博　士　論　文

Orbit parametrizations of theta characteristics on hypersurfaces

（超曲面上のシータ・キャラクタリスティックの
軌道によるパラメータ付け）

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To my family.

Abstract

We investigate a bijection between tuples of symmetric matrices and coherent sheaves on geometrically reduced hypersurfaces with rigidifying data. The coherent sheaves give a generalization of *theta characteristics,* the square roots of the canonical bundle of smooth curves ([**22**]). We call these sheaves theta characteristics on the hypersurfaces ([**5**, Definition 4.2.8]).

This bijection gives a generalization of results of Beauville and Ho ([**1**], [**15**]). Ho gave a bijection between orbits of triples of symmetric matrices and non-effective theta characteristics on smooth curves without any rigidifying data. Hence it is a new phenomenon that, when we consider non-effective theta characteristics on singular curves, we have to take such rigidifying data in consideration. We give an algebraic parametrization of the set of rigidifying data on a fixed theta characteristic on a geometrically reduced hypersurface.

As a related topic, we introduce some results on the *symmetric determinantal representations* of plane curves. The set of natural equivalence classes of such representations is in bijection with the set of equivalence classes of non-effective theta characteristics on the curve with rigidifying data. As consequences of this interpretation, we introduce three results: the non-existence of symmetric determinantal representations of Fermat curves of prime degree over the field of rational numbers ([**20**]), the failure of the local-global principle for symmetric determinantal representations on smooth plane quartic curves ([**18**]) and an exceptional behaviour over global fields of characteristic two ([**19**]).

Another topic is a description of the projective automorphism groups of complete intersections of quadrics. Beauville proved in [**2**] that the projective automorphism group of a smooth complete intersection of three quadrics is described as an extension of a subgroup of projective automorphism groups of the discriminant curve by an elementary two-group. We extend this result to more general complete intersections of any number of quadrics, and give an algebraic interpretation of the elementary two-group above.

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Contents

CHAPTER 1

Introduction

Theta characteristics, the square roots of the canonical bundles on projective smooth curves, are interesting objects in algebraic geometry and invariant theory. They appear in several different kinds of classical problems such as bitangents of plane quartics, determinantal representations and Appolonius' problem (cf. [**12**], [**5**, Chapter 4]). This thesis treats mainly the problem of determinantal representations.

It is well-known that, over a field of characteristic different from two, there is a natural bijection between certain smooth complete intersections of three quadrics and smooth plane curves with non-effective theta characteristics ([**1**, Chapitre 6], [**15**, Chapter 4]). (In fact, Beauville also treated the case of nodal plane curves in [**1**].) The purpose of this thesis is to generalize this bijection. We study more general hypersurfaces of any dimension, which may have singularities, over a field of arbitrary characteristic. There is a hidden datum, the duality (quasi-)isomorphisms of theta characteristics, to extend the bijection to more general cases. In this chapter, we briefly introduce the concepts and results in this thesis.

1.1. Tuples of symmetric matrices

The main results of this thesis are formulated in terms of linear orbits of $(m+1)$-tuples of symmetric matrices of size $n+1$ instead of complete intersections of $m+1$ quadrics in the projective space $\mathbb{P}^n$. Of course, this formulation gives a simple generalization when $n > m \geq 2$ and the characteristic of the base field is different from two. However, there is an essential difference in characteristic two or $n \leq m$. For example, if $n \leq m$, the intersection

of general $m+1$ quadrics is empty. Hence we cannot recover the $m+1$ quadrics from the intersection.

We fix a field k of arbitrary characteristic and integers $m \geq 2$ and $n \geq 1$. Let

$$\begin{aligned} W &:= k^{m+1} \otimes \mathrm{Sym}_2 k^{n+1} \\ &:= \left\{ M = (M_0, M_1, \ldots, M_m) \;\middle|\; M_i \in \mathrm{Mat}_{n+1}(k),\ {}^tM_i = M_i \ (i = 0, 1, \ldots, m) \right\} \end{aligned}$$

be the k-vector space of $(m+1)$-tuples of symmetric matrices of size $n+1$ with entries in k. For an element

$$M = (M_0, M_1, \ldots, M_m) \in W,$$

we define its *discriminant polynomial* by

$$\mathrm{disc}(M) := \det(X_0 M_0 + X_1 M_1 + \cdots + X_m M_m) \in k[X_0, X_1, \ldots, X_m].$$

If $\mathrm{disc}(M) \neq 0$, the discriminant polynomial $\mathrm{disc}(M)$ is a homogeneous polynomial of degree $n+1$ in $m+1$ variables $X_0, X_1, \ldots, X_m$.

The k-vector space W has a natural right action of the product of general linear groups $\mathrm{GL}_{m+1}(k) \times \mathrm{GL}_{n+1}(k)$. Concretely, for $A = (a_{i,j}) \in \mathrm{GL}_{m+1}(k), P \in \mathrm{GL}_{n+1}(k)$ and $M = (M_0, M_1, \ldots, M_m) \in W$, we set

$$M \cdot (A, P) := \left(\sum_{i=0}^{m} a_{i,0} {}^tP M_i P,\ \sum_{i=0}^{m} a_{i,1} {}^tP M_i P,\ \ldots,\ \sum_{i=0}^{m} a_{i,m} {}^tP M_i P \right).$$

When $(A, P) \in (k^\times I_{m+1}) \times \mathrm{GL}_{n+1}(k)$, where I_{m+1} is the identity matrix of size $m+1$, this action preserves the discriminant polynomial of an element of W up to the multiplication

by an element of k. Concretely, we have

$$\operatorname{disc}(M \cdot (A, P)) = \det(A) \det(P)^2 \operatorname{disc}(M).$$

We define two subsets

$$W_{\mathrm{gr}} \subset W_{\mathrm{nv}} \subset W$$

as follows. Let W_{nv} be the subset of $(m+1)$-tuples of symmetric matrices whose discriminant polynomial $\operatorname{disc}(M)$ is non-zero, and W_{gr} the subset of W_{nv} consisting of elements which have no multiple factors over an algebraic closure of k. (Here, the subscript "nv" stands for "non-vanishing" and the subscript "gr" stands for "geometrically reduced".) For an element $M \in W_{\mathrm{nv}}$, the equation $(\operatorname{disc}(M) = 0)$ defines a hypersurface

$$S \subset \mathbb{P}^m$$

of degree $n+1$ over k. The hypersurface $S \subset \mathbb{P}^m$ is geometrically reduced if and only if $M \in W_{\mathrm{gr}}$. The subsets W_{gr} and W_{nv} are stable under the action of $\mathrm{GL}_{m+1}(k) \times \mathrm{GL}_{n+1}(k)$.

1.2. Theta characteristics on hypersurfaces

To extend the classical bijection between the complete intersections of three quadrics and non-effective theta characteristics on plane curves, we must extend the definition of theta characteristics.

To explain the necessity, recall the definition of theta characteristics on a smooth curve C. It is a line bundle $\mathcal{L}$ such that $\mathcal{L} \otimes \mathcal{L}$ is isomorphic to the canonical bundle ω_C. With this definition, we will find some problems when we try to extend the bijection even if we consider a reduced Gorenstein curve with at worst ADE singularities. For example, the

union of two lines ($xy = 0$) in the projective plane $\mathbb{P}^2$ has no such line bundles even over algebraically closed fields (see [**23**]). However, there is a symmetric matrix

$$M(x, y) := \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$$

such that $\det(M(x, y)) = xy$. So to extend the bijection, we must modify the definition of theta characteristics.

How should we extend it? First, we have to give up to find a line bundle $\mathcal{L}$. Then instead of

$$\mathcal{L} \otimes \mathcal{L} \cong \omega_C,$$

we use

$$\mathcal{L} \cong \mathscr{H}om_C(\mathcal{L}, \omega_C)\,.$$

They are equivalent when $\mathcal{L}$ is a line bundle, but not equivalent in general.

Next, what is the appropriate condition for the sheaf $\mathcal{L}$? From our viewpoint, we need some cohomological properties on the sheaf $\mathcal{L}$. For example, the definition of Piontkowski ([**23**]) requires that the theta characteristic $\mathcal{L}$ on a geometrically reduced Gorenstein curve C is a *torsion-free* $\mathcal{O}_C$-module, and it is a free module of rank one on each generic point of C. This definition works well on geometrically reduced plane curves of course, but we need more conditions if we consider on geometrically reduced hypersurfaces.

Definition 1.2.1 (See Definition 4.1.6; [5, Definition 4.2.8]**).** Let $S \subset \mathbb{P}^m$ be a geometrically reduced hypersurface over k. A *theta characteristic* $\mathcal{M}$ on S is a coherent $\mathcal{O}_S$-module such that

- $\mathcal{M}$ is arithmetically Cohen–Macaulay, and

- $\mathcal{M}$ is pure of dimension $m-1$, and
- $\mathrm{length}(\mathcal{M}_\eta) = 1$ for each generic point η of S, and
- there is a quasi-isomorphism of $\mathcal{O}_S$-modules

$$\lambda\colon \mathcal{M} \xrightarrow{\sim} \mathrm{R}\mathscr{H}om_S(\mathcal{M}(2-m), \omega_S[1-m])\,.$$

A theta characteristic $\mathcal{M}$ on S is said to be *effective* (resp. *non-effective*) if $H^0(S, \mathcal{M}) \neq 0$ (resp. $H^0(S, \mathcal{M}) = 0$).

This definition actually gives a generalization of theta characteristics, and works well to construct the desired bijection (see Section 4.1.2).

For an element $M \in W_{\mathrm{gr}}$, we can construct an injective morphism of $\mathcal{O}_{\mathbb{P}^m}$-modules

$$M(X)\colon \bigoplus_{i=0}^{n} \mathcal{O}_{\mathbb{P}^m}(-2) \longrightarrow \bigoplus_{i=0}^{n} \mathcal{O}_{\mathbb{P}^m}(-1).$$

Then the cokernel $\mathcal{M}$ of the map $M(X)$ defines a non-effective theta characteristic on the hypersurface $S \subset \mathbb{P}^m$ defined by $\det(M(X)) = 0$ (cf. [**2**, Section 1], [**5**, Chapter 4]). Rigidifying this construction, we establish our bijection.

1.3. The bijection on theta characteristics

Let $\mathrm{TC}_{m+1,n+1}(k)$ be the set of equivalence classes of triples $(S, \mathcal{M}, \lambda)$ which consist of a geometrically reduced hypersurface $S \subset \mathbb{P}^m$ over k of degree $n+1$, a non-effective theta characteristic $\mathcal{M}$ on S and the quasi-isomorphism

$$\lambda\colon \mathcal{M} \xrightarrow{\sim} \mathrm{R}\mathscr{H}om_S(\mathcal{M}(2-m), \omega_S[1-m])\,.$$

Here, two triples $(S, \mathcal{M}, \lambda), (S', \mathcal{M}', \lambda')$ are *equivalent* if $S = S'$ and there exists an isomorphism $\rho\colon \mathcal{M} \xrightarrow{\sim} \mathcal{M}'$ of $\mathcal{O}_S$-modules satisfying a compatibility condition

$$^t\rho \circ \lambda' \circ \rho = u\lambda$$

for some $u \in k^\times$. The desired bijection is stated as follows:

Theorem 1.3.1 (See Corollary 4.2.5). Let k be a field of arbitrary characteristic. Let m, n be integers satisfying $m \geq 2$ and $n \geq 1$. There is a natural bijection between the $(k^\times I_{m+1}) \times \mathrm{GL}_{n+1}(k)$-orbits in W_{gr} and $\mathrm{TC}_{m+1,n+1}(k)$.

For a given geometrically reduced hypersurface $S \subset \mathbb{P}^m$ of degree $n+1$, the $(k^\times I_{m+1}) \times \mathrm{GL}_{n+1}(k)$-equivalence classes of $m+1$ tuples of symmetric matrices $M \in W_{\mathrm{gr}}$ such that the equation $(\mathrm{disc}(M) = 0)$ defines $S \subset \mathbb{P}^m$ are in bijection with the equivalence classes of $(\mathcal{M}, \lambda)$. Two problems occur: are there non-effective theta characteristics $\mathcal{M}$ on C defined over k? If so, how many λ are there?

The answer to the first question is "not always even if C is a smooth plane curve". For example, some plane conics, the degree two curves in $\mathbb{P}^2$, do not admit non-effective theta characteristics defined over k. Hence such conics have no $M \in W_{\mathrm{gr}}$ whose $\det(M(X))$ define the conics. For more higher dimensional cases, the situation becomes worse: there is a surface of degree four in $\mathbb{P}^3_{\mathbb{C}}$ with no non-effective theta characteristics ([**5**, Example 4.2.23]). We give other examples in Chapter 6.

The answer to the second question is more interesting. There is a commutative étale k-subalgebra L of $\mathrm{Mat}_{n+1}(k) \times \mathrm{Mat}_{n+1}(k)$ with the following properties:

Theorem 1.3.2. The set of the symmetric quasi-isomorphisms λ on fixed $\mathcal{M}$ has a simply transitive action of $L^\times$. Moreover, the set of the equivalence class of triples $(S, \mathcal{M}, \lambda)$ with fixed $\mathcal{M}$ has a simply transitive action of $L^\times / k^\times L^{\times 2}$.

In the second statement of Theorem 1.3.2, the group $L^\times / k^\times L^{\times 2}$ is the quotient of the multiplicative group $L^\times$ by the subgroup

$$k^\times L^{\times 2} := \left\{ ab^2 \;\middle|\; a \in k^\times, b \in L^\times \right\}.$$

We also prove a statement for tuples of symmetric matrices defining hypersurfaces which are not necessarily geometrically reduced (see Corollary 3.3.2, Corollary 3.3.3 and Proposition 3.4.3). From Theorem 1.3.2, we can recover some results of Beauville and Ho:

Corollary 1.3.3 (See Corollary 4.2.7). Assume that *at least one* of the following conditions is satisfied:

- the base field k is separably closed of characteristic different from two, or
- the base field k is perfect of characteristic two, or
- the hypersurface $S \subset \mathbb{P}^m$ is geometrically integral.

Then the set of equivalence classes of triples $(S, \mathcal{M}, \lambda)$ with fixed $\mathcal{M}$ consists of only one element.

In fact, it is easy to see that the group $L^\times / k^\times L^{\times 2}$ becomes trivial if at least one of the above conditions is satisfied.

Beauville proved Corollary 1.3.3 when k is algebraically closed of characteristic of k different from two, $m = 2, n \geq 3$ and a matrix $M \in W_{\mathrm{gr}}$ corresponding to a triple $(S, \mathcal{M}, \lambda)$ for some λ (hence every elements in the fiber) defines a smooth complete intersection of

three quadrics in $\mathbb{P}^n$ ([**1**, Proposition 6.19]). Ho proved Corollary 1.3.3 when $m = 2, n \geq 2$, k is a field where $3n(n-1)$ is invertible and $S \subset \mathbb{P}^2$ is a smooth plane curve ([**15**, Corollary 4.18]). However, there can be infinitely many equivalence classes of triples $[(S, \mathcal{M}, \lambda)]$ on some pair $(S, \mathcal{M})$.

1.4. Projective automorphism groups of complete intersections of quadrics

Another result of this thesis concerns the projective automorphism groups of complete intersection of quadrics. Let $(S, \mathcal{M}, \lambda) \in \mathrm{TC}_{m+1,n+1}(k)$ be a triple and $M \in W_{\mathrm{gr}}$ be the corresponding element (Theorem 1.3.2). Define the subvariety X_Q of $\mathbb{P}^n$ by

$$X_Q := \left\{ x \in \mathbb{P}^n \;\middle|\; {}^t x M_0 x = {}^t x M_1 x = \cdots = {}^t x M_m x = 0 \right\}.$$

The projective equivalence class of X_Q depends only on the $\mathrm{GL}_{m+1}(k) \times \mathrm{GL}_{n+1}(k)$-orbit of M.

The choice of M defines a duality quasi-isomorphism λ of $\mathcal{M}$, and the étale k-algebra L appearing in Theorem 1.3.2. Let G be the group defined as the kernel of

$$L^\times/k^\times \longrightarrow L^\times/k^\times \quad ; \quad a \mapsto a^2.$$

Define $\mathrm{Aut}_{\mathbb{P}^m}(S, \mathcal{M}, \lambda)$ as the subgroup of projective automorphism group of S which element $\nu \in \mathrm{Aut}(\mathbb{P}^m)$ gives the projective automorphism of S and the triple $(S, \nu^*\mathcal{M}, \nu^*\lambda)$ is equivalent to $(S, \mathcal{M}, \lambda)$. Then we have the following exact sequence.

Theorem 1.4.1 (See Theorem 5.2.1). Take $M \in W_{\mathrm{gr}}$ which defines a complete intersection X_Q of $m+1$ quadrics. Then there exists the short exact sequence

$$0 \longrightarrow G \longrightarrow \mathrm{Aut}_{\mathbb{P}^n}(X_Q) \longrightarrow \mathrm{Aut}_{\mathbb{P}^m}(S, \mathcal{M}, \lambda) \longrightarrow 0.$$

We also prove the analogue of this result for $M \in W_{\mathrm{nv}}$.

1.5. Future works

There are many problems related in this work. We first state some simple problems by itemization.

- Some parts of our proof do not work when $m = 2$ (for example, see Proposition 2.2.8). However, many techniques seem to work similarly. Can we have similar results in this case? Can we relate our work to a work of Bhargava, Gross and Wang ([**3**])?
- In [**2**, Chapter 5], Beauville gives a interpretation of skew-symmetric matrices of even size as a vector bundle on plane curves of rank two. Modify this to a bijection over a general field, and extend them. What does correspond to skew-symmetric matrices of odd size? (The last problem is pointed out by Akihiko Yukie.)
- In [**15**, Chapter 5], Ho considered the analogue results of curves in $\mathbb{P}^1 \times \mathbb{P}^1$. Are there analogue of our bijection in this case?
- Given a homogeneous polynomial $f \in k[X_0, X_1, \ldots, X_m]$. In above, we treat the matrix $M \in W$ such that $\det(M(X))$ coincides with f *up to multiplication by elements of* $k^\times$. Can we determine the matrix $M \in W$ with *just an equality*,

$$\det(M(X)) = f?$$

In $m = 2$ case, Bhargava, Gross and Shankar gave a characterization ([**3**, Theorem 23]).

The most important motivation of this work comes from Arithmetic Invariant Theory. In [**26**], Wang treated the pair of symmetric matrices and related them to a class of torsors of a hyperelliptic Jacobian. Using this, Shankar and Wang gave the upper bound of average rank of hyperelliptic Jacobians [**24**].

In the case of triple of symmetric matrices, Beauville ([**1**]) pointed out that there exists an abelian variety which relates deeply to the triples of symmetric matrices. It is the *Prym variety* occuring from n étale double covering of plane curves. It seems to exist orbital parametrization of torsors of such Prym varieties by triples of symmetric matrices.

Notation. We work over a field k of arbitrary characteristic except in Chapter 5. In Chapter 5, we assume the characteristic of k is different from two. The k-vector space of symmetric matrices of size $n+1$ with entries in k is denoted by $\mathrm{Sym}_2 k^{n+1}$. Hence an element of $k^{m+1} \otimes \mathrm{Sym}_2 k^{n+1}$ is identified with an $(m+1)$-tuple of symmetric matrices of size $n+1$ with entries in k. For a scheme X over k of finite type, we write the set of singular points, smooth points and generic points on X as $\mathrm{Sing}(X), \mathrm{Sm}(X)$ and $\mathrm{Gen}(X)$, respectively. For a point $p \in X$, the local ring at p is denoted by $\mathcal{O}_{X,p}$. We use the symbol ω_X to denote the *dualizing complex* on X rather than the dualizing sheaf (cf. [**13**]). For an object $\mathcal{F} \in D(\mathrm{Coh}_X)$ in the derived category of complexes of coherent $\mathcal{O}_X$-modules, let $\mathcal{F}[n]$ denote the degree n shift defined by $(\mathcal{F}[n])_i := \mathcal{F}_{n+i}$. For a morphism h between complexes of $\mathcal{O}_X$-modules, we denote the induced morphism between cohomology by the same symbol h.

CHAPTER 2

Minimal resolutions of coherent sheaves on projective spaces

Free resolution is a standard tool to study the modules over polynomial rings over a field. Among them, the minimal free resolutions are important ones because of their uniqueness property. There is the corresponding notion on coherent sheaves on projective spaces, and it inherits many properties from resolutions of modules. In this chapter, we recall the definition and properties of minimal free resolutions of graded modules and coherent sheaves on projective spaces.

2.1. Minimal resolutions of modules

In this section, we recall the definitions and properties of free resolution of graded modules over connected graded k-algebras. Many properties of local rings can be extended to the graded algebras. We refer the details of notion in this section to [**7**].

2.1.1. Graded algebra. An $\mathbb{N}$-*graded algebra* or simply a *graded algebra* R means a commutative, associative, unital algebra R with a direct decomposition $R = \bigoplus_{i \geq 0} R_i$ as an abelian group which satisfies

$$R_i R_j \subset R_{i+j}$$

for any $i, j \geq 0$. By definition, R_0 is a subalgebra of R, and each graded part R_i is a R_0-submodule of R. Hence R is an R_0-algebra.

An R-module M is said to be $\mathbb{Z}$-*graded* or simply *graded* if M has a direct decomposition

$$M := \bigoplus_{i \in \mathbb{Z}} M_i$$

as an R_0-module, and satisfies

$$R_i M_j \subset M_{i+j}$$

for any $i, j \in \mathbb{Z}$ and $i \geq 0$. For a graded R-module M and an integer $j \in \mathbb{Z}$, $M(j)$ denotes the degree j *shift* of M. It has the same R-module structure as M but

$$M(j)_n := M_{j+n}.$$

Example 2.1.1. Any graded free R-module of rank one has the form $R(j)$ for some $j \in \mathbb{Z}$. Hence any finitely generated graded free R-module M can be written as

$$M = \bigoplus_{i=0}^{n} R(e_i)$$

fo some integers $e_i \in \mathbb{Z}$.

Example 2.1.2. The R_0-submodule of R

$$R_+ := \bigoplus_{i \geq 0} R_i$$

is a graded R-ideal, called the *irrelevant ideal* of R. There is the canonical isomorphism

$$R/R_+ \xrightarrow{\sim} R_0.$$

Let M be a graded R-module. A *graded R-submodule* N of M is an R-submodule of M which satisfies

$$N = \bigoplus_{i \in \mathbb{Z}} N \cap M_i.$$

We write $N_i := N \cap M_i$. If N is a graded R-submodule of a graded R-module M, the quotient R-module M/N inherits the grade structure from M. In concrete terms, the i-th graded piece of M/N is given by

$$(M/N)_i = M_i/N_i.$$

As a consequence, the quotient of R by a graded ideal is naturally a graded algebra.

Let k be a field. A graded k-algebra is said to be *connected* if $R_0 = k$. In this case, the irrelevant ideal R_+ satisfies

$$R/R_+ \cong R_0 \xrightarrow{\sim} k,$$

so R_+ is a maximal ideal of R. Moreover, all graded ideals $I \neq R$ of R are included in this ideal.

Example 2.1.3. Let V be a finite dimensional k-vector space. For a positive integer $a \geq 0$, the a-th tensor $V^{\otimes a}$ of V has a natural action of the a-th symmetric group $\mathfrak{S}_a$. The covariant space

$$\begin{aligned} \mathrm{Sym}^a V &:= V^{\otimes a}_{\mathfrak{S}_a} \\ &= V^{\otimes a} / \left\langle \sigma v - v \;\middle|\; \sigma \in \mathfrak{S}_a, v \in V^{\otimes a} \right\rangle \end{aligned}$$

is called the *a-th symmetric power* of V. For convenience, we define

$$V^{\otimes 0} := \mathrm{Sym}^0 V := k.$$

The direct sum

$$\mathrm{Sym}^{\bullet} V := \bigoplus_{i \geq 0} \mathrm{Sym}^i V$$

of symmetric powers of V has a natural graded connected k-algebra structure. This algebra is called the *symmetric algebra* of V. It is isomorphic to the polynomial ring with $\dim_k V$ variables over k. In particular, the symmetric algebra is a Noetherian algebra.

Example 2.1.4. Another example can be found from symmetric subspaces. Let V be a finite dimensional k-vector space, and $a \in \mathbb{Z}_{\geq 0}$. The *$a$-th divided power* or the *a-th symmetric subspace* $\mathrm{Sym}_a V$ of V is the $\mathfrak{S}_a$-symmetric subspace of $V^{\otimes a}$, i.e.

$$\begin{aligned} \mathrm{Sym}_a V &:= (V^{\otimes a})^{\mathfrak{S}_a} \\ &= \left\{ v \in V^{\otimes a} \;\middle|\; \sigma v = v \quad (\sigma \in \mathfrak{S}_a) \right\}. \end{aligned}$$

For convenience, we define

$$\mathrm{Sym}_0 V := k.$$

The direct sum of symmetric subspaces

$$\mathrm{Sym}_{\bullet} V := \bigoplus_{i \geq 0} \mathrm{Sym}_i V$$

has a natural graded connected k-algebra structure. This algebra is called the *divided power* of V. It is not Noetherian if a field k has positive characteristic, but the algebra carries more detailed structure called *divided power algebra.* We do not treat this structure in this thesis.

All graded algebras appear later are connected Noetherian k-algebras for a fixed field k. The following lemma is a fundamental tool to investigate the structures of graded R-modules.

Lemma 2.1.5 (Nakayama lemma for graded rings). Let k be a field, and R a connected graded k-algebra. Let M be a finitely generated graded R-module, and N a graded R-submodule of M.

Assume that $N + R_+M = M$. Then $N = M$.

Proof. By replacing M by M/N, we may consider only the case that $N = 0$.

Assume that $M \neq 0$. Since M is finitely generated as an R-module, there exists the minimal index i such that $M_i \neq 0$. Then $M_i \cap R_+M = 0$ by assumption of i. Hence $M_i \longrightarrow M/R_+M$ is injective, and $M/R_+M \neq 0$. $\square$

The category of graded modules of a graded ring R consists an *abelian category*. Its morphisms are R-homomorphisms preserving grades. We use $\underline{(\mathrm{gr}.R\text{-mod})}$ to denote this category.

2.1.2. Complexes and projective modules. Let R be a commutative algebra. A *(chain) complex* $M_\bullet := \{M_i, \delta_i\}$ of R-modules is a sequence $\{M_i\}_{i\in\mathbb{Z}}$ of R-modules and R-homomorphisms

$$\delta_i \colon M_i \longrightarrow M_{i-1}$$

satisfying $\delta_{i-1} \circ \delta_i = 0$ for any $i \in \mathbb{Z}$. It is said to be *exact* if

$$\operatorname{Im} \delta_{i+1} = \operatorname{Ker} \delta_i$$

for any $i \in \mathbb{Z}$. Also, a *(cochain) complex* $M^\bullet := \{M^i, \delta^i\}$ of R-modules is a sequence $\{M^i\}_{i\in\mathbb{Z}}$ of R-modules and R-homomorphisms

$$\delta^i \colon M^i \longrightarrow M^{i+1}$$

satisfying $\delta^{i+1} \circ \delta^i = 0$ for any $i \in \mathbb{Z}$. It is said to be *exact* if

$$\operatorname{Ker} \delta^{i+1} = \operatorname{Im} \delta^i$$

for any $i \in \mathbb{Z}$.

For a chain complex $M_\bullet$ of R-modules and integer $j \in \mathbb{Z}$, we define the degree j *shift* $M[j]_\bullet$ by

$$M[j]_i := M_{i+j},$$

$$\delta[j]_i := \delta_{i+j}.$$

We define the shift of cochain complexes similarly. If $M_j = 0$ for $j \gg 0$ (resp. $j \ll 0$), we say the complex $M_\bullet$ is *left bounded* (resp. *right bounded*). Similarly, the cochain complex $M^\bullet$ is *left bounded* (resp. *right bounded*) if $M^j = 0$ for $j \ll 0$ (resp. $j \gg 0$). If the (co)chain complex is right bounded and left bounded, we say simply that $M_\bullet$ is *bounded.* For a cochain complex $M^\bullet = \{M^i, \delta^i\}$ of R-modules, we have a chain complex $M_\bullet = \{M_i := M^{-i}, \delta_i := \delta^{-i}\}$ of R-modules and vice versa. In the rest of this chapter, we only consider chain complexes.

The *homomorphism*

$$f = f_\bullet \colon M_\bullet \longrightarrow N_\bullet$$

of R-complexes is a collection of R-homomorphism

$$f_i \colon M_i \longrightarrow N_i$$

satisfying compatibility equalities

$$\delta_i \circ f_i = f_{i-1} \circ \delta_i.$$

The *isomorphism* $f \colon M_\bullet \xrightarrow{\sim} N_\bullet$ is a homomorphism of complexes f having the inverse morphism as a homomorphism of complexes. Note that it is a different matter from the *quasi-isomorphisms* which we will treat in Chapter 3.

Typical but important examples of exact complexes are given by *projective resolutions.* An R-module P is *projective* if the functor $\mathrm{Hom}_R(P, *)$ is exact. Here, we say the functor

$$F \colon \underline{(R\text{-mod})} \longrightarrow \underline{(R\text{-mod})}$$

is *exact* when, if a complex $M_\bullet = \{M_i, \delta_i\}$ is exact, then the complex $F(M_\bullet) = \{F(M_i), F(\delta_i)\}$ is also exact.

Example 2.1.6. The free modules over R are typical examples of projective R-modules. This is because there are functorial isomorphisms

$$\mathrm{Hom}_R(R, M) \xrightarrow{\sim} M \quad ; \quad f \mapsto f(1)$$

and

$$\mathrm{Hom}_R\left(\bigoplus_i N_i, M\right) \xrightarrow{\sim} \prod_i \mathrm{Hom}_R(N_i, M)$$

for R-modules M and N_i.

The similar argument works to prove that any direct summand of free R-modules is projective. In fact, it is easy to see that any projective R-module is a direct summand of a free R-module.

We introduce the following simple characterization of projective modules over a local ring.

Proposition 2.1.7 ([**7**, Theorem 19.2])**.** Any finitely generated projective module over a local ring R is a free module over R.

Let M be an R-module. The *projective resolution* of M is a complex $P_\bullet = \{P_i, \delta_i\}$ and a surjective R-homomorphism

$$\epsilon\colon P_0 \longrightarrow M$$

which satisfies the following conditions

- all R-modules P_i are projective, and
- $P_i = 0$ for any $i \leq -1$, and
- $\operatorname{Im} \delta_{i+1} = \operatorname{Ker} \delta_i$ for any $i > 0$, and
- ϵ induces the isomorphism $\operatorname{Coker} \delta_1 \xrightarrow{\sim} M$.

If, moreover, P_i are free R-modules for all i, the complex $P_\bullet$ is called a *free resolution* of R-module M. If $P_i = 0$ for enough large i, we call the projective resolution $P_\bullet$ is bounded.

For any R-module M, there exists such a resolution. In fact, let M be an R-module. Then the free R-module

$$P_0 := \bigoplus_{m \in M} Re_m$$

has a surjection ϵ onto M defined by $\epsilon(e_m) := m$. The kernel $\operatorname{Ker}\epsilon$ has also a surjection δ_0 from a free module P_1. By repeating this process, we obtain a free resolution of R-module M. However, the resolutions are not unique.

Example 2.1.8. The zero module 0 has a free resolution

$$\ldots \longrightarrow 0 \longrightarrow (P_1 =)R \xrightarrow{\mathrm{id}} (P_0 =)R \longrightarrow 0 \longrightarrow \ldots.$$

This is not isomorphic to the zero complex $\{M_i = 0, \delta_i = 0\}_{i\in\mathbb{Z}}$, another projective resolution of the zero module 0. This is an example of *trivial complexes*, i.e. it is an exact complex.

If a nonzero R-module M has a bounded projective resolution, there exists an integer d such that

- for any $n > d$, there exists a projective resolution $P_\bullet$ such that $P_n = 0$, but
- for any projective resolution $P_\bullet$, the d-th degree part P_d is a non-trivial R-module.

We call this integer d as the *projective dimension* of M, and write as $\operatorname{pd} M$. If M has no bounded projective resolution, we set $\operatorname{pd} M = \infty$. For an R-module, $\operatorname{pd} M = 0$ if and only if M is non-zero and projective.

Actually, the concepts in this subsection are similarly defined in any abelian category. Some objects of some abelian categories do not admit projective resolutions. However, all objects of the category of graded modules $\underline{(\mathrm{gr}.R\text{-mod})}$ have a graded free resolution. This is because, if $M_\bullet$ is a graded R-module, the free module

$$\bigoplus_{i\in\mathbb{Z}} \bigoplus_{m\in M_i} Re_m$$

has a canonical surjection to $M_\bullet$ preserving degrees. For an abelian category $\mathcal{C}$, we say that $\mathcal{C}$ has *enough projectives* if for any object $A \in \mathcal{C}$, there is a projective object $P \in \mathcal{C}$ and a surjection $\epsilon\colon P \longrightarrow A$. If an abelian category $\mathcal{C}$ has enough projectives, there is a projective resolution for any object in $\mathcal{C}$. Above discussion says that $\underline{(\mathrm{gr}.R\text{-mod})}$ has enough projectives.

The following proposition is an analogue of Proposition 2.1.7.

Proposition 2.1.9 ([**7**, Theorem 19.2])**.** Any finitely generated graded projective module over a connected graded k-algebra R is a graded free module over R.

The *global dimension* of a ring R is the maximum of projective dimension of R-modules, if exists. We define the global dimension of R is ∞ if the maximum does not exist. If R be a local ring with the residue field k or a connected graded k-algebra, the global dimension of R is equal to $\mathrm{pd}(k)$.

2.1.3. Depth and resolutions. Let M be an R-module. We say an element $a \in R$ is a *non-zero-divisor* on M if the multiplication-by-a morphism on M

$$\times a\colon M \longrightarrow M$$

is injective.

A tuple $(a_1, a_2, \ldots, a_r)$ of elements in R is said to be an *M-regular sequence* if the following conditions hold:

- a_1 is a non-zero-divisor on M, and
- for any $2 \leq i \leq r$, a_i is a non-zero-divisor on $M/(a_1, a_2, \ldots, a_{i-1})M$, and
- $M/(a_1, a_2, \ldots, a_r)M \neq 0$.

We say that for an ideal $I \subset R$, an M-regular sequence $(a_1, a_2, \ldots, a_r)$ is in I if all a_i are elements of I.

The *depth* $\operatorname{depth}(R, M)$ of R in an R-module M is defined by the maximal length of M-regular sequences. More generally, for any ideal $I \subset R$, the depth $\operatorname{depth}(I, M)$ of I in M is defined by the maximal length of M-regular sequence in I. If $IM \neq M$, then the condition

$$M/(a_1, a_2, \ldots, a_r)M \neq 0$$

is automatically satisfied. We usually write $\operatorname{depth}(I) := \operatorname{depth}(I, R)$ for simplicity.

Example 2.1.10 (Local rings). Let $(R, \mathfrak{m}, k)$ be a local ring. Take a finitely generated R-module $M \neq 0$. By Nakayama lemma, we have $\mathfrak{m}M \neq M$. The depth $\operatorname{depth}(\mathfrak{m}, M)$ of $\mathfrak{m}$ in M is usually written as $\operatorname{depth}(M)$.

Auslander and Buchsbaum showed the following theorem about $\operatorname{depth}(M)$.

Theorem 2.1.11 (Auslander–Buchsbaum formula, [7, Theorem 19.9]). Let $(R, \mathfrak{m}, k)$ be a local ring. If M is a finitely generated R-module and $\operatorname{pd} M < \infty$, then

$$\operatorname{pd}(M) = \operatorname{depth}(R) - \operatorname{depth}(M).$$

Example 2.1.12. Let R be a connected Noetherian graded k-algebra, and R_+ the irrelevant ideal. Take a finitely generated graded R-module $M = M_\bullet$. By Lemma 2.1.5, $R_+M \neq M$.

In this case, we can take the M-regular sequence of R_+ of maximal length as a sequence of homogeneous elements by the next proposition.

Proposition 2.1.13 (cf. [**7**, Proposition 18.4]**).** Let R be a connected Noetherian graded k-algebra, and $M = M_\bullet$ a finitely generated graded R-module. The following conditions are equivalent:

(1) for any $i < n$,

$$\mathrm{Ext}^i_R(k, M) = 0.$$

(2) there exists an M-regular sequence of length n and consisting of homogeneous elements.

Proof. (1) $\Rightarrow$ (2): If $\mathrm{depth}(R_+, M) = 0$, any elements in R_+ are zero divisor of M. Hence R_+ is an associated prime of M, so we have

$$\mathrm{Hom}_R(k, M) \neq 0.$$

Hence there exists a non-zero-divisor on M in R_+.

Since the associated primes of graded R-modules are graded prime ideals ([**7**, Proposition 3.12]), the prime avoidance lemma ([**7**, Lemma 3.3]) shows that there exists a homogeneous non-zero-divisor $a \in R_+$. This shows the case $n = 1$.

When $n > 1$, we have a homogeneous non-zero-divisor $a \in R_+$ by the $n = 1$ case. By the short exact sequence

$$0 \longrightarrow M \xrightarrow{\times a} M \longrightarrow M/aM \longrightarrow 0,$$

we have

$$\mathrm{Ext}^i_R(k, M/aM) = 0$$

for any $i < n-1$. By induction hypothesis, we have an M/aM-regular sequence $(a_2, a_3, \ldots, a_n)$ of length $n-1$ consisting of homogeneous elements in R_+. This shows M has a regular sequence of length n consisting of homogeneous elements of positive degree.

(2) $\Rightarrow$ (1): Take a homogeneous M-regular sequence $(a = a_1, a_2, \ldots, a_n)$. We have the following short exact sequence

$$0 \longrightarrow M \xrightarrow{\times a} M \longrightarrow M/aM \longrightarrow 0.$$

By the long exact sequence of cohomology, we see that

$$\operatorname{Ext}_R^i(k, M) = 0$$

for $i < n-1$. For $i = n-1$, we have a short exact sequence

$$0 = \operatorname{Ext}_R^{n-2}(k, M/aM) \longrightarrow \operatorname{Ext}_R^{n-1}(k, M) \xrightarrow{\times a} \operatorname{Ext}_R^{n-1}(k, M)\,.$$

Since $a \in R_+$, the R-module $\operatorname{Ext}_R^{n-1}(k, M) = \operatorname{Ext}_R^{n-1}(R/R^+, M)$ is annihilated by a. Hence we have

$$\operatorname{Ext}_R^{n-1}(k, M) = 0. \quad \square$$

There is an analogous result of Theorem 2.1.11 for graded R-modules.

Theorem 2.1.14 (Auslander–Buchsbaum formula for graded modules, [**7**, Exercise 19.8]**).** Let R be a connected graded k-algebra of finite type, and R_+ the irrelevant ideal of R. If M is a finitely generated graded R-module and $\operatorname{pd} M < \infty$, then

$$\operatorname{pd}(M) = \operatorname{depth}(R_+, R) - \operatorname{depth}(R_+, M).$$

Finally we introduce the notion of Cohen–Macaulay modules.

Definition 2.1.15. Let R be a Noetherian ring, and M an R-module. We say M is a *Cohen–Macaulay* R-module if, for any maximal ideal P of R, we have

$$\operatorname{depth}(M_P) = \dim(M_P) = \dim(R_P/\operatorname{Ann}(M_P)).$$

A Noetherian ring R is said to be *Cohen–Macaulay* if R is itself a Cohen–Macaulay R-module.

Example 2.1.16. Let k be a field. The polynomial ring $R = k[X_0, X_1, \ldots, X_m]$ of $m+1$ variables is a connected graded k-algebra. This is an example of Cohen–Macaulay rings.

If R is a Cohen–Macaulay ring and M is a finitely generated Cohen–Macaulay R-module, Auslander–Buchsbaum formula states

$$\operatorname{pd}(M_P) = \dim R_P - \dim M_P \tag{2.1.1}$$

for any prime ideal P of R with $\dim M_P < \infty$.

2.1.4. Free resolutions of graded modules. In this subsection, we introduce the minimal free resolution of graded modules. It is the simplest free resolution in a sense. We only treat the graded case, but there is analogue in the case of local rings. See [**7**, Chapter 20].

Definition 2.1.17 (Minimal resolutions of graded modules, [**7**, Chapter 19]**).** Let $M_\bullet = \{M_i, \delta_i\}$ be a complex over a connected graded k-algebra R. We say $M_\bullet$ is

minimal if, for any $i \in \mathbb{Z}$,

$$\mathrm{Im}(\delta_i \colon M_i \longrightarrow M_{i-1}) \subset R_+ M_{i-1}.$$

Note that any graded projective module over a connected graded k-algebra is a graded free module over R (Proposition 2.1.9). The next theorem states that the minimal graded resolution exists, and it is "minimal" over all graded projective resolutions in a sense.

Theorem 2.1.18 (The existence and uniqueness of minimal resolutions, cf. [7, Exercise 20.1]). Let R be a connected Noetherian graded k-algebra. For a finitely generated graded R-module M, a minimal graded free resolution of $M_\bullet$ exists and is unique up to isomorphism. Moreover, any free resolution is the direct sum of a trivial complex and a minimal free resolution.

Proof. Since $M = M_\bullet$ is finitely generated, $M \neq R_+M$ by Lemma 2.1.5 and M/R_+M is a finite dimensional k-vector space. Take an ordered k-basis $\overline{e} = \{\overline{e_0}, \overline{e_1}, \ldots, \overline{e_r}\}$ of M/R_+M and take a homogeneous lift $e = \{e_0, e_1, \ldots, e_r\}$ in M. Again by Lemma 2.1.5, M is generated by e.

Put

$$F_0 := \bigoplus_{i=0}^{r} R(\deg e_i) f_i,$$

where f_i are indeterminates. It is a graded free R-module. It has a surjective graded R-homomorphism

$$\epsilon \colon F_0 \longrightarrow M$$

defined by $\epsilon(f_i) = e_i$. The kernel K of this surjection is again finitely generated since R is Noetherian. Hence we can apply this construction again to K and we obtain a free

resolution of M. By construction, we have $K \subset R_+F_0$, so the free resolution is minimal. This shows the existence of minimal resolutions.

We take another graded free R-module F_0' and a graded surjection

$$\epsilon' \colon F_0' \longrightarrow M.$$

We would like to show that there exists a split surjection

$$\rho' \colon F_0' \longrightarrow F_0.$$

By the projectivity of F_0, there exists a graded R-homomorphism

$$\rho \colon F_0 \longrightarrow F_0'$$

such that $\epsilon' \circ \rho = \epsilon$. By a similar reason, we have a graded R-homomorphism

$$\rho' \colon F_0' \longrightarrow F_0$$

such that $\epsilon \circ \rho' = \epsilon'$. The composite $\rho' \circ \rho$ is a graded endomorphism with $\epsilon \circ \rho' \circ \rho = \epsilon$. This shows

$$\rho' \circ \rho \equiv 1 \mod R_+.$$

For any $i \in \mathbb{Z}$, we have the exact sequence of k-vector spaces

$$0 \longrightarrow \bigoplus_{j \geq 1} R_j (F_0)_{i-j} \longrightarrow (F_0)_i \longrightarrow (F_0/R_+F_0)_i \longrightarrow 0.$$

Using this sequence and by induction on i, we can show that the restriction of $\rho' \circ \rho$ to $(F_0)_i$ is an isomorphism for any $i \geq 0$. Thus we have $\nu = \rho' \circ \rho$ gives a graded automorphism of F_0. Hence $\nu^{-1} \circ \rho'$ is a split surjection and $\rho(F_0)$ is a direct summand of F_0'. If we put $F_0' = \rho(F_0) \otimes F_0''$, then we can show that $\mathrm{Ker}(\epsilon') = \mathrm{Ker}(\epsilon) \otimes F_0''$ and

$$F_1' = F_0'' \oplus \rho_1(F_1) \oplus F_1''$$

for some splitting injection $\rho_1 \colon F_1 \longrightarrow F_1'$. Hence the resolution $F_\bullet'$ is a direct sum of a minimal resolution of M and a trivial complex. $\square$

In general, the isomorphisms between minimal resolutions of M are not determined by the induced automorphism of M. However, if the minimal resolution is pure, it is uniquely determined.

Definition 2.1.19. The minimal free resolution $F_\bullet$ is said to be *pure* if it is generated by a single degree; in other words, there exists an integer $d_i \in \mathbb{Z}$ for any $i \geq 0$ such that

$$F_i = R \cdot (F_i)_d.$$

Lemma 2.1.20. If the minimal resolution of a graded R-module M is pure, then any endomorphism of M is uniquely lifted to endomorphism of the minimal resolution as a complex of R-modules.

Proof. By purity of minimal resolution, there is an integer $d \in \mathbb{Z}$ such that the graded R-module M is generated by M_d Then the zeroth graded free module F_0 of the minimal free resolution of M is also generated by the degree d part $(F_0)_d$. We have isomorphisms

of k-vector spaces

$$M_d \cong M/\mathfrak{m}_R M \cong F_0/\mathfrak{m}_R F_0 \cong (F_0)_d.$$

The endomorphism of M preserving degrees causes an endomorphism of M_d and therefore an endomorphism of $(F_0)_d$. Since F_0 is generated by $(F_0)_d$, any endomorphism of $(F_0)_d$ is uniquely extended to F_0 as an endomorphism of graded R-modules. By applying the similar arguments to other part of minimal free resolution, we have a unique endomorphism of each F_i of the minimal resolution of M. □

Finally, we remark that the polynomial rings over a field k have a special property.

Theorem 2.1.21 (Hilbert Syzygy theorem, [**7**, Corollary 19.7]**).** The polynomial ring $R = k[X_0, X_1, \ldots, X_m]$ over a field k in $m+1$ variables $X_0, X_1, \ldots, X_m$ has global dimension $m+1$.

Hence all finitely generated modules over polynomial rings over a field k have finite projective dimension and we can use the Auslander–Buchsbaum formula. Moreover, since polynomial rings over a field k are Cohen–Macaulay, we use freely the formula (2.1.1).

2.2. Minimal resolutions of $\mathcal{O}_{\mathbb{P}^m}$-modules

The next topic is the minimal resolutions of coherent $\mathcal{O}_{\mathbb{P}^m}$-modules. In this section, we freely use the cohomology functor of global sections, Ext functor and the abelian category of coherent modules on $\mathbb{P}^m$. The references of this topic are [**8**], [**2**], and we recall briefly in Section 3.1.

Let $m \geq 1$ be an integer, and k a field. We fix a projective coordinate $X_0, X_1, \ldots, X_m$ of $\mathbb{P}^m$. Then the graded k-algebra

$$R := \bigoplus_{j \geq 0} H^0(\mathbb{P}^m, \mathcal{O}_{\mathbb{P}^m}(j))$$

is isomorphic to the polynomial ring $k[X_0, X_1, \ldots, X_m]$ of $m+1$ variables as a graded k-algebra. For a coherent $\mathcal{O}_{\mathbb{P}^m}$-module $\mathcal{F}$, we define a graded R-module $\Gamma_*(\mathcal{F})$ as

$$\Gamma_*(\mathcal{F}) := \bigoplus_{j \in \mathbb{Z}} H^0(\mathbb{P}^m, \mathcal{F}(j)) \,.$$

By [**14**, Proposition 5.15], the coherent $\mathcal{O}_{\mathbb{P}^m}$-module $\Gamma_*(\mathcal{F})^\sim$ defined by $\Gamma_*(\mathcal{F})$ has a canonical isomorphism $\Gamma_*(\mathcal{F})^\sim \xrightarrow{\sim} \mathcal{F}$. Hence the functor

$$\Gamma_* \colon \underline{(\mathrm{Coh}_{\mathbb{P}^m})} \longrightarrow \underline{(\mathrm{gr}.R\text{-mod})}$$

is exact and essentially injective. Here, we write the abelian category of coherent $\mathcal{O}_{\mathbb{P}^m}$-modules as $\underline{(\mathrm{Coh}_{\mathbb{P}^m})}$.

Remark 2.2.1. There are two points about the functors Γ_* and $\widetilde{\cdot}$ which need careful treatments. First, the functor $\widetilde{\cdot}$ is not essentially injective. Second, if $\mathcal{F}$ has an associated point of dimension zero, then $\Gamma_*(\mathcal{F})$ is not finitely generated.

In the following, we consider only the coherent $\mathcal{O}_{\mathbb{P}^m}$-module $\mathcal{F}$ without any associated points of dimension zero.

The *graded locally free resolution* of a coherent $\mathcal{O}_{\mathbb{P}^m}$-module $\mathcal{F}$ is a complex $\mathcal{V}_\bullet = \{\mathcal{V}_i, \delta_i\}$ and a morphism $\epsilon \colon \mathcal{V}_0 \longrightarrow \mathcal{F}$ which satisfies the following conditions:

- each graded piece $\mathcal{V}_i$ is a direct sum of line bundles on $\mathbb{P}^m$, and

- $\mathcal{V}_i = 0$ for any $i \leq -1$, and
- $\operatorname{Im} \delta_{i+1} = \operatorname{Ker} \delta_i$ for any $i > 0$, and
- ϵ induces the isomorphism $\operatorname{Coker}(\delta_1) \xrightarrow{\sim} \mathcal{F}$.

This is equivalent to say that the complex $\Gamma_*(\mathcal{V}_\bullet) = \{\Gamma_*(\mathcal{V}_i), \Gamma_*(\delta_i)\}$ and the graded R-homomorphism $\Gamma_*(\epsilon)$ give a graded free resolution of the graded R-module $\Gamma_*(\mathcal{F})$. From the essential injectivity of Γ_*, we can treat the graded locally free resolutions of $\mathcal{O}_{\mathbb{P}^m}$-modules as the graded free resolutions of graded R-modules.

A graded locally free resolution $\mathcal{V}_\bullet$ is said to be *minimal* if $\Gamma_*(\mathcal{V}_\bullet)$ is a minimal free resolution of $\Gamma_*(\mathcal{F})$. By Theorem 2.1.18, the minimal graded free resolution of $\mathcal{F}$ is unique up to isomorphism.

A minimal graded locally free resolution $\mathcal{V}_\bullet$ is said to be *pure* if $\Gamma_*(\mathcal{V}_\bullet)$ is a pure minimal free resolution of $\Gamma_*(\mathcal{F})$. This is equivalent to the condition that each $\mathcal{V}_i$ is isomorphic to a direct sum of finite copies of a line bundle. By Lemma 2.1.20, if $\mathcal{F}$ has a pure minimal resolution, the endomorphism of $\mathcal{F}$ is uniquely lifted to an endomorphism of the pure minimal resolution of $\mathcal{F}$.

2.2.1. Arithmetically Cohen–Macaulay $\mathcal{O}_{\mathbb{P}^m}$-modules. As the previous subsection, we treat the coherent $\mathcal{O}_{\mathbb{P}^m}$-modules without any associated points of dimension zero.

A coherent $\mathcal{O}_{\mathbb{P}^m}$-module $\mathcal{F}$ is said to be *Cohen–Macaulay* if, for any point $x \in \mathbb{P}^m$, the module $\mathcal{F}_x$ is a Cohen–Macaulay $\mathcal{O}_{\mathbb{P}^m,x}$-module. There is more restrictive condition for coherent $\mathcal{O}_{\mathbb{P}^m}$-modules.

Definition 2.2.2 (Arithmetically Cohen–Macaulay modules). The coherent $\mathcal{O}_{\mathbb{P}^m}$-module $\mathcal{F}$ is said to be *arithmetically Cohen–Macaulay* if the R-module $\Gamma_*(\mathcal{F})$ is Cohen–Macaulay.

The following proposition explains why the condition that $\mathcal{F}$ is arithmetically Cohen–Macaulay is more restrictive than $\mathcal{F}$ is Cohen–Macaulay.

Proposition 2.2.3 ([**2**, Proposition 1.2])**.** A coherent $\mathcal{O}_{\mathbb{P}^m}$-module $\mathcal{F}$ is arithmetically Cohen–Macaulay if and only if $\mathcal{F}$ satisfies the following conditions:

- $\mathcal{F}$ is a Cohen–Macaulay $\mathcal{O}_{\mathbb{P}^m}$-module, and
- $H^i(\mathbb{P}^m, \mathcal{F}(j)) = 0$ for $1 \leq i \leq \dim \mathrm{Supp}(\mathcal{F}) - 1$ and $j \in \mathbb{Z}$.

Here, we introduce an example of arithmetically Cohen–Macaulay modules. Before it, we recall a concept of purity of coherent $\mathcal{O}_{\mathbb{P}^m}$-modules.

Let $\mathcal{F}$ be a coherent $\mathcal{O}_{\mathbb{P}^m}$-module. The module $\mathcal{F}$ is said to be *pure of dimension d* if, for any submodule $\mathcal{G}$ of $\mathcal{F}$, the support $\mathrm{Supp}(\mathcal{G})$ of $\mathcal{G}$ has dimension d ([**16**, Definition 1.1.2]). We recall the following characterization of purity.

Proposition 2.2.4 ([**16**, Proposition 1.1.10])**.** For a coherent $\mathcal{O}_{\mathbb{P}^m}$-module $\mathcal{F}$ such that $\mathrm{Supp}(\mathcal{F})$ is of dimension d, the following conditions are equivalent.

- The module $\mathcal{F}$ is pure of dimension d.
- For all integers $q > m - d$, we have

$$\mathrm{codim}(\mathrm{Supp}(\mathscr{E}xt^q_{\mathbb{P}^m}(\mathcal{F}, \omega^{\mathrm{shf}}_{\mathbb{P}^m}))) \geq q + 1.$$

Here, $\omega^{\mathrm{shf}}_{\mathbb{P}^m}$ is a canonical line bundle on $\mathbb{P}^m$ which is isomorphic to $\mathcal{O}_{\mathbb{P}^m}(-1-m)$. We set $\mathrm{Supp}(0) = \emptyset$ and $\mathrm{codim}(\emptyset) = \infty$.

Proposition 2.2.5 (cf. [**2**, Theorem A])**.** Let $\mathcal{F}$ be a coherent $\mathcal{O}_{\mathbb{P}^m}$-module. Assume $m \geq 2$, $\mathcal{F}$ is arithmetically Cohen–Macaulay and pure of dimension $m-1$. Then $\mathcal{F}$ admits

a graded locally free resolution

$$0 \longrightarrow \bigoplus_{i=0}^{r} \mathcal{O}_{\mathbb{P}^m}(e_i) \xrightarrow{M} \bigoplus_{i=0}^{r} \mathcal{O}_{\mathbb{P}^m}(d_i) \xrightarrow{\epsilon} \mathcal{F} \longrightarrow 0. \tag{2.2.1}$$

Conversely, if

$$M\colon \bigoplus_{i=0}^{r} \mathcal{O}_{\mathbb{P}^m}(e_i) \longrightarrow \bigoplus_{i=0}^{r} \mathcal{O}_{\mathbb{P}^m}(d_i)$$

is an injective $\mathcal{O}_{\mathbb{P}^m}$-homomorphism, the cokernel of M is arithmetically Cohen–Macaulay and $\mathcal{F}$ is pure of dimension $m-1$. In fact, the support $\mathrm{Supp}(\mathcal{F})$ of $\mathcal{F}$ is a hypersurface defined by $(\det M = 0)$.

Proof. Suppose that $\mathcal{F}$ is an arithmetically Cohen–Macaulay pure $\mathcal{O}_{\mathbb{P}^m}$-module of dimension $m-1$. Then we see that $\Gamma_*(\mathcal{F})$ is a module of dimension m; i.e. $\dim R/\operatorname{Ann}_R(\Gamma_*(\mathcal{F}))$ is m. By Proposition 2.2.3, $\Gamma_*(\mathcal{F})$ is a Cohen–Macaulay R-module. Hence we have

$$\operatorname{depth}(\Gamma_*(\mathcal{F})) = \dim \Gamma_*(\mathcal{F}) = m.$$

By (2.1.1), we have

$$\operatorname{pd} \Gamma_*(\mathcal{F}) = 1.$$

Hence the graded R-module $\Gamma_*(\mathcal{F})$ has a graded locally free resolution of the form

$$0 \longrightarrow \bigoplus_{i=0}^{r} R(e_i) \xrightarrow{M} \bigoplus_{i=0}^{r} R(d_i) \xrightarrow{\epsilon} \Gamma_*(\mathcal{F}) \longrightarrow 0.$$

The corresponding complex of coherent $\mathcal{O}_{\mathbb{P}^m}$-modules gives a graded locally free resolution of the form (2.2.1).

Conversely, suppose that $\mathcal{F}$ has a graded locally free resolution of the form (2.2.1). For a point $x \in \mathrm{Supp}(\mathcal{F})$, the condition $\mathcal{F}_x \neq 0$ is equivalent to $\det M(x) = 0$. Hence the support of $\mathcal{F}$ is defined by $(\det M = 0)$.

Next we show that $\mathcal{F}$ is arithmetically Cohen–Macaulay. For every $x \in \mathbb{P}^m$, the projective dimension of $\mathcal{O}_{\mathbb{P}^m,x}$-module $\mathcal{F}_x$ is less than or equal to one by (2.2.1). By Auslander–Buchsbaum formula 2.1.14, for a point $x \in \mathrm{Supp}(\mathcal{F})$, we have

$$1 \geq \mathrm{pd}\, \mathcal{F}_x = \mathrm{depth}(\mathcal{O}_{\mathbb{P}^m,x}) - \mathrm{depth}(\mathcal{F}_x).$$

Since $\mathcal{O}_{\mathbb{P}^m,x}$ is regular, we have $\mathrm{depth}(\mathcal{O}_{\mathbb{P}^m,x}) = \dim(\mathcal{O}_{\mathbb{P}^m,x})$. Thus we obtain

$$\mathrm{depth}(\mathcal{F}_x) \geq \dim(\mathcal{O}_{\mathbb{P}^m,x}) - 1 = \dim(\mathcal{F}_x)$$

for any point $x \in \mathrm{Supp}(\mathcal{F})$. Here, the second equality is because $\mathrm{Supp}(\mathcal{F})$ is defined by the equation $(\det(M) = 0)$. Hence $\mathcal{F}$ is a Cohen–Macaulay $\mathcal{O}_{\mathbb{P}^m}$-module. By the long exact sequence of cohomology induced by (2.2.1), we have

$$H^i(\mathbb{P}^m, \mathcal{F}(j)) = 0$$

for $1 \leq i \leq \dim \mathrm{Supp}(\mathcal{F}) - 1$ and $j \in \mathbb{Z}$. Therefore, by Proposition 2.2.3, the coherent $\mathcal{O}_{\mathbb{P}^m}$-module $\mathcal{F}$ is arithmetically Cohen–Macaulay.

To show the purity of $\mathcal{F}$, we shall show that, for any $q > 1$,

$$\mathrm{codim}(\mathrm{Supp}(\mathscr{E}xt^q_{\mathbb{P}^m}(\mathcal{F}, \omega^{\mathrm{shf}}_{\mathbb{P}^m}))) \geq q + 1.$$

It is enough to show that

$$\mathscr{E}xt^q_{\mathbb{P}^m}\left(\mathcal{F}, \omega^{\mathrm{shf}}_{\mathbb{P}^m}\right) = 0$$

for any $q \geq 1$. By the long exact sequence of cohomology induced by (2.2.1), we have an exact sequence

$$\bigoplus_{i=0}^{d} \mathscr{E}xt^{q-1}_{\mathbb{P}^m}\left(\mathcal{O}_{\mathbb{P}^m}(e_i), \omega^{\mathrm{shf}}_{\mathbb{P}^m}\right) \longrightarrow \mathscr{E}xt^q_{\mathbb{P}^m}\left(\mathcal{F}, \omega^{\mathrm{shf}}_{\mathbb{P}^m}\right) \longrightarrow \bigoplus_{i=0}^{d} \mathscr{E}xt^q_{\mathbb{P}^m}\left(\mathcal{O}_{\mathbb{P}^m}(d_i), \omega^{\mathrm{shf}}_{\mathbb{P}^m}\right).$$

The left and right terms of this sequence vanish when $q > 1$ ([**14**, III, Proposition 6.3]). Hence we have

$$\mathscr{E}xt^q_{\mathbb{P}^m}\left(\mathcal{F}, \omega^{\mathrm{shf}}_{\mathbb{P}^m}\right) = 0.$$

Therefore, $\mathcal{F}$ is pure of dimension $m-1$. □

2.2.2. Castelnuovo–Mumford regularity. If $\mathcal{F}_\bullet = \{\bigoplus \mathcal{O}_{\mathbb{P}^m}(d_i), \delta_i\}$ is the minimal pure resolution of a coherent $\mathcal{O}_{\mathbb{P}^m}$-module, then we have

$$d_i > d_{i-1}$$

for any $i > 0$. It is natural to consider the meaning of the sequence of numbers $\{d_i - i\}$. The maximum of the sequence gives the Castelnuovo–Mumford regularity. We recall a fundamental property of the regularity in this subsection, and have an application of the concept.

Definition 2.2.6 (Castelnuovo–Mumford regularity, cf. [**7**, Section 20.5]**).** A coherent $\mathcal{O}_{\mathbb{P}^m}$-module $\mathcal{F}$ is said to be *r-regular* for an integer $m \in \mathbb{Z}$ if, for all $i > 0$, we have

$$H^i(\mathbb{P}^m, \mathcal{F}(r-i)) = 0.$$

The *Castelnuovo–Mumford regularity* (or simply *regularity*) of $\mathcal{F}$ is the maximum integer r such that $\mathcal{F}$ is r-regular.

This definition is different from the above definition in ring-theoretic terms. But they coincide if $\mathcal{F}$ has no associated points of dimension zero ([**8**, Section 4]). The next theorem is a fundamental consequence of regularity.

Theorem 2.2.7 (Castelnuovo, [**21**, Lecture 14]**).** Let $\mathcal{F}$ be an r-regular coherent $\mathcal{O}_{\mathbb{P}^m}$-module. Then

(1) the canonical multiplication map

$$H^0(\mathbb{P}^m, \mathcal{F}(j)) \otimes_k H^0(\mathbb{P}^m, \mathcal{O}_{\mathbb{P}^m}(1)) \longrightarrow H^0(\mathbb{P}^m, \mathcal{F}(j+1))$$

is surjective when $j \geq r$.

(2) $H^i(\mathbb{P}^m, \mathcal{F}(j)) = 0$ whenever $i > 0$ and $j > r - i$.

(3) If $j \geq r$, then $\mathcal{F}(j)$ is generated by its global sections.

As a simple application of this Castelnuovo's result, we can refine Proposition 2.2.5 as follows.

Proposition 2.2.8 ([**2**, Proposition 1.11]**).** Take an integer $m \geq 2$. Let $\mathcal{F}$ be a coherent $\mathcal{O}_{\mathbb{P}^m}$-module. Then the following conditions are equivalent:

- there exists the pure minimal resolution of the following form:

$$0 \longrightarrow \bigoplus_{i=0}^{r} \mathcal{O}_{\mathbb{P}^m}(-2) \xrightarrow{M} \bigoplus_{i=0}^{r} \mathcal{O}_{\mathbb{P}^m}(-1) \longrightarrow \mathcal{F} \longrightarrow 0. \tag{2.2.2}$$

- The module $\mathcal{F}$ is arithmetically Cohen–Macaulay, pure of dimension $m-1$, and

$$H^0(\mathbb{P}^m, \mathcal{F}) = H^{m-1}(\mathbb{P}^m, \mathcal{F}(2-m)) = 0.$$

Proof. If $\mathcal{F}$ admits an exact sequence of the form (2.2.2), the $\mathcal{O}_{\mathbb{P}^m}$-module $\mathcal{F}$ is arithmetically Cohen–Macaulay and pure of dimension $m-1$ by Proposition 2.2.5. By the long exact sequence of cohomology, we have

$$H^0(\mathbb{P}^m, \mathcal{F}) = H^{m-1}(\mathbb{P}^m, \mathcal{F}(2-m)) = 0.$$

Conversely, suppose that $\mathcal{F}$ is arithmetically Cohen–Macaulay, pure of dimension $m-1$ and

$$H^0(\mathbb{P}^m, \mathcal{F}) = H^{m-1}(\mathbb{P}^m, \mathcal{F}(2-m)) = 0.$$

Since $\mathcal{F}$ is arithmetically Cohen–Macaulay, we have

$$H^i(\mathbb{P}^m, \mathcal{F}(1-i)) = 0$$

for $1 \leq i \leq \dim \operatorname{Supp}(\mathcal{F}) - 1 = m-2$. Since $\mathcal{F}$ is pure of dimension $m-1$, we have

$$H^i(\mathbb{P}^m, \mathcal{F}(1-i)) = 0$$

for $i \geq m$. By assumption, we have

$$H^{m-1}(\mathbb{P}^m, \mathcal{F}(2-m)) = 0$$

Hence we have

$$H^i(\mathbb{P}^m, \mathcal{F}(1-i)) = 0$$

for any $i > 0$. In other words, $\mathcal{F}$ is 1-regular. Hence, by Theorem 2.2.7, $\mathcal{F}(1)$ is generated by its global sections and the multiplication map

$$H^0(\mathbb{P}^m, \mathcal{F}(j)) \otimes_k H^0(\mathbb{P}^m, \mathcal{O}_{\mathbb{P}^m}(1)) \longrightarrow H^0(\mathbb{P}^m, \mathcal{F}(j+1))$$

is surjective when $j \geq 1$. Since $H^0(\mathbb{P}^m, \mathcal{F}) = 0$, the R-module $\Gamma_*(\mathcal{F})$ is generated by the degree one part $H^0(\mathbb{P}^m, \mathcal{F}(1))$. This shows that the minimal resolution of $\mathcal{F}$ takes the form

$$0 \longrightarrow \bigoplus_{i=0}^{r} \mathcal{O}_{\mathbb{P}^m}(e_i) \xrightarrow{M} \bigoplus_{i=0}^{r} \mathcal{O}_{\mathbb{P}^m}(-1) \longrightarrow \mathcal{F} \longrightarrow 0$$

for some integers $e_i \leq -2$ for any i. By the long exact sequence of cohomology and the assumption $H^{m-1}(\mathbb{P}^m, \mathcal{F}(2-m)) = 0$, we have

$$H^m(\mathbb{P}^m, \mathcal{F}(e_i + 2 - m)) = 0$$

for any i. Hence we obtain $e_i + 2 - m \geq -m$ and $e_i \geq -2$. Thus we conclude that $e_i = -2$ for any i, and $\mathcal{F}$ has a graded locally free resolution of the desired form. $\square$

CHAPTER 3

The Grothendieck duality and a bijection between coherent modules and symmetric matrices

Grothendieck duality for coherent sheaves is a main tool to investigate coherent modules on schemes. We would like to establish some bijections between symmetric matrices and a class of coherent $\mathcal{O}_{\mathbb{P}^m}$-modules with duality data by using Grothendieck duality.

In Section 3.1, we recall basic notion from homological algebras such as injective resolutions, derived functors and derived categories. Then we describe the definitions and properties of dualizing complexes, exceptional inverse image functors and Grothendieck duality. Then, by using Grothendieck duality, we establish the fundamental bijection between $(m+1)$-tuples of symmetric matrices and certain coherent $\mathcal{O}_{\mathbb{P}^m}$-modules with some duality data in Section 3.2. After it, we study the action of $\mathrm{GL}_{m+1}(k) \times \mathrm{GL}_{n+1}(k)$ on each side of the bijection, and obtain the bijection between orbits in Section 3.3. In Section 3.4, we describe the algebraic structure of the set of duality data on a coherent $\mathcal{O}_{\mathbb{P}^m}$-module.

3.1. A brief review of Grothendieck duality

In this section, we recall some notion of derived category and Grothendieck duality.

3.1.1. Homology and Cohomology. In this subsection, we work over abelian categories $\mathscr{C} = \underline{(R\text{-mod})}, \underline{(\mathrm{gr}.R\text{-mod})}$.

For a chain complex $\mathcal{F}_\bullet = \{\mathcal{F}_i, \delta_i\}$, we define its *i-th homology group* $H_i(\mathcal{F}_\bullet)$ as

$$H_i(\mathcal{F}_\bullet) := \operatorname{Ker} \delta_i / \operatorname{Im} \delta_{i+1}.$$

Similarly, for a cochain complex $\mathcal{F}^\bullet = \{\mathcal{F}^i, \delta^i\}$, we define its *i-th cohomology group* $H^i(\mathcal{F}^\bullet)$ as

$$H^i(\mathcal{F}^\bullet) := \operatorname{Ker} \delta^i / \operatorname{Im} \delta^{i-1}.$$

A morphism of chain complexes

$$f = f_\bullet \colon \mathcal{F}_\bullet \longrightarrow \mathcal{G}_\bullet$$

induces morphisms of homology groups

$$H_i(f) : H_i(\mathcal{F}_\bullet) \longrightarrow H_i(\mathcal{G}_\bullet)\,,$$

and similar for morphisms of cochain complexes. If a morphism of (co) chain complexes f induces isomorphisms between (co)homology groups, we say that f is a *quasi-isomorphism*.

Example 3.1.1. Let $F_\bullet, G_\bullet$ be chain complexes of R-modules, and $F[n]_\bullet$ the degree n shift of $F_\bullet$ (see Subsection 2.1.2). Two morphisms

$$u, v \colon F_\bullet \longrightarrow G_\bullet$$

are said to be *homotopic* if there exists a morphism of complexes

$$h = h_\bullet \colon F_\bullet \longrightarrow G[1]_\bullet$$

such that

$$u_i - v_i = h_{i-1} \circ \delta_i + \delta_{i+1} \circ h_i.$$

If u, v are two homotopic morphisms, then the morphisms induced on homology modules by u, v are the same morphisms. In particular, a homomorphism homotopic to a quasi-isomorphism is also a quasi-isomorphism. Note that the relation that u, v are homotopic is an equivalence relation between morphisms of complexes.

Next, we would like to recall the dual notion of projective modules and projective resolutions. An object $I \in \mathscr{C}$ is said to be *injective* if the contravariant functor $\mathrm{Hom}_{\mathscr{C}}(*, I)$ is exact. The abelian category $\mathscr{C}$ said to have *enough injectives* if, for any object $M \in \mathscr{C}$, there is an injective object $I \in \mathscr{C}$ and an injection $\iota\colon M \hookrightarrow I$.

Let M be an object in $\mathscr{C}$. An *injective resolution* of M is a cochain complex $I_\bullet = \{I^i, \delta^i\}$ and an injection $\iota\colon M \longrightarrow I_0$ which satisfies the following conditions

- all R-modules I^i are injective, and
- $I^i = 0$ for any $i < 0$, and
- $\mathrm{Im}\,\delta^{i+1} = \mathrm{Ker}\,\delta^i$ for any $i > 0$, and
- ι induces an isomorphism $M \xrightarrow{\sim} \mathrm{Ker}\,\delta^0$.

If an abelian category $\mathcal{C}$ has enough injectives, any object in $\mathcal{C}$ has an injective resolution.

Proposition 3.1.2 ([**7**, Proposition-Definition A3.10, Exercise A3.5])**.** The category ($\underline{R\text{-mod}}$) and ($\underline{\mathrm{gr}.R\text{-mod}}$) has enough injectives. In particular, any R-module has an injective resolution, and any graded R-module has an injective resolution by graded injective R-modules.

Example 3.1.3. Take $(X, \mathcal{O}_X)$ a ringed space. The category of $\mathcal{O}_X$-modules $\underline{(\mathcal{O}_X\text{-mod})}$ does not have enough projectives in general ([**14**, III, Exercise 6.2]), but it has enough injectives ([**14**, III, Proposition 2.2]).

3.1.2. Derived functor. Next, we recall the definition of derived functor.

Let F be a functor from the category of $\underline{(R\text{-mod})}$ to an abelian category $\mathscr{C}$. We say the functor F is *left exact* (resp. *right exact*) if, for any short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0,$$

the sequence

$$0 \longrightarrow F(A) \longrightarrow F(B) \longrightarrow F(C)$$

is exact (resp. the sequence

$$F(A) \longrightarrow F(B) \longrightarrow F(C) \longrightarrow 0$$

is exact).

The *i-th right-derived functor* R^iF of F for $i \geq 0$ is defined for a left-exact functor F from $\underline{(R\text{-mod})}$ (or $\underline{(\text{gr. } R\text{-mod})}$) to any abelian category as follow. Let M be an R-module. Take an injective resolution $I^\bullet$ of M, and we define

$$R^iF(M) := H^i(F(I^\bullet))\,.$$

It does not depend on the choice of the injective resolution $I^\bullet$ of M.

Similarly, the *i-th left-derived functor* L_iF of F for $i \geq 0$ is defined for a right-exact functor F from $\underline{(R\text{-mod})}$ (or $\underline{(\text{gr. } R\text{-mod})}$) to any abelian category as follows: let M be an R-module. Take a projective resolution $P_\bullet$ of M, and we define

$$L_iF(M) := H_i(F(P_\bullet))\,.$$

It does not depend on the choice of the projective resolution $P_\bullet$ of M.

We have $L_0F = R^0F = F$. Derived functors L_iF, R^iF of an exact functor F is zero for $i \neq 0$.

Example 3.1.4. Take an R-module M, and consider the endo-functor

$$* \otimes_R M \colon \underline{(R\text{-mod})} \longrightarrow \underline{(R\text{-mod})}.$$

This is a right exact functor, so we have the left-derived functors called *Tor functor*:

$$\mathrm{Tor}_i^R(*, M) \colon \underline{(R\text{-mod})} \longrightarrow \underline{(R\text{-mod})}.$$

We can show

$$\mathrm{Tor}_R^i(M, *) \cong \mathrm{Tor}_R^i(*, M).$$

Example 3.1.5. Similarly for an R-module M, we consider the endo-functor

$$\mathrm{Hom}_R(M, *) : \underline{(R\text{-mod})} \longrightarrow \underline{(R\text{-mod})}.$$

This is a left exact covariant functor, so we have the right-derived functor called *Ext functor*:

$$\mathrm{Ext}_R^i(M, *) : \underline{(R\text{-mod})} \longrightarrow \underline{(R\text{-mod})}.$$

Simultaneously, the endo-functor

$$\mathrm{Hom}_R(*, N) : \underline{(R\text{-mod})}^{\mathrm{op}} \longrightarrow \underline{(R\text{-mod})}.$$

is a right exact contravariant functor for any R-module N, so we have the left-derived functor $L_i \operatorname{Hom}_R(*, N)$. Actually, we have

$$\operatorname{Ext}^i_R(M, N) \cong L_i \operatorname{Hom}_R(*, N)\,(M),$$

so we can compute the Ext group $\operatorname{Ext}^i_R(M, N)$ both from the projective resolution of M and from the injective resolution of N.

As another example, we consider some functors from the category $\underline{(\mathcal{O}_X\text{-mod})}$ of $\mathcal{O}_X$-modules for a scheme X. The category $\underline{(\mathcal{O}_X\text{-mod})}$ does not have enough projectives generally, but has enough injectives. Hence we can define the right-derived functors of left exact functors.

Example 3.1.6. The global section functor

$$\Gamma(X, -)\colon \underline{(\mathcal{O}_X\text{-mod})} \longrightarrow \underline{(\mathrm{Ab})}$$

gives a left exact functor to the category of abelian groups. Its derived functors

$$H^i(X, -) : \underline{(\mathcal{O}_X\text{-mod})} \longrightarrow \underline{(\mathrm{Ab})}$$

are called the *cohomology functors*.

Example 3.1.7. For a quasi-compact and quasi-separated morphism of scheme

$$f\colon X \longrightarrow Y,$$

we have the *inverse image functor* of f

$$f^*\colon \underline{(\mathcal{O}_Y\text{-mod})} \longrightarrow \underline{(\mathcal{O}_X\text{-mod})}$$

and the *direct image functor* of f

$$f_*\colon \underline{(\mathcal{O}_X\text{-mod})} \longrightarrow \underline{(\mathcal{O}_Y\text{-mod})}.$$

The inverse image functor is exact, and the direct image functor is left exact. The inverse image functor f^* carries quasi-coherent sheaves on Y to quasi-coherent sheaves on X. Moreover, if X, Y are Noetherian, f^* sends coherent sheaves on Y to coherent sheaves on X ([**14**, II, Proposition 5.8]).

We define the *higher direct image* of f

$$R^i f_*\colon \underline{(\mathcal{O}_X\text{-mod})} \longrightarrow \underline{(\mathcal{O}_Y\text{-mod})}.$$

These functors carry quasi-coherent sheaves on X to quasi-coherent sheaves on Y ([**14**, III, Corollary 8.6]). If X, Y are Noetherian schemes and f is proper, the images $R^i f_*(\mathcal{F})$ of a coherent sheaf $\mathcal{F}$ on X are again coherent sheaves on Y for any $i \geq 0$ (cf. [**14**, III, Theorem 8.8]).

Example 3.1.8. Let X be a scheme and $\mathcal{F}$ a quasi-coherent sheaf on X. The endofunctor

$$\mathscr{H}om_X(\mathcal{F}, *) : \underline{(\mathcal{O}_X\text{-mod})} \longrightarrow \underline{(\mathcal{O}_X\text{-mod})}$$

is left exact, hence it gives the right derived functors

$$\mathscr{E}xt^i_X(\mathcal{F}, *) : \underline{(\mathcal{O}_X\text{-mod})} \longrightarrow \underline{(\mathcal{O}_X\text{-mod})}$$

for any $i \geq 0$. If X is locally Noetherian and $\mathcal{F}$ is coherent, the functor sends quasi-coherent sheaves to quasi-coherent sheaves and coherent sheaves to coherent sheaves ([**EGA III**, 0, 12.3.3]).

3.1.3. Derived category. We give a brief review of derived categories in this section.

We write $Ch(\underline{(R\text{-mod})})$ as the category of cochain complexes in $\underline{(R\text{-mod})}$. Its objects are cochain complexes of R-modules and its morphisms are morphisms of complexes. It is an abelian category. Similarly, $Ch^+(\underline{(R\text{-mod})}), Ch^-(\underline{(R\text{-mod})}), Ch^b(\underline{(R\text{-mod})})$ denote the full subcategories of bounded above complexes, bounded below complexes and bounded complexes respectively. We use other two full subcategories $Ch^+(\mathrm{Inj}(\underline{(R\text{-mod})}))$ and $Ch^-(\mathrm{Proj}(\underline{(R\text{-mod})}))$ which denote the full subcategory of bounded below complexes of injective R-modules and bounded above complexes of projective R-modules respectively.

Next, we define the *homotopy category* $K(\underline{(R\text{-mod})})$ of complexes of R-modules. It has the same objects as the category of complexes $Ch(\underline{(R\text{-mod})})$ of R-modules, but the morphisms are homotopy classes of morphisms of complexes (for the definition of homotopic morphisms, see Example 3.1.1.) Then we find the homotopy category is not abelian in general, but additive. It has a structure of *triangulated category* with the degree shift functor

$$T\colon K(\underline{(R\text{-mod})}) \longrightarrow K(\underline{(R\text{-mod})}).$$

We do not state the definition of triangulated categories here. We can consider the full subcategories $K^+(\underline{(R\text{-mod})}), K^-(\underline{(R\text{-mod})}), K^b(\underline{(R\text{-mod})}), K^+(\mathrm{Inj}(\underline{(R\text{-mod})}))$ and $K^-(\mathrm{Proj}(\underline{(R\text{-mod})}))$.

The *derived category* $D(\underline{(R\text{-mod})})$ of complexes of R-modules is the localization of homotopy category $K(\underline{(R\text{-mod})})$ by the multiplicative system of quasi-isomorphisms. It is intuitively the category obtained by adding to $K(\underline{(R\text{-mod})})$ the formal inverse of quasi-isomorphisms. The derived category $D(\underline{(R\text{-mod})})$ has a natural functor of triangulated categories

$$Q\colon K(\underline{(R\text{-mod})}) \longrightarrow D(\underline{(R\text{-mod})}).$$

We can consider the full subcategories $D^+(\underline{(R\text{-mod})}), D^-(\underline{(R\text{-mod})}), D^b(\underline{(R\text{-mod})}), D^+(\mathrm{Inj}(\underline{(R\text{-mod})}))$ and $D^-(\mathrm{Proj}(\underline{(R\text{-mod})}))$.

For an abelian category $\mathcal{C}$, we can consider the homotopy categories $K(\mathcal{C})$, $K^+(\mathcal{C})$, $K^-(\mathcal{C})$, $K^b(\mathcal{C})$ of complexes in $\mathcal{C}$ and the derived category $D(\mathcal{C})$, $D^+(\mathcal{C})$, $D^-(\mathcal{C})$, $D^b(\mathcal{C})$ of complexes in $\mathcal{C}$. The homotopy categories of bounded below complexes of injectives are written as $K^+(\mathrm{Inj}(\mathcal{C}))$ and the homotopy categories of bounded above complexes of projectives are written as $K^-(\mathrm{Proj}(\mathcal{C}))$. We do not treat the construction here, but state a property.

Theorem 3.1.9 ([13, I, Proposition 4.7]**).** If $\mathcal{C}$ has enough injectives, the composite functor of triangulated category

$$K^+(\mathrm{Inj}(\mathcal{C})) \hookrightarrow K^+(\mathcal{C}) \xrightarrow{Q} D^+(\mathcal{C})$$

gives an equivalence of triangulated categories. Similarly, if $\mathcal{C}$ has enough projectives, the composite functor of triangulated category

$$K^-(\mathrm{Proj}(\mathcal{C})) \hookrightarrow K^-(\mathcal{C}) \xrightarrow{Q} D^-(\mathcal{C})$$

gives an equivalence of triangulated categories.

Next we describe the derived functor in terms of derived category.

Definition 3.1.10. Let us take abelian categories $\mathscr{A}, \mathscr{B}$. Let

$$F\colon K(\mathscr{A}) \longrightarrow K(\mathscr{B})$$

be a morphism of triangulated categories. Let $Q_{\mathscr{A}}, Q_{\mathscr{B}}$ be two canonical functor from homotopy categories to its derived categories. Then the *right-derived functor* of F is a pair of a morphism of triangulated categories

$$\mathrm{R}F\colon D(\mathscr{A}) \longrightarrow D(\mathscr{B})$$

and a natural transformation

$$\xi\colon Q_{\mathscr{A}} \circ F \Rightarrow \mathrm{R}F \circ Q_{\mathscr{B}}$$

satisfying the following property: if there exists another morphism

$$G\colon D(\mathscr{A}) \longrightarrow D(\mathscr{B})$$

of triangulated categories equipped with a natural transformation $\zeta\colon Q_{\mathscr{B}} \circ F \Rightarrow G \circ Q_{\mathscr{A}}$, there is a unique natural transformation $\eta\colon \mathrm{R}F \Rightarrow G$ so that

$$\eta_{Q_{\mathscr{B}}M} \circ \xi_M = \zeta_M.$$

Similarly, the *left derived functor* of F is a pair of a morphism

$$\mathrm{L}F\colon D(\mathscr{A}) \longrightarrow D(\mathscr{B})$$

of triangulated categories and a natural transformation

$$\xi\colon \mathrm{L}F \circ Q_{\mathscr{A}} \Rightarrow Q_{\mathscr{B}} \circ F$$

satisfying the following properties: if there exists another morphism

$$G\colon D(\mathscr{A}) \longrightarrow D(\mathscr{B})$$

of triangulated categories equipped with a natural transformation $\zeta\colon G \circ Q_{\mathscr{A}} \Rightarrow Q_{\mathscr{B}} \circ F$, there is a unique natural transformation $\eta\colon G \Rightarrow \mathrm{L}F$ so that

$$\xi_M \circ \eta_{Q_{\mathscr{A}} M} = \zeta_M.$$

Theorem 3.1.11 (Existence of derived functors, [**13**, I, Theorem 5.1]**).** Let $\mathscr{A}, \mathscr{B}$ be two abelian categories. Let

$$F\colon K^+(\mathscr{A}) \longrightarrow K(\mathscr{B})$$

be a morphism of triangulated categories. If $\mathscr{A}$ has enough injectives, then the right derived functor (R^+F, ξ) of F exists. Moreover, for a bounded below complex $I^\bullet$ of injectives, ξ gives an isomorphism

$$\xi_{I^\bullet}\colon Q_{\mathscr{B}} \circ F(I^\bullet) \xrightarrow{\sim} \mathrm{R}^+F \circ Q_{\mathscr{A}}(I^\bullet).$$

Dually, let

$$G\colon K^-(\mathscr{A}) \longrightarrow K(\mathscr{B})$$

be a morphism of triangulated categories. If $\mathscr{A}$ has enough projectives, then the left derived functor (L^-F, ξ) of F exists. Moreover, for a bounded above complex $P^\bullet$ of projectives, ξ

gives an isomorphism

$$\xi_{P^\bullet}\colon \mathrm{L}^- G \circ Q_{\mathscr{A}}(P^\bullet) \xrightarrow{\sim} Q_{\mathscr{B}} \circ G(P^\bullet).$$

This theorem says that the derived functor in the sense of Definition 3.1.10 coincides with the derived functor defined in Subsection 3.1.2.

Example 3.1.12. Take two cochain complexes $F = F^\bullet = \{F^i, \delta^i\}$ and $G = G^\bullet = \{G^i, \delta'^i\}$. The *total tensor cochain complex*

$$\mathrm{Tot}^\oplus(F \otimes_R G) = \{\mathrm{Tot}^\oplus(F \otimes_R G)^i, \delta''^i\}$$

of $F^\bullet, G^\bullet$ is defined as

$$\mathrm{Tot}^\oplus(F \otimes_R G)^i = \bigoplus_{j\in\mathbb{Z}} F^j \otimes G^{i-j},$$

$$\delta''^i|_{F^j\otimes G^{i-j}} = \delta^j \otimes 1 + (-1)^{i-j} 1 \otimes \delta'^{i-j}.$$

By the functor $\mathrm{Tot}^\oplus(F \otimes_R *)$, we obtain the *total tensor product*

$$F \otimes^L * = \mathrm{L}^-(\mathrm{Tot}^\oplus(F \otimes_R *))\colon D^-(\underline{(R\text{-mod})}) \longrightarrow D(\underline{(R\text{-mod})}).$$

Example 3.1.13. Take two cochain complexes $F = F^\bullet = \{F^i, \delta^i\}$ and $G = G^\bullet = \{G^i, \delta'^i\}$. The *total Hom cochain complex*

$$\mathrm{Hom}_R^\bullet(F, G) = \{\mathrm{Hom}_R^i(F, G), \delta''^i\}$$

is defined as

$$\mathrm{Hom}_R^i(F,G) = \prod_{j\in\mathbb{Z}} \mathrm{Hom}_R\big(F^j, G^{i-j}\big)\,,$$

$$(\delta''^i f)(p) = f(\delta^i p) + (-1)^{i+1}\delta'^j(f(p))$$

where $f \in \mathrm{Hom}_R(F^i, G^j)$ and $p \in F^i$. By the functor $\mathrm{Hom}_R^\bullet(*, G)$, we obtain

$$\mathrm{RHom}_R(*, G) = \mathrm{R}^+(\mathrm{Hom}_R^\bullet(*, G))\colon D^+(\underline{(R\text{-mod})}) \longrightarrow D(\underline{(R\text{-mod})}).$$

If we would like to consider the derived category of a scheme X, we sometimes need to consider that the subcategory of complexes of $\mathcal{O}_X$-modules whose cohomology modules are (quasi-)coherent $\mathcal{O}_X$-modules. From now, we briefly recall basic properties of the subcategory.

An abelian subcategory $\mathscr{A}'$ of an abelian category $\mathscr{A}$ is said to be *thick* if any extension of two objects in $\mathscr{A}'$ is an object in $\mathscr{A}'$. We define $K_{\mathscr{A}'}(\mathscr{A})$ the full subcategory of $K(\mathscr{A})$ consisting of complexes whose cohomology objects are objects in $\mathscr{A}'$. The category $D_{\mathscr{A}'}(\mathscr{A})$ is defined as the localization of $K_{\mathscr{A}'}(\mathscr{A})$ by quasi-isomorphisms. We define $D^+_{\mathscr{A}'}(\mathscr{A}), D^-_{\mathscr{A}'}(\mathscr{A}), D^b_{\mathscr{A}'}(\mathscr{A})$ similarly.

Example 3.1.14. Let X be a scheme, and consider the abelian category $(\underline{\mathcal{O}_X\text{-mod}})$ of $\mathcal{O}_X$-modules. We use $D(X)$ to denote the derived category of complexes of $\mathcal{O}_X$-modules.

The abelian subcategory (QCoh_X) of quasi-coherent $\mathcal{O}_X$-modules is a thick subcategory. Also the abelian subcategory (Coh_X) of coherent $\mathcal{O}_X$-modules is a thick subcategory. We denote $D_{qc}(X)$ (resp. $D_c(X)$) as $D_{(\mathrm{QCoh}_X)}(\underline{\mathcal{O}_X\text{-mod}})$ (resp. $D_{(\mathrm{Coh}_X)}(\underline{\mathcal{O}_X\text{-mod}})$).

There is a natural morphism of triangulated categories

$$D(\mathscr{A}') \longrightarrow D_{\mathscr{A}'}(\mathscr{A}).$$

The following proposition gives a sufficient condition that this morphism gives an equivalence.

Proposition 3.1.15 ([13, I, Proposition 4.8]**).** Let $\mathscr{A}$ be an abelian category, and $\mathscr{A}'$ a thick abelian subcategory. If any object in $\mathscr{A}'$ is injected to an injective object in $\mathscr{A}$, the morphism

$$D^+(\mathscr{A}') \longrightarrow D^+_{\mathscr{A}'}(\mathscr{A})$$

of triangulated category gives an equivalence of categories. In particular, if $\mathscr{A}$ has enough injectives, the above morphism is always an equivalence of categories.

In our case, we have two equivalences of categories

$$D^+(\mathrm{QCoh}_X) \xrightarrow{\sim} D^+_{qc}(X)$$

and

$$D^+(\mathrm{Coh}_X) \xrightarrow{\sim} D^+_c(X).$$

Hence we can take a complex representing the objects in $D^+_c(X)$ as a complex of coherent $\mathcal{O}_X$-modules. We finish this subsection with two examples of derived functors which are main tools in this section.

Example 3.1.16 ([13, II, Proposition 2.2]**).** Let $f\colon X \longrightarrow Y$ be a morphism between schemes X and Y. It induces $\mathrm{R}^+f_*\colon D^+(X) \longrightarrow D^+(Y)$. If Y is locally Noetherian and f is

a proper morphism, then we have

$$\mathrm{R}^{+}f_{*}\colon D^{+}(\mathrm{Coh}_{X}) \longrightarrow D^{+}(\mathrm{Coh}_{Y}).$$

Example 3.1.17 ([13, II, Proposition 3.3]**).** Let X be a locally Noetherian scheme. Let $\mathcal{G}^{\bullet}$ be a complex of coherent $\mathcal{O}_X$-modules. Assume that $\mathcal{G}$ is bounded below, and has finite injective dimension. Then we have a right derived functor of $\mathscr{H}om_X(*, \mathcal{G})$

$$\mathrm{R}^{+}\mathscr{H}om_X(*, \mathcal{G}) = \mathrm{R}^{+}(\mathrm{Hom}^{\bullet}_{R}(*, \mathcal{G}))\colon D^{-}_{c}(X) \longrightarrow D_{c}(X).$$

3.1.4. Dualizing complex. In this section, we give a formal definition of the dualizing complexes. Then we state some properties of the complex, the uniqueness and the existence.

Definition 3.1.18 (Dualizing complex, [13, Section V.2]**).** Let X be a locally Noetherian scheme, $D^{+}(\mathrm{QCoh}_X)$ the derived category of complexes bounded below of quasi-coherent $\mathcal{O}_X$-modules and $D^{+}(\mathrm{Coh}_X)$ the derived category of complexes bounded below of coherent $\mathcal{O}_X$-modules. A complex $\omega_X \in D^{+}(\mathrm{Coh}_X)$ is a *dualizing complex* for X if it satisfies the following conditions:

- ω_X has finite injective dimension, and
- for any complex $\mathcal{F}_{\bullet} \in D_c(X)$ the canonical morphism

$$\mathrm{can}_{\mathcal{F}_{\bullet},\omega_X}\colon \mathcal{F}_{\bullet} \xrightarrow{\sim} \mathrm{R}\mathscr{H}om_X(\mathrm{R}\mathscr{H}om_X(\mathcal{F}_{\bullet}, \omega_X), \omega_X)$$

  is a quasi-isomorphism which is functorial with respect to $\mathcal{F}$.

Theorem 3.1.19 (Uniqueness of dualizing complex, [13, V. Theorem 3.1]**).** Let X be a connected locally Noetherian scheme, and ω_X, ω'_X two dualizing complexes. Then

there is an invertible sheaf $\mathcal{L}$ on X and an integer $n \in \mathbb{Z}$, such that

$$\omega'_X \cong \omega_X \otimes \mathcal{L}[n].$$

The line bundle $\mathcal{L}$ on X and the integer n are uniquely determined by

$$\mathcal{L}[n] \cong \mathrm{R}\mathscr{H}om_X(\omega_X, \omega'_X)\,.$$

Theorem 3.1.20 ([**13**, V, Proposition 9.3])**.** When X is a Gorenstein scheme and of finite Krull dimension, then $\mathcal{O}_X$ is a dualizing complex.

However, dualizing complexes do not exist in general; for example, a ring spectrum $\operatorname{Spec} A$ for a non-catenary ring A does not have such complex([**13**, V, Section 10]). There are much of studies of the existence of dualizing complexes, but we only quote the following facts which are enough to our purpose.

Theorem 3.1.21 (Existence of dualizing complexes, [**13**, V, Section 10])**.** Let X be a locally Noetherian scheme, and assume one of the following conditions holds true:

- X is Gorenstein and has finite Krull dimension, or
- there exists a morphism $f\colon X \longrightarrow Y$ of finite type, where Y is a Noetherian scheme and has a dualizing complex.

Then X has a dualizing complex.

Since $\operatorname{Spec} k$ has a dualizing complex represented by a sheaf $\mathcal{O}_{\operatorname{Spec} k}$, we have the following corollary.

Corollary 3.1.22. Any scheme of finite type over k has a dualizing complex.

It is a problem to choose the dualizing complexes functorially. In the next section, we choose ones functorially defined by a functor called the exceptional inverse image.

3.1.5. Grothendieck duality for coherent modules. We would like to obtain a functor between derived categories of complexes bounded below on Noetherian schemes with some kind of duality. We give two basic examples.

Example 3.1.23 ([13, III, Section 2]**).** If $f\colon X \longrightarrow Y$ is a smooth morphism of relative dimension d, then the relative canonical bundle

$$\omega_{X/Y}^{\mathrm{shf}} = \bigwedge^{d} \Omega_{X/Y}$$

exists. We define

$$f^{\sharp}(\mathcal{F}) := f^{*}\mathcal{F} \otimes \omega_{X/Y}^{\mathrm{shf}}[-d] \in D^{+}(\mathrm{Coh}_X)$$

for $\mathcal{F} \in D^{+}(\mathrm{Coh}_Y)$. For two smooth morphisms $f\colon X \to Y$ and $g\colon Y \to Z$ between Noetherian schemes, we have $(gf)^{\sharp} \xrightarrow{\sim} f^{\sharp}g^{\sharp}$ functorially.

There exist the *trace isomorphism*

$$\mathrm{Tr}_f\colon\ \mathrm{R}f_{*}f^{\sharp}\mathcal{G} \xrightarrow{\sim} \mathcal{G}$$

and the *duality isomorphism*

$$\mathrm{GD}_f\colon\ \mathrm{R}f_{*}\,\mathrm{R}\mathscr{H}om_X\big(\mathcal{F}, f^{\sharp}\mathcal{G}\big) \xrightarrow{\sim} \mathrm{R}\mathscr{H}om_X(\mathrm{R}f_{*}\mathcal{F}, \mathcal{G})$$

for any $\mathcal{F} \in D^{-}(\mathrm{Coh}_X)$ and $\mathcal{G} \in D^{+}(\mathrm{Coh}_Y)$. These isomorphisms are functorial with respect to $\mathcal{F}, \mathcal{G}$.

Example 3.1.24 ([13, III, Section 6]**).** If $f\colon X \longrightarrow Y$ is a finite morphism of locally Noetherian schemes and $\mathcal{F} \in D^+(\mathrm{Coh}_Y)$, we define

$$f^\flat(\mathcal{F}) := f^* \operatorname{R}\mathscr{H}om_Y(f_*\mathcal{O}_X, \mathcal{F}) \in D^+(\mathrm{Coh}_X).$$

For two finite morphisms $f\colon X \to Y$ and $g\colon Y \to Z$ between Noetherian schemes, we have $(gf)^\flat \xrightarrow{\sim} f^\flat g^\flat$ functorially.

There exist the *trace isomorphism*

$$\mathrm{Tr}_f\colon \operatorname{R}f_* f^\flat \mathcal{G} \xrightarrow{\sim} \mathcal{G}$$

and the *duality isomorphism*

$$\mathrm{GD}_f\colon \operatorname{R}f_* \operatorname{R}\mathscr{H}om_X\big(\mathcal{F}, f^\flat \mathcal{G}\big) \xrightarrow{\sim} \operatorname{R}\mathscr{H}om_X(\operatorname{R}f_*\mathcal{F}, \mathcal{G})$$

for any $\mathcal{F} \in D^-(\mathrm{Coh}_X)$ and $\mathcal{G} \in D^+(\mathrm{Coh}_Y)$. These isomorphisms are functorial with respect to $\mathcal{F}, \mathcal{G}$.

Considering these example as the simple case, we define the functor $f^!$.

Theorem 3.1.25 (Exceptional inverse image,[13, VII. Corollary 3.4]**).** Let X, Y be Noetherian schemes with dualizing complexes, and $f\colon X \longrightarrow Y$ a proper morphism of finite type. Then there exists a functor

$$f^!\colon D^+(\mathrm{Coh}_Y) \longrightarrow D^+(\mathrm{Coh}_X).$$

with the *trace isomorphism*

$$\mathrm{Tr}_f\colon \operatorname{R}f_* f^! \mathcal{G} \xrightarrow{\sim} \mathcal{G}$$

and the *duality isomorphism*

$$\mathrm{GD}_f\colon\ \mathrm{R}f_*\,\mathrm{R}\mathscr{H}om_X\big(\mathcal{F}, f^!\mathcal{G}\big) \xrightarrow{\sim} \mathrm{R}\mathscr{H}om_X(\mathrm{R}f_*\mathcal{F}, \mathcal{G})$$

for any $\mathcal{F} \in D^-(\mathrm{Coh}_X)$ and $\mathcal{G} \in D^+(\mathrm{Coh}_Y)$ with the following properties.

- The trace isomorphism and the duality isomorphism are functorial with respect to $\mathcal{F}, \mathcal{G}$.
- When f is a finite morphism, then we have $f^! \xrightarrow{\sim} f^\flat$ functorially.
- When f is a smooth morphism, then we have $f^! \xrightarrow{\sim} f^\sharp$ functorially.
- For two morphisms $f\colon X \to Y$ and $g\colon Y \to Z$ of finite type between Noetherian schemes with dualizing complexes, we have $(gf)^! \xrightarrow{\sim} f^!g^!$ functorially.

Moreover, with some conditions, $f^!$ is uniquely determined.

In the following, we only treat a dualizing complex ω_X for a proper scheme X of finite type over k. We fix the quasi-isomorphism classes of the dualizing complex of X as

$$\omega_X := f^!(\mathcal{O}_{\operatorname{Spec} k})$$

where $f\colon X \longrightarrow \operatorname{Spec} k$ is the structure morphism. This gives a functorial choice of the dualizing complex over schemes of finite type over k. In other words, if $f\colon X \longrightarrow Y$ is a proper morphism between schemes of finite type over k, we have

$$\omega_X = f^!\omega_Y.$$

For example, we have

$$\omega_{\mathbb{P}^m} = \bigwedge^m \Omega_{\mathbb{P}^m}[m].$$

For a hypersurface $S \subset \mathbb{P}^m$ of degree d, we have a quasi-isomorphism

$$\omega_S \cong \mathcal{O}_S(-1-m+d)[1-m].$$

3.2. A bijection on certain coherent $\mathcal{O}_{\mathbb{P}^m}$-modules and symmetric matrices

As an application of Grothendieck duality, we construct a bijection between certain symmetric matrices and certain coherent $\mathcal{O}_{\mathbb{P}^m}$-modules.

3.2.1. Symmetric morphism between complexes. Let X be a scheme of finite type over k. For any complexes $\mathcal{F}, \mathcal{G}$ of coherent $\mathcal{O}_X$-modules, there is a canonical morphism

$$\mathrm{can}_{\mathcal{F},\mathcal{G}} \colon \mathcal{F} \longrightarrow \mathrm{R}\mathscr{H}om_X(\mathrm{R}\mathscr{H}om_X(\mathcal{F},\mathcal{G}),\mathcal{G}).$$

We sometimes write "can" if there is no confusion. We define more notion related to morphisms between complexes of coherent $\mathcal{O}_X$-modules.

Definition 3.2.1. Let X be a scheme of finite type over k, and

$$h \colon \mathcal{F} \longrightarrow \mathcal{G}$$

be a morphism between complexes of coherent $\mathcal{O}_X$-modules. The *transpose* ${}^t\!h$ of a morphism h is defined as

$${}^t\!h \colon \mathrm{R}\mathscr{H}om_X(\mathcal{G},\omega_X) \longrightarrow \mathrm{R}\mathscr{H}om_X(\mathcal{F},\omega_X) \quad ; \quad g \mapsto g \circ h.$$

When the morphism h is

$$h \colon \mathcal{F} \longrightarrow \mathrm{R}\mathscr{H}om_X(\mathcal{F}(i),\omega_X[j-\dim X])$$

for some integers i, j, the transpose of h has a form

$$ {}^t h\colon \mathrm{R}\mathscr{H}om_X(\mathrm{R}\mathscr{H}om_X(\mathcal{F}(i), \omega_X[j - \dim X]), \omega_X) \longrightarrow \mathrm{R}\mathscr{H}om_X(\mathcal{F}, \omega_X). $$

By abuse of notation, we denote the $(-i)$-th twist and the degree $(j - \dim X)$ shift of this morphism

$$ \mathrm{R}\mathscr{H}om_X(\mathrm{R}\mathscr{H}om_X(\mathcal{F}, \omega_X), \omega_X) \longrightarrow \mathrm{R}\mathscr{H}om_X(\mathcal{F}(i), \omega_X[j - \dim X]) $$

as the same symbol ${}^t h$. Composing with the canonical morphism $\mathrm{can}_{\mathcal{F},\omega_X}$, we have another morphism

$$ {}^t h \circ \mathrm{can}_{\mathcal{F},\omega_X} : \mathcal{F} \longrightarrow \mathrm{R}\mathscr{H}om_X(\mathcal{F}(i), \omega_X[j - \dim X]). $$

This morphism has the same domain and target as h, so we can compare the two morphisms.

Definition 3.2.2. A morphism

$$ h\colon \mathcal{F} \longrightarrow \mathrm{R}\mathscr{H}om_X(\mathcal{F}(i), \omega_X[j - \dim X]) $$

for some integers i, j between complexes of coherent $\mathcal{O}_X$-modules is said to be *symmetric* if it satisfies

$$ {}^t h \circ \mathrm{can}_{\mathcal{F},\omega_X} = h. $$

3.2.2. A refinement of Proposition 2.2.8. Now we would like to refine Proposition 2.2.8 to the case that the coherent $\mathcal{O}_{\mathbb{P}^m}$-module $\mathcal{F}$ has another datum of quasi-isomorphism

$$ \lambda\colon \mathcal{F} \xrightarrow{\sim} \mathrm{R}\mathscr{H}om_{\mathbb{P}^m}(\mathcal{F}(2 - m), \omega_{\mathbb{P}^m}[1 - m]). $$

Then the matrix appearing in the resolution (2.2.2) can be taken as a symmetric matrix.

Lemma 3.2.3. Let $\mathcal{M}$ be a coherent $\mathcal{O}_{\mathbb{P}^m}$-module with a quasi-isomorphism

$$\lambda\colon \mathcal{M} \xrightarrow{\sim} \mathrm{R}\mathscr{H}om_{\mathbb{P}^m}(\mathcal{M}(2-m), \omega_{\mathbb{P}^m}[1-m])\,.$$

Then we have

$$\dim_k H^i(\mathbb{P}^m, \mathcal{M}(j)) = \dim_k H^{m-i-1}(\mathbb{P}^m, \mathcal{M}(2-m-j))$$

for any i, j.

Proof (Proof of Lemma 3.2.3). By Grothendieck duality for the structure morphism $f\colon \mathbb{P}^m \to \operatorname{Spec} k$, we obtain

$$\begin{aligned}\mathrm{R}f_*(\mathcal{M}(j)) &\xrightarrow[\mathrm{R}f_*\lambda]{\sim} \mathrm{R}f_*\,\mathrm{R}\mathscr{H}om_{\mathbb{P}^m}(\mathcal{M}(2-m-j), \omega_{\mathbb{P}^m}[1-m]) \\ &\xrightarrow[\mathrm{GD}_f]{\sim} \mathrm{R}\mathscr{H}om_{\operatorname{Spec} k}(\mathrm{R}f_*\mathcal{M}(2-m-j), \mathcal{O}_{\operatorname{Spec} k}[1-m])\,.\end{aligned}$$

Taking the cohomology, we have the desired equality. □

Proposition 3.2.4. Let $\mathcal{M}$ be a coherent $\mathcal{O}_{\mathbb{P}^m}$-module and $t \in \mathbb{Z}$ an integer. Fix a quasi-isomorphism

$$c\colon \omega_{\mathbb{P}^m} \xrightarrow{\sim} \mathcal{O}_{\mathbb{P}^m}(-1-m)[m].$$

Assume that $\mathcal{M}$ satisfies the following conditions:

- $\mathcal{M}$ is arithmetically Cohen–Macaulay, and
- $\mathcal{M}$ is pure of dimension $m-1$, and
- $H^0(\mathbb{P}^m, \mathcal{M}) = 0$.

If there is a symmetric quasi-isomorphism of complexes of coherent $\mathcal{O}_{\mathbb{P}^m}$-modules

$$\lambda\colon \mathcal{M} \xrightarrow{\sim} \mathrm{R}\mathscr{H}om_{\mathbb{P}^m}(\mathcal{M}(2-m), \omega_{\mathbb{P}^m}[1-m]),$$

M can be taken as a symmetric matrix. Moreover, if we fix a ordered k-basis $s = \{s_0, s_1, \ldots, s_m\}$ of $H^0(\mathbb{P}^m, \mathcal{M}(1))$, then M is uniquely determined.

Proof. By Lemma 3.2.3, we have

$$H^{m-1}(\mathbb{P}^m, \mathcal{M}(2-m)) = 0.$$

By Proposition 2.2.8, the coherent $\mathcal{O}_{\mathbb{P}^m}$-module $\mathcal{M}$ has a short exact sequence

$$0 \longrightarrow \mathcal{G} \xrightarrow{\iota} \bigoplus_{i=0}^{r} \mathcal{O}_{\mathbb{P}^m}(-1) \xrightarrow{p} \mathcal{M} \longrightarrow 0. \tag{3.2.1}$$

Here we write $\mathrm{Ker}(p)$ as $\mathcal{G}$ and the embedding

$$\iota\colon \mathcal{G} \hookrightarrow \bigoplus_{i=0}^{r} \mathcal{O}_{\mathbb{P}^m}(-1).$$

By the sequence (2.2.2), the $\mathcal{O}_{\mathbb{P}^m}$-module $\mathcal{G}$ is isomorphic to $\bigoplus_{i=0}^{r} \mathcal{O}_{\mathbb{P}^m}(-2)$. This gives a pure minimal resolution of $\mathcal{M}$.

Let us write the contravariant endo-functor of derived category $D^b(\mathrm{Coh}_X)$ of bounded complexes of coherent $\mathcal{O}_{\mathbb{P}^m}$-modules

$$\mathcal{F} \mapsto \mathrm{R}\mathscr{H}om_{\mathbb{P}^m}(\mathcal{F}(2-m), \omega_{\mathbb{P}^m}[-m])$$

as D in short. Note that there exists a natural transformation $\eta\colon \mathrm{id} \xrightarrow{\sim} D \circ D$. By assumption, we have

$$\lambda\colon \mathcal{M} \xrightarrow{\sim} D\mathcal{M}[1].$$

This shows the complex $D\mathcal{M}$ is represented by a degree one shift of coherent $\mathcal{O}_{\mathbb{P}^m}$-module. We write the $\mathcal{O}_{\mathbb{P}^m}$-module as $D\mathcal{M}[1]$ by abuse of notation.

On the other hand, we have the canonical quasi-isomorphism

$$\begin{aligned} D\mathcal{O}_{\mathbb{P}^m}(-2) &= \mathrm{R}\mathscr{H}om_{\mathbb{P}^m}(\mathcal{O}_{\mathbb{P}^m}(-m), \omega_{\mathbb{P}^m}[-m]) \\ &\xrightarrow{\sim} \mathrm{R}\mathscr{H}om_{\mathbb{P}^m}(\mathcal{O}_{\mathbb{P}^m}, \omega_{\mathbb{P}^m}(m)[-m]) \\ &\xrightarrow{\sim} \omega_{\mathbb{P}^m}(m)[-m]. \end{aligned}$$

Similarly, we have the canonical quasi-isomorphism

$$D\mathcal{O}_{\mathbb{P}^m}(-1) \xrightarrow{\sim} \omega_{\mathbb{P}^m}(m-1)[-m].$$

Recall that the dualizing complex $\omega_{\mathbb{P}^m}$ of $\mathbb{P}^m$ is supported on degree m. Hence $D\mathcal{G}$ and $D(\bigoplus_{i=0}^r \mathcal{O}_{\mathbb{P}^m}(-1))$ is supported on degree zero. Thus we have the short exact sequence

$$0 \longrightarrow D\left(\bigoplus_{i=0}^{r} \mathcal{O}_{\mathbb{P}^m}(-1)\right) \xrightarrow{D\iota} D\mathcal{G} \xrightarrow{Dp} D\mathcal{M}[1] \longrightarrow 0.$$

Since

$$\omega_{\mathbb{P}^m} \cong \mathcal{O}_{\mathbb{P}^m}(-1-m)[m],$$

we see that $D\mathcal{G} \cong \bigoplus_{i=0}^{r} \mathcal{O}_{\mathbb{P}^m}(-1)$ non-canonically. However, the (quasi-)isomorphism

$$\rho\colon \bigoplus_{i=0}^{r} \mathcal{O}_{\mathbb{P}^m}(-1) \xrightarrow{\sim} D\mathcal{G}$$

satisfying $Dp \circ \rho = \lambda \circ p$ is unique by Lemma 2.1.20. We have the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{G} \arrow[r, "\iota"] \arrow[d, "\rho'"', "\wr"] & \bigoplus_{i=0}^{n} \mathcal{O}_{\mathbb{P}^m}(-1) \arrow[r, "p"] \arrow[d, "\rho"', "\wr"] & \mathcal{M} \arrow[r] \arrow[d, "\lambda"', "\wr"] & 0 \\
0 \arrow[r] & D\left(\bigoplus_{i=0}^{n} \mathcal{O}_{\mathbb{P}^m}(-1)\right) \arrow[r, "D\iota"] & D\mathcal{G} \arrow[r, "Dp"] & D\mathcal{M}[1] \arrow[r] & 0
\end{tikzcd}$$

for a (quasi-)isomorphism ρ'. Applying the functor D to this diagram, we have

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{G} \arrow[r, "\iota"] \arrow[d, "{}^t\rho\circ\mathrm{can}"', "\wr"] & \bigoplus_{i=0}^{n} \mathcal{O}_{\mathbb{P}^m}(-1) \arrow[r, "p"] \arrow[d, "{}^t\rho'\circ\mathrm{can}"', "\wr"] & \mathcal{M} \arrow[r] \arrow[d, "{}^t\lambda\circ\mathrm{can}"', "\wr"] & 0 \\
0 \arrow[r] & D\left(\bigoplus_{i=0}^{n} \mathcal{O}_{\mathbb{P}^m}(-1)\right) \arrow[r, "D\iota"] & D\mathcal{G} \arrow[r, "Dp"] & D\mathcal{M}[1] \arrow[r] & 0.
\end{tikzcd}$$

By the symmetricity of λ, we have

$${}^t\lambda \circ \mathrm{can} = \lambda.$$

Since each row in the above diagram gives the pure minimal resolution, we have

$${}^t\rho' \circ \mathrm{can} = \rho, \quad {}^t\rho \circ \mathrm{can} = \rho'.$$

Hence we can modify the commutative diagram as follows:

$$\begin{tikzcd}
0 \arrow[r] & D\left(\bigoplus_{i=0}^{n} \mathcal{O}_{\mathbb{P}^m}(-1)\right) \arrow[r, "(*)"] \arrow[d, equal] & \bigoplus_{i=0}^{n} \mathcal{O}_{\mathbb{P}^m}(-1) \arrow[r, "p"] \arrow[d, equal] & \mathcal{M} \arrow[r] \arrow[d, "\lambda"', "\wr"] & 0 \\
0 \arrow[r] & D\left(\bigoplus_{i=0}^{n} \mathcal{O}_{\mathbb{P}^m}(-1)\right) \arrow[r, "\rho^{-1}\circ D\iota"] & \bigoplus_{i=0}^{n} \mathcal{O}_{\mathbb{P}^m}(-1) \arrow[r, "Dp\circ\rho"] & D\mathcal{M}[1] \arrow[r] & 0,
\end{tikzcd}$$

where (*) is $\iota\circ D\rho^{-1}\circ\mathrm{can}$. Using the fixed quasi-isomorphism c, we can rewrite this diagram as

$$\begin{tikzcd}
0 \arrow[r] & \bigoplus_{i=0}^{n} \mathcal{O}_{\mathbb{P}^m}(-2) \arrow[r, "M"] \arrow[d, equal] & \bigoplus_{i=0}^{n} \mathcal{O}_{\mathbb{P}^m}(-1) \arrow[r, "p"] \arrow[d, equal] & \mathcal{M} \arrow[r] \arrow[d, "\lambda"', "\wr"] & 0 \\
0 \arrow[r] & \bigoplus_{i=0}^{n} \mathcal{O}_{\mathbb{P}^m}(-2) \arrow[r, "{}^tM"] & \bigoplus_{i=0}^{n} \mathcal{O}_{\mathbb{P}^m}(-1) \arrow[r, "Dp\circ\rho"] & D\mathcal{M}[1] \arrow[r] & 0.
\end{tikzcd}$$

This shows $M = {}^tM$ is a symmetric matrix of size $n+1$ with entries in $H^0(\mathbb{P}^m, \mathcal{M}(1))$.

We shall show the last statement. Fix an ordered basis $\{s_0, s_1, \ldots, s_m\}$ of the k-vector space $H^0(\mathbb{P}^m, \mathcal{M}(1))$. Put the standard ordered $\mathcal{O}_{\mathbb{P}^m}$-basis $\{e_0, e_1, \ldots, e_r\}$ of the middle term

$$\left(\bigoplus_{i=0}^{r} \mathcal{O}_{\mathbb{P}^m}(-1)\right)(1).$$

Then the condition that

$$p(e_i) = s_i$$

determines p uniquely, and the above construction of M determines the matrix M uniquely.

$\square$

By Proposition 3.2.4, we construct a map Ψ sending the triple $(\mathcal{M}, \lambda, s)$ to a symmetric matrix M of size $n+1$ with entries in

$$H^0(\mathbb{P}^m, \mathcal{O}_{\mathbb{P}^m}(1)) = kX_0 + kX_1 + \cdots + kX_m,$$

where $X_0, X_1, \ldots, X_m$ is a projective coordinate of $\mathbb{P}^m$. Hence we can describe M as

$$M = X_0M_0 + X_1M_1 + \cdots + X_mM_m$$

where M_i is a symmetric matrix of size $n+1$ with entries in k for each $0 \leq i \leq m$. This shows that the matrix M can be identified with an $(m+1)$-tuple of symmetric matrices of size $n+1$ with entries in k. We fix the projective coordinates $X_0, X_1, \ldots, X_m$ of $\mathbb{P}^m$.

Remark 3.2.5. The map Ψ_c depends on the quasi-isomorphism

$$c\colon \omega_{\mathbb{P}^m} \xrightarrow{\sim} \mathcal{O}_{\mathbb{P}^m}(-1-m)[m].$$

We fix the quasi-isomorphism in the following sections.

3.2.3. A bijection between coherent sheaves and symmetric matrices. From now, we show that the map Ψ induces a bijection between the set of equivalence classes of triples $(\mathcal{F}, \lambda, s)$ and a subset of $(m+1)$-tuples of symmetric matrices with $\mathrm{disc}(M) \neq 0$. To do this, we introduce some notation of symmetric matrices and define the equivalence on triples.

First, we denote the k-vector space of symmetric matrices of size $n+1$ with entries in k as $\mathrm{Sym}_2 k^{n+1}$. Then the k-vector space of ordered $(m+1)$-tuples of symmetric matrices

of size $n+1$ with entries in k is identified to the space

$$W := k^{m+1} \otimes \mathrm{Sym}_2 k^{n+1}.$$

For an element $M \in W$, we write

$$M(X) := X_0 M_0 + X_1 M_1 + \cdots + X_m M_m,$$

where $X_0, X_1, \ldots, X_m$ are $m+1$ variables. The matrix $M(X)$ is a symmetric matrix of size $n+1$ with entries in $H^0(\mathbb{P}^m, \mathcal{O}_{\mathbb{P}^m}(1))$. In the following, we identify the two matrices.

Definition 3.2.6. For an element $M = (M_0, M_1, \ldots, M_m) \in W$, the *discriminant polynomial* $\mathrm{disc}(M)$ of M is defined by

$$\mathrm{disc}(M) := \det(M(X)) = \det(X_0 M_0 + X_1 M_1 + \cdots + X_m M_m).$$

If it is nonzero, then $\mathrm{disc}(M)$ is a homogeneous polynomial of degree $n+1$ of $m+1$ variables $X_0, X_1, \ldots, X_m$.

Note that the map

$$M = M(X) \colon \bigoplus_{i=0}^{r} \mathcal{O}_{\mathbb{P}^m}(-2) \longrightarrow \bigoplus_{i=0}^{r} \mathcal{O}_{\mathbb{P}^m}(-1)$$

is injective if and only if

$$\mathrm{disc}(M) \neq 0.$$

We define the following subset.

Definition 3.2.7. Let W_{nv} be the subset of W which consists of elements with non-zero discriminant polynomials.

Next we define the equivalence of triples.

Definition 3.2.8. In the following, a *triple* $(\mathcal{M}, \lambda, s)$ in $\mathbb{P}^m$ of degree $n+1$ or simply a *triple* consists of three data:

- $\mathcal{M}$ is a coherent $\mathcal{O}_{\mathbb{P}^m}$-module satisfying
  - $\mathcal{M}$ is arithmetically Cohen–Macaulay, and
  - $\mathcal{M}$ is pure of dimension $m-1$, and
  - $H^0(\mathbb{P}^m, \mathcal{M}) = 0$ and $\dim_k H^0(\mathbb{P}^m, \mathcal{M}(1)) = n+1$,
- λ is a symmetric quasi-isomorphism between complexes of coherent $\mathcal{O}_{\mathbb{P}^m}$-modules

$$\lambda\colon \mathcal{M} \xrightarrow{\sim} \mathrm{R}\mathscr{H}om_{\mathbb{P}^m}(\mathcal{M}(2-m), \omega_{\mathbb{P}^m}[1-m])\,,$$

  and
- $s = \{s_0, s_1, \ldots, s_n\}$ is an ordered k-basis of $H^0(\mathcal{M}(1))$.

Let $(\mathcal{M}, \lambda, s)$ and $(\mathcal{M}', \lambda', s')$ be triples. The two triples are said to be *equivalent* if there exists an isomorphism $\rho\colon \mathcal{M}' \xrightarrow{\sim} \mathcal{M}$ of $\mathcal{O}_{\mathbb{P}^m}$-modules satisfying

$${}^t\rho \circ \lambda \circ \rho = \lambda', \quad \rho(s'_i) = s_i$$

for any $0 \le i \le n$.

Definition 3.2.9. Let $V_{m+1,n+1}$ be the set of equivalence classes of triples in $\mathbb{P}^m$ of degree $n+1$.

Now we can state our first bijection.

Theorem 3.2.10. The map Ψ_c induces a bijective map

$$\psi_c \colon V_{m+1,n+1} \longrightarrow W_{\mathrm{nv}}.$$

Proof. First, we shall show the equivalent triples give the same matrix. Let $(\mathcal{M}, \lambda, s)$ and $(\mathcal{M}', \lambda', s')$ be equivalent triples. By the definition of equivalence of triples, we have an isomorphism

$$\rho \colon \mathcal{M} \xrightarrow{\sim} \mathcal{M}'$$

satisfying

$${}^t\rho \circ \lambda' \circ \rho = \lambda, \quad \rho(s_i) = s_i'$$

for each $0 \leq i \leq n$.

We have the minimal resolution of $\mathcal{M}'$

$$0 \longrightarrow \bigoplus_{i=0}^{n} \mathcal{O}_{\mathbb{P}^m}(-2) \xrightarrow{M'} \bigoplus_{i=0}^{n} \mathcal{O}_{\mathbb{P}^m}(-1) \xrightarrow{p'} \mathcal{M}' \longrightarrow 0,$$

where M' is a symmetric matrix of size $n+1$ with entries in $H^0(\mathbb{P}^m, \mathcal{O}_{\mathbb{P}^m}(1))$. By Lemma 2.1.20 and $\rho(s_i) = s_i'$ for each $0 \leq i \leq n$, we have the following commutative diagrams:

$$\begin{tikzcd}
0 \arrow[r] & \bigoplus_{i=0}^{n} \mathcal{O}_{\mathbb{P}^m}(-2) \arrow[r, "M"] \arrow[d, "f", "\wr"'] & \bigoplus_{i=0}^{n} \mathcal{O}_{\mathbb{P}^m}(-1) \arrow[r, "p"] \arrow[d, equal] & \mathcal{M} \arrow[r] \arrow[d, "\rho", "\wr"'] & 0 \\
0 \arrow[r] & \bigoplus_{i=0}^{n} \mathcal{O}_{\mathbb{P}^m}(-2) \arrow[r, "M'"] & \bigoplus_{i=0}^{n} \mathcal{O}_{\mathbb{P}^m}(-1) \arrow[r, "p'"] & \mathcal{M}' \arrow[r] & 0,
\end{tikzcd}$$

where f is an isomorphism of $\mathcal{O}_{\mathbb{P}^m}$-modules. Applying D to this diagram and using c, we have another diagram

$$\begin{tikzcd}
0 \arrow[r] & \bigoplus_{i=0}^{n} \mathcal{O}_{\mathbb{P}^m}(-2) \arrow[r, "M"] \arrow[d, equal] & \bigoplus_{i=0}^{n} \mathcal{O}_{\mathbb{P}^m}(-1) \arrow[r, "q"] & D\mathcal{M}[1] \arrow[r] & 0 \\
0 \arrow[r] & \bigoplus_{i=0}^{n} \mathcal{O}_{\mathbb{P}^m}(-2) \arrow[r, "M'"] & \bigoplus_{i=0}^{n} \mathcal{O}_{\mathbb{P}^m}(-1) \arrow[r, "q'"] \arrow[u, "{}^t f", "\wr\|"'] & D\mathcal{M}'[1] \arrow[r] \arrow[u, "{}^t\rho", "\wr\|"'] & 0,
\end{tikzcd}$$

where $q = \lambda \circ p$ and $q' = \lambda' \circ p'$. Hence we calculate

$$\begin{aligned}
{}^t\rho \circ q' &= {}^t\rho \circ \lambda' \circ p' \\
&= \lambda \circ \rho^{-1} \circ p' \\
&= \lambda \circ p \\
&= q.
\end{aligned}$$

On the other hand, we have

$${}^t\rho \circ q' = q \circ {}^t f.$$

Hence we have $q = q \circ {}^t f$, and by Lemma 2.1.20, we have $f = \mathrm{id}$. Thus $M' = M$, and any equivalent triple to $(\mathcal{M}, \lambda, s)$ gives the same matrix M.

Next we show the injectivity of ψ_c. Let $(\mathcal{M}, \lambda, s)$ and $(\mathcal{M}', \lambda', s')$ be two triples such that

$$\Psi_c(\mathcal{M}, \lambda, s) = \Psi_c(\mathcal{M}', \lambda', s') = M.$$

Then there exists an isomorphism

$$\rho\colon \mathcal{M} \xrightarrow{\sim} \mathcal{M}'$$

that makes the following diagram commute:

$$\begin{tikzcd}
0 \arrow[r] & \bigoplus_{i=0}^{n} \mathcal{O}_{\mathbb{P}^m}(-2) \arrow[r, "M"] \arrow[d, equal] & \bigoplus_{i=0}^{n} \mathcal{O}_{\mathbb{P}^m}(-1) \arrow[r, "p"] \arrow[d, equal] & \mathcal{M} \arrow[r] \arrow[d, "\rho", "\wr"'] & 0 \\
0 \arrow[r] & \bigoplus_{i=0}^{n} \mathcal{O}_{\mathbb{P}^m}(-2) \arrow[r, "M"] & \bigoplus_{i=0}^{n} \mathcal{O}_{\mathbb{P}^m}(-1) \arrow[r, "p'"] & \mathcal{M}' \arrow[r] & 0.
\end{tikzcd}$$

We must show that

$${}^t\rho \circ \lambda' \circ \rho = \lambda, \quad \rho(s_i) = s_i'$$

for each $0 \leq i \leq n$. If we write the standard basis $\{e_0, e_1, \ldots, e_m\}$ of a twist of the middle term

$$\left(\bigoplus_{i=0}^{n} \mathcal{O}_{\mathbb{P}^m}(-1)\right)(1),$$

then $p(e_i) = s_i$ and $p'(e_i) = s_i'$. Hence $\rho(s_i) = s_i'$ for any $0 \leq i \leq n$.

Dualizing the diagram above, we have

$$\begin{tikzcd}
0 \arrow[r] & \bigoplus_{i=0}^{n} \mathcal{O}_{\mathbb{P}^m}(-2) \arrow[r, "M"] \arrow[d, equal] & \bigoplus_{i=0}^{n} \mathcal{O}_{\mathbb{P}^m}(-1) \arrow[r, "q"] \arrow[d, equal] & D\mathcal{M}[1] \arrow[r] & 0 \\
0 \arrow[r] & \bigoplus_{i=0}^{n} \mathcal{O}_{\mathbb{P}^m}(-2) \arrow[r, "M"] & \bigoplus_{i=0}^{n} \mathcal{O}_{\mathbb{P}^m}(-1) \arrow[r, "q'"] & D\mathcal{M}'[1] \arrow[r] \arrow[u, "{}^t\rho", "\wr"'] & 0,
\end{tikzcd}$$

where $q = \lambda \circ p$ and $q' = \lambda' \circ p'$. Then we calculate

$$\begin{aligned}
{}^t\rho \circ \lambda' \circ \rho(s_i) &= {}^t\rho \circ \lambda'(s'_i) \\
&= {}^t\rho \circ \lambda' \circ p'(e_i) \\
&= {}^t\rho \circ q'(e_i) \\
&= q(e_i) \\
&= \lambda \circ p(e_i) \\
&= \lambda(s_i).
\end{aligned}$$

Since $\mathcal{M}$ and $\mathcal{M}'$ is 1-regular (see the proof of Proposition 2.2.8), $\mathcal{M}(1), \mathcal{M}'(1)$ is generated by $s_0, s_1, \ldots, s_n$ and $s'_0, s'_1, \ldots, s'_n$. Hence we have

$${}^t\rho \circ \lambda' \circ \rho = \lambda.$$

This shows two triples $(\mathcal{M}, \lambda, s)$ and $(\mathcal{M}', \lambda', s')$ are equivalent. Hence ψ_c is injective.

We must the map ψ_c is surjective. To show it, we must construct a triple from a matrix $M \in W_{\mathrm{nv}}$.

By Proposition 2.2.8, we have a coherent $\mathcal{O}_{\mathbb{P}^m}$-module $\mathcal{M}$ and the short exact sequence

$$0 \longrightarrow \bigoplus_{i=0}^{n} \mathcal{O}_{\mathbb{P}^m}(-2) \xrightarrow{M} \bigoplus_{i=0}^{n} \mathcal{O}_{\mathbb{P}^m}(-1) \xrightarrow{p} \mathcal{M} \longrightarrow 0.$$

Moreover, by Proposition 2.2.8, $\mathcal{M}$ is arithmetically Cohen–Macaulay and pure of dimension $m-1$. As the ordered k-basis $s = \{s_0, s_1, \ldots, s_n\}$, take the image of the standard basis

$\{e_0, e_1, \ldots, e_m\}$ of a twist of the middle term

$$\left(\bigoplus_{i=0}^{n} \mathcal{O}_{\mathbb{P}^m}(-1)\right)(1).$$

Hence we only have to show that there exists a symmetric quasi-isomorphism

$$\lambda\colon \mathcal{M} \xrightarrow{\sim} D\mathcal{M}[1] = \mathrm{R}\mathscr{H}om_{\mathbb{P}^m}(\mathcal{F}(2-m), \omega_{\mathbb{P}^m}[1-m])\,.$$

By the long exact sequence of cohomology, we have

$$\mathscr{E}xt^{i-m}_{\mathbb{P}^m}(\mathcal{M}, \omega_{\mathbb{P}^m}) = 0$$

for $i \geq 2$. Since $\mathcal{M}$ is pure of dimension $m-1$, we have

$$\mathscr{E}xt^{-m}_{\mathbb{P}^m}(\mathcal{M}, \omega_{\mathbb{P}^m}) = 0.$$

Hence $D\mathcal{M}[1]$ is a degree one shift of a coherent $\mathcal{O}_{\mathbb{P}^m}$-module. By abusing the notation, we write the coherent $\mathcal{O}_{\mathbb{P}^m}$-module as $D\mathcal{M}[1]$. Applying the functor D to the above sequence and using c, we have

$$0 \longrightarrow \bigoplus_{i=0}^{n} \mathcal{O}_{\mathbb{P}^m}(-2) \xrightarrow{M} \bigoplus_{i=0}^{n} \mathcal{O}_{\mathbb{P}^m}(-1) \xrightarrow{q} D\mathcal{M}[1] \longrightarrow 0.$$

Hence there exists a (quasi-)isomorphism

$$\lambda\colon \mathcal{M} \xrightarrow{\sim} D\mathcal{M}[1]$$

satisfying $q = \lambda \circ p$. We have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \bigoplus_{i=0}^{n} \mathcal{O}_{\mathbb{P}^m}(-2) \arrow[r, "M"] \arrow[d, equal] & \bigoplus_{i=0}^{n} \mathcal{O}_{\mathbb{P}^m}(-1) \arrow[r, "p"] \arrow[d, equal] & \mathcal{M} \arrow[r] \arrow[d, "\lambda"', "\wr"] & 0 \\
0 \arrow[r] & \bigoplus_{i=0}^{n} \mathcal{O}_{\mathbb{P}^m}(-2) \arrow[r, "M"] & \bigoplus_{i=0}^{n} \mathcal{O}_{\mathbb{P}^m}(-1) \arrow[r, "q"] & D\mathcal{M}[1] \arrow[r] & 0.
\end{tikzcd} \tag{3.2.2}$$

If we apply the functor D and use c, we obtain another commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \bigoplus_{i=0}^{n} \mathcal{O}_{\mathbb{P}^m}(-2) \arrow[r, "M"] \arrow[d, equal] & \bigoplus_{i=0}^{n} \mathcal{O}_{\mathbb{P}^m}(-1) \arrow[r, "p"] \arrow[d, equal] & \mathcal{M} \arrow[r] \arrow[d, "{}^t\lambda\circ\mathrm{can}"', "\wr"] & 0 \\
0 \arrow[r] & \bigoplus_{i=0}^{n} \mathcal{O}_{\mathbb{P}^m}(-2) \arrow[r, "M"] & \bigoplus_{i=0}^{n} \mathcal{O}_{\mathbb{P}^m}(-1) \arrow[r, "q"] & D\mathcal{M}[1] \arrow[r] & 0.
\end{tikzcd}$$

This shows $\lambda = {}^t\lambda \circ \mathrm{can}$, and λ is symmetric.

By the diagram (3.2.2), we have

$$\psi_c([(\mathcal{M}, \lambda, s)]) = M.$$

Hence the map ψ_c is surjective. This finishes the proof. $\square$

We write the inverse map of ψ_c

$$\phi_c \colon W_{\mathrm{nv}} \longrightarrow V_{m+1,n+1}.$$

3.3. Actions of general linear groups

The k-vector space of $(m+1)$-tuples of symmetric matrices $W = k^{m+1} \otimes \mathrm{Sym}_2 k^{n+1}$ has a natural right action of $\mathrm{GL}_{m+1}(k) \times \mathrm{GL}_{n+1}(k)$. In concrete terms, take $M = (M_0, M_1, \ldots, M_m) \in W$, $A = (a_{i,j}) \in \mathrm{GL}_{m+1}(k)$ and $P \in \mathrm{GL}_{n+1}(k)$, we define

$$M \cdot (A, P) := \left(\sum_{i=0}^{n} a_{i,0} {}^tPM_iP,\ \sum_{i=0}^{n} a_{i,1} {}^tPM_iP,\ \ldots,\ \sum_{i=0}^{n} a_{i,n} {}^tPM_iP \right).$$

This action preserves the subset W_{nv}. In this section, we study the corresponding action on $V_{m+1,n+1}$.

3.3.1. Some notations. Let I_r be the identity matrix of size r. To ease notation, we define

$$M \cdot A := M \cdot (A, I_{n+1}),$$

$$M \cdot P := M \cdot (I_{m+1}, P)$$

for $A = (a_{i,j}) \in \mathrm{GL}_{m+1}(k)$ and $P \in \mathrm{GL}_{n+1}(k)$.

We define the left action of $\mathrm{GL}_{m+1}(k)$. For $A = (a_{i,j}) \in \mathrm{GL}_{m+1}(k)$,

$$A \cdot X := \left(\sum_{j=0}^{n} a_{0,j} X_j,\ \sum_{j=0}^{n} a_{1,j} X_j,\ \ldots,\ \sum_{j=0}^{n} a_{n,j} X_j \right).$$

Then we find

$$(M \cdot (A, P))(X) = (M \cdot P)(A \cdot X) = {}^tPM(A \cdot X)P.$$

If $A = aI_{m+1}$ for some $a \in k^\times$, we have

$$\mathrm{disc}(M \cdot (aI_{m+1}, P)) = a^{m+1} \det(P)^2 \,\mathrm{disc}(M).$$

To define the right action of $\mathrm{GL}_{m+1}(k) \times \mathrm{GL}_{n+1}(k)$ on $V_{m+1,n+1}$ which makes ψ_c, ϕ_c equivariant, we only have to define the corresponding action of $\mathrm{GL}_{m+1}(k)$ and $\mathrm{GL}_{n+1}(k)$ on $V_{m+1,n+1}$ separately.

3.3.2. Action of $\mathrm{GL}_{n+1}(k)$. We take a triple $(\mathcal{M}, \lambda, s)$ in $\mathbb{P}^m$ of degree $n+1$. Put $M := \phi_c([(\mathcal{M}, \lambda, s)])$. We consider $\phi_c(M \cdot P) = \phi_c({}^tPMP)$. We take a triple $(\mathcal{M}', \lambda', s')$ representing $\phi_c({}^tPMP)$. Since ${}^tPMP \circ P^{-1} = {}^tP \circ M$, there exists an isomorphism

$$\rho\colon \mathcal{M} \xrightarrow{\sim} \mathcal{M}'$$

such that the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & \bigoplus_{i=0}^{n} \mathcal{O}_{\mathbb{P}^m}(-2) \arrow[r, "M"] \arrow[d, "P^{-1}"] & \bigoplus_{i=0}^{n} \mathcal{O}_{\mathbb{P}^m}(-1) \arrow[r, "p"] \arrow[d, "{}^tP"] & \mathcal{M} \arrow[r] \arrow[d, "\rho", "\wr"'] & 0 \\
0 \arrow[r] & \bigoplus_{i=0}^{n} \mathcal{O}_{\mathbb{P}^m}(-2) \arrow[r, "{}^tPMP"] & \bigoplus_{i=0}^{n} \mathcal{O}_{\mathbb{P}^m}(-1) \arrow[r, "p'"] & \mathcal{M}' \arrow[r] & 0.
\end{tikzcd}$$

Applying the functor D and using c, we have

$$\begin{tikzcd}
0 \arrow[r] & \bigoplus_{i=0}^{n} \mathcal{O}_{\mathbb{P}^m}(-2) \arrow[r, "M"] \arrow[d, "P^{-1}"] & \bigoplus_{i=0}^{n} \mathcal{O}_{\mathbb{P}^m}(-1) \arrow[r, "q"] \arrow[d, "{}^tP"] & D\mathcal{M} \arrow[r] \arrow[d, "{}^t\rho^{-1}", "\wr"'] & 0 \\
0 \arrow[r] & \bigoplus_{i=0}^{n} \mathcal{O}_{\mathbb{P}^m}(-2) \arrow[r, "{}^tPMP"] & \bigoplus_{i=0}^{n} \mathcal{O}_{\mathbb{P}^m}(-1) \arrow[r, "q'"] & D\mathcal{M}' \arrow[r] & 0,
\end{tikzcd}$$

where $q = \lambda \circ p$ and $q' = \lambda' \circ p'$. Since the left squares in the above two diagrams are the same, we have

$${}^t\rho^{-1} \circ \lambda = \lambda' \circ \rho \quad \Leftrightarrow \quad {}^t\rho \circ \lambda' \circ \rho = \lambda.$$

Since

$$\begin{aligned}\rho^{-1} \circ p'(e_i) &= p \circ {}^tP(e_i) \\ &= (p(e)^tP)_i = (s^tP)_i,\end{aligned}$$

we see that $(\mathcal{M}, \lambda, s^tP)$ is equivalent to $(\mathcal{M}', \lambda', s')$. Hence we have $\phi_c(M \cdot P) = [(\mathcal{M}, \lambda, s^tP)]$.

3.3.3. Action of $\mathrm{GL}_{m+1}(k)$. Next we consider $\phi_c(M \cdot A)$. From $A \in \mathrm{GL}_{m+1}(k)$, we define a projective automorphism

$$\nu_A \colon \mathbb{P}^m \xrightarrow{\sim} \mathbb{P}^m \quad ; \quad [u_0 : u_1 : \cdots : u_m] \mapsto \left[\sum_{j=0}^m a_{0,j}u_j : \sum_{j=0}^m a_{1,j}u_j : \cdots : \sum_{j=0}^m a_{m,j}u_j\right].$$

This gives

$$\nu_A^*(X)_i = \sum_{j=0}^m a_{i,j}X_j = (A \cdot X)_i$$

and

$$\nu_A^*(M(X)_{i,j}) = M(A \cdot X)_{i,j} = (M \cdot A)(X)_{i,j}.$$

Hence $\nu_A^*(M)(X) = (M \cdot A)(X)$. We write $\nu_A^*(M) = M \cdot A$. Applying ν_A^* to the minimal resolution

$$0 \longrightarrow \bigoplus_{i=0}^n \mathcal{O}_{\mathbb{P}^m}(-2) \xrightarrow{M} \bigoplus_{i=0}^n \mathcal{O}_{\mathbb{P}^m}(-1) \xrightarrow{p} \mathcal{M} \longrightarrow 0.$$

of $\mathcal{M}$, we have

$$0 \longrightarrow \bigoplus_{i=0}^n \mathcal{O}_{\mathbb{P}^m}(-2) \xrightarrow{M \cdot A} \bigoplus_{i=0}^n \mathcal{O}_{\mathbb{P}^m}(-1) \xrightarrow{\nu_A^* p} \nu_A^*\mathcal{M} \longrightarrow 0.$$

Thus we have

$$\phi_c(M \cdot A) = [(\nu_A^* \mathcal{M}, \nu_A^* \lambda, \nu_A^* s)].$$

To conclude, we have

$$\phi_c(M \cdot (A, P)) = [(\nu_A^* \mathcal{M}, \nu_A^* \lambda, \nu_A^* s^t P)].$$

Hence we define

$$[(\mathcal{M}, \lambda, s)] \cdot (A, P) = [(\nu_A^* \mathcal{M}, \nu_A^* \lambda, \nu_A^* s^t P)].$$

Then ϕ_c and ψ_c are equivariant with respect to $\mathrm{GL}_{m+1}(k) \times \mathrm{GL}_{n+1}(k)$.

3.3.4. Bijection between orbits and sheaves. We consider the special case when $A = aI_{m+1} \in \mathrm{GL}_{m+1}(k)$. Then $\phi_c(M \cdot aI_{m+1}) = \phi_c(aM)$. If we put $\phi_c(M) = [(\mathcal{M}, \lambda, s)]$, then we can take the representative of $\phi_c(aM)$ as $(\mathcal{M}, a^{-1}\lambda, s)$.

In fact, for aM, we find the short exact sequence

$$0 \longrightarrow \bigoplus_{i=0}^{n} \mathcal{O}_{\mathbb{P}^m}(-2) \xrightarrow{aM} \bigoplus_{i=0}^{n} \mathcal{O}_{\mathbb{P}^m}(-1) \xrightarrow{p} \mathcal{M} \longrightarrow 0.$$

This shows the coherent $\mathcal{O}_{\mathbb{P}^m}$-module $\mathcal{M}$ and the basis s are the same as in the case of M. However, when we apply the functor D to the exact sequence, we have

$$0 \longrightarrow \bigoplus_{i=0}^{n} \mathcal{O}_{\mathbb{P}^m}(-2) \xrightarrow{aM} \bigoplus_{i=0}^{n} \mathcal{O}_{\mathbb{P}^m}(-1) \xrightarrow{a^{-1}q} D\mathcal{M} \longrightarrow 0,$$

where $q = \lambda \circ p$. Hence λ changes to $a^{-1}\lambda$.

We define some notion to simplify our bijections. We call simply a *pair* that $(\mathcal{M}, \lambda)$ consists of the following data:

- $\mathcal{M}$ is a coherent $\mathcal{O}_{\mathbb{P}^m}$-module which satisfies
  - $\mathcal{M}$ is arithmetically Cohen–Macaulay, and
  - $\mathcal{M}$ is pure of dimension $m-1$, and
  - $H^0(\mathbb{P}^m, \mathcal{M}) = 0$ and $\dim_k H^0(\mathbb{P}^m, \mathcal{M}(1)) = n+1$.
- λ is a symmetric quasi-isomorphism

$$\lambda\colon \mathcal{M} \xrightarrow{\sim} \mathrm{R}\mathscr{H}om_{\mathbb{P}^m}(\mathcal{M}(2-m), \omega_{\mathbb{P}^m}[1-m])\,.$$

Definition 3.3.1. Let $(\mathcal{M}, \lambda), (\mathcal{M}', \lambda')$ be pairs satisfying the conditions stated above.

(1) We write $(\mathcal{M}, \lambda) \sim_1 (\mathcal{M}', \lambda')$ if there exists an isomorphism $\rho\colon \mathcal{M} \xrightarrow{\sim} \mathcal{M}'$ satisfying

$${}^t\rho \circ \lambda' \circ \rho = u\lambda$$

for some $u \in k^\times$.

(2) We write $(\mathcal{M}, \lambda) \sim_2 (\mathcal{M}', \lambda')$ if there exist isomorphisms $\nu\colon \mathbb{P}^m \xrightarrow{\sim} \mathbb{P}^m$ and $\rho\colon \nu^*\mathcal{M} \xrightarrow{\sim} \mathcal{M}'$ satisfying

$${}^t\rho \circ \lambda' \circ \rho = \nu^*\lambda.$$

To consider the set of $\mathrm{GL}_{n+1}(k)$-orbits of $V_{m+1,n+1}$ is equivalent to forget the basis s from triples. To consider the set of $k^\times I_{m+1}$-orbits of $V_{m+1,n+1}$ is equivalent to consider the quasi-isomorphism λ up to multiplication by $a \in k^\times$. Hence we obtain the following bijection:

Corollary 3.3.2. There exists a natural bijection between the following two sets.

- The set of $(k^\times I_{m+1}) \times \mathrm{GL}_{n+1}(k)$-orbits of $(m+1)$-tuples of symmetric matrices $M = (M_0, M_1, \ldots, M_m) \in W_{\mathrm{nv}}$ of size $n+1$ with entries in k with $\mathrm{disc}(M) \neq 0$.

- The set of equivalence classes of pairs $(\mathcal{M}, \lambda)$ with respect to $\sim_1$.

The orbits of the whole group $\mathrm{GL}_{m+1}(k) \times \mathrm{GL}_{n+1}(k)$ give another bijection:

Corollary 3.3.3. There exists a natural bijection between the following two sets.

- The set of $\mathrm{GL}_{m+1}(k) \times \mathrm{GL}_{n+1}(k)$-orbits of $(m+1)$-tuples of symmetric $(n+1) \times (n+1)$-matrices $M = (M_0, M_1, \ldots, M_m)$ with entries in k with $\mathrm{disc}(M) \neq 0$.
- The set of equivalence classes of pairs $(\mathcal{M}, \lambda)$ with respect to $\sim_2$.

3.4. A description of endomorphisms and fibers

Next we consider the set of symmetric quasi-isomorphisms with respect to $\mathcal{M}$. We would like to parametrize symmetric quasi-isomorphisms. Take a pair $(\mathcal{M}, \lambda)$. Let $L_0 := \mathrm{End}_{\mathbb{P}^m}(\mathcal{M})$. Then L_0 is a finite-dimensional k-algebra, and the k-vector space $\mathrm{Hom}_{\mathbb{P}^m}(\mathcal{M}, D\mathcal{M}[1])$ has two way of simply transitive action of L_0. Precisely, for $l \in L_0$,

$$\eta \mapsto \eta \circ l$$

and

$$\eta \mapsto {}^t l \circ \eta.$$

If η is symmetric, we have ${}^t(\eta \circ l) = {}^t l \circ \eta$. Henceforth $\eta \circ l$ is symmetric if and only if

$$ {}^t l \circ \eta = \eta \circ l. \tag{3.4.1}$$

Let L denote the k-subspace of L_0 defined by

$$\left\{ l \in L_0 \;\middle|\; {}^t l \circ \lambda = \lambda \circ l \right\}.$$

Then the map

$$L \longrightarrow \mathrm{Hom}_{\mathbb{P}^m}(\mathcal{M}, D\mathcal{M}[1]) \quad ; \quad l \mapsto \lambda \circ l$$

gives a bijection between the subspace $L \subset L_0$ and the subspace of symmetric homomorphisms.

Take an element M in the $(k^\times I_{m+1}) \times \mathrm{GL}_{n+1}(k)$-orbit of W_{nv} corresponding to a pair $(\mathcal{M}, \lambda)$ (see Corollary 3.3.2). In terms of matrix algebra $\mathrm{Mat}_{n+1}(k)$, we can describe L_0 and L as follows.

Proposition 3.4.1. The k-algebra L_0 is isomorphic to

$$\left\{(P, P') \in \mathrm{Mat}_{n+1}(k) \times \mathrm{Mat}_{n+1}(k) \;\middle|\; {}^tPM = MP'\right\} \tag{3.4.2}$$

where the product of $\mathrm{Mat}_{n+1}(k) \times \mathrm{Mat}_{n+1}(k)$ is defined by

$$(P_1, P_1') \circ (P_2, P_2') = (P_2P_1, P_1'P_2').$$

The anti-homomorphism $(P, P') \mapsto (P', P)$ gives an anti-endomorphism σ of L_0 as a k-algebra, and the fixed part L_0^σ is identified to L.

Remark 3.4.2. The fixed part $L = L_0^\sigma$ is not a k-subalgebra in general. In fact, for two different symmetric quasi-isomorphisms λ and λ', it is a completely different matter for l that $\lambda \circ l$ is symmetric and that $\lambda' \circ l$ is symmetric.

Proof. We first show the existence of an inclusion $L_0 \hookrightarrow \mathrm{Mat}_{n+1}(k) \times \mathrm{Mat}_{n+1}(k)$. Take $l \in L_0$ and fix a basis s of $H^0(\mathbb{P}^m, \mathcal{M}(1))$. Then, since the endomorphism of $\mathcal{M}$ is uniquely lifted to the endomorphism of the pure minimal resolution of $\mathcal{M}$ by Lemma 2.1.20, we find

the unique elements $P, P' \in \mathrm{Mat}_{n+1}(k)$ which make the following diagram commute:

$$
\begin{tikzcd}
0 \arrow[r] & \bigoplus_{i=0}^{n} \mathcal{O}_{\mathbb{P}^m}(-2) \arrow[r, "M"] \arrow[d, "P'"] & \bigoplus_{i=0}^{n} \mathcal{O}_{\mathbb{P}^m}(-1) \arrow[r, "p"] \arrow[d, "{}^tP"] & \mathcal{M} \arrow[r] \arrow[dd, "l"] & 0 \\
0 \arrow[r] & \bigoplus_{i=0}^{n} \mathcal{O}_{\mathbb{P}^m}(-2) \arrow[r, "M"] & \bigoplus_{i=0}^{n} \mathcal{O}_{\mathbb{P}^m}(-1) \arrow[r, "p"] & \mathcal{M} \arrow[r] & 0.
\end{tikzcd}
\tag{3.4.3}
$$

This proves that L_0 is embedded into $\mathrm{Mat}_{n+1}(k) \times \mathrm{Mat}_{n+1}(k)$ as a k-subalgebra. The image satisfies ${}^tPM = MP'$. Conversely, if we take (P, P') satisfying ${}^tPM = MP'$, we obtain a morphism l which makes the diagram (3.4.3) commute. Thus the image of L_0 consists of the set (3.4.2). Transposing

$$
{}^tPM = MP',
$$

we have ${}^tP'M = MP$, and we see that (P', P) is also an element of L_0. Thus

$$
\sigma \colon L_0 \longrightarrow L_0
$$

gives an automorphism as a k-vector space. It is easy to see that σ gives an anti-endomorphism of L_0.

Let us take l from L. Dualizing the diagram (3.4.3), we obtain

$$
\begin{tikzcd}
0 \arrow[r] & \bigoplus_{i=0}^{n} \mathcal{O}_{\mathbb{P}^m}(-2) \arrow[r, "M"] & \bigoplus_{i=0}^{n} \mathcal{O}_{\mathbb{P}^m}(-1) \arrow[r, "q"] & D\mathcal{M}[1] \arrow[r] & 0 \\
0 \arrow[r] & \bigoplus_{i=0}^{n} \mathcal{O}_{\mathbb{P}^m}(-2) \arrow[r, "M"] \arrow[u, "P"] & \bigoplus_{i=0}^{n} \mathcal{O}_{\mathbb{P}^m}(-1) \arrow[r, "q"] \arrow[u, "{}^tP'"] & D\mathcal{M}[1] \arrow[r] \arrow[u, "{}^tl"] & 0.
\end{tikzcd}
$$

Now we would like to say $P' = P$. Since $q = \lambda \circ p$, we have

$$\begin{aligned} {}^t l \circ q &= {}^t l \circ \lambda \circ p \\ &= \lambda \circ l \circ p \\ &= \lambda \circ p \circ {}^t P. \end{aligned}$$

Meanwhile, we have

$$\begin{aligned} {}^t l \circ q &= q \circ {}^t P' \\ &= \lambda \circ p \circ {}^t P'. \end{aligned}$$

Hence $p \circ {}^t P = p \circ {}^t P'$, and

$$s^t P = s^t P'.$$

This shows $P' = P$. Thus L is contained in L_0^σ. Conversely, if we take $(P, P) \in L_0^\sigma$, we can see that the corresponding endomorphism l is symmetric. $\square$

Next, we study when two pairs $(\mathcal{M}, \lambda), (\mathcal{M}, l\lambda)$ are equivalent in the sense of $\sim_1$ and $\sim_2$ respectively.

We treat $\sim_1$ first. Assume that $(\mathcal{M}, \lambda) \sim_1 (\mathcal{M}, l\lambda)$. By the definition of equivalence, there exist an automorphism $\rho \colon \mathcal{M} \longrightarrow \mathcal{M}$ and a constant $a \in k^\times$ such that

$$l\lambda = a^t \rho \circ \lambda \circ \rho.$$

Let $(P, P') \in L_0$ be the element corresponds to $\rho \in \mathrm{Aut}_{\mathbb{P}^m}(\mathcal{M})$, and $(R, R) \in L$ corresponds to $l \in L$. Then $\sigma((P, P')) = (P', P)$ corresponds to $\sigma(\rho) := \lambda^{-1} \circ {}^t\rho \circ \lambda$. Hence we have

$$l = a\sigma(\rho)\rho \quad \Leftrightarrow \quad R = aPP'. \tag{3.4.4}$$

Conversely, if we can write $l = a\sigma(\rho)\rho \in L_0$ for some $\rho \in L$, we can go backward and obtain $(\mathcal{M}, \lambda) \sim_1 (\mathcal{M}, l\lambda)$.

Next we treat the other equivalence $\sim_2$. Assume that $(\mathcal{M}, \lambda) \sim_2 (\mathcal{M}, l\lambda)$. Take any basis s of $H^0(\mathbb{P}^m, \mathcal{M}(1))$. Put $M := \psi_c([(\mathcal{M}, \lambda, s)])$. Then there exist an element $A \in \mathrm{GL}_{m+1}(k)$ and an automorphism $\rho\colon \mathcal{M} \longrightarrow \mathcal{M}$ such that

$$l\lambda = {}^t\rho \circ \nu^*\lambda \circ \rho. \tag{3.4.5}$$

Take $(P, P') \in L_0$ corresponding to ρ and $(R, R) \in L$ corresponding to l. Then (3.4.5) means

$$MR = {}^tPMP \cdot A. \tag{3.4.6}$$

We see that the converse holds. For $l = (R, R) \in L$ such that there exist $(P, P') \in L_0, A \in \mathrm{GL}_{m+1}(k)$ satisfying (3.4.6), then by going backward, we see that $(\mathcal{M}, \lambda) \sim_2 (\mathcal{M}, l\lambda)$. Thus we conclude the following proposition about the fiber of equivalence:

Proposition 3.4.3. Take $M \in W_{\mathrm{nv}}$, and we write $\phi_c(M) = [(\mathcal{M}, \lambda, s)]$. The following statements hold.

(i) The k-algebra $L_0 = \mathrm{End}_{\mathbb{P}^m}(\mathcal{M})$ acts on $\mathrm{Hom}_{\mathbb{P}^m}(\mathcal{M}, D\mathcal{M}[1])$ simply transitively in two ways.

(ii) The k-subspace L defined above is bijective to k-subspace of symmetric homomorphisms of $\mathrm{Hom}_{\mathbb{P}^m}(\mathcal{M}, D\mathcal{M}[1])$ via $l \mapsto \lambda \circ l$.

(iii) We define the subset L_1 of $L \cap L_0^\times$ such that for $l \in L_1$, there exist $a \in k^\times, \rho \in L$ such that (3.4.4) holds. Then L_1 is bijective to λ' which satisfies $(\mathcal{M}, \lambda) \sim_1 (\mathcal{M}, \lambda')$.

(iv) We define the subset L_2 of $L \cap L_0^\times$ consists of l such that there exist $A \in \mathrm{GL}_{m+1}(k), \rho \in L$ satisfying (3.4.6). Then L_2 is bijective to λ' which satisfies $(\mathcal{M}, \lambda) \sim_2 (\mathcal{M}, \lambda')$.

CHAPTER 4

Theta characteristics on hypersurfaces

In this chapter, we define a class of coherent modules on hypersurface in projective spaces which we call *theta characteristics*. This concept is usually defined over smooth curves due to Mumford [**22**], and extended to possibly singular curves by several people, including Harris, Piontkowski ([**12**], [**23**]). Our definition is the essentially same as the definition due to Dolgachev ([**5**, Definition 4.2.9]).

From our viewpoint, it is convenient to define theta characteristics in terms of derived category. As proved in Subsection 4.1.2, those definition are equivalent to a sheaf-theoretic definition expected from Piontkowski's definition.

We will first review the definition of theta characteristics on smooth curves and some examples in Subsection 4.1.1. After them, we define theta characteristics on hypersurfaces, and give examples and properties in Subsection 4.1.2. Then we state a bijection between theta characteristics and symmetric matrices in Section 4.2, and prove Theorem 1.3.2 and Corollary 1.3.3.

We fix a quasi-isomorphism

$$c\colon \omega_{\mathbb{P}^m} \xrightarrow{\sim} \mathcal{O}_{\mathbb{P}^m}(-1-m)[m]$$

in this chapter.

4.1. Theta characteristics

In this section, we recall the definition of theta characteristics on smooth curves, and define theta characteristics on hypersurfaces.

4.1.1. Theta characteristics on smooth curves. The following is the definition of theta characteristics on smooth curves.

Definition 4.1.1 (Classical definition of theta characteristics, [22]). Let C be a smooth curve over a field k, and ω_C^{shf} the canonical line bundle on C. A line bundle $\mathcal{L}$ on C is called a *theta characteristic* on C if there is an isomorphism of $\mathcal{O}_C$-modules

$$\mathcal{L} \otimes \mathcal{L} \xrightarrow{\sim} \omega_C^{\mathrm{shf}}.$$

A theta characteristic $\mathcal{L}$ on C is called *even* (resp. *odd*) if the dimension of the space of global sections $\dim_k H^0(C, \mathcal{L})$ is even (resp. odd). Also, a theta characteristic $\mathcal{L}$ on C is called *effective* (resp. *non-effective*) if $H^0(C, \mathcal{L}) \neq 0$ (resp. $H^0(C, \mathcal{L}) = 0$).

Let us consider some low-degree curves. Let C be a smooth curve over k, and $g = \dim_k H^1(C, \mathcal{O}_C)$ the genus of C.

Example 4.1.2 (Brauer–Severi varieties). When $g = 0$, the curve C is geometrically isomorphic to $\mathbb{P}^1_{\overline{k}}$. Over a general field k, however, it is not necessarily isomorphic to $\mathbb{P}^1_k$. The genus zero curves over k are called the *Brauer–Severi varieties* of dimension one.

Let us study first the case of $C = \mathbb{P}^1_k$. Then the Picard group $\mathrm{Pic}(C)$ is isomorphic to $\mathbb{Z}$, and it is generated by $\mathcal{O}_{\mathbb{P}^1_k}(1)$. The canonical line bundle ω_C^{shf} is isomorphic to $\mathcal{O}_{\mathbb{P}^1_k}(-2)$, so the only isomorphism class of theta characteristics is the tautological bundle $\mathcal{O}_{\mathbb{P}^1_k}(-1)$. It is non-effective, hence even.

Next, we study a non-trivial Brauer–Severi variety C of dimension one. The canonical sheaf ω_C^{shf} is a line bundle on C defined over k of degree minus two. The k-valued points $\mathrm{Pic}_{C/k}(k)$ of Picard variety $\mathrm{Pic}_{C/k}$ is isomorphic to $\mathbb{Z}$, and generated by the point corresponding to the line bundle on C of degree minus one. However, there is no line bundle on C of degree minus one defined over k. By this observation, C has no theta characteristics defined over k. For details, see Proposition 6.2.1.

Example 4.1.3 (Elliptic curves). When $g = 1$, the curve C is geometrically isomorphic to an elliptic curve E. Actually, the Jacobian variety $E = \mathrm{Jac}(C)$ of C is an elliptic curve defined over k, and C is geometrically isomorphic to E.

The canonical bundle ω_C^{shf} on C is isomorphic to $\mathcal{O}_C$ in this case. Hence theta characteristics on C correspond to a k-rational point of two-torsion points $E[2](k)$ of E. For a degree zero line bundle $\mathcal{L}$ on a smooth curve C, it is isomorphic to $\mathcal{O}_C$ if and only if $H^0(C, \mathcal{L}) \neq 0$. Therefore, $\mathcal{O}_C$ is the only effective theta characteristic on C, and others are non-effective. We can show that actually any k-rational point of $E[2]$ gives a line bundle on C defined over k. Hence non-effective theta characteristics exists if and only if $E[2](k) \neq 0$. For details, see Proposition 6.2.2.

Example 4.1.4. When $g = 3$ and C is not hyperelliptic, C is a plane quartic curve. The canonical embedding gives an embedding to $\mathbb{P}^2$, so we have $\omega_C^{\mathrm{shf}} = \mathcal{O}_C(1)$.

In this case, effective theta characteristics have an interesting geometric interpretation. *Bitangents* of a plane quartic curve C are lines l in $\mathbb{P}^2$ such that there exist (not necessarily distinct) points $P, Q \in C$

$$l \cdot C = 2[P] + 2[Q]$$

as a divisor of C. Then the line bundle

$$\mathcal{O}_C(P+Q)$$

gives an effective theta-characteristic. We can show that such theta characteristics are all of effective theta characteristics.

Proposition 4.1.5. Effective theta characteristics on smooth quartic curves are odd.

Proof. Let $C \subset \mathbb{P}^2$ be a smooth quartic curve over k. The canonical line bundle gives the closed embedding

$$\iota\colon C \hookrightarrow \mathbb{P}^2$$

([**14**, IV, Proposition 5.2]).

Let $\mathcal{L}$ be an even theta characteristic on C. If it is effective, we have

$$\dim_k H^0(C, \mathcal{L}) = 2.$$

The complete linear system defined by $\mathcal{L}$ gives a degree two morphism $\pi\colon C \longrightarrow \mathbb{P}^1$. Then ι must coincide with the composite of π and Segre embedding

$$\sigma\colon \mathbb{P}^1 \hookrightarrow \mathbb{P}^2.$$

If so, the ι is a degree two morphism. This contradicts to the fact that ι is an embedding. Hence even theta characteristic $\mathcal{L}$ must be non-effective. $\square$

4.1.2. Theta characteristics on hypersurfaces. Now we generalize the definition of theta characteristics.

Definition 4.1.6 ([**5**, Definition 4.2.9])**.** Let $S \subset \mathbb{P}^m$ be a geometrically reduced hypersurface over k. A *theta characteristic* $\mathcal{M}$ on S is a coherent $\mathcal{O}_S$-module such that

- $\mathcal{M}$ is arithmetically Cohen–Macaulay, and
- $\mathcal{M}$ is pure of dimension $m-1$, and
- $\mathrm{length}(\mathcal{M}_\eta) = 1$ for each generic point $\eta \in \mathrm{Gen}(S)$, and
- there is a quasi-isomorphism of $\mathcal{O}_S$-modules

$$\lambda \colon \mathcal{M} \xrightarrow{\sim} \mathrm{R}\mathscr{H}om_S(\mathcal{M}(2-m), \omega_S[1-m])\,.$$

A theta characteristic $\mathcal{M}$ on S is said to be *effective* (resp. *non-effective*) if $H^0(S, \mathcal{M}) \neq 0$ (resp. $H^0(S, \mathcal{M}) = 0$).

Remark 4.1.7. The quasi-isomorphism λ is automatically symmetric. In fact, it is enough to show that any morphism

$$v \colon \mathcal{M} \to \mathrm{R}\mathscr{H}om_S(\mathcal{M}(2-m), \omega_S[-m+1])$$

is symmetric. Consider a morphism

$$h_v = {}^t v \circ \mathrm{can} - v.$$

Since $\mathcal{M}$ is locally free of rank one on $\mathrm{Sm}(S)$, h_v is zero on $\mathrm{Sm}(S)$. So the $\mathcal{O}_S$-submodule $\mathrm{Im}(h_v)$ of $\mathcal{M}$ is supported on a subscheme of S whose dimension is strictly less than $\dim S = m-1$. However, since $\mathcal{M}$ is pure of dimension $m-1$, we conclude $\mathrm{Im}(h_v) = 0$ and v is symmetric. See also Lemma 4.2.6.

We check that, when S is a smooth plane curve C over k, this definition of theta characteristics coincides with the classical definition. Let $\mathcal{M}$ be a theta characteristic on C in the sense of Definition 4.1.6. Since C is a smooth curve, Cohen–Macaulay $\mathcal{O}_C$-modules of dimension one are locally free modules. Hence $\mathcal{M}$ is a locally free $\mathcal{O}_C$-module. By the assumption $\mathrm{length}(\mathcal{M}_\eta) = 1$ for the generic point of C, the coherent $\mathcal{O}_C$-module $\mathcal{M}$ is a line bundle on C. In this case, the symmetric quasi-isomorphism

$$\lambda\colon \mathcal{M} \xrightarrow{\sim} \mathrm{R}\mathscr{H}om_C(\mathcal{M}, \omega_C[-1])\,.$$

This gives the isomorphism of coherent $\mathcal{O}_C$-modules

$$\lambda^{\mathrm{shf}}\colon \mathcal{M} \xrightarrow{\sim} \mathscr{H}om_C\big(\mathcal{M}, \omega_C^{\mathrm{shf}}\big)\,,$$

or

$$\mathcal{M} \otimes \mathcal{M} \xrightarrow{\sim} \omega_C^{\mathrm{shf}}.$$

Here, we denote the dualizing sheaf of C by ω_C^{shf}. This shows $\mathcal{M}$ is a theta characteristic on C in the classical sense (see Definition 4.1.1).

Conversely, let $\mathcal{M}$ be a theta characteristic on a smooth plane curve C in the classical sense. This is the case of $m = 2$, and then $\mathcal{M}$ is arithmetically Cohen–Macaulay if and only if it is Cohen–Macaulay. Since $\mathcal{M}$ is a line bundle on C, it is Cohen–Macaulay and pure of dimension one. The condition

$$\mathcal{M} \otimes \mathcal{M} \xrightarrow{\sim} \omega_C^{\mathrm{shf}}$$

is equivalent to the existence of an isomorphism

$$\lambda^{\mathrm{shf}} \colon \mathcal{M} \xrightarrow{\sim} \mathscr{H}om_C\big(\mathcal{M}, \omega_C^{\mathrm{shf}}\big)\,.$$

Since $\mathcal{M}$ is a line bundle on C, there is a quasi-isomorphism

$$\mathrm{R}\mathscr{H}om_C(\mathcal{M}, \omega_C[-1]) \cong \mathscr{H}om_C\big(\mathcal{M}, \omega_C^{\mathrm{shf}}\big)\,.$$

We obtain

$$\lambda \colon \mathcal{M} \xrightarrow{\sim} \mathrm{R}\mathscr{H}om_C(\mathcal{M}, \omega_C[-1])\,.$$

Here we see an example of theta characteristic in our sense on a not smooth curve.

Example 4.1.8 (cf. [23]). Take a geometrically reduced plane curve $C \subset \mathbb{P}^2$. For the normalization $\pi \colon N \to C$, the push-forward of a theta characteristic $\mathcal{L}$ on N is a theta characteristic on C. In fact, take a quasi-isomorphism

$$\lambda \colon \mathcal{L} \xrightarrow{\sim} \mathrm{R}\mathscr{H}om_N(\mathcal{L}, \omega_N[-1])\,.$$

By Grothendieck duality, we have a quasi-isomorphism on C

$$\lambda_{\pi_*\mathcal{L}} \colon \pi_*\mathcal{L} \xrightarrow[\pi_*\lambda_{\mathcal{L}}]{\sim} \pi_*\,\mathrm{R}\mathscr{H}om_N(\mathcal{L}, \omega_N[-1]) \xrightarrow[\mathrm{GD}_\pi]{\sim} \mathrm{R}\mathscr{H}om_C(\pi_*\mathcal{L}, \omega_C[-1])\,.$$

4.2. Bijection between theta characteristic and symmetric matrices

By our bijection in Theorem 3.2.10, theta characteristics on geometrically reduced hypersurfaces $S \subset \mathbb{P}^m$ correspond to some $(m+1)$-tuples of symmetric matrices with entries in k. In this section, we determine what tuples correspond to theta characteristics

on S. To do this, we begin with general settings, and prepare some propositions from intersection theory.

4.2.1. Bijection considering multiplicities and degrees. Recall that, for a coherent $\mathcal{O}_{\mathbb{P}^m}$-module $\mathcal{F}$, there is a polynomial $P_{\mathcal{F}}(t)$ of degree less than or equal to $\dim \mathrm{Supp}(\mathcal{F})$ satisfying

$$\begin{aligned} P_{\mathcal{F}}(t) &:= \chi(\mathcal{F}(t)) \\ &= \sum_{i=0}^{m} (-1)^i \dim_k H^i(\mathbb{P}^m, \mathcal{F}(t)) \,. \end{aligned}$$

The polynomial $P_{\mathcal{F}}(t)$ is called the *Hilbert polynomial* of $\mathcal{F}$.

Lemma 4.2.1 ([9, Example 2.5.2]**).** Let $\mathcal{M}$ be a coherent $\mathcal{O}_{\mathbb{P}^m}$-module with $\dim \mathrm{Supp}(\mathcal{M}) = m-1$. The coefficient of t^{m-1} in the Hilbert polynomial $P_{\mathcal{M}}(t)$ is equal to

$$\frac{1}{(m-1)!} \sum_{\eta \in \mathrm{Gen}(\mathrm{Supp}(\mathcal{M}))} \deg[\eta] \cdot \mathrm{length}_{\mathcal{O}_{\mathbb{P}^m,\eta}}(\mathcal{M}_\eta),$$

where $[\eta]$ denotes the algebraic cycle on $\mathbb{P}^m$ corresponding to the generic point η of an irreducible component of $\mathrm{Supp}(\mathcal{M})$.

Proof. If we have a short exact sequence

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$$

and if the assertion of the lemma holds for $\mathcal{F}'$ and $\mathcal{F}''$, the assertion of the lemma for $\mathcal{F}$ also holds by the additivity of Hilbert polynomial and length.

Let $\iota\colon Z \hookrightarrow \mathbb{P}^m$ be an irreducible component of $\mathrm{Supp}(\mathcal{M})$ of dimension $m-1$, and η the generic point of Z. Then we have the following short exact sequence:

$$0 \longrightarrow \mathrm{Ker}(\varphi) \longrightarrow \mathcal{M} \xrightarrow{\varphi} \iota_*\iota^*\mathcal{M} \longrightarrow 0.$$

The last term $\iota_*\iota^*\mathcal{M}$ is supported on Z, and we have

$$\mathrm{length}_{\mathcal{O}_{\mathbb{P}^m,\eta}}(\mathrm{Ker}(\varphi)) < \mathrm{length}_{\mathcal{O}_{\mathbb{P}^m,\eta}}(\mathcal{M}).$$

If $\dim \mathrm{Supp}(\mathcal{M}) < m-1$, then the coefficient of t^{m-1} is zero. Hence we may assume that $\mathcal{M}$ is supported on an irreducible hypersurface $Z \subset \mathbb{P}^m$.

The local ring $\mathcal{O}_{\mathbb{P}^m,\eta}$ is a discrete valuation ring. Since $\iota^*\mathcal{M}$ admits a free resolution of finite length, we can assume that $\mathcal{M}$ is isomorphic to $\iota_*\mathcal{O}_{\mathbb{P}^m,\eta}$. In that case, the required equality is just the definition of the degree of Z. $\square$

Lemma 4.2.2. Let $\mathcal{M}$ be a coherent $\mathcal{O}_{\mathbb{P}^m}$-module satisfying the following conditions:

- $\mathcal{M}$ is arithmetically Cohen–Macaulay, and
- $\mathcal{M}$ is pure of dimension $m-1$, and
- $H^0(\mathbb{P}^m, \mathcal{M}) = 0$, and
- there exists a quasi-isomorphism of $\mathcal{O}_{\mathbb{P}^m}$-modules

$$\lambda\colon \mathcal{M} \xrightarrow{\sim} \mathrm{R}\mathscr{H}om_{\mathbb{P}^m}(\mathcal{M}(2-m), \omega_{\mathbb{P}^m}[-m+1]).$$

Then we have

$$\dim H^0(\mathbb{P}^m, \mathcal{M}(t)) = \left(\sum_{\eta \in \mathrm{Gen}(\mathrm{Supp}(\mathcal{M}))} \deg[\eta] \cdot \mathrm{length}_{\mathcal{O}_{\mathbb{P}^m,\eta}}(\mathcal{M}_\eta) \right) \binom{t+m-2}{m-1}$$

for $t \geq 0$.

Proof. By Proposition 3.2.4, the coherent $\mathcal{O}_{\mathbb{P}^m}$-module $\mathcal{M}$ admits a short exact sequence of the following form:

$$0 \longrightarrow \bigoplus_{i=0}^{n} \mathcal{O}_{\mathbb{P}^m}(-2) \longrightarrow \bigoplus_{i=0}^{n} \mathcal{O}_{\mathbb{P}^m}(-1) \longrightarrow \mathcal{M} \longrightarrow 0.$$

By the additivity of the Hilbert polynomials, we have

$$P_{\mathcal{M}}(t) = (n+1)\left(P_{\mathcal{O}_{\mathbb{P}^m}}(t-1) - P_{\mathcal{O}_{\mathbb{P}^m}}(t-2)\right).$$

The Hilbert polynomial $P_{\mathcal{O}_{\mathbb{P}^m}}(t)$ is known as

$$P_{\mathcal{O}_{\mathbb{P}^m}}(t) = \binom{t+m}{m}.$$

Hence we have

$$\begin{aligned} P_{\mathcal{M}}(t) &= (n+1)\left(\binom{t+m-1}{m} - \binom{t+m-2}{m}\right) \\ &= (n+1)\binom{t+m-2}{m-1}. \end{aligned}$$

By Lemma 4.2.1, we obtain the desired equality. □

By Theorem 3.2.10 and Lemma 4.2.2, we obtain the following proposition.

Proposition 4.2.3. Let $S_1, S_2, \ldots, S_r$ be a collection of distinct geometrically irreducible hypersurfaces in $\mathbb{P}^m$, $\eta_1, \eta_2, \ldots, \eta_r$ their generic points, $F_1, F_2, \ldots, F_r$ their defining equations and $n_1, n_2, \ldots, n_r$ non-negative integers. Put $\mathbf{n} = (n_1, n_2, \ldots, n_r)$ and write

$|\mathbf{n}| = \sum_i n_i \deg[\eta_i] - 1$. Let $S := \bigcup_i S_i$. Then there exists a natural bijection between the following two sets.

- The set $U_{S,\mathbf{n}}$ of $(m+1)$-tuples of symmetric matrices $M = (M_0, M_1, \ldots, M_m)$ of size $(|\mathbf{n}|+1)$ with entries in k satisfying

$$\mathrm{disc}(M) = uF_1^{n_1} \cdot F_2^{n_2} \cdot \cdots \cdot F_r^{n_r}$$

  for some $u \in k^\times$.

- The set $V_{S,\mathbf{n}}$ of equivalence classes of triples $(\mathcal{M}, \lambda, s)$, where
  - $\mathcal{M}$ is a coherent $\mathcal{O}_{\mathbb{P}^m}$-module satisfying the following conditions:
    - $\mathcal{M}$ is arithmetically Cohen–Macaulay, and
    - $\mathcal{M}$ is pure of dimension $m-1$, and
    - $\mathrm{Supp}(\mathcal{M}) \subset S$, and
    - $\mathrm{length}_{\mathcal{O}_{\mathbb{P}^m,\eta_i}}(\mathcal{M}_{\eta_i}) = n_i$, and
    - $H^0(\mathbb{P}^m, \mathcal{M}) = 0$ and $\dim H^0(\mathbb{P}^m, \mathcal{M}(1)) = |\mathbf{n}| + 1$.
  - λ is a symmetric quasi-isomorphism

$$\lambda \colon \mathcal{M} \xrightarrow{\sim} \mathrm{R}\mathscr{H}om_S(\mathcal{M}(2-m), \omega_S[-m+1])\,.$$

  - $s = \{s_0, s_1, \ldots, s_{|\mathbf{n}|}\}$ is an ordered k-basis of $H^0(\mathbb{P}^m, \mathcal{M}(1))$.

  Here, two triples $(\mathcal{M}, \lambda, s), (\mathcal{M}', \lambda', s')$ are said to be *equivalent* if there exists an isomorphism $\rho \colon \mathcal{M}' \xrightarrow{\sim} \mathcal{M}$ of $\mathcal{O}_{\mathbb{P}^m}$-modules satisfying
  - ${}^t\rho \circ \lambda \circ \rho = \lambda'$, and
  - $\rho(s'_i) = s_i$ for any $0 \le i \le |\mathbf{n}|$.

Proof. Let us take an element $M \in U_{S,\mathbf{n}}$. Then, as before, we have the following exact sequence

$$0 \longrightarrow \bigoplus_{i=0}^{|\mathbf{n}|} \mathcal{O}_{\mathbb{P}^m}(-2) \xrightarrow{M} \bigoplus_{i=0}^{|\mathbf{n}|} \mathcal{O}_{\mathbb{P}^m}(-1)e_i \xrightarrow{p} \mathcal{M} \longrightarrow 0.$$

Localizing this sequence on the generic point η_i of S_i, we have the short exact sequence of $\mathcal{O}_{\mathbb{P}^m,\eta_i}$-modules:

$$0 \longrightarrow \bigoplus_{i=0}^{|\mathbf{n}|} \mathcal{O}_{\mathbb{P}^m,\eta_i}(-2) \xrightarrow{M} \bigoplus_{i=0}^{|\mathbf{n}|} \mathcal{O}_{\mathbb{P}^m,\eta_i}(-1)e_i \xrightarrow{p} \mathcal{M}_{\eta_i} \longrightarrow 0.$$

By [**9**, Lemma A.2.6], we have

$$\begin{aligned}\operatorname{length}_{\mathcal{O}_{\mathbb{P}^m,\eta_i}}(\mathcal{M}_{\eta_i}) &= \operatorname{length}_{\mathcal{O}_{\mathbb{P}^m,\eta_i}}(\mathcal{O}_{\mathbb{P}^m,\eta_i}/\operatorname{disc}(M)) \\ &= \operatorname{ord}_{F_i}(\operatorname{disc}(M)) \\ &= n_i.\end{aligned}$$

Hence $\phi_c(M) \in V_{S,\mathbf{n}}$.

Conversely, take a representative $(\mathcal{M}, \lambda, s)$ of an element of $V_{S,\mathbf{n}}$. Again by [**9**, Lemma A.2.6], we have

$$\operatorname{ord}_{F_i}(\operatorname{disc}(M)) = n_i.$$

Since we assume that

$$\dim H^0(\mathbb{P}^m, \mathcal{M}(1)) = |\mathbf{n}| + 1,$$

we have

$$\begin{aligned}\deg \operatorname{disc}(M) &= |\mathbf{n}| + 1 \\ &= \sum_i n_i \deg[\eta_i] \\ &= \sum_i n_i \deg F_i.\end{aligned}$$

Hence we have

$$\operatorname{disc}(M) = uF_1^{n_1} \cdot F_2^{n_2} \cdot \cdots \cdot F_r^{n_r}$$

for some $u \in k^\times$. Thus $\psi_c([(\mathcal{M}, \lambda, s)]) \in U_{S,\mathbf{n}}$. □

4.2.2. Bijections between theta characteristics and symmetric matrices. Now we define the subset W_{gr} of W_{nv}. It corresponds to theta characteristics on geometrically reduced hypersurfaces.

Definition 4.2.4. The subset $W_{\mathrm{gr}} \subset W_{\mathrm{nv}}$ consists of elements whose discriminant polynomial has no multiple factor over an algebraic closure $\overline{k}$ of k.

This subset is stable under the $\mathrm{GL}_{m+1}(k) \times \mathrm{GL}_{n+1}(k)$-action.

As a special case of Proposition 4.2.3, we have the following corollary which determines what matrices correspond to theta characteristics on a geometrically reduced hypersurface $S \subset \mathbb{P}^m$.

Corollary 4.2.5 (Theorem 1.3.1). Let $S \subset \mathbb{P}^m$ be a geometrically reduced hypersurface over k. Then there exists a bijection between the following two sets.

- The set of $(k^\times I_{m+1}) \times \mathrm{GL}_{n+1}(k)$-orbits of $(m+1)$-tuples of symmetric matrices $M = (M_0, M_1, \ldots, M_m)$ of size $n+1$ with entries in k such that the equation $(\mathrm{disc}(M) = 0)$ defines S.
- The set $\mathrm{TC}_{m+1,n+1}(k)_S$ of equivalence classes of pairs $(\mathcal{M}, \lambda)$ with respect to $\sim_1$, where
  - $\mathcal{M}$ is a theta characteristic on S, and
  - λ is a quasi-isomorphism

$$\lambda\colon \mathcal{M} \xrightarrow{\sim} \mathrm{R}\mathscr{H}om_S(\mathcal{M}(2-m), \omega_S[1-m])\,.$$

Proof. We only have to show that $\mathcal{M}$ can be considered as a coherent $\mathcal{O}_S$-module. It is enough to show that

$$\mathrm{disc}(M)\mathcal{M} = 0.$$

Take the free resolution of $\mathcal{M}$, and consider the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \bigoplus_{i=0}^{n} \mathcal{O}_{\mathbb{P}^m}(-2) \arrow[r, "M"] \arrow[d, "\times \mathrm{disc}(M)"] & \bigoplus_{i=0}^{n} \mathcal{O}_{\mathbb{P}^m}(-1) \arrow[r, "q"] \arrow[d, "\times \mathrm{disc}(M)"] & \mathcal{M} \arrow[r] \arrow[d, "\times \mathrm{disc}(M)"] & 0 \\
0 \arrow[r] & \bigoplus_{i=0}^{n} \mathcal{O}_{\mathbb{P}^m}(-2) \arrow[r, "M"] & \bigoplus_{i=0}^{n} \mathcal{O}_{\mathbb{P}^m}(-1) \arrow[r, "q"] & \mathcal{M} \arrow[r] & 0.
\end{tikzcd}$$

To prove the condition $\mathrm{disc}(M)\mathcal{M} = 0$, it suffices to show

$$\mathrm{disc}(M) \bigoplus_{i=0}^{n} \mathcal{O}_{\mathbb{P}^m}(-n-2) \subset M \bigoplus_{i=0}^{n} \mathcal{O}_{\mathbb{P}^m}(-2).$$

Take the adjugate matrix $\mathrm{adj}(M)$ of M, then

$$M\bigoplus_{i=0}^{n}\mathcal{O}_{\mathbb{P}^m}(-2)\supset M\mathrm{adj}(M)\bigoplus_{i=0}^{n}\mathcal{O}_{\mathbb{P}^m}(-n-2)=\mathrm{disc}(M)\bigoplus_{i=0}^{n}\mathcal{O}_{\mathbb{P}^m}(-n-2).\quad\square$$

4.2.3. Proof of the main theorem. We define the set $\mathrm{TC}_{m+1,n+1}(k)$ as

$$\mathrm{TC}_{m+1,n+1}(k):=\bigcup_{S}\mathrm{TC}_{m+1,n+1}(k)_S$$

where S runs over all geometrically reduced hypersurfaces in $\mathbb{P}^m$ defined over k. We also define the map

$$\Phi_{m+1,n+1}\colon W_{\mathrm{gr}}\to\mathrm{TC}_{m+1,n+1}(k)$$

so that $\Phi_{m+1,n+1}(M)$ is the equivalence class $[(S,\mathcal{M},\lambda)]\in\mathrm{TC}_{m+1,n+1}(k)$ of triples $(S,\mathcal{M},\lambda)$, where $\phi_c(M)=[(\mathcal{M},\lambda,s)]$ and $\mathrm{Supp}(\mathcal{M})=S$. The surjectivity follows from Corollary 4.2.5, so we only have to study the fiber of $\Phi_{m+1,n+1}$. In other words, we investigate how many equivalence classes of quasi-isomorphisms λ with respect to $\sim_1$.

Lemma 4.2.6. Take any hypersurface $S\subset\mathbb{P}^m$ over k, and let $\mathcal{M}$ be a coherent $\mathcal{O}_S$-module satisfying the conditions in Proposition 4.2.3. The endomorphism sheaf $\mathscr{E}nd_S(\mathcal{M})$ is embedded into $\prod_{\eta\in\mathrm{Gen}(S)}\mathscr{E}nd_S(i_{\eta,*}\mathcal{M}_\eta)$. In particular, if S is geometrically reduced and $\mathcal{M}$ is a theta characteristic on S, the $\mathcal{O}_S$-algebra $\mathscr{E}nd_S(\mathcal{M})$ is commutative and the k-algebra of its global sections $L_0=\mathrm{End}_S(\mathcal{M})=L$ is an étale k-algebra of finite dimension.

Proof. Take any non-zero element f in $\mathscr{E}nd_S(\mathcal{M})\,(U)=\mathrm{End}_U(\mathcal{M}(U))$ for an open subscheme $i_U\colon U\hookrightarrow S$. Then f also gives endomorphisms f_U of $i_U^*\mathcal{M}$ and f_η of $i_{\eta,*}\mathcal{M}_\eta$ for $\eta\in\mathrm{Gen}(U)$.

If f_η is the zero endomorphism for any $\eta \in \mathrm{Gen}(U)$, the image $\mathrm{Im}(f_U) \subset i_U^*\mathcal{M}$ is supported on a subscheme of U of dimension less than $m-1$. Since $i_U^*\mathcal{M}$ is pure of dimension $m-1$, and we have $f = 0$.

If S is geometrically reduced and $\mathcal{M}$ is a theta characteristic on S, $\mathscr{E}nd_S(i_{\eta,*}\mathcal{M}_\eta)$ is isomorphic to $i_{\eta,*}\mathcal{O}_{S,\eta}$ since $\mathcal{M}_\eta$ is a free $\mathcal{O}_{S,\eta}$-module of rank one. In particular, $\mathscr{E}nd_S(i_{\eta,*}\mathcal{M}_\eta)$ is commutative and moreover a discrete valuation ring. Hence the ring of global endomorphism $\mathrm{End}_S(i_{\eta,*}\mathcal{M}_\eta)$ is commutative, and the subring L_0 is commutative.

On the other hand, for any element $l \in L_0$, the difference $\sigma(l) - l$ between l and $\sigma(l)$ is zero at least over smooth points $\mathrm{Sm}(S)$. Hence $\mathrm{Im}(\sigma(l) - l)$ has a dimension less than $m-1$, and by purity of $\mathcal{M}$, we have $\sigma(l) = l$. This shows $L_0 = L$.

Finally we shall show that L_0 is an étale k-algebra if S is geometrically reduced. By descent theory, we may assume that k is algebraically closed. The algebra L_0 is a finite dimensional k-subalgebra of the product of discrete valuation rings $\prod_{\eta\in\mathrm{Gen}(S)} \mathrm{End}_S(i_{\eta,*}\mathcal{M}_\eta)$. Hence L_0 has no nilpotent elements, and L_0 is an étale k-algebra. □

Proof (Proof of Theorem 1.3.2). We fix a geometrically reduced hypersurface S over k, and a theta characteristic $\mathcal{M}$ on S. The set of symmetric quasi-isomorphisms

$$\mathcal{M} \xrightarrow{\sim} \mathrm{R}\mathscr{H}om_S(\mathcal{M}(2-m), \omega_S[1-m])$$

admits a simply transitive action of $L^\times$ by Proposition 3.4.3(i) and Lemma 4.2.6. By Lemma 4.2.6, we see that $L_1 = k^\times L^{\times 2}$. Hence the equivalence class with respect to $\sim_1$ admits a simply transitive action of $L^\times/k^\times L^{\times 2}$. This finishes the proof of Theorem 1.3.2.

□

Corollary 4.2.7. Assume that *at least one* of the following conditions is satisfied:

- the base field k is separably closed of characteristic different from two, or
- the base field k is perfect of characteristic two, or
- the hypersurface $S \subset \mathbb{P}^m$ is geometrically integral.

Then the set of equivalence classes of triples $(S, \mathcal{M}, \lambda)$ with fixed $\mathcal{M}$ consists of only one element.

Proof. Recall that $L = L_0$ is an étale k-algebra. If we assume the first or second condition, the algebra L is a product of k. Then we have $L^\times = L^{\times 2}$. Hence the group $L^\times / k^\times L^{\times 2}$ is a trivial group. Since the equivalence classes of triples has a simply transitive action of trivial group, it is a singleton.

We assume the third condition. We may assume that k is algebraically closed. Then L is a finite dimensional étale k-algebra embedded to a discrete valuation ring, hence $L = k$. Thus we have $L^\times = k^\times$, and the group $L^\times / k^\times L^{\times 2}$ is a trivial group. Therefore, the fiber is a singleton. □

4.2.4. Symmetric determinantal representations. By Corollary 4.2.7, we have an immediate application to symmetric determinantal representation.

Definition 4.2.8. Let $S \subset \mathbb{P}^m$ be a hypersurface over k and F a defining equation of S. The *symmetric determinantal representation* of S is an $(m+1)$-tuple $M = (M_0, M_1, \ldots, M_m) \in W$ of symmetric matrices of size $n+1$ with entries in k satisfying

$$\mathrm{disc}(M) = uF(X_0, X_1, \ldots, X_m)$$

for some $u \in k^\times$.

Two symmetric determinantal representations M, M' of S are said to be *equivalent* if there exist a constant $a \in k^\times$ and a matrix $P \in \mathrm{GL}_{n+1}(k)$ such that

$$aM \cdot P (= a^t PMP) = M'.$$

Take a geometrically reduced hypersurface $S \subset \mathbb{P}^m$. By definition, the set of equivalence classes of symmetric determinantal representations of S is bijective to the set of $(k^\times I_{m+1}) \times \mathrm{GL}_{n+1}(k)$-orbits in W_{gr}. By Corollary 4.2.5, we have

Proposition 4.2.9. The set of equivalence classes of symmetric determinantal representations of S is bijective to $\mathrm{TC}_{m+1,n+1}(k)_S$.

Hence, in order to determine the set of the equivalence classes of symmetric determinantal representations, it suffices to check two points.

- Collect all non-effective theta characteristics on S defined over k.
- For any non-effective theta characteristic $\mathcal{M}$ on S, calculate the algebra $L = \mathrm{End}_S(\mathcal{M})$ and the group $L^\times / k^\times L^{\times 2}$.

It seems rare that we can check two steps completely. However, we can check the existence of the symmetric determinantal representations of S by checking the existence of non-effective theta characteristics on S. On the other hand, we can show the infiniteness of equivalence classes of symmetric determinantal representations of S via the investigation of the group $L^\times / k^\times L^{\times 2}$ for some non-effective theta characteristics $\mathcal{M}$ on S. For those examples, see Chapter 6.

CHAPTER 5

Projective automorphism groups of complete intersections of quadrics

In Proposition 3.4.3, we give an orbit interpretation of symmetric quasi-isomorphisms of a theta characteristic $\mathcal{M}$ on S via the k-algebra $L_0 = \mathrm{End}_S(\mathcal{M})$. The algebra L_0 has another application: a description of the projective automorphism group of complete intersections of quadrics. We investigate them in this chapter.

Preliminaries. In this chapter, we assume that k is a field of *characteristic different from two.* We fix a quasi-isomorphism

$$c\colon \omega_{\mathbb{P}^m} \xrightarrow{\sim} \mathcal{O}_{\mathbb{P}^m}(-1-m)[m].$$

We identify the point of $\mathbb{P}^n$ with the one-dimensional subspaces of k^{n+1}, the k-vector space spanned by $(n+1)$-dimensional column vectors.

5.1. Quadrics

In this section, we recall some facts on quadrics. For a reference, see [**10**, Chapter 6].

5.1.1. Quadrics and Gram matrices. A *quadric* Q in $\mathbb{P}^n$ over k is a hypersurface of degree two. It is defined by a homogeneous polynomial $q = q_Q \in k[X_0, X_1, \ldots, X_n]$ of degree two, i.e. a *quadratic form* in $n+1$ variables $X_0, X_1, \ldots, X_n$.

Over a field k of characteristic different from two, any quadratic form q_Q has the corresponding symmetric matrix M_Q. The matrix M_Q is called the *Gram matrix* for the

quadratic form q_Q. Concretely, each quadratic form q_Q is uniquely represented as

$$q_Q(X_0, X_1, \ldots, X_n) = \sum_{0 \le i \le j \le n} a_{i,j} X_i X_j.$$

Then the Gram matrix M_Q for q_Q is

$$\begin{pmatrix} a_{0,0} & \frac{a_{0,1}}{2} & \cdots & \frac{a_{0,n}}{2} \\ \frac{a_{0,1}}{2} & a_{1,1} & \cdots & \frac{a_{1,n}}{2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{a_{0,n}}{2} & \frac{a_{1,n}}{2} & \cdots & a_{n,n} \end{pmatrix}.$$

This satisfies

$${}^t x M_Q x = q_Q(x).$$

Symmetric matrices of size $n+1$ with entries in k correspond bijectively to quadratic forms in $n+1$ variables. We identify the two interpretations.

Recall that the symbol $\mathrm{Sym}_2 k^{n+1}$ denotes the set of symmetric matrices of size $n+1$ with entries in k. Since q_Q is nonzero for a quadric Q and determined up to multiplication by elements of $k^\times$, the projective space $\mathbb{P}(\mathrm{Sym}_2 k^{n+1})$ of one-dimensional subspaces in $\mathrm{Sym}_2 k^{n+1}$ is in bijection with the set of all quadrics in $\mathbb{P}^n$.

The tangent space of Q at $x \in Q$ is identified with the subspace

$$\left\{ y \in k^{n+1} \;\middle|\; {}^t y M_Q x \; (= {}^t x M_Q y) = 0 \right\}.$$

Hence a point $x \in Q$ is singular if and only if $M_Q x = 0$. A quadric Q is singular if and only if $\det(M_Q) = 0$. The singular locus $\mathrm{Sing}(Q)$ is the linear subspace defined by $(M_Q x = 0)$. It is a $(\mathrm{corank}(M_Q) - 1)$-dimensional linear subvariety of $\mathbb{P}^n$, where the corank of M_Q is

defined by

$$\mathrm{corank}(M_Q) := n + 1 - \mathrm{rank}(M_Q).$$

The locus of singular quadrics in $\mathbb{P}(\mathrm{Sym}_2 k^{n+1})$ is a hypersurface Δ defined by $(\det M_Q = 0)$. It is known that the degeneracy locus of the hypersurface Δ corresponds to the quadrics whose coranks are greater than one.

5.1.2. Linear systems of quadrics. Let us consider the linear subvarieties of $\mathbb{P}(\mathrm{Sym}_2 k^{n+1})$. They are called *linear systems of quadrics* in $\mathbb{P}^n$.

Remark 5.1.1. When the dimension of a linear system is one (resp. two, three), we call the linear system a *pencil* (resp. *net*, *web*) of quadrics.

Take an m-dimensional linear system $\Pi_Q \subset \mathbb{P}(\mathrm{Sym}_2 k^{n+1})$ of quadrics. Let $Q = (Q_0, Q_1, \ldots, Q_m) \in \Pi_Q$ be an $(m+1)$-tuple of quadrics which is not contained in any hyperplane in Π_Q. We write

$$\Pi_Q = \langle Q_0, Q_1, \ldots, Q_m \rangle.$$

The base locus X_Q of the linear system Π_Q is defined by

$$X_Q = Q_0 \cap Q_1 \cap \cdots \cap Q_m.$$

The $(m+1)$-tuple Q of quadrics gives an $(m+1)$-tuple of symmetric matrices

$$M_Q = (M_0, M_1, \ldots, M_m).$$

If X_Q is a complete intersection, i.e. $\dim(X_Q) = n - m - 1 \geq 0$ (see [**14**, II, Exercise 8.4]), then the linear system Π_Q is characterized as the quadrics containing X_Q as closed

subschemes by the following lemma. Note that the following lemma holds true even when $\dim X = 0$ or X has singularities.

Lemma 5.1.2. Let n, r, d be positive integers satisfying $n \geq r \geq 1$. Let $F_1, \ldots, F_r \in k[X_0, X_1, \ldots, X_n]$ be non-zero homogeneous polynomials of degree d defining a complete intersection of r hypersurfaces of degree d in $\mathbb{P}^n$. The closed subscheme

$$X := \{(u_0 : u_1 : \ldots : u_n) \in \mathbb{P}^n \mid F_i(u_0, u_1, \ldots, u_n) = 0 \ (i = 1, \ldots, r)\}$$

of $\mathbb{P}^n$ is of dimension $n - r$. Then any non-zero homogeneous polynomial of degree d vanishing on X is written as a k-linear combination of $F_1, \ldots, F_r$.

Proof. It is enough to prove that the kernel of the map $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) \longrightarrow H^0(X, \mathcal{O}_X(d))$ is spanned by $F_1, \ldots, F_r$. We shall prove it by induction on r. When $r = 1$, the assertion follows from the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n} \xrightarrow{\times F_1} \mathcal{O}_{\mathbb{P}^n}(d) \longrightarrow \mathcal{O}_X(d) \longrightarrow 0.$$

When $r > 1$, we put

$$Y := \{(u_0 : u_1 : \ldots : u_n) \in \mathbb{P}^n \mid F_i(u_0, u_1, \ldots, u_n) = 0 \ (i = 1, \ldots, r-1)\}.$$

Then Y is a complete intersection of $r - 1$ hypersurfaces of degree d containing X. From the exact sequence

$$0 \longrightarrow \mathcal{O}_Y \xrightarrow{\times F_r} \mathcal{O}_Y(d) \longrightarrow \mathcal{O}_X(d) \longrightarrow 0,$$

we see that the kernel of $H^0(Y, \mathbb{P}_Y(d)) \longrightarrow H^0(X, \mathcal{O}_X(d))$ is spanned by the image of F_r. Since the kernel of $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) \longrightarrow H^0(Y, \mathcal{O}_Y(d))$ is spanned by $F_1, \ldots, F_{r-1}$ by induction hypothesis, we conclude that the kernel of $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) \longrightarrow H^0(X, \mathcal{O}_Y(d))$ is spanned by $F_1, \ldots, F_{r-1}, F_r$. $\square$

Hence a complete intersection of $m+1$ quadrics in $\mathbb{P}^n$ gives a $\mathrm{GL}_{m+1}(k) \times (k^\times I_{n+1})$-orbit of $W = k^{m+1} \otimes \mathrm{Sym}_2 k^{n+1}$.

Moreover, the projective equivalence class of X_Q defines a $\mathrm{GL}_{m+1}(k) \times \mathrm{GL}_{n+1}(k)$-orbit of W. Here, two subschemes $Y \subset \mathbb{P}^n$ and $Z \subset \mathbb{P}^n$ are *projectively equivalent* if there exists an automorphism

$$\nu \colon \mathbb{P}^n \xrightarrow{\sim} \mathbb{P}^n$$

such that it induces an isomorphism of subschemes $Y \xrightarrow{\sim} Z$. This interpretation permits us to consider the projective equivalence classes of complete intersections of $m+1$ quadrics in $\mathbb{P}^n$ as $\mathrm{GL}_{m+1}(k) \times \mathrm{GL}_{n+1}(k)$-orbits of W.

5.2. A description of the projective automorphism groups

We fix an element $M_Q \in W_{\mathrm{nv}}$, and assume that M_Q defines a complete intersection X_Q of $m+1$ quadrics in $\mathbb{P}^n$. Let us take a representative $(\mathcal{M}, \lambda, s)$ of $\phi_c(M_Q)$. We put $L_0 = \mathrm{End}_{\Delta_Q}(\mathcal{M})$ and use symbols defined in Section 3.4 freely. We consider the norm map Nm of L_0 defined as

$$\mathrm{Nm} \colon L_0 \longrightarrow L \quad ; \quad l \mapsto \sigma(l)l.$$

Obviously $\mathrm{Nm}(k^\times) \subset k^\times$, so the norm map induces

$$\overline{\mathrm{Nm}} \colon L_0^\times / k^\times \longrightarrow (L \cap L_0^\times)/k^\times.$$

The projective automorphism group of X_Q is defined by

$$\mathrm{Aut}_{\mathbb{P}^n}(X_Q) := \{g \in \mathrm{Aut}(\mathbb{P}^n) \mid gX_Q = X_Q\}.$$

We define another group $\mathrm{Aut}_{\Pi_Q}(\Delta_Q, \mathcal{M}, \lambda)$ as

$$\{A \in \mathrm{GL}_{m+1}(k) \mid \nu_A(\Delta_Q) = \Delta_Q, (\nu_A^*\mathcal{M}, \nu_A^*\lambda) \sim_1 (\mathcal{M}, \lambda)\} / k^\times I_{m+1}.$$

Now we can state the main theorem in this chapter.

Theorem 5.2.1. Take $M \in W_{\mathrm{nv}}$ which defines a complete intersection X_Q of $m+1$ quadrics. Write $\phi_c(M) = [(\mathcal{M}, \lambda, s)] \in V_{m+1,n+1}$. Then there exists the following short exact sequence

$$0 \longrightarrow \mathrm{Ker}(\overline{\mathrm{Nm}}) \longrightarrow \mathrm{Aut}_{\mathbb{P}^n}(X_Q) \longrightarrow \mathrm{Aut}_{\Pi_Q}(\Delta_Q, \mathcal{M}, \lambda) \longrightarrow 0.$$

Beauville proved this theorem when $m = 2, n \geq 3, k$ is algebraically closed, and X_Q is a smooth complete intersection of three quadrics ([**1**, Proposition 6.19]).

Remark 5.2.2. Our proof concludes a stronger assertion. Besides the assumptions $L_0 = L$ and L_0 is commutative, we assume that the following condition for X_Q holds: $\Pi_Q \cong \mathbb{P}^m$, and if a quadric Q' contains the intersection X_Q as a closed subscheme, the quadric Q' is an element of Π_Q. Then we also have the short exact sequence as in Theorem 5.2.1.

5.2.1. A description of the projective automorphism groups. In this subsection we give a short exact sequence of groups. It provides us an interpretation of the projective

automorphism group $\mathrm{Aut}_{\mathbb{P}^n}(X_Q)$ of a complete intersection X_Q of quadrics as an extension of two groups. Then we prove Theorem 5.2.1.

Take X_Q a complete intersection of $m+1$ quadrics, and fix $m+1$ quadrics

$$Q = (Q_0, Q_1, \ldots, Q_m)$$

defining X_Q. As before, we fix a quasi-isomorphism of complexes of coherent $\mathcal{O}_{\mathbb{P}^m}$-modules

$$c\colon \omega_{\Pi_Q} \xrightarrow{\sim} \mathcal{O}_{\Pi_Q}(-1-m)[m].$$

Assume that the corresponding orbit contained in W_{nv}, and take a representative $M \in W_{\mathrm{nv}}$. Let us consider the projective automorphism group $\mathrm{Aut}_{\mathbb{P}^n}(X_Q)$ of X_Q.

Note that $\mathrm{Aut}(\mathbb{P}^n) \cong \mathrm{PGL}_{n+1}(k)$. Since X_Q is a complete intersection of Q, if we take $P \in \mathrm{GL}_{n+1}(k)$ such that $\overline{P} \in \mathrm{Aut}_{\mathbb{P}^n}(X_Q)$, then the matrix P preserves Π_Q. Hence there exists a matrix $A \in \mathrm{GL}_{m+1}(k)$ such that

$${}^tPM_QP = M_Q \cdot A.$$

Thus we have

$$\mathrm{Aut}_{\mathbb{P}^n}(X_Q) \cong \left\{ P \in \mathrm{GL}_{m+1}(k) \;\middle|\; {}^tPM_QP = M_Q \cdot A \quad (\exists A \in \mathrm{GL}_{m+1}(k)) \right\} / (k^\times I_{n+1}).$$

Let us define the groups $G_Q, F'_Q, E_Q, F_Q, P_Q, H_Q$ by

$$G_Q := \left\{(A,P) \in \mathrm{GL}_{m+1}(k) \times \mathrm{GL}_{n+1}(k) \;\middle|\; {}^tPM_QP = M_Q \cdot A\right\},$$

$$F'_Q := \left\{(u,P) \in k^\times \times \mathrm{GL}_{n+1}(k) \;\middle|\; {}^tPM_QP = uM_Q\right\} \lhd G_Q,$$

$$E_Q := \left\{P \in \mathrm{GL}_{n+1}(k) \;\middle|\; {}^tPM_QP = M_Q \cdot A \quad (\exists A \in \mathrm{GL}_{m+1}(k))\right\},$$

$$F_Q := \left\{P \in \mathrm{GL}_{n+1}(k) \;\middle|\; {}^tPM_QP = uM_Q \quad (\exists u \in k^\times)\right\} \lhd E_Q,$$

$$P_Q := \left\{(a^2 I_{m+1}, aI_{n+1}) \;\middle|\; a \in k^\times\right\} \subset G_Q,$$

$$H_Q := E_Q/F_Q.$$

The group F'_Q is a normal subgroup of G_Q and F_Q is a normal subgroup of E_Q. Obviously, E_Q has a canonical surjection onto $\mathrm{Aut}_{\mathbb{P}^n}(X_Q)$. Additionally, we find G_Q is isomorphic to E_Q.

Lemma 5.2.3. The second projection $\mathrm{pr}\colon G_Q \longrightarrow E_Q$ is an isomorphism. In particular, it induces an isomorphism $F'_Q \xrightarrow{\sim} F_Q$.

Proof. For each $P \in E_Q$, a matrix $A \in \mathrm{GL}_{m+1}(k)$ with

$${}^tPM_QP = M_Q \cdot A$$

is determined by the action of P on the k-vector space spanned by $M_{Q_0}, M_{Q_1}, \ldots, M_{Q_m}$. Hence there exists a unique matrix A satisfies this condition. $\square$

By this lemma, we have $H_Q \cong G_Q/F'_Q$. The kernel of

$$\mathrm{pr}_1\colon G_Q \longrightarrow \mathrm{GL}_{m+1}(k)$$

is obviously contained in F'_Q, hence we also have

$$H_Q \cong \mathrm{pr}_1(G_Q)/\,\mathrm{pr}_1(F'_Q).$$

Lemma 5.2.4. The kernel of the composite map

$$G_Q \xrightarrow[\mathrm{pr}]{\sim} E_Q \longrightarrow \mathrm{Aut}_{\Pi_Q}(X_Q)$$

coincides with P_Q.

Proof. If (A, P) is an element in the kernel, we have $P = aI_{n+1}$ for some $a \in k^\times$. Then we have $a^2 M_Q = M_Q \cdot A$. Hence we have $A = a^2 I_{m+1}$. $\square$

Thus we have the following commutative diagram with exact rows and exact columns:

$$\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & & \\
 & P_Q \arrow[r, equal] \arrow[d] & P_Q \arrow[d] & & \\
1 \arrow[r] & F_Q \arrow[r] \arrow[d] & E_Q = G_Q \arrow[r] \arrow[d] & H_Q \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & F_Q/P_Q \arrow[r] \arrow[d] & \mathrm{Aut}_{\mathbb{P}^n}(X_Q) \arrow[r] \arrow[d] & H_Q \arrow[r] & 1 \\
 & 1 & 1. & &
\end{tikzcd}$$

In order to prove Theorem 5.2.1, it is enough to find an interpretation of the groups F_Q/P_Q and H_Q.

Lemma 5.2.5. We have

$$F_Q \cong \left\{ l \in L_0^\times \;\middle|\; \sigma(l)l \in k \right\}.$$

Proof. Take $P \in F_Q$ and $u \in k^\times$ satisfying ${}^tPM_QP = uM_Q$. Since we have ${}^tPM_Q = uM_QP^{-1}$, we have $l := (P, uP^{-1}) \in L_0$ by Proposition 3.4.1. In particular, $\sigma(l)l = u \in k^\times$. Conversely, if $l = (P, P') \in L_0$ satisfies $\sigma(l)l = u \in k^\times$, we immediately see $P = uP'^{-1}$. Hence we have ${}^tPM_QP = uM_Q$. □

Proof (Proof of Theorem 5.2.1). By Lemma 5.2.5, we have

$$F_Q/P_Q \cong \mathrm{Ker}(\overline{\mathrm{Nm}})$$

because $\overline{\mathrm{Nm}}\colon L_0^\times/k^\times \longrightarrow L_0^\times/k^\times$ is defined by $\overline{\mathrm{Nm}}(l) = \sigma(l)l$.

Next we consider the group H_Q. Let $(A, P) \in G_Q$ be an element. We have isomorphisms

$$\nu_A\colon \Pi_Q \xrightarrow{\sim} \Pi_Q,$$

$$\rho\colon \nu_A^*\mathcal{M} \xrightarrow{\sim} \mathcal{M}$$

satisfying ${}^t\rho \circ \lambda \circ \rho = \nu_A^*\lambda$. This shows $\nu_A \in \mathrm{Aut}(\Pi_Q)$ induces a projective automorphism of $\Delta_Q = \mathrm{Supp}(\mathcal{M}) \subset \Pi_Q$. Hence the equivalence class $\overline{A} \in \mathrm{PGL}_{m+1}(k)$ of A is an element of

$$\mathrm{Aut}_{\Pi_Q}(\Delta_Q, \mathcal{M}, \lambda) := \left\{ A \in \mathrm{GL}_{m+1}(k) \;\middle|\; \begin{array}{l} \nu_A(\Delta_Q) = \Delta_Q, \\ \\ (\nu_A^*\mathcal{M}, \nu_A^*\lambda) \sim_1 (\mathcal{M}, \lambda) \end{array} \right\} / (k^\times I_{m+1}),$$

If (A, P) is an element of F'_Q, then we have $\nu_A = \mathrm{id}_{\Pi_Q}$ as a projective automorphism. Conversely, if $\nu_A = \mathrm{id}_{\Pi_Q}$ then we can write $A = aI_{m+1}$ for some $a \in k^\times$. Hence $A \in \mathrm{pr}_1(F'_Q)$. Thus $H_Q = \mathrm{pr}_1(G_Q)/\mathrm{pr}_1(F'_Q)$ is a subgroup of $\mathrm{Aut}_{\Pi_Q}(\Delta_Q, \mathcal{M}, \lambda)$.

If we take an element $A \in \mathrm{GL}_{m+1}(k)$ whose equivalence class $\overline{A} \in \mathrm{PGL}_{m+1}(k)$ is an element of $\mathrm{Aut}_{\Pi_Q}(\Delta_Q, \mathcal{M}, \lambda)$, there exists an isomorphism

$$\rho\colon \nu_A^*\mathcal{M} \xrightarrow{\sim} \mathcal{M}$$

satisfying ${}^t\rho \circ \lambda \circ \rho = \nu_A^*\lambda$. By a similar argument to the proof of Proposition 3.4.3, we have a matrix $P \in \mathrm{GL}_{n+1}(k)$ such that ${}^tPM_QP = M_Q \cdot A$. This shows $(A, P) \in G_Q$ and $\nu_A \in \mathrm{pr}_1(G_Q)/\mathrm{pr}_1(F'_Q)$. Thus we finish the proof of Theorem 5.2.1. $\square$

CHAPTER 6

Symmetric determinantal representations of plane curves

As a related topic of theta characteristics, we study the symmetric determinantal representations of hypersurfaces. We recall first the definition of symmetric determinantal representations and its equivalence in Section 6.1. We also give some examples of symmetric determinantal representations in Section 6.1. We state a characterization of the existence for low-degree smooth plane curves in Section 6.2. In Section 6.3, we introduce some results on symmetric determinantal representations of smooth plane curves over global fields. These results are joint works with Tetsushi Ito.

6.1. Symmetric determinantal representations and examples

Recall the definition of symmetric determinantal representations from Section 4.2.4.

Definition 6.1.1. Let $S \subset \mathbb{P}^m$ be a hypersurface over k and F a defining equation of S. The *symmetric determinantal representation* of S is an $(m+1)$-tuple $M = (M_0, M_1, \ldots, M_m) \in W$ of symmetric matrices of size $n+1$ with entries in k satisfying

$$\mathrm{disc}(M) = uF(X_0, X_1, \ldots, X_m)$$

for some $u \in k^\times$.

Two symmetric determinantal representations M, M' of S are said to be *equivalent* if there exist a constant $a \in k^\times$ and a matrix $P \in \mathrm{GL}_{n+1}(k)$ such that

$$aM \cdot P \ (= a^t PMP) = M'.$$

We showed the bijection between the equivalence classes of symmetric determinantal representations and $\mathrm{TC}_{m+1,n+1}(k)_S$ in Proposition 4.2.9.

Classically, Hesse showed there are three equivalence classes of symmetric determinantal representations of smooth cubics over $\mathbb{C}$. Moreover, he showed there are thirty-six equivalence classes of symmetric determinantal representations of smooth quartics over $\mathbb{C}$. Since there are at most $2^{g-1}(2^g+1)$ non-effective theta characteristics on a smooth curve of genus g over an algebraically closed field, there are finitely many equivalence classes over $\mathbb{C}$.

Moreover, if C is a smooth plane curve over any field k, the equivalence classes of symmetric determinantal representations of C are in bijection with non-effective theta characteristics on C by Corollary 4.2.7 and Proposition 4.2.9. Since theta characteristics on C are line bundles, if two theta characteristics $\mathcal{M}, \mathcal{M}'$ on C defined over k are isomorphic over $\overline{k}$, then they are isomorphic over k by Hilbert's theorem 90. Hence we obtain:

Proposition 6.1.2. The number of equivalence classes of symmetric determinantal representations of smooth plane curves C of genus g are at most $2^{g-1}(2^g+1)$.

The case of singular curves are more complicated. Beauville showed there is at least one symmetric determinantal representation of plane curves over $\mathbb{C}$ in [**2**]. However the number of equivalence classes of symmetric determinantal representations may become infinite even if we consider over $\mathbb{C}$. Interestingly, it is finite if the curve C defined over $\mathbb{C}$ has at worst ADE singularities (see [**23**]).

We give some examples of symmetric determinantal representations over $\mathbb{Q}$.

Example 6.1.3 (Hesse cubic curves). The *Hesse cubic curves* over $\mathbb{Q}$ are plane curves defined by the equation

$$X_0^3 + X_1^3 + X_2^3 - aX_0X_1X_2 = 0$$

for $a \in \mathbb{Q}$. If there is a $b \in \mathbb{Q}$ such that

$$2b^3 + ab^2 + 1 = 0,$$

we have a symmetric determinantal representation

$$X_0^3 + X_1^3 + X_2^3 - aX_0X_1X_2 = -\frac{1}{b^2}\det\begin{pmatrix} X_0 & bX_2 & bX_1 \\ bX_2 & X_1 & bX_0 \\ bX_1 & bX_0 & X_2 \end{pmatrix}.$$

Example 6.1.4 (Klein quartic curve). The *Klein quartic curve* over $\mathbb{Q}$ is the smooth plane quartic over $\mathbb{Q}$ defined by

$$X_0^3X_1 + X_1^3X_2 + X_2^3X_0 = 0.$$

It was already known to Klein that this curve has a symmetric determinantal representation over $\mathbb{Q}$;

$$X_0^3X_1 + X_1^3X_2 + X_2^3X_0 = -\det\begin{pmatrix} X_0 & 0 & 0 & X_1 \\ 0 & X_1 & 0 & X_2 \\ 0 & 0 & X_2 & X_0 \\ X_1 & X_2 & X_0 & 0 \end{pmatrix}.$$

Actually, it gives the only equivalence class of symmetric determinantal representations of the Klein quartic curve over $\mathbb{Q}$. For a proof, see [**20**, Theorem 1.3].

Example 6.1.5 (Edge quartics). Edge treats in [**6**] a family of plane quartics, which we call the *family of Edge quartic curves*. It is parametrized by $a \in \mathbb{Q}$ and defined by

$$(1-a^2)^2(X_0^4+X_1^4+X_2^4)-2(1+a^4)(X_0^2X_1^2+X_1^2X_2^2+X_2^2X_0^2)$$
$$= \det\begin{pmatrix} 0 & X_0-aX_1 & X_2-aX_0 & X_1+aX_2 \\ X_0-aX_1 & 0 & X_1-aX_2 & X_2+aX_0 \\ X_2-aX_0 & X_1-aX_2 & 0 & X_0+aX_1 \\ X_1+aX_2 & X_2+aX_0 & X_0+aX_1 & 0 \end{pmatrix} = 0.$$

If $a = \zeta_8$ is a primitive eighth root of unity in $\overline{\mathbb{Q}}$, we have a symmetric determinantal representation of the Fermat curve

$$F_4\colon X^4+Y^4+Z^4=0$$

of degree four over $\mathbb{Q}(\zeta_8)$.

6.2. Low-degree curves

In this section, we give a simple condition of symmetric determinantal representations of low degree smooth curves.

We recall basic properties of the Picard group and the Picard scheme (for details, see [**4**]). The *Picard group* $\mathrm{Pic}(C) := H^1\big(C, \mathcal{O}_C^\times\big)$ of C is the group of isomorphism classes of line bundles on C. If the base field k is algebraically closed, it is isomorphic to the group $\mathrm{Pic}_{C/k}(k)$ of k-valued points of the *Picard variety* $\mathrm{Pic}_{C/k}$ representing the relative Picard

functor. However, over a general field k, there is a difference between the two groups $\mathrm{Pic}_{C/k}(k)$ and $\mathrm{Pic}(C)$. In fact, there is the following exact sequence:

$$0 \longrightarrow \mathrm{Pic}(C) \longrightarrow \mathrm{Pic}_{C/k}(k) \xrightarrow{i} \mathrm{Br}(k) \xrightarrow{j} \mathrm{Br}(C) \tag{6.2.1}$$

where $\mathrm{Br}(k)$ (resp. $\mathrm{Br}(C)$) is the *Brauer group* of k (resp. C).

Denote the 2-torsion subgroup of the Picard group $\mathrm{Pic}(C)$ as $\mathrm{Pic}(C)[2]$. This is the set of isomorphism classes of line bundles $\mathcal{L}$ with $\mathcal{L} \otimes \mathcal{L} \cong \mathcal{O}_C$. If there is a theta characteristic on C, this group acts simply transitively on the set $\mathrm{TC}(k)_C$ of isomorphism classes of theta characteristics on C defined over k.

Proposition 6.2.1. Let $C \subset \mathbb{P}^2$ be a smooth conic over k. The following are equivalent.

(1) C has a k-rational point.

(2) C is isomorphic to $\mathbb{P}^1_k$.

(3) C has a symmetric determinantal representation over k.

Proof (Sketch: for details, see [18, Subsection 4.2]**).** The projective smooth curve C has genus 0. Then

$$\mathrm{Pic}_{C/k}(k) = \mathrm{Pic}_{C/k}(k^{\mathrm{sep}})^{G_k} \cong \mathbb{Z}.$$

Since the canonical bundle ω_C^{shf} is a line bundle of degree minus two defined over k. Hence $\mathrm{Pic}(C)$ is the subgroup of $\mathrm{Pic}_{C/k}(k)$ with index at most two, and the theta characteristic is, if exists, a line bundle on C of degree minus one. This says that if a theta characteristic on C exisits, it is unique and automatically non-effective, and it corresponds to the unique equivalence class of symmetric determinantal representations over k. Its isomorphism class defines a k-rational point of $\mathrm{Pic}_{C/k}(k)$.

(2) $\Leftrightarrow$ (3) : If there exists a theta characteristic $\mathcal{L}$ on C, we have $\mathrm{Pic}(C) \cong \mathrm{Pic}_{C/k}(k)$. So there is a line bundle $\mathcal{L}^{-1}$ of degree one. By Riemann–Roch theorem, the linear system defined by $\mathcal{L}^{-1}$ gives an isomorphism $C \xrightarrow{\sim} \mathbb{P}^1_k$. Conversely, if $C \cong \mathbb{P}^1_k$, the line bundle $\mathcal{O}_{\mathbb{P}^1_k}(-1)$ defines a theta characteristic on C.

(1) $\Leftrightarrow$ (2) : If C is isomorphic to $\mathbb{P}^1_k$, then C has a k-rational point. Conversely, if C has a k-rational point P, then the line bundle $\mathcal{O}_C(P)$ is a degree one line bundle, and it defines an isomorphism $C \xrightarrow{\sim} \mathbb{P}^1_k$. $\square$

Proposition 6.2.2. Let $C \subset \mathbb{P}^2$ be a smooth cubic over k. The following are equivalent.

(1) The group subscheme $\mathrm{Pic}_{C/k}[2]$ of two-torsion points in the Picard scheme $\mathrm{Pic}_{C/k}$ of C has a k-rational point not equal to the class of the trivial line bundle $\mathcal{O}_C$.

(2) C has a symmetric determinantal representation over k.

Proof (Sketch; for details, see [18, Subsection 4.3]). Since ω_C^{shf} is isomorphic to $\mathcal{O}_C$, the subset of $\mathrm{Pic}(C)$ of theta characteristics on C is actually the same as $\mathrm{Pic}(C)[2]$. Hence by Proposition 4.2.9, C has a symmetric determinantal representation over k if and only if there exists a line bundle $\mathcal{L}$ on C defined over k with $\mathcal{L} \otimes \mathcal{L} \cong \mathcal{O}_C$.

(2) $\Rightarrow$ (1) : This direction is easy. If there is a non-effective theta characteristic $\mathcal{L}$ on C over k, then $\mathrm{Pic}(C)[2]$ is a nontrivial group. By the sequence (6.2.1), $\mathrm{Pic}_{C/k}(k)[2]$ is nontrivial, so there is a nontrivial k-rational point of $\mathrm{Pic}_{C/k}[2]$.

(1) $\Rightarrow$ (2) : Let $\alpha \in \mathrm{Pic}_{C/k}(k)[2]$ be a non-zero two-torsion k-rational point. By the exact sequence (6.2.1), we have to show that $i(\alpha) \in \mathrm{Br}(k)$ is trivial.

We can find a finite extension M/k of odd degree such that $C(M) \neq \emptyset$. Then we can show that

$$i_M \colon \mathrm{Pic}_{C/k}(M) \longrightarrow \mathrm{Br}(M)$$

is the trivial homomorphism. By the commutative diagram

$$\begin{tikzcd} \mathrm{Pic}_{C/k}(M) \arrow[r, "i_M"] & \mathrm{Br}(M) \\ \mathrm{Pic}_{C/k}(k) \arrow[u] \arrow[r, "i"'] & \mathrm{Br}(k), \arrow[u, "\mathrm{Res}_{M/k}"'] \end{tikzcd}$$

where $\mathrm{Res}_{M/k}$ is the restriction map, we have

$$\mathrm{Res}_{M/k}(i(\alpha)) = 0.$$

Then we have

$$[M : k]i(\alpha) = \mathrm{Cor}_{M/k} \circ \mathrm{Res}_{M/k}(i(\alpha)) = 0,$$

where $\mathrm{Cor}_{M/k}$ is the corestriction map. Since $i(\alpha)$ is a two-torsion element, this shows that $i(\alpha) = 0$. $\square$

6.3. Symmetric determinantal representations over global fields

Following the interests from arithmetical viewpoint, we would like to consider the symmetric determinantal representations over global fields.

6.3.1. Fermat curves of prime degree. We consider the Fermat curve over $\mathbb{Q}$ in this subsection. Before it, we argue the existence of non-effective theta characteristics defined over k on smooth plane curves of odd degree.

Let C be a smooth plane curve over k. We assume that C has an odd degree d. Then we have a canonical effective theta characteristic $\mathcal{O}_C(\frac{d-3}{2})$ since

$$\omega_C^{\mathrm{shf}} \cong \mathcal{O}_C(d-3).$$

The theta characteristic is defined over k. Hence, if there is no k-rational point 2-torsion point of the Jacobian variety $\mathrm{Jac}(C)$, there is no theta characteristic on C defined over k and no symmetric determinantal representations of C over k.

As an application of this argument, we show the following theorem in [**20**].

Theorem 6.3.1 ([20]). Fix any prime $p \geq 2$. There is no symmetric determinantal representation of the Fermat curve of degree p

$$F_p \colon X^p + Y^p + Z^p = 0$$

over $\mathbb{Q}$.

When $p = 2$, we can check by Proposition 6.2.1. For $p \neq 2$ and 7, then the following theorem suffices to show the proposition.

Theorem 6.3.2 ([20, Corollary 3.3]**, cf. [11]).** If $p \neq 7$, we have $\mathrm{Jac}(F_p)[2](\mathbb{Q}) \cong 0$. On the other hand, we have $\mathrm{Jac}(F_7)[2](\mathbb{Q}) \cong (\mathbb{Z}/2\mathbb{Z})^2$.

When $p = 7$, we can check directly that any theta characteristic defined over $\mathbb{Q}$ is effective. For details, see [**20**].

6.3.2. The local-global principle for symmetric determinantal representations: general characteristic. If the base field k is a global field, we can consider the following question: if a plane curve C defined over k has a symmetric determinantal representation over any local field of k, then does C have a symmetric determinantal representation over k? We investigate the question for smooth plane curves.

Definition 6.3.3. We say *the local-global principle for symmetric determinantal representation of C holds* if the following condition holds: if C has a symmetric determinantal

representation over k_v for any place v of k, then C has a symmetric determinantal representation over k.

In the case of smooth curves of degree at most three, all smooth curves satisfy the local-global principle for symmetric determinantal representation. However, in the case of smooth plane quartics, we have a counterexample for the local-global principle for symmetric determinantal representation.

Theorem 6.3.4 ([18]). Let k be a global field of characteristic different from two. Let C be a smooth plane curve defined over k.

- If the degree of C is less than or equal to three, then the local-global principle for symmetric determinantal representation for C holds.
- If the characteristic of the base field k is different from $2, 3, 5, 7, 11, 29$ and 1229, then there is a smooth plane quartic C over a finite extension of k such that the local-global principle for symmetric determinantal representation for C does not hold.

6.3.3. The local-global principle for symmetric determinantal representations: the case of characteristic two. In characteristic two, we obtain the following results contrary to the case of general characteristic.

Theorem 6.3.5 ([19]). Let k be a global field of characteristic two. Let C be a smooth plane curve defined over k. Then the local-global principle on symmetric determinantal representations for C holds true. Moreover, if C has an odd degree and there is a place v of k such that C has a symmetric determinantal representation over k_v, then C has a symmetric determinantal representation over k.

We sketch the proof. Over an algebraically closed field of characteristic two, any smooth curve C has a *canonical theta characteristic* ([**22**]). Any other theta characteristics on$C$ are effective ([**25**]), hence C has a symmetric determinantal representation over $\overline{k}$ if and only if the canonical theta characteristic is non-effective. If the canonical theta characteristic is defined over k, then we immediately see that the local-global principle holds.

The subtle point is the question whether the canonical theta characteristic is defined over k. We show that the canonical theta characteristic is defined over the separable closure of k by Greenberg's approximation theorem. Since the theta characteristic is canonical, the class gives a k-valued point of the Picard scheme. Hence the obstruction of the condition that the canonical theta characteristic on C is defined over k can be described by an element in the Brauer group of k. Using the global class field theory, we show the obstruction vanishes. For details, see [**19**].

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