}(2)_\alpha/\mathfrak{h} \cap \mathfrak{su}(2)_\alpha$. Here the notations $T_{[I]}$ denote the tangent spaces at the identity classes respectively.

It remains to prove (A.3). Let $Y \in \mathfrak{k}$. We denote by $e^\bullet$ the ordinary exponential $\sum_{n\geq 0} \frac{1}{n!}\bullet^n$ defined on $\mathfrak{gl}(\mathfrak{k})$. Since $\mathrm{Ad}\exp X' = e^{\mathrm{ad}X'}$ (see [**22**, p.128, (5)]), $Y \in \mathrm{Lie}(Z_{\widetilde{K}}(\exp X'))$ if and only if

$$\sum_{n\geq 1} \frac{1}{n!}(\mathrm{ad}X')^n(Y) = 0. \tag{A.4}$$

We write $Y = \sum_\beta v_\beta \in \mathfrak{k}_\mathbb{C} = \oplus_\beta L_\beta$. Since $(\mathrm{ad}X')^n(Y) = \sum_\beta \beta(X')^n v_\beta$, (A.4) holds if and only if $\sum_\beta (e^{\beta(X')} - 1)v_\beta = 0$ if and only if $v_\beta = 0$ or $e^{\beta(X')} = 1$ for each β. Note that $e^{\beta(X')} = 1$ amounts to $\beta(X') \in 2\pi\sqrt{-1}\mathbb{Z}$ and thus to $\beta \in \{0, \pm\alpha\}$ (as $X' = \pi\sqrt{-1}h_\alpha$). Therefore we have $\mathrm{Lie}(Z_{\widetilde{K}}(q)) = \mathfrak{h} + \mathfrak{su}(2)_\alpha$.

A similar argument shows $\mathrm{Lie}(Z_{\widetilde{K}}(\exp \mathbb{R}X')) = \mathfrak{h}$ because $e^{\beta(\mathbb{R}X')} = 1$ for all β with $v_\beta \neq 0$, which amounts to $\beta = 0$. This finishes the proof of (A.3).

Now we have the isomorphism (A.2). As a result we have $\pi_2(\Omega_{\mathrm{SU}(2)_\alpha}) \cong \pi_2(\Omega_{\widetilde{K}})$ and thus $\pi_3(\mathrm{SU}(2)_\alpha) \cong \pi_3(\widetilde{K})$ by the canonical isomorphism $\pi_2(\Omega_\bullet) \cong \pi_3(\bullet)$. $\square$

If α is not a long root, the canonical homomorphism is never an isomorphism. See Remark A.4 below.

Let P_K be a principal K-bundle over S^4. It is given by a transition from the lower hemisphere to the upper one. Such a transition corresponds to an element of $\pi_3(K)$. Thus an element of $\pi_3(K)$ determines P_K topologically in a unique way.

Let P_α be the induced principal SU(2)-bundle over S^4 by the canonical homomorphism $\pi_3(\mathrm{SU}(2)_\alpha) \to \pi_3(K)$. Via the homomorphism the element of $\pi_3(\mathrm{SU}(2)_\alpha)$ corresponding to P_α maps to the element of $\pi_3(K)$ corresponding to P_K. So if this does not happen, such P_α does not exist. But if α is a long root, the homomorphism is an isomorphism by Theorem A.3 and thus P_α always exists. And the instanton numbers of P_K, P_α are equal. Our first step (1) is attained.

Let us start (2): compare the second Chern numbers of associated vector bundles of P_K, P_α. We denote by F be a finite dimensional representation of $\mathfrak{k}$. Let $\mathrm{tr}_F^{\mathfrak{k}} \colon \mathfrak{k} \times \mathfrak{k} \to \mathbb{C}$, $(X, Y) \mapsto \mathrm{tr}(XY \colon F \to F)$ be the (bilinear) trace form on E.

Note that $\mathrm{tr}_F^{\mathfrak{k}} = \mathrm{tr}_{F_{\mathbb{C}}}^{\mathfrak{k}_{\mathbb{C}}}|_{\mathfrak{k}\times\mathfrak{k}}$. And $\mathrm{tr}_F^{\mathfrak{k}} = \mathrm{tr}_F^{\mathfrak{k}_{\mathbb{C}}}|_{\mathfrak{k}\times\mathfrak{k}}$, if F itself is a complex representation of $\mathfrak{k}$. Therefore we will use the simplified notation tr_F as the trace forms from $\mathfrak{k}, \mathfrak{k}_{\mathbb{C}}$ do not cause confusion.

By [**24**, Lemma 22.1], tr_F is a scalar multiple of $\mathrm{tr}_{F'}$ where F' is an irreducible representation. In particular if $F' = \mathfrak{k}$, we denote this scalar by $c_F := \mathrm{tr}_F/\mathrm{tr}_{\mathfrak{k}}$.

Let $\widetilde{F} := P_K \times_K F$. We have $p_1(\widetilde{F}) = c_F p_1(\widetilde{\mathfrak{k}})$ and $c_2(\widetilde{F}_{\mathbb{C}}) = c_F c_2(\widetilde{\mathfrak{k}}_{\mathbb{C}})$, since $\mathrm{tr}_{\widetilde{F}} F_A^2 = c_F \mathrm{tr}_{\mathfrak{k}} F_A^2$ where A is an $\mathrm{ad}\mathfrak{k}$-valued connection.

If $K = \mathrm{SU}(2)$ and $F = \mathbb{C}^2$ (the fundamental representation) then

$$c_2(\widetilde{\mathfrak{su}}(2)_{\mathbb{C}}) = c_F^{-1} c_2(\widetilde{F}) = 4c_2(\widetilde{F}) = 4n \tag{A.5}$$

where the last equality was deduced in (A.1). Thus we have $c_2(\widetilde{\mathfrak{k}}_{\mathbb{C}}) = c_2(P_\alpha \times_{\mathrm{SU}(2)} \mathfrak{k}_{\mathbb{C}}) = 4c_{\mathfrak{k}} n$ where $c_{\mathfrak{k}} = \mathrm{tr}_{\mathfrak{k}}/\mathrm{tr}_{\mathfrak{su}(2)}$.

Let us compute $c_{\mathfrak{k}}$ in the above in order to compute $c_2(\widetilde{\mathfrak{k}}_{\mathbb{C}})$. Let α be the highest root. It is known to be a long root (cf. [**24**, Lemma D, §10.4]). We can compute $c_{\mathfrak{k}} = \mathrm{tr}_{\mathfrak{k}_{\mathbb{C}}}(\mathrm{ad}_{h_\alpha}^2)/\mathrm{tr}_{\mathfrak{su}(2)_{\mathbb{C}}}(\mathrm{ad}_{h_\alpha}^2)$ explicitly by using the root space decomposition. Since $\mathrm{tr}_{L_\beta} \mathrm{ad}_{h_\alpha}$ is equal to $\frac{2(\beta,\alpha)}{(\alpha,\alpha)}$, it is 0 or ± 1 unless $\beta = \pm\alpha$. Thus we have

$$c_{\mathfrak{k}} = \frac{\frac{(\delta,\alpha)}{(\alpha,\alpha)} + 1}{2}$$

where δ is the sum of all positive roots. The numerator in the above is called the *dual Coxeter number* denoted by $h^\vee$. See [**5**, Ch. VI, §§4.7–4.13] for the expressions of α, δ in linear combinations of simple roots. Hence we get $h^\vee$ by direct computation. See Table 1 (l denotes the rank of K). This finishes the step (2).

Table 1. dual Coxeter number $h^\vee$

K	A_l	B_l	C_l	D_l	E_6	E_7	E_8	F_4	G_2
$h^\vee$	$l+1$	$2l-1$	$l+1$	$2l-2$	12	18	30	9	4

With the steps (1) and (2), we get back to the original goal: a relation between $c_2(\widetilde{F}_{\mathbb{C}})$ and the instanton number n. Here F is the fundamental representation of classical groups K. Since $c_2(\widetilde{F}_{\mathbb{C}}) = c_F c_2(\widetilde{\mathfrak{k}}_{\mathbb{C}}) = 2c_F h^\vee n$, we need to compute $c_F = \mathrm{tr}_F(a^2)/\mathrm{tr}_{\mathfrak{k}}(\mathrm{ad}_a^2)$. We will choose $a \in \mathfrak{k}$ in a way that we can compute c_F by hand without heavy computation.

Let $K := \mathrm{SO}(N, \mathbb{R})$ and $F := \mathbb{R}^N$ first. Let $e_{p,q}$ be the (p, q)-elementary matrix. Let $E_{p,q} := e_{p,q} - e_{q,p}$. Then $E_{p,q}$ form a basis of $\mathfrak{k}$ for $1 \leq p < q \leq N$. Let $a := E_{1,2}$. Then $c_F = \mathrm{tr}_F(a^2)/\mathrm{tr}_{\mathfrak{k}}(\mathrm{ad}_a^2) = \frac{-2}{-2(N-2)}$. Thus we obtain

$$c_2(\widetilde{F}_{\mathbb{C}}) = \begin{cases} 2n & \text{if } N \geq 5 \\ 4n & \text{if } N = 3. \end{cases}$$

Let $K := \mathrm{USp}(N/2)$ and $F := \mathbb{R}^N$, secondly. A basis of $\mathfrak{k}_{\mathbb{C}}\,(= \mathfrak{sp}(N/2))$ is the set of the matrices $e_{p,q} - e_{q+N/2,p+N/2}$ $(1 \leq p, q \leq N/2)$ and the two kinds of matrices $e_{p,q+N/2} + e_{q,p+N/2}$ and $e_{p+N/2,q} + e_{q+N/2,p}$ $(1 \leq p \leq q \leq N/2)$. Let $a := e_{1,1} - e_{1+N/2,1+N/2}$. Then $c_F = \mathrm{tr}_{F_{\mathbb{C}}}(a^2)/\mathrm{tr}_{\mathfrak{k}_{\mathbb{C}}}(\mathrm{ad}_a^2) = \frac{2}{4(N+2)}$. Thus $c_2(\widetilde{F}_{\mathbb{C}}) = \frac{2h^\vee n}{2(N+2)} = n$.

Let $K := \mathrm{SU}(N)$ and $F := \mathbb{C}^N$. We can show $c_2(\widetilde{F}) = n$ similarly. We omit the details.

Remark A.4. The canonical homomorphism $\pi_3(\mathrm{SU}(2)_\alpha) \to \pi_3(\widetilde{K})$ is not surjective in general. Let α (resp. α_0) be any root (resp. a long root). Suppose the instanton number n of P_K is in the image of the above homomorphism. Let n_α be its instanton number of P_α. By Theorem A.3, we have $n = n_{\alpha_0}$.

On the other hand we have $c_2(\widetilde{\mathfrak{k}_\mathbb{C}}) = c_\mathfrak{k} c_2(P_\alpha \times_{\mathrm{SU}(2)} \mathfrak{su}(2)_\mathbb{C}) = 4c_\mathfrak{k} n_\alpha$. So $c_\mathfrak{k} n_\alpha$ and hence $\mathrm{tr}_{\mathfrak{k}_\mathbb{C}}(\mathrm{ad}^2_{h_\alpha})n_\alpha \,(= \frac{4n_\alpha}{(\alpha,\alpha)})$ are independent from the choice of α. This implies $n_\alpha = \frac{(\alpha,\alpha)}{(\alpha_0,\alpha_0)} n$. Therefore if α is a short root, $n_\alpha = n/2$ or $n/3$ and thus 1 is not in the image of the canonical homomorphism.

A.3. Proofs of Lemmas A.1 and A.2

Let Φ be the root system of $\mathfrak{k}_\mathbb{C}$ and Δ be a base of Φ. Let L^+ (resp. L^-) be the semigroup of L generated by h_β (resp. $-h_\beta$) and 0 where $\beta \in \Delta$. For $z \in L$, we denote by $z \geq 0$ (resp. $z \gneq 0$) if $z \in L^+$ (resp. $z \in L^+ \setminus 0$). Let σ_β be the reflection with respect to the hyperplane $\beta^\perp$. So $\sigma_\beta(\gamma) = \gamma - 2\frac{(\gamma,\beta)}{(\beta,\beta)}\beta$.

We fix a long root α. By basic properties of Δ ([**24**, §§9.4,10.1]), we have

$$(h_\beta, h_{\beta'}) = 4\frac{(\beta,\beta')}{(\beta,\beta)(\beta',\beta')} \in \mathbb{Z}_{\leq 0}\frac{2}{(\alpha,\alpha)} \tag{A.6}$$

for any distinct $\beta, \beta' \in \Delta$.

Let $Y \in L$. We can write $Y = \sum_{\beta\in\Delta} m_\beta h_\beta$ as a unique linear combination where $m_\beta \in \mathbb{Z}$. By (A.6), we have

$$(Y,Y) \in \mathbb{Z}_{\geq 0}\frac{4}{(\alpha,\alpha)}.$$

Let $X = Y + \frac{1}{2}h_\alpha$ where $Y \in L$. This notation X differs by $\sqrt{-1}$ from the one in the statements of Lemmas A.1 and A.2. We use only $\sqrt{-1}\mathfrak{h}$ instead of $\mathfrak{h}$ via the multiplication by $\sqrt{-1}$ during this section. Since they are isometric up to sign, the lemmas can be adjusted in an obvious way.

Proof of Lemma A.1. We write $(Y, Y) = \frac{4n}{(\alpha,\alpha)}$ for $n \in \mathbb{Z}_{\geq 0}$. Then by the Cauchy-Schwarz inequality $|(Y, h_\alpha)| \leq \frac{4\sqrt{n}}{(\alpha,\alpha)}$, we have

$$(X, X) = (Y, Y) + (Y, h_\alpha) + \frac{1}{4}(h_\alpha, h_\alpha) \geq \frac{4(n - \sqrt{n}) + 1}{(\alpha, \alpha)}.$$

So, $(X, X) \geq \frac{1}{(\alpha,\alpha)}$ and the equality holds if and only if $Y = 0$ or $Y = -h_\alpha$. We are done. $\square$

The following remark is about a generalization of Lemma A.1 to short roots. It is irrelevant to Lemma A.2 and its proof, so can be safely skipped for those who are not interested in the case of short roots.

Remark A.5. In this remark we suppose α is not necessarily a long root. Instead, we fix a long root α_0.

Lemma A.1 is slightly generalized as follows: (X, X) attains the minimal value $\frac{1}{(\alpha,\alpha)}$ if and only if $X = \frac{1}{2}h_\gamma$ for some β with the same length with α, unless K is of type G_2.

Let us consider the case when K is of type G_2 first. If α is a short root, $(X, X) \geq \frac{1}{(\alpha,\alpha)}$ fails. For instance $X = h_\beta + \frac{1}{2}h_\alpha$ has strictly shorter length than $\frac{1}{2}h_\alpha$, where β is a long root with angle $\frac{5}{6}\pi$ from α.

Now we prove the generalized claim. So K is not of type G_2. The case when α is a long root was proven readily. So α is a short root. Note that a short root exists only when the type of K is one of B, C and F. Thus we have $\frac{(\alpha,\alpha)}{(\alpha_0,\alpha_0)} = \frac{1}{2}$.

As before we write $(Y, Y) = \frac{4n}{(\alpha_0,\alpha_0)}$ for some $n \in \mathbb{Z}_{\geq 0}$. By the Cauchy-Schwarz inequality $|(Y, h_\alpha)| \leq \frac{4\sqrt{2n}}{(\alpha_0,\alpha_0)}$, we obtain

$$(X, X) = (Y, Y) + (Y, h_\alpha) + \frac{1}{4}(h_\alpha, h_\alpha) \geq \frac{4(n - \sqrt{2n})}{(\alpha_0, \alpha_0)} + \frac{1}{4}(h_\alpha, h_\alpha). \tag{A.7}$$

Thus for $n \geq 3$, (X, X) is not minimal.

On the other hand when $Y \in \mathbb{Z}h_\alpha$, we have nothing to show since Lemma A.1 itself is true manifestly. So we assume $Y \notin \mathbb{Z}h_\alpha$.

Now we need to study the cases $n = 0, 1, 2$.

The cases $n = 0, 2$ do not happen. The reason is as follows. The equality of (A.7) holds if and only if the equality of the Cauchy-Schwarz inequality holds if and only if $Y \in \mathbb{Z}_{\leq 0}h_\alpha$. This contradicts to our assumption $Y \notin \mathbb{Z}h_\alpha$.

The case $n = 1$ can happen as follows. In the case the Cauchy-Schwarz inequality is $|(Y, h_\alpha)| \leq \frac{4\sqrt{2}}{(\alpha_0,\alpha_0)}$. By (A.6), it is refined as $|(Y, h_\alpha)| \leq \frac{4}{(\alpha_0,\alpha_0)}$. Plugging this into (A.7), we have $(X, X) \geq \frac{1}{4}(h_\alpha, h_\alpha)$. And the equality holds if and only if $(Y, Y) = \frac{4}{(\alpha_0,\alpha_0)}$ and $(Y, h_\alpha) = -\frac{4}{(\alpha_0,\alpha_0)}$. We find Y satisfying the latter two equalities from now on. In Lemma A.8 below, we will see that $(Y, Y) = \frac{4}{(\alpha_0,\alpha_0)}$ if and only if $Y = h_\beta$ for some long root β. Plugging $Y = h_\beta$ into the equality $(Y, h_\alpha) = -\frac{4}{(\alpha_0,\alpha_0)}$, we have $(h_\beta, h_\alpha) = -\frac{4}{(\alpha_0,\alpha_0)}$. This happens exactly when the angle between α, β is $\frac{3}{4}\pi$ and $\frac{(\beta,\beta)}{(\alpha,\alpha)} = 2$. In the case $\alpha + \beta$ is a short root and $X = \frac{1}{2}h_{\alpha+\beta}$ which follows from checking directly that $\gamma(h_\beta + \frac{1}{2}h_\alpha) = \frac{1}{2}\gamma(h_{\alpha+\beta})$ for all roots γ. Since $(X, X) = \frac{1}{4}(h_\alpha, h_\alpha)$ in the case, the claim is proven.

For the proof of Lemma A.2, we need a preliminary lemma.

Lemma A.6. *Let β_1, β_2 be any roots. Then we have the followings.*

(1) Φ *is not contained in* $\beta_1^\perp \cup \beta_2^\perp$.

(2) *If* $(\beta_1, \beta_2) \geq 0$, *there is a root* γ *such that* $(\gamma, \beta_1), (\gamma, \beta_2) > 0$.

The proof will appear at the end of this section.

We prove Lemma A.2 for some special cases:

Lemma A.7. *Let $m, n \in \mathbb{Z}$ and β be any root such that $\beta \neq \pm\alpha$. Let $X = mh_\beta + (n + \frac{1}{2})h_\alpha$. If $X \neq \frac{1}{2}h_\alpha$, we have $l \geq 4$.*

Proof. We may assume either $m, (n + \frac{1}{2}) \geq 0$ or $m, (n + \frac{1}{2}) \leq 0$, because we can replace β, m into $-\beta, -m$, respectively if necessary. We proceed the proof in the following three possible cases: (i) $(\alpha, \beta) = 0$, (ii) $(\alpha, \beta) > 0$, (iii) $(\alpha, \beta) < 0$.

Case (i). Let γ be a root such that $(\gamma, \alpha), (\gamma, \beta) > 0$ (Lemma A.6). Then we have $\beta(X) = 2m$, $\alpha(X) = 2n + 1$ and $|\gamma(X)| = |m + (n + \frac{1}{2})|$. If $m \neq 0$ then $|\beta(X)|, |\gamma(X)| > 1$ which implies $l \geq 4$. If $m = 0$ then $|\alpha(X)| > 2$ which implies $l \geq 4$.

Case (ii): $(\alpha, \beta) > 0$. We have $|\beta(X)| \geq |2m + (n + \frac{1}{2})|$ and $\alpha(X) = m + 2n + 1$. Thus $l \geq 4$ as the case $m = n = 0$ does not happen.

Case (iii): $(\alpha,\beta)<0$. We use the followings:

$$\alpha(X)=\begin{cases} -3m+2(n+\frac{1}{2}) & \text{if } \frac{(\alpha,\alpha)}{(\beta,\beta)}=3 \\ -2m+2(n+\frac{1}{2}) & \text{if } \frac{(\alpha,\alpha)}{(\beta,\beta)}=2 \\ -m+2(n+\frac{1}{2}) & \text{if } \frac{(\alpha,\alpha)}{(\beta,\beta)}=1 \end{cases}$$

$$\beta(X)=2m-(n+\frac{1}{2})$$

$$(\alpha+\beta)(X)=\begin{cases} -m+(n+\frac{1}{2}) & \text{if } \frac{(\alpha,\alpha)}{(\beta,\beta)}=3 \\ n+\frac{1}{2} & \text{if } \frac{(\alpha,\alpha)}{(\beta,\beta)}=2 \\ m+(n+\frac{1}{2}) & \text{if } \frac{(\alpha,\alpha)}{(\beta,\beta)}=1 \end{cases}$$

$$(\alpha+2\beta)(X)=\begin{cases} m & \text{if } \frac{(\alpha,\alpha)}{(\beta,\beta)}=3 \\ 2m & \text{if } \frac{(\alpha,\alpha)}{(\beta,\beta)}=2 \end{cases}$$

$$(\alpha+3\beta)(X)=3m-(n+\frac{1}{2}) \qquad \text{if } \frac{(\alpha,\alpha)}{(\beta,\beta)}=3.$$

$$(2\alpha+3\beta)(X)=3(n+\frac{1}{2}) \qquad \text{if } \frac{(\alpha,\alpha)}{(\beta,\beta)}=3.$$

Note that $\alpha,\beta,\alpha+\beta,\alpha+2\beta,\alpha+3\beta$ and $2\alpha+3\beta$ are all roots in respective case in the above. See [**24**, Theorem 8.5] for the first five roots; see [**24**, Theorem 11.4] for the last root (the G_2 case). We check $l\geq 4$ by computing the absolute values of the right hand sides of the above identities for some m,n. By looking at the coefficients in the above we see that the case $m\leq 0, n\leq -1$ is essentially same as the case $m\geq 0, n\geq 0$. So we assume $m\geq 0, n\geq 0$.

Let us prove the case $\frac{(\alpha,\alpha)}{(\beta,\beta)}=3$ (G_2 case). If $n\geq 1$, then $(2\alpha+3\beta)(X)=\frac{9}{2}$, which asserts $l\geq 4$. If $n=0$ and $m\geq 1$ then $\alpha(X)\leq -2$ and $\beta(X)\geq\frac{3}{2}$, which asserts $l\geq 4$.

Let us prove the case $\frac{(\alpha,\alpha)}{(\beta,\beta)} = 2$ (BCF case). If $m, n \geq 1$ then $(\alpha+\beta)(X) \geq \frac{3}{2}$ and $(\alpha+2\beta)(X) \geq 2$, which asserts $l \geq 4$. If $m \geq 1$ and $n = 0$ then $\beta(X) \geq \frac{3}{2}$ and $(\alpha+2\beta)(X) \geq 2$, which asserts $l \geq 4$. If $m = 0$ and $n \geq 1$ then $\alpha(X) \geq 3$, which asserts $l \geq 4$.

Let us prove the case $\frac{(\alpha,\alpha)}{(\beta,\beta)} = 1$. If $m, n \geq 1$ then $(\alpha+\beta)(X) \geq \frac{5}{2}$, which asserts $l \geq 4$. If $m \geq 1$ and $n = 0$ then $\beta(X), (\alpha+\beta)(X) \geq \frac{3}{2}$, which asserts $l \geq 4$. If $m = 0$ and $n \geq 1$ then $\alpha(X) \geq 3$, which asserts $l \geq 4$.

This completes the proof. $\square$

We need further set-up to prove Lemma A.2 for general cases. Let us write $Y = \sum_{\beta\in\Delta} m_\beta h_\beta$ as a unique linear combination. We define

$$Y_+ := \sum_{m_\beta > 0} m_\beta h_\beta, \quad Y_- := -\sum_{m_\beta < 0} m_\beta h_\beta$$

in L, and

$$y := \sum_{\beta\in\Delta} m_\beta \beta, \quad y_+ := \sum_{m_\beta > 0} m_\beta \beta, \quad y_- := -\sum_{m_\beta < 0} m_\beta \beta$$

in the root lattice. It is obvious that $Y = Y_+ - Y_-$ and $y = y_+ - y_-$.

The *support of* Y, denoted by $\mathrm{Supp}(Y)$, is defined to be the set $\{\beta \in \Delta |\, m_\beta \neq 0\}$. We also define $\mathrm{Supp}(y) := \mathrm{Supp}(Y)$ the *support of* y. We observe that $\mathrm{Supp}(Y_+)$ and $\mathrm{Supp}(Y_-)$ are mutually disjoint and thus

(A.8) $\quad (\beta, \beta') \leq 0$ for any $\beta \in \mathrm{Supp}(Y_+)$ and $\beta' \in \mathrm{Supp}(Y_-)$.

Suppose $Y_- = 0$ and $Y_+ \neq 0$ in this paragraph. Let β be any root in $\mathrm{Supp}(Y)$ (= $\mathrm{Supp}(Y_+)$). A maximal root γ containing β in y always exists, i.e., $\beta \leq \gamma \leq y$ and for any $\beta' \in \mathrm{Supp}(Y)$, either $\gamma + \beta'$ is not a root or $\gamma + \beta' \not\leq y$. Clearly a maximal root γ containing β in y is also a maximal root containing β' in y for any other $\beta' \in \mathrm{Supp}(\gamma)$. For brevity we call such γ as a maximal root without specifying β or y, if no confusion arises. For a maximal root γ, we have

$$(\gamma, \beta') \geq 0 \text{ for any } \beta' \in \mathrm{Supp}(y - \gamma) \tag{A.9}$$

([**24**, Lemma 9.4]).

Proof of Lemma A.2. We prove the lemma for the following three possible cases: (i) $Y_+, Y_- \neq 0$, (ii) $Y_- = 0$, (iii) $Y_+ = 0$.

Case (i). If both $\mathrm{Supp}(Y_\pm)$ are one-point sets respectively and one of them is $\{\alpha\}$, we already proved the statement in Lemma A.7. So we assume this does not happen.

Suppose first that none of $\mathrm{Supp}(Y_\pm)$ is $\{\alpha\}$. We choose maximal roots $\gamma_\pm$ in $y_\pm$ such that $\gamma_\pm \neq a$, respectively. So $\gamma_\pm(h_\alpha) \in \{0, \pm 1\}$. Using (A.8) and (A.9), we have $\gamma_+(Y_+) \geq 2, \gamma_+(Y_-) \leq 0$ and $\gamma_-(Y_+) \leq 0, \gamma_-(Y_-) \geq 2$. Therefore we have

$$\begin{aligned}
\gamma_+(X) &= \gamma_+(Y_+) - \gamma_+(Y_-) + \gamma_+(\frac{1}{2}h_\alpha) \geq \frac{3}{2} \\
\gamma_-(X) &= \gamma_-(Y_+) - \gamma_-(Y_-) + \gamma_-(\frac{1}{2}h_\alpha) \leq -\frac{3}{2}.
\end{aligned} \tag{A.10}$$

Since $\gamma_\pm$ are not mutually proportional (as they have disjoint supports), we have $l \geq 4$.

Suppose secondly that one of $\mathrm{Supp}(Y_\pm)$ is $\{\alpha\}$, say $\mathrm{Supp}(Y_-) = \{\alpha\}$. We choose a maximal root γ_+ in y_+. Let $m \in \mathbb{Z}_{\geq 1}$ such that $y_+ - m\gamma_+ \geq 0$ but $y_+ - (m+1)\gamma_+ \not\geq$

0. If $y_+ - m\gamma_+ = 0$ then $y = m\gamma_+ - n\alpha$ and thus $Y = mh_{\gamma_+} - nh_\alpha$ for some $n \geq 1$. This case was proven in Lemma A.7. If $y_+ - m\gamma_+ \neq 0$, we choose a maximal root γ'_+ in $y_+ - m\gamma_+$. As was proved in (A.10), we have $\gamma_+(X), \gamma'_+(X) \geq \frac{3}{2}$. Therefore $l \geq 4$.

The case $\mathrm{Supp}(Y_+) = \{\alpha\}$ is similarly proven.

Case (ii): $Y_- = 0$ (i.e., $Y = Y_+$). The case when $\mathrm{Supp}(y)$ is a one-point set is proven by Lemma A.7.

So we assume $\#\mathrm{Supp}(y) \geq 2$. We choose a maximal root γ_+ in y_+ such that $\gamma_+ \neq \alpha$. Let $m \in \mathbb{Z}_{\geq 1}$ such that $y_+ - m\gamma_+ \geq 0$ but $y_+ - (m+1)\gamma_+ \not\geq 0$. If $y_+ - m\gamma_+$ is a multiple of α, we have $y_+ = m\gamma_+ + m'\alpha$. This case was proven by Lemma A.7. So we may assume there is a maximal root γ'_+ in $y_+ - m\gamma_+$ other than α. As was proved in (A.10), we have $\gamma_+(X), \gamma'_+(X) \geq \frac{3}{2}$. Therefore $l \geq 4$.

Case (iii): $Y_+ = 0$. This case is similarly proven as in Case (ii).

This completes the proof of Lemma A.2. $\square$

Lemma A.8. (1) *If $(Y, Y) = \frac{4}{(\alpha,\alpha)}$, we have $Y = h_\beta$ for some long root β.*

(2) *If $(Y, Y) = \frac{8}{(\alpha,\alpha)}$, we have either $Y = h_\beta$ for some short root β with $(\beta, \beta) = \frac{1}{2}(\alpha, \alpha)$, or $Y = h_\beta + h_{\beta'}$ for some long roots β, β' with $(\beta, \beta') = 0$.*

Proof. (1) By (A.6) and (A.8), we have $2(Y^+, Y^-) \in \mathbb{Z}_{\leq 0}\frac{4}{(\alpha,\alpha)}$. Since each term in $(Y, Y) = (Y^+, Y^+) + (Y^-, Y^-) - 2(Y^+, Y^-)$ is contained in $\mathbb{Z}_{\geq 0}\frac{4}{(\alpha,\alpha)}$, one of $Y^\pm$ should be 0.

Suppose $Y^- = 0$. Let γ be a maximal positive root in $y = y_+$. We have

$$(Y,Y) = (h_\gamma, h_\gamma) + (Y - h_\gamma, Y - h_\gamma) + 2(h_\gamma, Y - h_\gamma) = \frac{4}{(\alpha,\alpha)}.$$

Each term in the above is nonnegative by (A.9). Therefore by (A.6), we have $m = 1$, γ is a long root and $Y = h_\gamma$.

In the case $Y^+ = 0$, similarly we have $Y = -h_\gamma \, (= h_{-\gamma})$ for a long root γ.

(2) Using $(Y,Y) = (Y^+,Y^+) + (Y^-,Y^-) - 2(Y^+,Y^-) = \frac{8}{(\alpha,\alpha)}$, we have only three possible cases:

(i) $(Y^+,Y^+) = (Y^-,Y^-) = \frac{4}{(\alpha,\alpha)}, (Y^+,Y^-) = 0$ or

(ii) $Y^- = 0, (Y,Y) = \frac{8}{(\alpha,\alpha)}$ or

(iii) $Y^+ = 0, (Y,Y) = \frac{8}{(\alpha,\alpha)}$.

Case (i). The first item of the lemma assures $Y^+ = h_\beta$ and $Y^- = h_{\beta'}$ for some long roots β, β'. We also have $(\beta,\beta') = 0$ by $(Y^+,Y^-) = 0$. Thus we have $Y = h_\beta + h_{\beta'}$ for some long roots β, β' with $(\beta,\beta') = 0$.

Case (ii). Let γ be a maximal root in y. Since

$$(Y,Y) = (h_\gamma, h_\gamma) + (Y - h_\gamma, Y - h_\gamma) + 2(h_\gamma, Y - h_\gamma) = \frac{8}{(\alpha,\alpha)},$$

(A.6) and (A.9) force either $Y = h_\gamma$ and $(h_\gamma,h_\gamma) = \frac{8}{(\alpha,\alpha)}$ or $(h_\gamma,h_\gamma) = (Y - h_\gamma, Y - h_\gamma) = \frac{4}{(\alpha,\alpha)}$ and $(h_\gamma, Y - h_\gamma) = 0$. In the first case we have $Y = h_\beta$ for a short root with $\frac{(\beta,\beta)}{(\alpha,\alpha)} = \frac{1}{2}$. In the second case the first item of the lemma asserts $Y = h_\gamma + h_\beta$ where both β, γ are long roots. And $(h_\gamma, Y - h_\gamma) = 0$ implies $(\gamma,\beta) = 0$.

The case (iii) is similar to (ii). This completes the proof.

Proof of Lemma A.6. (1) If $\beta_1 = \pm\beta_2$ then the lemma comes from the fact that Φ is an irreducible root system. So we assume $\beta_1 \neq \pm\beta_2$. We take a partition of Δ as follows:

$$\Delta_1 := \Delta \cap \beta_1^{\perp} \setminus \beta_2^{\perp}, \quad \Delta_2 := \Delta \cap \beta_2^{\perp} \setminus \beta_1^{\perp}, \quad \Delta_{12} := \Delta \cap \beta_1^{\perp} \cap \beta_2^{\perp}.$$

It is obvious that both Δ_1, Δ_2 are nonempty.

There exist $\gamma_1 \in \Delta_1$, $\gamma_2 \in \Delta_2$ and distinct $\delta_1, \delta_2, ..., \delta_n \in \Delta_{12}$ such that among the inner products of two distinct roots in $\{\gamma_1, \delta_1, \delta_2, ..., \delta_n, \gamma_2\}$, the possible nonzero ones are exactly $(\gamma_1, \delta_1), (\delta_1, \delta_2), ..., (\delta_{n-1}, \delta_n)$ and (δ_n, γ_2). This is possible because the reduced Dynkin diagram is a tree. Here the reduced diagram is obtained by replacing the multiple oriented edges to a multiplicity 1 non-oriented edge respectively. Since these nonzero products are strictly negative we have $(\gamma_1 + \sum_{i=1}^{n} \delta_i, \gamma_2) < 0$ and $(\gamma_1 + \sum_{i=1}^{m-1} \delta_i, \delta_m) < 0$ for all $m \leq n$. Thus the sum $\gamma := \gamma_1 + \sum_{i=1}^{n} \delta_i + \gamma_2$ is a root by [**24**, Lemma 9.4].

We have $(\gamma, \beta_1) = (\gamma_2, \beta_1) \neq 0$ and $(\gamma, \beta_2) = (\gamma_1, \beta_2) \neq 0$. Thus γ is not contained in $\beta_1^{\perp} \cup \beta_2^{\perp}$. This finishes the proof of (1).

(2) By (1), there is a root γ such that $(\gamma, \beta_1) > 0$ and $(\gamma, \beta_2) \neq 0$. If $(\gamma, \beta_2) > 0$ then we are done. Otherwise, we claim that $\sigma_{\beta_2}(\gamma)$ is the desired root. First we have $(\sigma_{\beta_2}(\gamma), \beta_2) = (\gamma - \frac{2(\gamma,\beta_2)}{(\beta_2,\beta_2)}\beta_2, \beta_2) = -(\gamma, \beta_2) > 0$. Since $(\beta_1, \beta_2) \geq 0$, we have $(\sigma_{\beta_2}(\gamma), \beta_1) = (\gamma, \beta_1) - \frac{2(\gamma,\beta_2)}{(\beta_2,\beta_2)}(\beta_2, \beta_1) > 0$. This proves the claim. $\square$

REMARK A.9. We remark that the second item of Lemma A.6 does not hold for $(\alpha, \beta) < 0$. E.g., F_2.

APPENDIX B

Hyperkähler moment map

We give the explicit forms of the hyperkähler structure and the moment map (cf. [**41**, §3.2]). The hyperkähler moment map 0 equation is nothing but the real ADHM equation ([**1**]).

Let $V := \mathbb{C}^k$ and $W := \mathbb{C}^N$ with the standard symplectic and orthogonal forms $(\ ,\ )_V$ and $(\ ,\ )_W$ respectively. Let $\mathbf{M} := \mathbf{M}_{(V,W)}$. We declare that the standard bases $\{e_1, e_2, ..., e_k\}$ and $\{e_1, e_2, ..., e_N\}$ are symplectic or orthogonal bases. Let $\mathbf{N} := \{(B_1, B_2, i, j) \in \mathbf{M} |\, B_1 = B_1^*, B_2 = B_2^*, j = i^*\}$ as before.

Let $\mathrm{Mat}_{m,n}$ be the space of $(m \times n)$ matrices. If $m = n$, we write $\mathrm{Mat}_{m,n} = \mathrm{Mat}_n$. Let Id_n be the identity matrix of Mat_n. We identify $\mathrm{Hom}(V, V) = \mathrm{Mat}_k$, $\mathrm{Hom}(W, V) = \mathrm{Mat}_{k,N}$ and $\mathrm{Hom}(V, W) = \mathrm{Mat}_{N,k}$ by the standard bases of V and W. Let

$$S := \begin{pmatrix} 0 & \mathrm{Id}_{k/2} \\ -\mathrm{Id}_{k/2} & 0 \end{pmatrix}.$$

Then

$$B^* = -SB^tS, \quad i^* = i^tS, \quad j^* = -Sj^t \tag{B.1}$$

where $B \in \mathrm{Hom}(V, V)$, $i \in \mathrm{Hom}(W, V)$ and $j \in \mathrm{Hom}(V, W)$.

We denote the hermitian adjoint of A by $A^\dagger := \overline{A}^t$ where $A \in \mathrm{Mat}_{m,n}$. We use the hermitian inner product on $\mathrm{Mat}_{m,n}$ given by $\mathrm{tr}(AA^\dagger)$. Hence we have the hermitian inner product on $\mathbf{M}$.

Let us define the hyperkähler structure on $\mathbf{N}$. First we recall the quaternion structure on $\mathbf{M}$ ([**41**, p.41]). Let I be the standard complex structure on $\mathbf{M}$. Let J be another complex structure given by $J(B_1, B_2, i, j) := (B_2^\dagger, -B_1^\dagger, j^\dagger, -i^\dagger)$ where $(B_1, B_2, i, j) \in \mathbf{M}$. Let $K := IJ$ (a complex structure). Then the complex structures I, J, K define a quaternion structure on $\mathbf{M}$.

Lemma B.1. $\mathbf{N}$ *is a quarternion subspace of* $\mathbf{M}$.

Proof. $I(\mathbf{N}) \subset \mathbf{N}$ is clear. It suffices to prove $J(\mathbf{N}) \subset \mathbf{N}$. We need to check ${B_n^\dagger}^* = B_n^\dagger$ and $(j^\dagger)^* = -i^\dagger$ where $B \in \mathrm{Hom}(V, V)$ for $n = 1, 2$ where $(B_1, B_2, i, j) \in \mathbf{N}$. This comes from the direct computation using (B.1). $\square$

The hermitian inner product on $\mathbf{M}$ defines a Riemannian metric g. A (flat) hyperkähler structure on $\mathbf{M}$ is defined with respect to g and I, J, K as follows (see [**41**, Definition 3.29]). Let $\omega_I, \omega_J, \omega_K$ be Kähler forms defined by

$$w_I(v, w) := g(Iv, w),\ w_J(v, w) := g(Jv, w),\ w_K(v, w) := g(Kv, w) \text{ for } v, w \in \mathbf{M}$$

respectively. Let us denote the restricted Kähler forms on $\mathbf{N}$ by the same notation $\omega_I, \omega_J, \omega_K$. Hence $\mathbf{N}$ is a hyperkähler subspace of $\mathbf{M}$.

Let us define moment maps first in a general context.

Definition B.2. Let $\mathcal{V}$ be a real (resp. complex) vector space with a symplectic (resp. holomorphic symplectic) form ω. Let G be a Lie group acting on $\mathcal{V}$. We assume this action is Hamiltonian (i.e., preserves the symplectic form ω). Let $\mathfrak{g} := \mathrm{Lie}(G)$. A *moment map* (resp. *holomorphic moment map*) is a map $\mu^{\omega}_{\mathcal{V},G} \colon \mathcal{V} \to \mathfrak{g}^\vee$ such that $\langle d(\mu^{\omega}_{\mathcal{V},G})_x(v), g\rangle = \omega(g.x, v)$. Here $g.x$ is defined via the Lie algebra action on $\mathcal{V}$.

We impose the condition $\mu^{\omega}_{\mathcal{V},G}(0) = 0$ once and for all. Thus the moment map exists uniquely. Let us construct the moment map $\mu^{\omega_I}_{\mathbf{N},\mathrm{USp}(V)} \colon \mathbf{N} \to \mathfrak{usp}(V)^\vee$ with respect to the Kähler (symplectic) form ω_I where $\mathfrak{usp}(V) = \mathrm{Lie}(\mathrm{USp}(V))$. It is known that the moment map $\mu^{\omega_I}_{\mathbf{M},U(V)} \colon \mathbf{M} \to \mathfrak{u}(V)^\vee$ with respect to $U(V)$ is given by

$$(B_1, B_2, i, j) \mapsto \frac{\sqrt{-1}}{2}\mathrm{tr}(([B_1, B_1^\dagger] + [B_2, B_2^\dagger] + ii^\dagger - jj^\dagger)\bullet) \tag{B.2}$$

(see [**41**, (3.16)]).

It is clear from the definition of the moment map that $\mu^{\omega_I}_{\mathbf{M},\mathrm{USp}(V)} = \iota^\vee \mu^{\omega_I}_{\mathbf{M},U(V)}$ where $\iota \colon \mathfrak{usp}(V) \to \mathfrak{u}(V)$ is the inclusion. Therefore $\mu^{\omega_I}_{\mathbf{M},\mathrm{USp}(V)}$ is of the same form (B.2). Now composed with the inclusion $\mathbf{N} \subset \mathbf{M}$, $\mu^{\omega_I}_{\mathbf{N},\mathrm{USp}(V)}$ is also of the same form (B.2).

Similarly the explicit forms of $\mu^{\omega_J}_{\mathbf{N},\mathrm{USp}(V)}$ and $\mu^{\omega_K}_{\mathbf{N},\mathrm{USp}(V)}$ coincide those of $\mu^{\omega_J}_{\mathbf{M},U(V)}$ and $\mu^{\omega_K}_{\mathbf{N},U(V)}$ respectively. We can write their explicit forms using (B.2). But they can be also expressed as parts of one holomorphic moment map as follows. Let $\omega := \omega_J + \sqrt{-1}\omega_K$. By direct calculation we have $\omega((B_1, B_2, i, j), (B_1', B_2', i', j')) =$

$\mathrm{tr}(B_1B_2' - B_2B_1' + ij' - ji')$. This implies ω is a holomorphic symplectic form. See also [**41**, Remark 3.31]. Hence the holomorphic moment map $\mu^{\omega}_{\mathbf{N},\mathrm{Sp}(V)}$ is given by $\mu^{\omega}_{\mathbf{N},\mathrm{Sp}(V)}(B_1, B_2, i, j) = \mathrm{tr}(([B_1, B_2] + ij)\bullet)$. Note that $\mu^{\omega}_{\mathbf{N},\mathrm{Sp}(V)}$ is our moment map μ after identifying $\mathfrak{sp}(V)$ with its dual via trace. Now the original moment maps $\mu^{\omega_J}_{\mathbf{N},\mathrm{USp}(V)}, \mu^{\omega_K}_{\mathbf{N},\mathrm{USp}(V)}$ can be viewed as the real and imaginary parts of μ respectively after identifying $\mathfrak{sp}(V) = \mathfrak{usp}(V)_{\mathbb{C}}$.

We define the *hyperkähler moment map* as

$$\mu^{\mathrm{hk}}_{\mathbf{N},\mathrm{USp}} := (\mu^{\omega_I}_{\mathbf{N},\mathrm{USp}(V)}, \mu^{\omega_J}_{\mathbf{N},\mathrm{USp}(V)}, \mu^{\omega_K}_{\mathbf{N},\mathrm{USp}(V)}) : \mathbf{N} \to \mathbb{R}^3 \otimes \mathfrak{usp}(V)^{\vee},$$

(cf. [**41**, (3.33)]). As was observed above the part $(\mu^{\omega_J}_{\mathbf{N},\mathrm{USp}(V)}, \mu^{\omega_K}_{\mathbf{N},\mathrm{USp}(V)})$ in $\mu^{\mathrm{hk}}_{\mathbf{N},\mathrm{USp}(V)}$ can be identified with μ. We apply Kempf-Ness' theorem [**28**] to an affine $\mathrm{Sp}(V)$-variety $\mu^{-1}(0)$. Then we have a canonical set-theoretic bijection

$$(\mu^{\mathrm{hk}}_{\mathbf{N},\mathrm{USp}(V)})^{-1}(0)/\mathrm{USp}(V) = \left(\mu^{-1}(0) \cap {\mu^{\omega_I}_{\mathbf{N},\mathrm{USp}(V)}}^{-1}(0)\right)/\mathrm{USp}(V) \to \mu^{-1}(0)/\!\!/\mathrm{Sp}(V).$$

A similar argument equally works for $\mathrm{O}(V)$, $\mathrm{O}(V_{\mathbb{R}})$ and the same $\mathbf{N}$ where $V_{\mathbb{R}}$ denotes the real vector space in V.

APPENDIX C

Finite dimesionality of weight spaces

The main purpose of this chapter is to prove that each weight space of $\mathbb{C}[\mu^{-1}(0)]^{G(V)}$ with respect to T is finite-dimensional.

Let $\epsilon(m) = 0$ (resp. $\epsilon(m) = 1$) if m is even integer (resp. odd integer). If $\varepsilon = -1$ then using a symplectic basis of V, we identify $V = \mathbb{C}^k$. If $\varepsilon = +1$ then using the basis

$$\{f_1 \pm \sqrt{-1}f_2, f_3 \pm \sqrt{-1}f_4, ..., f_{k-\epsilon(k)-1} \pm \sqrt{-1}f_{k-\epsilon(k)}, (f_k)\}$$

we identify $V = \mathbb{C}^k$ where $\{f_1, f_2, ..., f_k\}$ is an orthogonal basis of V. Here the notation (f_k) denotes f_k only when k is odd (vacuous otherwise). By a similar way we identify $W = \mathbb{C}^N$. Let $\lfloor a \rfloor$ be the maximal integer in $\mathbb{Z}_{\leq a}$ where $a \in \mathbb{R}$. We fix maximal tori of $G(V)$ and $G(W)$ as

$$T_{G(V)} = \{\mathrm{diag}(z_1, z_2, ..., z_{\lfloor k/2 \rfloor}) \oplus \mathrm{diag}(z_1^{-1}, z_2^{-1}, ..., z_{\lfloor k/2 \rfloor}^{-1}) |\, z_1, z_2, ..., z_{\lfloor k/2 \rfloor} \in \mathbb{C}^*\},$$

$$T_{G(W)} = \{\mathrm{diag}(t_1, t_2, ..., t_{\lfloor N/2 \rfloor}) \oplus \mathrm{diag}(t_1^{-1}, t_2^{-1}, ..., t_{\lfloor N/2 \rfloor}^{-1}) |\, t_1, t_2, ..., t_{\lfloor N/2 \rfloor} \in \mathbb{C}^*\}$$

respectively.

Now we identify the rings of characters of $T_{G(V)}$, $T_{G(W)}$ and $(\mathbb{C}^*)^2$

$$R(T_{G(V)}) = \mathbb{Z}[z_1^{\pm 1}, z_2^{\pm 1}, ..., z_{\lfloor k/2 \rfloor}^{\pm 1}],$$

$$R(T_{G(W)}) = \mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}, ..., t_{\lfloor N/2 \rfloor}^{\pm 1}]$$

$$R((\mathbb{C}^*)^2) = \mathbb{Z}[q_1^{\pm 1}, q_2^{\pm 1}]$$

respectively.

Let us prove

$$\mathcal{X}_{\mathbb{C}[\mu^{-1}(0)]^{G(V)}} \in \hat{R}(T) := R(T_{G(W)})[[q_1^{-1}, q_2^{-1}]] \tag{C.1}$$

where $T = T_{G(W)} \times (\mathbb{C}^*)^2$ and $\mathcal{X}_{\mathbb{C}[\mu^{-1}(0)]^{G(V)}}$ is the formal T-character. The idea is to use the following two:

(1) $\mu^{-1}(0)/\!\!/G(V)$ is a closed $(\mathbb{C}^*)^2 \times G(W)$-subscheme of $\mathbf{N}/\!\!/G(V)$;

(2) the (surjective) GIT quotient $\mathbf{N} \to \mathbf{N}/\!\!/G(V)$ is $(\mathbb{C}^*)^2 \times G(W)$-equivariant.

So each weight space of $\mathbb{C}[\mu^{-1}(0)]^{G(V)}$ is a subquotient of the weight space of the same weight of $\mathbb{C}[\mathbf{N}]$. So the proof will be done if we show a stronger claim:

$$\mathcal{X}_{\mathbb{C}[\mathbf{N}]} \in \hat{R}(T).$$

The claim will follow from an even stronger claim:

$$\mathcal{X}^{(\mathbb{C}^*)^2}_{\mathbb{C}[\mathbf{N}]} \in \mathbb{Z}[[q_1^{-\frac{1}{2}}, q_2^{-\frac{1}{2}}]]$$

where $\mathcal{X}^{(\mathbb{C}^*)^2}_{\mathbb{C}[\mathbf{N}]}$ denotes the formal $(\mathbb{C}^*)^2$-character of $\mathbb{C}[\mathbf{N}]$. This is because all the monomials in $\mathcal{X}_{\mathbb{C}[\mathbf{N}]}$ have nonnegative integer coefficients.

Let us check the last claim. The decomposition $\mathbf{N} = \mathfrak{p}(V) \oplus \mathfrak{p}(V) \oplus \mathrm{Hom}(W, V)$ is the weight decomposition with respect to $(\mathbb{C}^*)^2$. The first two direct summands $\mathfrak{p}(V)$ of $\mathbf{N}$ are of weight q_1 and q_2 respectively. The last summand $\mathrm{Hom}(W, V)$ is of weight $(q_1 q_2)^{\frac{1}{2}}$. Since $\mathbb{C}[\mathbf{N}] = S(\mathfrak{p}(V)^\vee) \otimes S(\mathfrak{p}(V)^\vee) \otimes S(\mathrm{Hom}(W, V)^\vee)$ where S

denotes the symmetric product, we have

$$\mathcal{X}_{\mathbb{C}[\mathbf{N}]}^{(\mathbb{C}^*)^2} = \left(\sum_{n\geq 0} q_1^{-n}\right)^{\frac{1}{2}k(k+\varepsilon)} \left(\sum_{n\geq 0} q_2^{-n}\right)^{\frac{1}{2}k(k+\varepsilon)} \left(\sum_{n\geq 0} (q_1q_2)^{-\frac{n}{2}}\right)^{kN}.$$

This proves the last claim.

To complete the proof of (C.1) we recall from §1.2.2 that $i \in \mathrm{Hom}(W,V)$ always appears together with i^* in $\mathbb{C}[\mathbf{N}]^{G(W)}$. Hence any monomials with non-integer exponents in $\mathcal{X}_{\mathbb{C}[\mu^{-1}(0)]^{G(W)}}$ have coefficient 0. This proves (C.1).

Remark C.1. (1) $\hat{R}(T)$ is a ring.

(2) The origin 0 is the unique $(\mathbb{C}^*)^2$-fixed point of $\mathbf{N}$.

APPENDIX D

Method of associated cones

We adjust Kraft's method of associated cones in [**31**, II.4.2] to apply it to the moment map μ. It was mentioned by Panyushev [**50**]. It is a slight modification from Kraft's original arguments in [**31**, II.4.2]. The goal is to prove the following theorem:

THEOREM D.1. *Let V_1, V_2 be finite dimensional vector spaces over $\mathbb{C}$ with $d_0 := \dim V_1 - \dim V_2 \geq 0$. Suppose there is a $\mathbb{C}^*$-action on V_1 (resp. V_2) such that $t.v = tv$ (resp. $t.v = t^{n_0}v$ for some $n_0 \in \mathbb{Z}_{>0}$). Suppose $\phi\colon V_1 \to V_2$ is a $\mathbb{C}^*$-equivariant affine morphism.*

(1) *ϕ is flat if $\phi^{-1}(0) \neq \emptyset$ and $\dim \phi^{-1}(0) = d_0$.*

If ϕ is flat, we have the followings.

(2) *If $\phi^{-1}(0)$ is reduced and irreducible, so is any nonempty fibre $\phi^{-1}(v)$ where $v \in V_2$.*

(3) *If we further assume $\phi^{-1}(0)$ is normal, so is any nonempty fibre $\phi^{-1}(v)$ where $v \in V_2$.*

In the original setting in [**31**, II.4.2], ϕ is a $\mathbb{C}^*$-equivariant GIT quotient map. The arguments in [loc. cit.] are also true under our setting with only slight modification.

The proof will appear after preliminary steps. In the case of μ we have $n_0 = 2$.

Note that the $\mathbb{C}^*$-equivariant assumption in the theorem asserts $\phi^{-1}(0) \neq \emptyset$ since $\phi(0) = 0$.

We will use Chevalley's semicontinuity theorem ([**19**, Theorem 13.1.3]): $\dim_x \varphi^{-1}(\varphi(x))$ is an upper-semicontinuous function on X where $\varphi\colon X \to Y$ is a morphism between irreducible schemes X, Y (of finite type) and $\dim_x$ denotes the dimension at $x \in X$.

Let $R := \mathbb{C}[V_1]$. It decomposes as $R = \bigoplus_{d=0}^{\infty} R_d$ by the usual homogeneous degree denoted by deg. Let $\deg 0 = -\infty$ as a rule. Let $\operatorname{gr} f := f_h$ if $f \neq 0$ and $f = \sum_{n=0}^{h} f_h$ where $h = \deg f$. If $f = 0$, let $\operatorname{gr} f := 0$.

D.1. Associated graded rings and ideals

Definition D.2. Let $T \subset R$ be a subspace. Let $\operatorname{gr} T$ be the subspace spanned by $\operatorname{gr} f$ for all $f \in T$.

Lemma D.3. (1) *If I is an ideal of R, $\operatorname{gr} I$ is also an ideal of R.*

(2) *If I is a homogenous ideal, $\operatorname{gr} I = I$.*

(3) $\operatorname{gr}(fT) = (\operatorname{gr} f)(\operatorname{gr} T)$ *for any $f \in R$ and subspace $T \subset R$.*

(4) *If $T \subset T'$ where T and T' are subspaces of R, $\operatorname{gr} T \subset \operatorname{gr} T'$. Moreover if $T \subsetneq T'$, $\operatorname{gr} T \subsetneq \operatorname{gr} T'$.*

(5) $\operatorname{gr}\sqrt{I} \subset \sqrt{\operatorname{gr} I}$ *where I is any ideal of R.*

Proof. Use $\operatorname{gr} fg = (\operatorname{gr} f)(\operatorname{gr} g)$. Then (1), (2) and (3) are clear.

(4) The first assertion is clear. Let $f \in T' \setminus T$ such that $\deg f = \min\{\deg g | \, g \in T' \setminus T\}$. Let $h := \deg f$. We will show $f_h \notin \operatorname{gr} T$. Suppose the contrary. There

exists $g \in T$ such that $\operatorname{gr} g = f_h$. We have $f - g \in T' \setminus T$ and $\deg(f-g) < h$. This contradicts to the minimality of the degree of f in $T' \setminus T$.

(5) Let $f \in R$ such that $f^n \in I$ for some $n > 0$. Since $\operatorname{gr} f^n = (\operatorname{gr} f)^n \in \operatorname{gr} I$, we have $\operatorname{gr} f \in \sqrt{\operatorname{gr} I}$. $\square$

Let $R_{\leq d} := \bigoplus_{n=0}^{d} R_n$. Then $R = \cup_d R_{\leq d}$ and R is a filtered algebra.

DEFINITION D.4. Let $I \subset R$ be an ideal. Let $\overline{R} := R/I$ the quotient ring. Then $\overline{R}$ has an induced filtration from R by $\overline{R}_{\leq d} := (R_{\leq d} + I)/I$. If $f \in \overline{R}_{\leq d} \setminus \overline{R}_{\leq d-1}$ then we define $\deg f := d$. Let $\deg 0 := -\infty$ as a rule. Let $\operatorname{gr} \overline{R} := \bigoplus_{d \geq 0} \overline{R}_{\leq d}/\overline{R}_{\leq d-1}$.

LEMMA D.5. $\operatorname{gr} \overline{R}$ *is a* $\mathbb{Z}_{\geq 0}$*-graded ring.*

PROOF. This is direct to check. $\square$

DEFINITION D.6. We define $\rho \colon R \to \operatorname{gr} \overline{R}$ by $r \mapsto \sum_d [r_d + I]_d$ where $r = \sum_d r_d \in R = \bigoplus_d R_d$ and $[r_d + I]_d$ is the class in $\overline{R}_{\leq d}/\overline{R}_{\leq d-1}$.

Note ρ is a surjective graded ring homomorphism.

PROPOSITION D.7. ([**31**, Lemma, p.134]) *We have* $\operatorname{Ker} \rho = \operatorname{gr} I$. *Hence we have the induced ring isomorphism* $R/\operatorname{gr} I \to \operatorname{gr}\overline{R}$*.*

PROOF. We have $\operatorname{Ker} \rho \cap R_d = (R_{\leq d-1} + I) \cap R_d$ for each $d \geq 0$. We need to show that this is equal to $(\operatorname{gr} I)_d$. Let $f \in \operatorname{Ker}\rho \cap R_d$. Then there exists $g \in R_{\leq d-1}$ such that $f + g \in I$. Since $\deg(f+g) = \deg f$, we have $f = \operatorname{gr} f = \operatorname{gr}(f+g)$. This means $f \in (\operatorname{gr} I)_d$ and proves one inclusion. Let $f \in (\operatorname{gr} I)_d$. Then there exists $g \in I$ such that $\operatorname{gr} g = f$. Thus $\deg g = d$ and $g_d = f$. Therefore $f = g - \sum_{n=0}^{d-1} g_n \in I + R_{\leq d-1}$. This proves the opposite inclusion. $\square$

Suppose $\overline{R}$ is an integral domain in the rest of this section. Let $K := \operatorname{Quot}(\overline{R})$ the quotient field. We define the degree on K as follows.

Definition D.8. Let $f/g \in K$ where $f \in \overline{R}$ and $g \in \overline{R} \setminus 0$. Let $\deg f/g := \deg f - \deg g$.

It is easy to check deg on K is well-defined.

Definition D.9. Let $K_{\leq d} := \{f \in K | \deg f \leq d\}$. Let $\operatorname{gr} K := \bigoplus_{d \in \mathbb{Z}} K_{\leq d}/K_{\leq d-1}$. We define $\operatorname{gr} f := f + K_{\leq d-1} \in K_{\leq d}/K_{\leq d-1} \subset \operatorname{gr} K$ where $f \in K$ with $f \in K_{\leq d} \setminus K_{\leq d-1}$.

Then $\operatorname{gr} K$ is a $\mathbb{Z}$-graded ring.

The following lemma is clear.

Lemma D.10. *For each $d \geq 0$, we have $\overline{R} \cap K_{\leq d} = \overline{R}_{\leq d}$. Hence $\operatorname{gr} \overline{R}$ is a $\mathbb{Z}$-graded subring of $\operatorname{gr} K$.* $\square$

D.2. Associated cones

Let I_v be the ideal of R defining $\phi^{-1}(v)$. The coordinates of ϕ are e elements $\phi_1, \phi_2, ..., \phi_e \in R_{n_0}$ where $e := \dim V_2$, since ϕ is $\mathbb{C}^*$-equivariant. Let $v = (v_1, ..., v_e)$. Then I_v is generated by $\phi_1 - v_1, \phi_2 - v_2, ..., \phi_e - v_e$. Thus $\operatorname{gr} I_v = I_0$ for any $v \in V_2$.

DEFINITION D.11. Let X be a closed subscheme of $\phi^{-1}(v)$ for some $v \neq 0$. Let $\mathrm{Cone}(X) := \mathrm{Spec}\,(R/\mathrm{gr}\, I_X)$ where I_X is the defining ideal of X in V_1. $\mathrm{Cone}(X)$ is called the *associated cone* of X.

Let $R[T] := R \otimes_{\mathbb{C}} \mathbb{C}[T]$ where T is an indeterminate. Then $R[T]$ is a $\mathbb{Z}$-graded ring by $R[T]_d := \bigoplus_{n=0}^{d} R_n T^{d-n}$. We give a $\mathbb{C}^*$-action on $\mathrm{Spec}\, R[T] = V_1 \times \mathbb{C}$ by $t.(x, x') = (tx, tx')$.

DEFINITION D.12. Let $f = \sum_{n=0}^{d} f_n \in R$ where $d = \deg f$. We define $f_T := \sum_{n=0}^{d} f_n T^{d-n} \in R[T]$. Let $\widetilde{I}_X$ be the ideal of $R[T]$ generated by f_T for all $f \in I_X$. Let $\widetilde{X} := \mathrm{Spec}\,(R[T]/\widetilde{I}_X)$. Let $\pi\colon \widetilde{X} \to \mathbb{C}$ a (surjective) morphism induced by the inclusion $\mathbb{C}[T] \to R[T]$.

Note that $\widetilde{I}_X$ is a homogenous ideal. Thus $\mathrm{gr}\,\widetilde{I}_X = \widetilde{I}_X$.

The following lemma is immediate.

LEMMA D.13. *We have canonical isomorphisms*

(1) $\pi^{-1}(0) \cong \mathrm{Cone}(X)$.

(2) $\pi^{-1}(t) \cong X \;\; (t \neq 0)$. $\square$

LEMMA D.14. (1) $\mathrm{Cone}(X)$ *is a closed subscheme of* $\phi^{-1}(0)$.

(2) *If X is reduced and irreducible,* $\dim_0 \mathrm{Cone}(X) = \dim \mathrm{Cone}(X) = \dim X$.

PROOF. (1) This follows from $I_0 = \mathrm{gr}\, I_v \subset \mathrm{gr}\, I_X$.

(2) We claim $\widetilde{X} = \widetilde{X}_{\mathrm{red}} = \overline{X}$ where $\overline{X}$ is the Zariski closure of $\mathbb{C}^*.(X \times \{1\})$ in $V_1 \times \mathbb{C}$. Since $\widetilde{X}_{\mathrm{red}}$ is a Zariski closed cone in $V_1 \times \mathbb{C}$ and $X \times \{1\} \subset \widetilde{X}_{\mathrm{red}}$, we have

$\overline{X} \subset \widetilde{X}_{\mathrm{red}}$. So we have

$$\overline{X} \subset \widetilde{X}_{\mathrm{red}} \subset \widetilde{X}.$$

It remains to prove $\widetilde{X} \subset \overline{X}$, equivalently $I_{\overline{X}} \subset \widetilde{I}_X$. Let $F \in I_{\overline{X}}$. Let $F = \sum_{n=0}^{d} F_n$ where $F_n \in R[T]_n$. Since $t.F = \sum_{n=0}^{d} t^{-n} F_n$ and $t.F \in I_{\overline{X}}$ for any $t \in \mathbb{C}^*$, $F_n \in I_{\overline{X}}$ for all $0 \leq n \leq d$.

We write $F_n = \sum_{m=0}^{n} f_m T^{n-m}$ where $f_m \in R_m$. Then $f := \sum_{m=0}^{n} f_m$ vanishes on X and thus $f \in I_X$. Therefore $F_n \in \widetilde{I}_X$. This proves the claim.

We prove (2) using the claim. Since $\dim \mathbb{C}^*.(X \times \{1\}) = \dim X + 1$, we have $\dim \overline{X} = \dim X + 1$. Since π is surjective and $\overline{X}$ is irreducible we have $\dim \mathrm{Cone}(X) \leq \dim X$ by Lemma D.13. By Chevalley's semicontinuity we have $\dim_0 \mathrm{Cone}(X) \geq \dim X$. As a result $\dim_0 \mathrm{Cone}(X) = \dim \mathrm{Cone}(X) = \dim X$ since $\dim \mathrm{Cone}(X) \geq \dim_0 \mathrm{Cone}(X)$. $\square$

D.3. Proof of Theorem D.1

We start to prove our main theorem in this chapter.

(1) By the assumption $\dim \phi^{-1}(0) = d_0$, every irreducible components of $\phi^{-1}(0)$ are all of dimension d_0.

The flatness will be shown if $\dim_x \phi^{-1}(v) \leq d_0$ holds for any $x \in \phi^{-1}(v)$ and $v \in V_2$ ([**20**, Exer. III.10.9]). Suppose $\phi^{-1}(v) \neq \emptyset$. Let $x \in \phi^{-1}(v)$. Then $\dim_{tx} \phi^{-1}(t^{n_0} v)\,(= \dim_{tx} \phi^{-1}(\phi(tx)))$ is an upper-semicontinuous function with respect to $t \in \mathbb{C}$ by Chevalley's semicontinuity theorem. Since $\dim_{tx} \phi^{-1}(t^{n_0} v)$ is constant on $\mathbb{C}^*$, $\dim_x \phi^{-1}(v) \leq \dim \phi^{-1}(0) = d_0$.

(2) Let X be any irreducible component of a nonempty fibre $\phi^{-1}(v)$. Then $\dim X = d_0$ by the flatness of ϕ. Since $I_v \subset I_X$, we have

$$I_0 = \operatorname{gr} I_v \subset \operatorname{gr} I_X \subset \sqrt{\operatorname{gr} I_X}. \tag{D.1}$$

Equivalently we have $\operatorname{Cone}(X)_{\mathrm{red}} \subset \operatorname{Cone}(X) \subset \phi^{-1}(0)$. By Lemma D.14, $\operatorname{Cone}(X)$ is a d_0-dimensional subscheme of the (reduced and irreducible) fibre $\phi^{-1}(0)$ of dimension d_0. Therefore we obtain $\operatorname{Cone}(X)_{\mathrm{red}} = \phi^{-1}(0)$. Thus all the inclusions in (D.1) are equalities. By Lemma D.3 (4) we have $I_v = I_X$. This means $\phi^{-1}(v) = X$. Since X is reduced, so is $\phi^{-1}(v)$.

(3) Let $I := I_v$ and $\overline{R} := R/I$. By (2), $\overline{R}$ is an integral domain.

We can define the quotient field $\operatorname{Quot}(\operatorname{gr} \overline{R})$ of $\operatorname{gr} \overline{R}$ since $\operatorname{gr} \overline{R} = R/I_0$ is an integral domain by the assumption. We claim that there is a canonical inclusion $\operatorname{gr} K \subset \operatorname{Quot}(\operatorname{gr} \overline{R})$. Let $s \in K_{\le d}/K_{\le d-1} \subset \operatorname{gr} K$. So we write $s = f/g + K_{\le d-1}$ where $f, g \in \overline{R}$ and $\deg f = h + d$, $\deg g = h$ for some h. Thus $(\operatorname{gr} g)s = (g + K_{\le h-1})(f/g + K_{\le d-1}) = f + K_{\le d+h-1} = \operatorname{gr} f$. So $s = \operatorname{gr} f / \operatorname{gr} g \in \operatorname{Quot}(\operatorname{gr}\overline{R})$.

Let S be the normal closure of $\overline{R}$ in K. Let $S_{\le d} := S \cap K_{\le d}$. Let $\operatorname{gr} S := \bigoplus_{d \in \mathbb{Z}} S_{\le d}/S_{\le d-1}$. We have $\operatorname{gr} \overline{R} \subset \operatorname{gr} S \subset \operatorname{gr} K$. By the above claim we get

$$\operatorname{gr} \overline{R} \subset \operatorname{gr} S \subset \operatorname{gr} K \subset \operatorname{Quot}(\operatorname{gr}\overline{R}).$$

We finish the proof of (3) by showing $S = \overline{R}$. Since S is a finitely generated $\overline{R}$-module ([**20**, Theorem I.3.9A]), there exists $f \in \overline{R} \setminus 0$ such that $fS \subset \overline{R}$. Since fS is an ideal of $\overline{R}$, $\operatorname{gr} fS$ is an ideal of $\operatorname{gr} \overline{R}$ by Lemma D.3 (1). By Lemma D.3 (3), $(\operatorname{gr} f)(\operatorname{gr} S)$ is an ideal of $\operatorname{gr} \overline{R}$. Since $\operatorname{gr} \overline{R}\,(= R/I_0)$ is a Noetherian ring, the ideal $(\operatorname{gr} f)(\operatorname{gr} S)$ is a finitely generated $\operatorname{gr} \overline{R}$-module. Since the multiplication map $\operatorname{gr} f \colon \operatorname{gr} S \to (\operatorname{gr} f)(\operatorname{gr} S)$ is a $\operatorname{gr} \overline{R}$-module isomorphism, $\operatorname{gr} S$ is a finitely generated $\operatorname{gr} \overline{R}$-module. We obtain $\operatorname{gr} \overline{R} = \operatorname{gr} S$ since $\operatorname{gr} \overline{R}$ is a normal domain by the assumption. By Lemma D.3 (4) we have $S = \overline{R}$. This completes the proof of the theorem.

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