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Identifying All Preorders on the Subdistribution Monad

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Abstract

The countable valuation monad, the countable distribution monad, and the countable subdistribution monad are often used in the coalgebraic treatment of discrete probabilistic transition systems. We identify preorders on them using a technique based on the preorder $\top\top$-lifting and elementary facts about preorders on real intervals preserved by convex combinations. We show that there are exactly 15, 5, and 41 preorders on the countable valuation monad, the countable distribution monad, and the countable subdistribution monad respectively. We also give concrete definitions of these preorders. By applying Hesselink and Thijs's/ Hughes and Jacobs's construction to some preorder on the countable subdistribution monad, we obtain probabilistic bisimulation between Markov chains ignoring states with deadlocks.

Keywords: coalgebras, preorders, monads, probabilistic transition systems, probabilistic bisimulation

1 Introduction

We completely identify preorders on the countable valuation monad $\mathcal{V}$, the countable distribution monad $\mathcal{D}_{=1}$, and the countable subdistribution monad $\mathcal{D}$ on **Set** respectively. We list the main results of this paper:

- There are exactly 15 preorders on the monad $\mathcal{V}$, and they are generated from 4 preorders $\sqsubseteq^0$, $\sqsubseteq^1$, $\sqsubseteq^2$, and $\sqsubseteq^3$ (Section 4).
- There are exactly 5 preorders on the monad $\mathcal{D}_{=1}$, and they are generated from the equality $\mathrm{Eq}^{\mathcal{D}_{=1}}$ and the support-inclusion $\sqsubseteq^s$ (Section 5).
- There are exactly 41 preorders on the monad $\mathcal{D}$, and they are generated from 5 preorders $\sqsubseteq^r$, $\sqsubseteq^s$, $\sqsubseteq^d$, $\sqsubseteq^m$, and $\sqsubseteq^M$ (Section 6).
- To identify preorders on $\mathcal{V}$, it is enough to analyse preorders at the singleton type. To identify preorders on $\mathcal{D}$ and $\mathcal{D}_{=1}$, it is enough to analyse preorders at

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the Boolean type.

Our task is identifying the class $\mathbf{Pre}(T)$ of preorders on a monad T ($T = \mathcal{V}, \mathcal{D}_{\leq 1}, \mathcal{D}$). We focus on the component $\sqsubseteq_I$ of each $\sqsubseteq \in \mathbf{Pre}(T)$ at a set I. The component $\sqsubseteq_I$ is a preorder on TI that satisfies *congruence* and *substitutivity*. We denote by $\mathbf{CSPre}(T, I)$ the set of such preorders on TI. We introduce the mapping $(-)_I \colon \mathbf{Pre}(T) \to \mathbf{CSPre}(T, I)$ that extracts components at I from preorders on T. We calculate preorders on T from $\mathbf{CSPre}(T, I)$ by the left adjoint $\langle - \rangle^I$ and the right adjoint $[-]^I$ of the mapping $(-)_I$, and we analyse the sandwiching situation $\langle \preceq \rangle^I \trianglelefteq \sqsubseteq \trianglelefteq [\preceq]^I$ for each $\preceq \in \mathbf{CSPre}(T, I)$, where $\trianglelefteq$ is the component-wise inclusion order for preorders on T.

We identify $\mathbf{Pre}(\mathcal{V})$, $\mathbf{Pre}(\mathcal{D}_{\leq 1})$, and $\mathbf{Pre}(\mathcal{D})$ as the following steps:

(i) We identify the sets $\mathbf{CSPre}(\mathcal{V}, 1)$, $\mathbf{CSPre}(\mathcal{D}_{\leq 1}, 2)$, and $\mathbf{CSPre}(\mathcal{D}, 1)$. Then, the class $\mathbf{Pre}(\mathcal{V})$ is identified by applying [8, Lemma 7].

(ii) We calculate the mappings $\langle - \rangle^I$ and $[-]^I$ for $(T, I) = (\mathcal{D}_{\leq 1}, 2)$ and $(T, I) = (\mathcal{D}, 1)$. We then identify $\mathbf{Pre}(\mathcal{D}_{\leq 1})$ by proving $\langle - \rangle^2 = [-]^2$. To finish identifying $\mathbf{Pre}(\mathcal{D})$, we analyse the remaining preorders $\sqsubseteq \in \mathbf{Pre}(\mathcal{D})$ such that $\langle \sqsubseteq_1 \rangle^1 \ntrianglelefteq \sqsubseteq \ntrianglelefteq [\sqsubseteq_1]^1$ by using preorders on $\mathcal{D}_{\leq 1}$.

In [8], Katsumata and the author developed a method to identity preorders on monads, but it is not applied well to the monads $\mathcal{V}$, $\mathcal{D}_{\leq 1}$, and $\mathcal{D}$. In this paper, we introduce the following new ideas to identify $\mathbf{Pre}(\mathcal{V})$, $\mathbf{Pre}(\mathcal{D}_{\leq 1})$, and $\mathbf{Pre}(\mathcal{D})$: in (i) of the above steps, we use Lemma 1.1 to identify congruent and substitutive preorders on the *infinite* sets $\mathcal{V}1$, $\mathcal{D}_{\leq 1}2$, and $\mathcal{D}1$. In (ii), we introduce the left adjoint $\langle - \rangle^I$ of the mapping $(-)_I$, and we use the sandwiching situation $\langle \preceq \rangle^I \trianglelefteq \sqsubseteq \trianglelefteq [\preceq]^I$ to identify $\mathbf{Pre}(\mathcal{D}_{\leq 1})$ and $\mathbf{Pre}(\mathcal{D})$.

This work is motivated by a mathematical interest. The author has not found interesting applications of the main results of this work yet, but at least, we have the following contribution: By applying preorders on $\mathcal{D}$ to methods in [5,7,8], we discuss coalgebraic simulations between probabilistic transition systems, and obtain probabilistic bisimulations *ignoring states with deadlocks* between Markov chains (Section 7).

1.1 Background

Preorders on monads are equivalent to pointwise *preorder enrichments on their Kleisli categories*. A suitable partial order on a monad gives a coalgebraic trace semantics [4] and forward/backward simulations between coalgebras [3]. In the studies [5,7], simulations between coalgebras are given from preorders on coalgebra functors systematically. Many of them involve preorders on monads (e.g. the inclusion order $\mathcal{P}(A \times -)$).

In the study [10], precongruences on a typed language with nondeterminism (`or`) and a divergent term are determined completely, and they are almost equivalent to preorders on the composite monad $\mathcal{P}\mathcal{L}$ of the powerset monad $\mathcal{P}$ and the monad $\mathcal{L}$ given by $\mathcal{L} = 1 + \mathrm{Id}$ [8]. From this point of view, in other words, our work is seen as

the variant of [10] for probabilistic languages: behavioural precongruences on the language with subprobabilistic choice $\sum_{i\in I} p_i(-_i)$ and the probabilistic conditional expression for the ground type X correspond to congruent substitutive preorders on $\mathcal{D}X$.

1.2 Preliminaries

Throughout this paper, we work on the category **Set** of sets and functions. For a monad (T, η, μ) on **Set** and a function $f\colon X \to TY$, the *Kleisli Lifting* $f^\sharp\colon TX \to TY$ of f is the composition $f^\sharp = \mu \circ T(f)$.

For each set X, we denote by $\top_X$ the trivial relation $X \times X$ on X, and denote by Eq_X the equality/diagonal relation on X. We denote by R^{op} the opposite relation of R.

We will use the *complete semiring* $([0,\infty], +, \cdot, 0, 1)$ for the countable valuation monad; it has arbitrary summations, and an infinite sum is the least upper bound with respect to the standard order $\leq$ of all finite partial sums [2, Volume A, pp. 124–125, denoted by $\mathscr{R}_+$].

The following lemma is crucial to analyse preorders.

Lemma 1.1 *Let $0 < N < \infty$. If $\preceq$ is a preorder on the interval $[0, N]$ that is preserved by* convex combinations*; in other words, the preorder $\preceq$ satisfies*

$$(p_1 \preceq q_1 \wedge p_2 \preceq q_2 \wedge t \in [0,1]) \implies tp_1 + (1-t)p_2 \preceq tq_1 + (1-t)q_2$$

then $p \preceq q$ for some $0 < p < q < N$ implies $r \preceq s$ for each $0 < r < s < N$.

Proof. Suppose $p \preceq q$ and $0 < p < q < N$. First, we construct the monotone decreasing sequence $\{a_n\}_{n\in\mathbb{N}}$ and the monotone increasing sequence $\{b_n\}_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} a_n = 0$, $\lim_{n\to\infty} b_n = N$, and $0 < a_n < b_n < N$ and $a_n \preceq b_n$ for each $n \in \mathbb{N}$.

Let $\alpha = p/q$. We define the sequence $\{a_n\}_{n\in\mathbb{N}}$ by $a_n = \alpha^n p = p^{n+1}/q^n$. Since $0 < \alpha < 1$, the sequence $\{a_n\}_{n\in\mathbb{N}}$ is monotone decreasing, and it converges to 0. Since $\preceq$ is preserved by convex combinations, and $0 \preceq 0$ holds from the reflexivity of $\preceq$, for each $n \in \mathbb{N}$ we obtain

$$a_{n+1} = \alpha^{n+1}p + (1-\alpha^{n+1}) \cdot 0 \preceq \alpha^{n+1}q + (1-\alpha^{n+1}) \cdot 0 = a_n.$$

Let $\beta = (N-q)/(N-p)$. We define the sequence $\{b_n\}_{n\in\mathbb{N}}$ by $b_n = \beta^n p + (1-\beta^n)N$. Since $0 < \beta < 1$, $p < N$, and $q < N$, the sequence $\{b_n\}_{n\in\mathbb{N}}$ is monotone increasing, and it converges to N. Since $N \preceq N$ holds, and $\preceq$ is preserved by convex combinations, for each $n \in \mathbb{N}$ we obtain

$$b_n = \beta^n p + (1-\beta^n)N \preceq \beta^n q + (1-\beta^n)N = b_{n+1}.$$

Since $a_0 = p = b_0$, we obtain $0 < a_n < b_n < N$ and $a_n \preceq b_n$ for each $n \in \mathbb{N}$.

Next, we suppose $0 < r < s < N$. There is $m \in \mathbb{N}$ such that $a_m < r < s < b_m$ since $\lim_{n\to\infty} a_n = 0$ and $\lim_{n\to\infty} b_n = N$. Let

$$\gamma = \frac{s-r}{b_m - a_m}, \quad c = \frac{rb_m - sa_m}{(r-s) + (b_m - a_m)}.$$

It is obvious that $0 < \gamma < 1$ and $0 < c$ hold. We prove $c < N$ as follows:

$$N((r-s) + (b_m - a_m)) - (rb_m - sa_m) = (N-s)(N-a_m) - (N-b_m)(N-r) > 0.$$

Since $c \preceq c$ and $a_m \preceq b_m$ hold, and $\preceq$ is preserved by convex combinations,

$$r = \gamma a_m + (1-\gamma)c \preceq \gamma b_m + (1-\gamma)c = s.$$

□

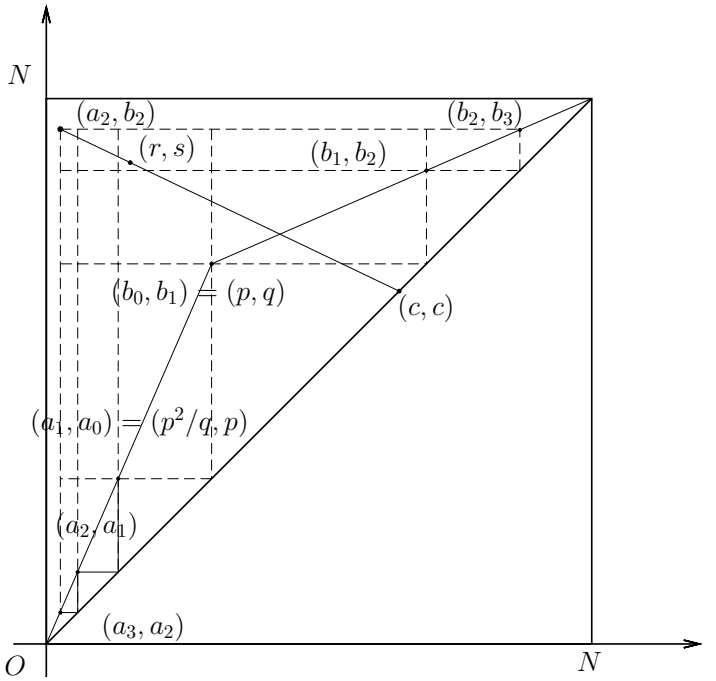

Fig. 1. The picture of proof of Lemma 1.1 (in the case of $m = 2$)

2 Monads for Probabilistic Branching

We first introduce some notations: the *sum* $d[U]$ of $d\colon X \to [0,\infty]$ over $U \subseteq X$ is defined to be $\sum_{x\in U} d(x)$. The *support* of $d\colon X \to [0,\infty]$ is defined by $\mathrm{supp}(d) = \{\, x \in X \mid d(x) \neq 0 \,\}$. The *zero distribution* $\mathtt{0}$ is defined by $\mathtt{0}(x) = 0$. The *Dirac distribution* δ_x is defined by $\delta_x(x) = 1$ and $\delta_x(y) = 0$ $(x \neq y)$.

Next, we define the three monads $\mathcal{V}$, $\mathcal{D}$, and $\mathcal{D}_{=1}$ on **Set** as follows:

Definition 2.1 • We denote by $(\mathcal{V}, \eta^{\mathcal{V}}, \mu^{\mathcal{V}})$ the *countable valuation monad* that is defined as follows: the functor part $\mathcal{V}$ is defined by $\mathcal{V}X = \{\, d\colon X \to [0,\infty] \mid \omega \geq |\mathrm{supp}(d)| \,\}$ for each set X and $\mathcal{V}f(d)(y) = \sum_{x \in f^{-1}(y)} d(x)$ for each $f\colon X \to Y$ and $y \in Y$. The unit and multiplication are defined by $\eta^{\mathcal{V}}_X(x) = \delta_x$ and $(\mu^{\mathcal{V}}_X(\xi))(x) = \sum_{d \in \mathcal{V}X} \xi(d) \cdot d(x)$ $(x \in X)$.

- The countable *subdistribution monad* $(\mathcal{D}, \eta^{\mathcal{D}}, \mu^{\mathcal{D}})$ is defined as follows: for each set X, $\mathcal{D}X = \{\, d\colon X \to [0,1] \mid d[X] \leq 1 \,\}$, and the unit and the multiplication are inherited from the countable valuation monad.
- The countable *distribution monad* $(\mathcal{D}_{=1}, \eta^{\mathcal{D}_{=1}}, \mu^{\mathcal{D}_{=1}})$ is defined as follows: for each set X, $\mathcal{D}_{=1}X = \{\, d\colon X \to [0,1] \mid d[X] = 1 \,\}$, and the unit and the multiplication are inherited from the subdistribution monad.

We remark that the condition $\omega \geq |\mathrm{supp}(d)|$ is automatically obtained from $d[X] = 1$ ($d[X] \leq 1$) in the definitions of the (sub)distribution monad.

The probabilistic branching is characterised coalgebraically by $\mathcal{D}$:

- A Markov chain is characterised as $\xi_1\colon X \to \mathcal{D}X$.
- A probabilistic transition system is characterised as $\xi_2\colon X \to \mathcal{D}(A \times X)$.
- A Segala automaton [11] is characterised as $\xi_3\colon X \to \mathcal{P}\mathcal{D}(1 + A \times X)$.

Since $\mathcal{D}X \cong \mathcal{D}_{=1}(1 + X)$, we obtain the notion of *deadlocks* in the probabilistic branching. For example, a Markov chain $\xi\colon X \to \mathcal{D}X$ has a deadlock at a state $x \in X$ when $\xi(x)[X] < 1$. For further examples, see [12].

3 The Class of Preorders on a Monad

We introduce some results of [8], which we use to identify preorders on monads. We fix a monad (T, η, μ) on **Set**. We denote it by T for simplicity.

We define the congruence and substitutivity of preorders on TI and preorders on the monad T, the latter of which correspond bijectively to pointwise preorder enrichments of the Kleisli category $\mathbf{Set}_T$ of T.

Definition 3.1 Let I be a set, and let $\preceq$ be a preorder on TI. (i) We call $\preceq$ *congruent* if $(\forall j \in J. f(j) \preceq g(j)) \implies (\forall x \in TJ. f^\sharp(x) \preceq g^\sharp(x))$ for each set J and functions $f, g\colon J \to TI$. (ii) We call $\preceq$ *substitutive* if $f^\sharp$ is a monotone function on $(TI, \preceq)$ for each $f\colon I \to TI$.

We write $(\mathbf{CSPre}(T, I), \subseteq)$ for the set of congruent and substitutive preorders on TI, ordered by inclusions. It is closed under opposites and intersections, and it has the greatest and least preorders $\top_{TI}$ and Eq_{TI} respectively.

Definition 3.2 ([8, Definition 3]) A *preorder* $\sqsubseteq$ on a monad T is an assignment of a preorder $\sqsubseteq_I$ on TI to each set I such that (i) each $\sqsubseteq_I$ is congruent, and (ii) for each $f\colon J \to TI$, $f^\sharp$ is a monotone function from $(TJ, \sqsubseteq_J)$ to $(TI, \sqsubseteq_I)$ (we also call this property *substitutivity*).

For example, the assignment $\sqsubseteq$ that is defined by $A \sqsubseteq_X B \iff A \subseteq B$ is indeed

a preorder on the powerset monad $\mathcal{P}$.

We write $(\mathbf{Pre}(T), \trianglelefteq)$ for the class of preorders on T, ordered by the partial order $\trianglelefteq$ defined by $\sqsubseteq \trianglelefteq \sqsubseteq' \overset{\text{def.}}{\iff} \forall I. \sqsubseteq_I \subseteq \sqsubseteq'_I$. It is closed under these opposites and intersections, which are defined by $(\sqsubseteq^{\mathrm{op}})_X = (\sqsubseteq_X)^{\mathrm{op}}$ and $(\bigcap_{\lambda\in\Lambda} \sqsubseteq^\lambda)_X = \bigcap_{\lambda\in\Lambda} \sqsubseteq^\lambda_X$, and it has the least and greatest preorders: the equality Eq^T defined by $\mathrm{Eq}^T_X = \mathrm{Eq}_{TX}$ and the trivial preorder $\top^T$ defined by $\top^T_X = \top_{TX}$.

For each preorder $\sqsubseteq$ on T, we call $\sqsubseteq_I$ the *evaluation* at I of $\sqsubseteq$. The evaluation mapping $(-)_I \colon \sqsubseteq \mapsto \sqsubseteq_I$ is a monotone mapping from $(\mathbf{Pre}(T), \trianglelefteq)$ to $(\mathbf{CSPre}(T, I), \subseteq)$. It has both the right and left adjoints.

$$\begin{tikzcd}
(\mathbf{CSPre}(T,I), \subseteq) \arrow[d, bend right, "\langle - \rangle^I"'] \arrow[d, bend left, "{[-]^I}"] \arrow[d, phantom, "\dashv (-)_I \dashv"] \\
(\mathbf{Pre}(T), \trianglelefteq) \arrow[u, bend right, "(-)_I"']
\end{tikzcd}$$

Fig. 2. Right and left adjoints of the evaluation mapping $(-)_I \colon \sqsubseteq \mapsto \sqsubseteq_I$

The right adjoint $[-]^I$ of the evaluation mapping $(-)_I$ is defined by

$$x \, [\preceq]^I_X \, y \iff \forall f \colon X \to TI. f^\sharp(x) \preceq f^\sharp(y).$$

The mapping $[-]^I$ is monotone, and it preserves opposites and intersections. We remark that it preserves the empty-intersection, that is, $[\top_{TI}]^I = \top^T$.

Proposition 3.3 ([8, Theorem 3]) *For each I, $(-)_I \dashv [-]^I$ and $[-]^I_I = \mathrm{Id}$.*

Hence, the preorder $[\preceq]^I$ on T is the *greatest* one whose evaluation at I equals $\preceq$ for each $\preceq \in \mathbf{CSPre}(T, I)$.

The left adjoint $\langle - \rangle^I$ of the evaluation mapping $(-)_I$ is defined by

$$\langle \preceq \rangle^I = \bigcap \{ \sqsubseteq \in \mathbf{Pre}(T) \mid \sqsubseteq_I = \preceq \}.$$

The preorder $\langle \preceq \rangle^I$ on T is the *least* one whose evaluation at I equals $\preceq$ for each $\preceq \in \mathbf{CSPre}(T, I)$ since $\mathbf{Pre}(T)$ is closed under intersections, and $[\preceq]^I_I = \preceq$ holds. By using this, we easily obtain that the mapping $\langle - \rangle^I$ is monotone, that it preserves opposites, and that the adjunction $\langle - \rangle^I \dashv (-)_I$ holds.

Lemma 3.4 *Let $\preceq \in \mathbf{CSPre}(T, I)$. If $[\preceq]^I = \langle \preceq \rangle^I$ then the preorder $[\preceq]^I$ the* unique *preorder whose evaluation at I equals $\preceq$.*

We here introduce the opposite-intersection operators on $\mathbf{Pre}(T)$ and $\mathbf{CSPre}(T, I)$. The one on $\mathbf{CSPre}(T, I)$ is given as follows:

$$\mathcal{C}^{\mathbf{CSPre}(T,I)}_{\cap,\mathrm{op}}(\mathbb{K}) = \left\{ \bigcap \mathbb{L} \cap \left(\bigcap \mathbb{M} \right)^{\mathrm{op}} \;\middle|\; \mathbb{L}, \mathbb{M} \subseteq \mathbb{K} \right\} \text{ where } \mathbb{K} \subseteq \mathbf{CSPre}(T, I)$$

The opposite-intersection closure operator on $\mathbf{Pre}(T)$ is given in a similar way as the above (we denote it by $\mathcal{C}_{\cap,\mathfrak{op}}^{\mathbf{Pre}(T)}$). We often write $\mathcal{C}_{\cap,\mathfrak{op}}$ for simplicity.

3.1 Main Results

Theorem 3.5 *Preorders on $\mathcal{V}$, $\mathcal{D}_{\leq 1}$, and $\mathcal{D}$ are identified as follows:*

(i) $\mathbf{Pre}(\mathcal{V}) = \mathcal{C}_{\cap,\mathfrak{op}}\{\sqsubseteq^0, \sqsubseteq^1, \sqsubseteq^2, \sqsubseteq^3\} \cong \mathbf{CSPre}(\mathcal{V}, 1) \cong 15$ *where*

$$\begin{aligned}
d_1 \sqsubseteq_X^0 d_2 &\overset{\text{def}}{\iff} \mathrm{supp}(d_1) \subseteq \mathrm{supp}(d_2)\\
d_1 \sqsubseteq_X^1 d_2 &\overset{\text{def}}{\iff} \forall x \in X.(d_1(x) \leq d_2(x))\\
d_1 \sqsubseteq_X^2 d_2 &\overset{\text{def}}{\iff} \forall x \in X.(d_1(x) = d_2(x) \vee d_2(x) = \infty)\\
d_1 \sqsubseteq_X^3 d_2 &\overset{\text{def}}{\iff} \forall x \in X.(d_1(x) \leq d_2(x) \wedge (d_1(x) = 0 \implies d_2(x) \in \{\infty, 0\})).
\end{aligned}$$

(ii) $\mathbf{Pre}(\mathcal{D}_{\leq 1}) = \mathcal{C}_{\cap,\mathfrak{op}}\{\sqsubseteq^s, \mathrm{Eq}^{\mathcal{D}_{\leq 1}}\} \cong \mathbf{CSPre}(\mathcal{D}_{\leq 1}, 2) \cong 5$ *where*

$$d_1 \sqsubseteq_X^s d_2 \overset{\text{def}}{\iff} \mathrm{supp}(d_1) \subseteq \mathrm{supp}(d_2).$$

(iii) $\mathbf{Pre}(\mathcal{D}) = \mathcal{C}_{\cap,\mathfrak{op}}\{\sqsubseteq^r, \sqsubseteq^s, \sqsubseteq^d, \sqsubseteq^m, \sqsubseteq^M\} \cong \mathbf{CSPre}(\mathcal{D}, 2) \cong 41$ *where*

$$\begin{aligned}
d_1 \sqsubseteq_X^r d_2 &\overset{\text{def}}{\iff} \forall x \in X.d_1(x) \leq d_2(x),\\
d_1 \sqsubseteq_X^s d_2 &\overset{\text{def}}{\iff} \mathrm{supp}(d_1) \subseteq \mathrm{supp}(d_2),\\
d_1 \sqsubseteq_X^d d_2 &\overset{\text{def}}{\iff} (d_1[X] = 1 \implies d_2[\mathrm{supp}(d_1)] = 1),\\
d_1 \sqsubseteq_X^m d_2 &\overset{\text{def}}{\iff} (d_1[X] = 1 \implies d_2 = d_1),\\
d_1 \sqsubseteq_X^M d_2 &\overset{\text{def}}{\iff} (d_1[X] = 1 \implies (d_2[X] = 1 \wedge \mathrm{supp}(d_1) = \mathrm{supp}(d_2))).
\end{aligned}$$

We prove (i), (ii), and (iii) of Theorem 3.5 in Section 4, 5, and 6.

4 Preorders on the Countable Valuation Monad

Preorders on a semiring-valued *finite* multiset monad are pointwise [8, Lemma 7 and Theorem 8]. The following lemma holds by applying this fact to the monad $\mathcal{V}$ with a slight change of cardinality of supports to *countable*.

Lemma 4.1 *Each $\sqsubseteq \in \mathbf{Pre}(\mathcal{V})$ satisfies $d_1 \sqsubseteq_X d_2 \iff \forall x \in X.d_1(x) \sqsubseteq_1 d_2(x)$, where $1 = \{*\}$. Moreover, $\mathbf{CSPre}(\mathcal{V}, 1) \cong \mathbf{Pre}(\mathcal{V})$.*

Hence, it suffices to identify $\mathbf{CSPre}(\mathcal{V}, 1)$ to identify $\mathbf{Pre}(\mathcal{V})$. We regard $\mathcal{V}1$ as $[0, \infty]$ by the correspondence between each $d \in \mathcal{V}1$ and the value $d(*) \in [0, \infty]$. For each $\preceq \in \mathbf{CSPre}(\mathcal{V}, 1)$, the substitutivity of $\preceq$ is equivalent to

$$(p \preceq q \wedge t \in [0, \infty]) \implies tp \preceq tq,$$

and the congruence of $\preceq$ is equivalent to

$$\forall i \in I.(p_i \preceq q_i \wedge t_i \in [0, \infty]) \implies \sum_{i \in I} p_i t_i \preceq \sum_{i \in I} q_i t_i.$$

Hence, each $\preceq \in \mathbf{CSPre}(\mathcal{V}, 1)$ is preserved by convex combinations.

We partition the set $\mathcal{V}1 \times \mathcal{V}1 \cong [0,\infty] \times [0,\infty]$ into $\mathrm{Eq}_{\mathcal{V}1}$, $R_0 = \{\, (0,q) \mid q \in (0,\infty) \,\}$, $R_1 = \{\, (p,q) \mid 0 < p < q < \infty \,\}$, $R_2 = \{(0,\infty)\}$, $R_3 = \{\, (p,\infty) \mid p \in (0,\infty) \,\}$, $R_4 = {R_0}^{\mathrm{op}}$, $R_5 = {R_1}^{\mathrm{op}}$, $R_6 = {R_2}^{\mathrm{op}}$, and $R_7 = {R_3}^{\mathrm{op}}$.

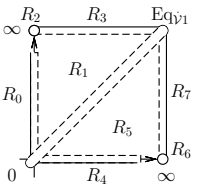

Fig. 3. The partitions $\mathrm{Eq}_{\mathcal{V}1}$, R_0, R_1,..., R_7 of $\mathcal{V}1 \times \mathcal{V}1$

By using Lemma 1.1, we obtain Lemma 4.2 and 4.3.

Lemma 4.2 *Let $\preceq \in \mathbf{CSPre}(\mathcal{V}, 1)$. We obtain the following properties:*

(i) *$p \preceq \infty$ for some $0 < p < \infty$ if and only if $r \preceq \infty$ for all $0 < r \leq \infty$. This is equivalent to $R_3 \cap \preceq \neq \emptyset \implies R_3 \subseteq \preceq$.*

(ii) *$0 \preceq \infty$ if and only if $r \preceq s$ for all $0 \leq r \leq \infty$. This is equivalent to $R_2 \cap \preceq \neq \emptyset \implies R_2 \cup R_3 \subseteq \preceq$.*

(iii) *$p \preceq q$ for some $0 < p < q < \infty$ if and only if $r \preceq s$ for all $0 < r < s \leq \infty$. This is equivalent to $R_1 \cap \preceq \neq \emptyset \implies R_1 \cup R_3 \subseteq \preceq$.*

(iv) *$0 \preceq q$ for some $0 < q < \infty$ if and only if $r \preceq s$ for all $0 \leq r < s \leq \infty$. This is equivalent to $R_0 \cap \preceq \neq \emptyset \implies R_0 \cup R_1 \cup R_2 \cup R_3 \subseteq \preceq$.*

Lemma 4.3 *Let $\preceq \in \mathbf{CSPre}(\mathcal{V}, 1)$. We obtain $\preceq = \mathrm{Eq}_{\mathcal{V}1} \cup \bigcup_{i \in I} R_i$ where $I = \{\, i \in \{0, 1, \ldots, 7\} \mid R_i \cap \preceq \neq \emptyset \,\}$.*

We prepare the following congruent substitutive preorders on $\mathcal{V}1$:

- $p \preceq^0 q \overset{\text{def.}}{\iff} (p > 0 \implies q > 0)$
- $p \preceq^1 q \overset{\text{def.}}{\iff} (p \leq q)$
- $p \preceq^2 q \overset{\text{def.}}{\iff} (p = q) \vee (q = \infty)$
- $p \preceq^3 q \overset{\text{def.}}{\iff} (p \leq q) \wedge (p = 0 \implies q \in \{\infty, 0\})$

Proposition 4.4 *We obtain $\mathbf{CSPre}(\mathcal{V}, 1) = \mathcal{C}_{\cap,\mathrm{op}} \left\{ \preceq^0, \preceq^1, \preceq^2, \preceq^3 \right\} \cong 15$.*

Proof (Sketch). Let $\preceq \in \mathbf{CSPre}(\mathcal{V}, 1)$. We define $R(p_0, p_1, \ldots, p_7) = \mathrm{Eq}_{\mathcal{V}1} \cup \bigcup \{\, R_i \mid p_i = \text{true} \,\}$. By Lemma 4.3, we obtain $\preceq = R(p_0, p_1, \ldots, p_7)$ where $p_i \iff R_i \cap \preceq \neq \emptyset$ $(i \in \{0, 1, \ldots, 7\})$. From Lemma 4.2 and the transitivity of $\preceq$, the octuple $(p_0, p_1, \ldots, p_7)$ should satisfy the following formula:

$$
\begin{aligned}
P = \ & (p_0 \implies p_1 \wedge p_2) \wedge (p_1 \vee p_2 \implies p_3) \\
& \wedge (p_3 \wedge p_7 \implies p_1 \wedge p_5) \wedge (p_2 \wedge p_7 \implies p_0) \wedge (p_3 \wedge p_6 \implies p_4) \\
& \wedge (p_4 \implies p_5 \wedge p_6) \wedge (p_5 \vee p_6 \implies p_7).
\end{aligned}
$$

We remark that the last 2 clauses of P are given by applying the opposite order $\preceq^{\mathrm{op}}$ to Lemma 4.2. It is easy to check that there are exactly 15 satisfying assignments of P and that the following inclusion holds:

$$15 \cong \{\, R(p_0, p_1, \ldots, p_7) \mid (p_0, p_1, \ldots, p_7) \text{ satisfies } P \,\} \subseteq \mathcal{C}_{\cap,\mathrm{op}}\{\preceq^0, \preceq^1, \preceq^2, \preceq^3\}.$$

Since $\mathbf{CSPre}(\mathcal{V}, 1) \subseteq \{\, R(p_0, p_1, \ldots, p_7) \mid (p_0, p_1, \ldots, p_7) \text{ satisfies } P \,\}$ and $\preceq^0, \preceq^1, \preceq^2, \preceq^3 \in \mathbf{CSPre}(\mathcal{V}, 1)$, we conclude this proposition. □

Theorem 4.5 (Theorem 3.5(i)) *Let $\sqsubseteq^i$ be the pointwise ordering generated from $\preceq^i$ ($i \in \{0, 1, 2, 3\}$). We obtain $\mathbf{Pre}(\mathcal{V}) = \mathcal{C}_{\cap,\mathrm{op}}\left\{\sqsubseteq^0, \sqsubseteq^1, \sqsubseteq^2, \sqsubseteq^3\right\} \cong 15$.*

Proof. It is proved immediately from Lemma 4.1 and Proposition 4.4. □

5 Preorders on the Distribution Monad

First, we identify $\mathbf{CSPre}(\mathcal{D}_{=1}, 2)$ where $2 = \{\mathbf{0}, \mathbf{1}\}$. We regard $\mathcal{D}_{=1}2$ as $[0, 1]$ by the correspondence between each $d = d(\mathbf{0})\delta_{\mathbf{0}} + (1 - d(\mathbf{0}))\delta_{\mathbf{1}} \in \mathcal{D}_{=1}2$ and the value $d(\mathbf{0}) \in [0, 1]$. For each $\preceq \in \mathbf{CSPre}(\mathcal{D}_{=1}, 2)$, the substitutivity of $\preceq$ is equivalent to

$$p \preceq q \implies \forall t, u \in [0, 1].((t - u)p + u \preceq (t - u)q + u),$$

and the congruence of $\preceq$ is equivalent to

$$(\forall i \in I.(p_i \preceq q_i) \wedge \sum_{i \in I} t_i = 1) \implies \sum_{i \in I} p_i t_i \preceq \sum_{i \in I} q_i t_i.$$

Hence, each $\preceq \in \mathbf{CSPre}(\mathcal{V}, 1)$ is preserved by convex combinations.

We partition the set $\mathcal{D}_{=1}2 \times \mathcal{D}_{=1}2 \cong [0, 1] \times [0, 1]$ into $\mathrm{Eq}_{\mathcal{D}_{=1}2}$, $R_0 = \{\, (0, 1), (1, 0) \,\}$, $R_1 = \{\, (p, q) \mid p \in \{0, 1\}, 0 < q < 1 \,\}$, $R_2 = \{\, (p, q) \mid p, q \in (0, 1), p \neq q \,\}$, and $R_3 = {R_1}^{\mathrm{op}}$.

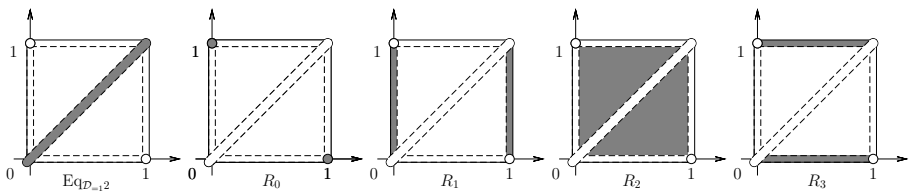

Fig. 4. The partitions $\mathrm{Eq}_{\mathcal{D}_{=1}2}$, R_0, R_1, R_2, and R_3 of $\mathcal{D}_{=1}2 \times \mathcal{D}_{=1}2$

By using Lemma 1.1, we obtain Lemma 5.1 and 5.2.

Lemma 5.1 *Let $\preceq \in \mathbf{CSPre}(\mathcal{D}_{=1}, 2)$. We obtain the following properties:*

(i) *$p \preceq q$ for some $0 < p < q < 1$ if and only if $r \preceq s$ for all $r, s \in (0, 1)$. This is equivalent to $R_2 \cap \preceq \neq \emptyset \implies R_2 \subseteq \preceq$.*

(ii) *$0 \preceq q$ for some $0 < q < 1$ if and only if $r \preceq s$ for all $(r, s) \in [0, 1] \times (0, 1)$. This is equivalent to $R_1 \cap \preceq \neq \emptyset \implies R_1 \cup R_2 \subseteq \preceq$.*

(iii) $0 \preceq 1$ *if and only if* $r \preceq s$ *for all* $r, s \in [0,1]$.
This is equivalent to $R_0 \cap \preceq \neq \emptyset \implies R_0 \cup R_1 \cup R_2 \cup R_3 \subseteq \preceq$.

Lemma 5.2 *Let* $\preceq \in \mathbf{CSPre}(\mathcal{D}_{\leq 1}, 2)$. *We obtain* $\preceq = \mathrm{Eq}_{\mathcal{D}_{\leq 1}2} \cup \bigcup_{i \in I} R_i$ *where* $I = \{\, i \in \{0,1,2,3\} \mid R_i \cap \preceq \neq \emptyset \,\}$.

Proposition 5.3 . *We have the following identification:*

$$\mathbf{CSPre}(\mathcal{D}_{\leq 1}, 2) = \mathcal{C}_{\cap,\mathrm{op}}\{\preceq^s, \mathrm{Eq}_{\mathcal{D}_{\leq 1}2}\} = \{\top_{\mathcal{D}_{\leq 1}2}, \mathrm{Eq}_{\mathcal{D}_{\leq 1}2}, \preceq^s, \preceq^{s\mathrm{op}}, \preceq^s \cap \preceq^{s\mathrm{op}}\} \cong 5,$$

where $p \preceq^s q \overset{\mathrm{def}}{\iff} (p \neq q) \implies (0 < q < 1)$.

Proof (Sketch). Analogous to Lemma 4.4, by Lemma 5.1 and 5.2 and the transitivity of $\preceq$, for each $\preceq \in \mathbf{CSPre}(\mathcal{D}_{\leq 1}, 2)$, there is a quadruple (p_0, p_1, p_2, p_3) of truth values which satisfies the following formula:

$$P = (p_0 \implies p_1 \wedge p_2 \wedge p_3) \wedge (p_1 \implies p_2) \wedge (p_1 \wedge p_3 \implies p_0) \wedge (p_3 \implies p_2)$$

and the union $R(p_0, p_1, p_2, p_3) = \mathrm{Eq}_{\mathcal{D}_{\leq 1}2} \cup \bigcup \{\, R_i \mid p_i = \mathrm{true} \,\}$ is equal to the given preorder $\preceq$. It is easy to check that there are exactly 5 satisfying assignments (p_0, p_1, p_2, p_3) of P and that the following inclusion holds:

$$5 \cong \{\, R(p_0, p_1, p_2, p_3) \mid (p_0, p_1, p_2, p_3) \text{ satisfies } P \,\} \subseteq \mathcal{C}_{\cap,\mathrm{op}}\{\preceq^s, \mathrm{Eq}_{\mathcal{D}_{\leq 1}2}\}.$$

Since $\mathbf{CSPre}(\mathcal{D}_{\leq 1}, 2) \subseteq \{\, R(p_0, p_1, p_2, p_3) \mid (p_0, p_1, p_2, p_3) \text{ satisfies } P \,\}$ and $\preceq^s, \mathrm{Eq}_{\mathcal{D}_{\leq 1}2} \in \mathbf{CSPre}(\mathcal{D}_{\leq 1}, 2)$, we conclude this proposition. □

Next, we calculate the mapping $[-]^2 \colon \mathbf{CSPre}(\mathcal{D}_{\leq 1}, 2) \to \mathbf{Pre}(\mathcal{D}_{\leq 1})$. Since it preserves intersections and opposites, and $\mathbf{CSPre}(\mathcal{D}_{\leq 1}, 2) = \mathcal{C}_{\cap,\mathrm{op}}\{\preceq^s, \mathrm{Eq}_{\mathcal{D}_{\leq 1}2}\}$, it suffices to identify the preorders $[\mathrm{Eq}_{\mathcal{D}_{\leq 1}2}]^2$ and $[\preceq^s]^2$ (we denote it by $\sqsubseteq^s$).

Proposition 5.4 *The preorders* $[\mathrm{Eq}_{\mathcal{D}_{\leq 1}2}]^2$ *and* $\sqsubseteq^s$ *are identified as follows:*

(i) $d_1 \, [\mathrm{Eq}_{\mathcal{D}_{\leq 1}2}]^2_X \, d_2 \iff d_1 = d_2$.
(ii) $d_1 \sqsubseteq^s_X d_2 \iff \mathrm{supp}(d_1) \subseteq \mathrm{supp}(d_2)$.

Next, we calculate the mapping $\langle - \rangle^2 : \mathbf{CSPre}(\mathcal{D}_{\leq 1}, 2) \to \mathbf{Pre}(\mathcal{D}_{\leq 1})$.

Lemma 5.5 *Let* $\preceq \in \mathbf{CSPre}(\mathcal{D}_{\leq 1}, 2)$ *and* $\alpha \in [0,1]$. *If* $d_1, d_2 \in \mathcal{D}_{\leq 1}X$ *satisfy the following condition: for each* $y \in X$ *such that* $d_1(y) > d_2(y)$,

$$\left(\alpha + (1-\alpha)\frac{d_2(y)}{d_1(y)}\right)\delta_{\mathbf{0}} + (1-\alpha)\left(1 - \frac{d_2(y)}{d_1(y)}\right)\delta_{\mathbf{1}} \preceq \frac{d_2(y)}{d_1(y)}\delta_{\mathbf{0}} + \left(1 - \frac{d_2(y)}{d_1(y)}\right)\delta_{\mathbf{1}}$$

then $(\alpha d_1 + (1-\alpha)d_2) \, \langle \preceq \rangle^2_X \, d_2$ *holds.*

Proof. Let $Y = \{\, x \in X \mid d_1(x) > d_2(x) \,\}$. We assume $d_1 \neq d_2$ without loss of generality. This implies $Y \neq \emptyset$, $X \setminus Y \neq \emptyset$, and $\sum_{x \in Y} d_2(x) - d_1(x) > 0$. We obtain $\sum_{x \in X \setminus Y} d_2(x) - d_1(x) = \sum_{x \in Y} d_1(x) - d_2(x)$ since $d_1[X] = d_2[X] = 1$.

Hence, the following distribution $d_3 \in \mathcal{D}_{=1}X$ is well-defined:

$$d_3 = \frac{1}{\sum_{x \in X \setminus Y}(d_2(x) - d_1(x))} \sum_{x \in X \setminus Y} (d_2(x) - d_1(x))\delta_x.$$

From the assumption of this lemma, for each $y \in Y$ we obtain

$$\left(\alpha + (1-\alpha)\frac{d_2(y)}{d_1(y)}\right)\delta_{\mathbf{0}} + (1-\alpha)\left(1 - \frac{d_2(y)}{d_1(y)}\right)\delta_{\mathbf{1}} \preceq \frac{d_2(y)}{d_1(y)}\delta_{\mathbf{0}} + \left(1 - \frac{d_2(y)}{d_1(y)}\right)\delta_{\mathbf{1}}.$$

We denote by c_y and c'_y the left-hand and right-hand side of the above inequality respectively for each $y \in Y$. We define the mapping $f_y \colon 2 \to \mathcal{D}_{=1}X$ by $f_y(\mathbf{0}) = \delta_y$ and $f_y(\mathbf{1}) = d_3$ for each $y \in Y$.

From the substitutivity of $\langle \preceq \rangle^2$, we obtain $f_y^\sharp(c_y) \langle \preceq \rangle_X^2 f_y^\sharp(c'_y)$ for each $y \in Y$. We define $e_y = f_y^\sharp(c_y)$ and $e'_y = f_y^\sharp(c'_y)$ for each $y \in Y$. They are calculated as

$$\begin{aligned} e_y &= \left(\alpha + (1-\alpha)\frac{d_2(y)}{d_1(y)}\right)\delta_y + (1-\alpha)\left(1 - \frac{d_2(y)}{d_1(y)}\right)d_3, \\ e'_y &= \frac{d_2(y)}{d_1(y)}\delta_y + \left(1 - \frac{d_2(y)}{d_1(y)}\right)d_3. \end{aligned}$$

We define $g, g' \colon X \to \mathcal{D}_{=1}X$ by $g(y) = e_y$ and $g'(y) = e'_y$ for each $y \in Y$, and $g(x) = g'(x) = \delta_x$ for each $x \in X \setminus Y$. We obtain $g(x) \langle \preceq \rangle_X^2 g'(x)$ for each $x \in X$. From the the congruence of $\langle \preceq \rangle^2$, we obtain $g^\sharp(d_1) \langle \preceq \rangle_X^2 g'^\sharp(d_1)$.

We obtain $g^\sharp(d_1) = \alpha d_1 + (1-\alpha)d_2$ by

$$\begin{aligned} g^\sharp(d_1) &= \sum_{y \in Y} d_1(y)e_y + \sum_{x \in X \setminus Y} d_1(x)\delta_x \\ &= \sum_{y \in Y}(\alpha d_1(y) + (1-\alpha)d_2(y))\delta_y + (1-\alpha)\sum_{y \in Y}(d_1(y) - d_2(y))d_3 + \sum_{x \in X \setminus Y} d_1(x)\delta_x \\ &= \sum_{y \in Y}(\alpha d_1(y) + (1-\alpha)d_2(y))\delta_y + (1-\alpha)\sum_{x \in X \setminus Y}(d_2(x) - d_1(x))\delta_x + \sum_{x \in X \setminus Y} d_1(x)\delta_x \\ &= \alpha d_1 + (1-\alpha)d_2. \end{aligned}$$

Similarly (apply $\alpha = 0$ to the above calculation), we obtain $g'^\sharp(d_1) = d_2$. Therefore, we conclude $(\alpha d_1 + (1-\alpha)d_2) = g^\sharp(d_1) \langle \preceq \rangle_X^2 g'^\sharp(d_1) = d_2$. □

Proposition 5.6 *The mapping $\langle - \rangle^2$ equals the mapping $[-]^2$.*

Proof (Sketch). We prove the case $\preceq\, = \,\preceq^s \cap \preceq^{s\mathrm{op}}$, and omit the other cases. (Case: $\preceq\, = \,\preceq^s \cap \preceq^{s\mathrm{op}}$) Suppose $d_1[\preceq]_X^2 d_2$. By Lemma 5.4, it is equivalent to $\mathrm{supp}(d_1) = \mathrm{supp}(d_2)$. This implies for each $y \in X$ such that $d_1(y) > d_2(y)$,

$$\left(\frac{d_1(y) + d_2(y)}{2d_1(y)}\delta_{\mathbf{0}} + \frac{d_1(y) - d_2(y)}{2d_1(y)}\delta_{\mathbf{1}}\right) \preceq \left(\frac{d_2(y)}{d_1(y)}\delta_{\mathbf{0}} + \frac{d_1(y) - d_2(y)}{d_1(y)}\delta_{\mathbf{1}}\right).$$

By Lemma 5.5 with $\alpha = 1/2$, we obtain $(d_1 + d_2)/2 \langle \preceq \rangle_X^2 d_2$. Similarly, we also have $d_1 \langle \preceq \rangle_X^2 (d_1 + d_2)/2$. Thus, $d_1 \langle \preceq \rangle_X^2 d_2$. Therefore, $[\preceq]^2 = \langle \preceq \rangle^2$ holds. □

Theorem 5.7 (Theorem 3.5(ii)) *We obtain the following identification:*

$$\mathbf{Pre}(\mathcal{D}_{\leq 1}) = \mathcal{C}_{\cap,\mathfrak{P}}\{\sqsubseteq^s, \mathrm{Eq}^{\mathcal{D}_{\leq 1}}\} = \{\top^{\mathcal{D}_{\leq 1}}, \mathrm{Eq}^{\mathcal{D}_{\leq 1}}, \sqsubseteq^s, \sqsubseteq^{s\mathfrak{P}}, \sqsubseteq^s \cap \sqsubseteq^{s\mathfrak{P}}\} \cong 5.$$

Proof. It is proved from Lemma 3.4, Proposition 5.4, 5.3, and 5.6. □

6 Preorders on the Subdistribution Monad

First, we identify $\mathbf{CSPre}(\mathcal{D}, 1)$. We regard $\mathcal{D}1$ as $[0, 1]$ by the correspondence between each $d \in \mathcal{D}1$ and the value $d(*) \in [0, 1]$. For each $\preceq \in \mathbf{CSPre}(\mathcal{D}, 1)$, the substitutivity of $\preceq$ is equivalent to

$$(p \preceq q \wedge t \in [0, 1]) \implies tp \preceq tq,$$

and the congruence of $\preceq$ is equivalent to

$$\forall i \in I.(p_i \preceq q_i) \wedge \sum_{i \in I} t_i \leq 1 \implies \sum_{i \in I} p_i t_i \preceq \sum_{i \in I} q_i t_i.$$

Hence, each $\preceq \in \mathbf{CSPre}(\mathcal{D}, 1)$ is preserved by convex combinations.

We partition the set $\mathcal{D}1 \times \mathcal{D}1 \cong [0, 1] \times [0, 1]$ into $\mathrm{Eq}_{\mathcal{D}1}$, $R_0 = \{(0, 1)\}$, $R_1 = \{\, (0, q) \mid 0 < q < 1 \,\}$, $R_2 = \{\, (p, 1) \mid 0 < p < 1 \,\}$, $R_3 = \{\, (p, q) \mid 0 < p < q < 1 \,\}$, $R_4 = {R_0}^{\mathfrak{P}}$, $R_5 = {R_1}^{\mathfrak{P}}$, $R_6 = {R_2}^{\mathfrak{P}}$, and $R_7 = {R_3}^{\mathfrak{P}}$.

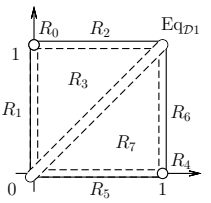

Fig. 5. The partition $\mathrm{Eq}_{\mathcal{D}1}$, R_0, R_1,..., R_7 of $\mathcal{D}1 \times \mathcal{D}1$

By using Lemma 1.1, we obtain Lemma 6.1 and 6.2.

Lemma 6.1 *Let $\preceq \in \mathbf{CSPre}(\mathcal{D}, 1)$. We obtain the following properties:*

(i) *$p \preceq q$ for some $0 < p < q < 1$ if and only if $r \preceq s$ for all $0 < r < s < 1$. This is equivalent to $R_3 \cap \preceq \neq \emptyset \implies R_3 \subseteq \preceq$.*

(ii) *$0 \preceq q$ for some $0 < q < 1$ if and only if $r \preceq s$ for all $0 \leq r < s < 1$. This is equivalent to $R_1 \cap \preceq \neq \emptyset \implies R_1 \cup R_3 \subseteq \preceq$.*

(iii) *$p \preceq 1$ for some $0 < p < 1$ if and only if $r \preceq s$ for all $0 < r < s \leq 1$. This is equivalent to $R_2 \cap \preceq \neq \emptyset \implies R_2 \cup R_3 \subseteq \preceq$.*

(iv) *$0 \preceq 1$ if and only if $r \preceq s$ for all $0 \leq r < s \leq 1$. This is equivalent to $R_0 \cap \preceq \neq \emptyset \implies R_0 \cup R_1 \cup R_2 \cup R_3 \subseteq \preceq$.*

Lemma 6.2 *Let* $\preceq \in \mathbf{CSPre}(\mathcal{D}, 1)$. *We obtain* $\preceq = \mathrm{Eq}_{\mathcal{D}1} \cup \bigcup_{i \in I} R_i$ *where* $I = \{\, i \in \{0, 1, \ldots, 7\} \mid R_i \cap \preceq \neq \emptyset \,\}$.

We prepare the following congruent substitutive preorders on $\mathcal{D}1$:

- $p \preceq^r q \overset{\text{def}}{\iff} p \leq q$
- $p \preceq^s q \overset{\text{def}}{\iff} p > 0 \implies q > 0$
- $p \preceq^d q \overset{\text{def}}{\iff} p = 1 \implies q = 1$

The superscripts r, s, and d stand for real values, supports, and deadlocks of distributions respectively. We let $\preceq^{sd} = \preceq^s \cap \preceq^d$ for simplicity.

Proposition 6.3 *We obtain* $\mathbf{CSPre}(\mathcal{D}, 1) = \mathcal{C}_{\cap,\Phi}\{\preceq^r, \preceq^s, \preceq^d\} \cong 25$.

Proof (Sketch). Analogous to Lemma 4.4, by Lemma 6.1 and 6.2 and the transitivity of $\preceq$, for each $\preceq \in \mathbf{CSPre}(\mathcal{D}, 1)$, there is an octuple $(p_0, p_1, \ldots, p_7)$ of truth values which satisfies the formula $P = P' \wedge P''$ where

$$P' = (p_0 \iff (p_1 \wedge p_2)) \wedge ((p_1 \vee p_2) \implies p_3),$$
$$P'' = (p_4 \iff (p_5 \wedge p_6)) \wedge ((p_5 \vee p_6) \implies p_7)$$

and the union $R(p_0, p_1, \ldots, p_7) = \mathrm{Eq}_{\mathcal{D}1} \cup \bigcup \{\, R_i \mid p_i = \text{true} \,\}$ is equal to the given preorder $\preceq$. It is easy to check that there are 25 satisfying assignments $(p_0, p_1, \ldots, p_7)$ of P and that the following inclusion holds:

$$25 \cong \{\, R(p_0, p_1, \ldots, p_7) \mid (p_0, p_1, \ldots, p_7) \text{ satisfies } P \,\} \subseteq \mathcal{C}_{\cap,\Phi}\{\preceq^r, \preceq^s, \preceq^d\}.$$

Since $\mathbf{CSPre}(\mathcal{D}, 1) \subseteq \{\, R(p_0, p_1, \ldots, p_7) \mid (p_0, p_1, \ldots, p_7) \text{ satisfies } P \,\}$ and $\preceq^r, \preceq^s, \preceq^d \in \mathbf{CSPre}(\mathcal{D}, 1)$, we conclude this proposition. □

Next, we calculate the mapping $[-]^1 \colon \mathbf{CSPre}(\mathcal{D}, 1) \to \mathbf{Pre}(\mathcal{D})$. Since it preserves intersections and opposites, and $\mathbf{CSPre}(\mathcal{D}, 1) = \mathcal{C}_{\cap,\Phi}\{\preceq^r, \preceq^s, \preceq^d\}$ holds, it suffices to identify the preorders $[\preceq^r]^1$, $[\preceq^s]^1$, and $[\preceq^d]^1$ (e.g. $[\preceq^d \cap \preceq^{s\Phi}]^1 = [\preceq^d]^1 \cap [\preceq^s]^{1\Phi}$). Let $\sqsubseteq^r = [\preceq^r]^1$, $\sqsubseteq^s = [\preceq^s]^1$, and $\sqsubseteq^d = [\preceq^d]^1$.

Proposition 6.4 *The preorders* $\sqsubseteq^r$, $\sqsubseteq^s$, *and* $\sqsubseteq^d$ *are identified as follows:*

(i) $d_1 \sqsubseteq^r_X d_2 \iff \forall x \in X. d_1(x) \leq d_2(x)$.

(ii) $d_1 \sqsubseteq^s_X d_2 \iff \mathrm{supp}(d_1) \subseteq \mathrm{supp}(d_2)$.

(iii) $d_1 \sqsubseteq^d_X d_2 \iff (d_1[X] = 1 \implies d_2[\mathrm{supp}(d_1)] = 1)$.

Next, we calculate the mapping $\langle - \rangle^1 : \mathbf{CSPre}(\mathcal{D}, 1) \to \mathbf{Pre}(\mathcal{D})$. Generally speaking, $\langle - \rangle^I : \mathbf{CSPre}(T, I) \to \mathbf{Pre}(T)$ needs not preserve intersections, but the mapping $\langle - \rangle^1 : \mathbf{CSPre}(\mathcal{D}, 1) \to \mathbf{Pre}(\mathcal{D})$ preserves intersections.

Proposition 6.5 *The mapping* $\langle - \rangle^1$ *satisfies the following:*

- *The mapping* $\langle - \rangle^1$ *preserves intersections and opposites.*

- $\langle \preceq^r \rangle^1 = \sqsubseteq^r$, $\langle \preceq^s \rangle^1 = \sqsubseteq^s$, *and* $\langle \preceq^d \rangle^1 = \sqsubseteq^m$ *where* $\sqsubseteq^m$ *is defined by*

$$d_1 \sqsubseteq^m_X d_2 \overset{\text{def.}}{\iff} (d_1[X] = 1 \implies d_1 = d_2).$$

By Proposition 6.3 and 6.5, we obtain that the preorder $\langle \preceq \rangle^1$ is identified completely for each $\preceq \in \mathbf{CSPre}(\mathcal{D}, 1)$ (e.g. $\langle \preceq^d \cap \preceq^{s\Phi} \rangle^1 = \sqsubseteq^m \cap \sqsubseteq^{s\Phi}$).

The following lemma and is crucial to identify the mapping $\langle - \rangle^1$.

Lemma 6.6 *Let* $\preceq \in \mathbf{CSPre}(\mathcal{D}, 1)$. *If* $d_1, d_2 \in \mathcal{D}X$ *satisfy the condition:*

$$\forall x \in \mathrm{supp}(d_1). \left(\frac{1 + d_1[X]}{2} \preceq \frac{(1 + d_1[X])}{2} \frac{\min(d_1, d_2)(x)}{d_1(x)} \right) \tag{1}$$

then we obtain $d_1 \langle \preceq \rangle^1_X \min(d_1, d_2)$.

Here, $\min(d_1, d_2) \in \mathcal{D}X$ is defined by $\min(d_1, d_2)(x) = \min(d_1(x), d_2(x))$.

Proof. We may assume $d_1 \neq 0$ since $\min(d_1, d_2) = 0$ whenever $d_1 = 0$. We recall $\langle \preceq \rangle^1_1 = \preceq$. From the substitutivity of $\langle \preceq \rangle^1$, for each $x \in \mathrm{supp}(d_1)$,

$$\frac{1 + d_1[X]}{2} \delta_x \langle \preceq \rangle^1_X \frac{(1 + d_1[X])}{2} \frac{\min(d_1, d_2)(x)}{d_1(x)} \delta_x.$$

We define the functions $f, g \colon X \to \mathcal{D}X$ as follows: for each $x \in \mathrm{supp}(d_1)$,

$$f(x) = \frac{1 + d_1[X]}{2} \delta_x \quad \text{and} \quad g(x) = \frac{(1 + d_1[X])}{2} \frac{\min(d_1, d_2)(x)}{d_1(x)} \delta_x$$

and $f(x) = g(x) = 0$ for each $x \in X \setminus \mathrm{supp}(d_1)$. It is obvious $f(x) \langle \preceq \rangle^1_X g(x)$ for each $x \in X$. From the congruence of $\langle \preceq \rangle^1$, we obtain

$$d_1 = f^\sharp \left(\frac{2}{1 + d_1[X]} d_1 \right) \langle \preceq \rangle^1_X g^\sharp \left(\frac{2}{1 + d_1[X]} d_1 \right) = \min(d_1, d_2).$$

We remark that $2d_1/(1 + d_1[X]) \in \mathcal{D}X$ because $2d_1[X]/(1 + d_1[X]) \leq 1$. □

Proof of Proposition 6.5 (Sketch). First, we prove $\sqsubseteq^m \in \mathbf{Pre}(\mathcal{D})$. Since $\sqsubseteq^m_1 = \preceq^d$, the image of the mapping $(-)_1$ under $\mathcal{C}_{\cap,\Phi}\{\sqsubseteq^r, \sqsubseteq^s, \sqsubseteq^m\}$ is $\mathbf{CSPre}(\mathcal{D}, 1)$. Next, we check $d_1 \langle \sqsubseteq_1 \rangle^1_X \min(d_1, d_2) \langle \sqsubseteq_1 \rangle^1_X d_2$ for each $d_1 \sqsubseteq_X d_2$ by applying Lemma 6.6 for each $\sqsubseteq \in \mathcal{C}_{\cap,\Phi}\{\sqsubseteq^r, \sqsubseteq^s, \sqsubseteq^m\}$.

For instance, we check the following case.

(case: $\sqsubseteq = \sqsubseteq^m$) We have $\sqsubseteq_1 = \preceq^d$. Suppose $d_1 \sqsubseteq_X d_2$, that is, $d_1[X] = 1 \implies d_1 = d_2$. Thus, we may assume $d_1[X] < 1$. This implies $(1 + d_1[X])/2 < 1$, and hence $\frac{1+d_1[X]}{2} \preceq^d \frac{(1+d_2[X])}{2} \frac{\min(d_1,d_2)(x)}{d_1(x)}$ for each $x \in \mathrm{supp}(d_1)$. By Lemma 6.6, $d_1 \langle \preceq^d \rangle^1 \min(d_1, d_2)$. Next, we have $\frac{(1+d_2[X])}{2} \frac{\min(d_1,d_2)(x)}{d_2(x)} \preceq^d \frac{1+d_2[X]}{2}$ for each $x \in \mathrm{supp}(d_2)$ since $\min(d_1, d_2) \leq d_2$. By Lemma 6.6, $\min(d_1, d_2) \langle \preceq^d \rangle^1 d_2$. □

We see that the mappings $[-]^1$ and $\langle - \rangle^1$ coincide on the subset $\mathcal{C}_{\cap,\mathrm{op}}\{\preceq^r, \preceq^s\}$ of $\mathbf{CSPre}(\mathcal{D}, 1)$, and that they differ from each other. Hence, there is a preorder $\sqsubseteq$ on $\mathcal{D}$ such that $\langle \preceq \rangle^1 \lneq \sqsubseteq \lneq [\preceq]^1$, and hence $\sqsubseteq_1 = \preceq$. The following proposition tells that there are exactly 4 such preorders.

Proposition 6.7 *Let $\preceq \in \mathbf{CSPre}(\mathcal{D}, 1)$. If $\preceq$ is one of $\preceq^d$, $\preceq^{d^{\mathrm{op}}}$, $\preceq^d \cap \preceq^{s^{\mathrm{op}}}$, and $\preceq^{d^{\mathrm{op}}} \cap \preceq^s$, a preorder $\sqsubseteq \in \mathbf{Pre}(\mathcal{D})$ such that $\langle \preceq \rangle^1 \lneq \sqsubseteq \lneq [\preceq]^1$ is determined* uniquely *as follows:*

$$\preceq = \preceq^d \implies \sqsubseteq = \sqsubseteq^M, \qquad \preceq = \preceq^{d^{\mathrm{op}}} \implies \sqsubseteq = \sqsubseteq^{M^{\mathrm{op}}},$$
$$\preceq = \preceq^d \cap \preceq^{s^{\mathrm{op}}} \implies \sqsubseteq = \sqsubseteq^M \cap \sqsubseteq^{s^{\mathrm{op}}}, \quad \preceq = \preceq^{d^{\mathrm{op}}} \cap \preceq^s \implies \sqsubseteq = \sqsubseteq^{M^{\mathrm{op}}} \cap \sqsubseteq^s,$$

where, the preorder $\sqsubseteq^M \in \mathbf{Pre}(\mathcal{D})$ is defined by

$$d_1 \sqsubseteq_X d_2 \overset{\text{def.}}{\iff} (d_1[X] = 1 \implies (d_2[X] = 1 \wedge \mathrm{supp}(d_1) = \mathrm{supp}(d_2))).$$

Otherwise, a preorder $\sqsubseteq \in \mathbf{Pre}(\mathcal{D})$ such that $\langle \preceq \rangle^1 \lneq \sqsubseteq \lneq [\preceq]^1$ does not exist.

To prove this proposition, we introduce the following restriction mapping C. Let $\tau \colon \mathcal{D}_{\leq 1} \Rightarrow \mathcal{D}$ be the natural transformation defined by $\tau_X(d) = d$ for each $d \in \mathcal{D}_{\leq 1}X$. For each $\sqsubseteq \in \mathbf{Pre}(\mathcal{D})$, we define its restriction $C(\sqsubseteq)$ by

$$C(\sqsubseteq)_X = \{\, (d_1, d_2) \in \mathcal{D}_{\leq 1}X \times \mathcal{D}_{\leq 1}X \mid \tau_X(d_1) \sqsubseteq_X \tau_X(d_2) \,\}.$$

The following lemma shows that the restriction $C(-)$ is a monotone mapping from $(\mathbf{Pre}(\mathcal{D}), \trianglelefteq)$ to $(\mathbf{Pre}(\mathcal{D}_{\leq 1}), \trianglelefteq)$ since the monotonicity of C is obvious.

Lemma 6.8 *For each $\sqsubseteq \in \mathbf{Pre}(\mathcal{D})$, $C(\sqsubseteq)$ is indeed a preorder on $\mathcal{D}_{\leq 1}$.*

Lemma 6.9 *Let $\preceq \in \mathbf{CSPre}(\mathcal{D}, 1)$ and $\sqsubseteq \in \mathbf{Pre}(\mathcal{D})$ with $\langle \preceq \rangle^1 \lneq \sqsubseteq \lneq [\preceq]^1$.*

(i) $(d_1[X] < 1 \vee d_2[X] < 1) \implies (d_1\, [\preceq]^1_X\, d_2 \iff d_1 \sqsubseteq_X d_2 \iff d_1 \langle \preceq \rangle^1_X d_2)$

(ii) $C(\langle \preceq \rangle^1) \lneq C(\sqsubseteq) \lneq C([\preceq]^1)$

Proof. (proof of (i)) We first prove $d_1\, [\preceq]^1_X\, d_2 \iff d_1\, \langle \preceq \rangle^1_X\, d_2$ whenever $d_1[X] < 1$ or $d_2[X] < 1$. Suppose a pair $d_1[\preceq]^1_X d_2$ such that $d_1[X] < 1$ or $d_2[X] < 1$. Since the mappings $\langle - \rangle^1$, $[-]^1$, and $C(-)$ preserve intersections and opposites, it suffices to check $d_1\, \langle \preceq \rangle^1_X\, d_2$ in the following 3 cases:

- (case: $\preceq = \preceq^r$) Since $\langle \preceq^r \rangle^1 = [\preceq^r]^1$, it is obvious that $d_1\, \langle \preceq \rangle^1_X\, d_2$.
- (case: $\preceq = \preceq^s$) Since $\langle \preceq^s \rangle^1 = [\preceq^s]^1$, it is obvious that $d_1\, \langle \preceq \rangle^1_X\, d_2$.
- (case: $\preceq = \preceq^d$) Suppose $d_1\, [\preceq^d]^1_X\, d_2$, that is, $d_2[X] < 1 \implies d_1[X] < 1$. Since $d_1[X] < 1$ or $d_2[X] < 1$, we obtain $d_1[X] < 1$. This implies $d_1\, \langle \preceq^d \rangle^1_X\, d_2$.

Since $\langle \preceq \rangle^1 \trianglelefteq \sqsubseteq \trianglelefteq [\preceq]^1$, for each $\preceq \in \mathbf{CSPre}(\mathcal{D}, 1)$, we conclude

$$(d_1[X] < 1 \text{ or } d_2[X] < 1) \implies (d_1\, [\preceq]^1_X\, d_2 \iff d_1 \sqsubseteq_X d_2 \iff d_1\, \langle \preceq \rangle^1_X\, d_2).$$

(proof of (ii)) From (i) of this lemma, $\langle \preceq \rangle^1 \lneq \sqsubseteq \lneq [\preceq]^1$ implies the following:

- $d_1[X] = d_2[X] = 1$ and $(d_1, d_2) \notin \langle \preceq \rangle^1_X$ holds for some X and $d_1 \sqsubseteq_X d_2$.
- $d_3[Y] = d_4[Y] = 1$ and $(d_3, d_4) \notin \sqsubseteq_Y$ holds for some Y and $d_3\ [\preceq]^1_Y\ d_4$.

The former implies $C(\langle \preceq \rangle^1)_X \trianglelefteq\!\!\!\!\neq C(\sqsubseteq)_X$ because there is $d' \in \mathcal{D}_{=1}X$ such that $\tau(d') = d$ for each $d \in \mathcal{D}X$ such that $d[X] = 1$, and the latter implies $C(\sqsubseteq)_Y \trianglelefteq\!\!\!\!\neq C([\preceq]^1)_Y$ similarly. These imply $C(\langle \preceq \rangle^1) \trianglelefteq\!\!\!\!\neq C(\sqsubseteq) \trianglelefteq\!\!\!\!\neq C([\preceq]^1)$. □

Hence, each preorder $\sqsubseteq \in \mathbf{Pre}(\mathcal{D})$ such that $\langle \preceq \rangle^1 \trianglelefteq\!\!\!\!\neq \sqsubseteq \trianglelefteq\!\!\!\!\neq [\preceq]^1$ is determined by preorders on $\mathcal{D}_{=1}$ between $C(\langle \preceq \rangle^1)$ and $C([\preceq]^1)$ and the preorder $[\preceq]^1$. Then, we obtain the preorder $\sqsubseteq^M$, which is the unique preorder between $\sqsubseteq^m$ and $\sqsubseteq^d$. It is easy to check $\sqsubseteq^M$ is indeed a preorder on $\mathcal{D}$.

Proof of Proposition 6.7 (Sketch). In the first 4 cases, $C(\langle \preceq \rangle^1) = \mathrm{Eq}^{\mathcal{D}_{=1}}$ and $C([\preceq]^1) \in \{\sqsubseteq^s, \sqsubseteq^{s\mathrm{op}}\}$. Thus, $C(\sqsubseteq) = \sqsubseteq^s \cap \sqsubseteq^{s\mathrm{op}}$ by Lemma 6.9 (ii). Hence, the preorder $\sqsubseteq$ is determined uniquely by Lemma 6.9 (i). Otherwise, $\langle \preceq \rangle^1 \trianglelefteq\!\!\!\!\neq \sqsubseteq \trianglelefteq\!\!\!\!\neq [\preceq]^1$ contradicts Lemma 6.9 (ii) since $C([\preceq]^1) = \sqsubseteq^s \cap \sqsubseteq^{s\mathrm{op}}$ or $C([\preceq]^1) = C(\langle \preceq \rangle^1)$ holds. □

We have finished identifying $\mathbf{Pre}(\mathcal{D})$.

Theorem 6.10 (Theorem 3.5(iii)) *The set* $\mathbf{Pre}(\mathcal{D})$ *is identified as Table 1 below. Moreover, we obtain* $\mathbf{Pre}(\mathcal{D}) = \mathcal{C}_{\cap,\mathrm{op}}\{\sqsubseteq^r, \sqsubseteq^s, \sqsubseteq^d, \sqsubseteq^m, \sqsubseteq^M\} \cong 41$.

$\preceq \in \mathbf{CSPre}(\mathcal{D}, 1)$	$\sqsubseteq \in \mathbf{Pre}(\mathcal{D})$ such that $\sqsubseteq_1 = \preceq$
$\top_{\mathcal{D}1}$	$\top^{\mathcal{D}}$
$\mathrm{Eq}_{\mathcal{D}1}$	$\mathrm{Eq}^{\mathcal{D}}$
$\preceq^r$	$\sqsubseteq^r$
$\preceq^r \cap \preceq^{s\mathrm{op}}$	$\sqsubseteq^r \cap \sqsubseteq^{s\mathrm{op}}$
$\preceq^r \cap \preceq^{d\mathrm{op}}$	$\sqsubseteq^r \cap \sqsubseteq^{d\mathrm{op}}$
$\preceq^r \cap \preceq^{s\mathrm{op}} \cap \preceq^{d\mathrm{op}}$	$\sqsubseteq^r \cap \sqsubseteq^{s\mathrm{op}} \cap \sqsubseteq^{d\mathrm{op}}$
$\preceq^s$	$\sqsubseteq^s$
$\preceq^s \cap \preceq^{s\mathrm{op}}$	$\sqsubseteq^s \cap \sqsubseteq^{s\mathrm{op}}$
$\preceq^d \cap \preceq^s$	$\sqsubseteq^m \cap \sqsubseteq^s,\ \ \sqsubseteq^d \cap \sqsubseteq^s$
$\preceq^d \cap \preceq^{d\mathrm{op}}$	$\sqsubseteq^m \cap \sqsubseteq^{m\mathrm{op}},\ \ \sqsubseteq^d \cap \sqsubseteq^{d\mathrm{op}}$
$\preceq^d \cap \preceq^{d\mathrm{op}} \cap \preceq^s$	$\sqsubseteq^m \cap \sqsubseteq^{m\mathrm{op}} \cap \sqsubseteq^s,\ \ \sqsubseteq^d \cap \sqsubseteq^{d\mathrm{op}} \cap \sqsubseteq^s$
$\preceq^d \cap \preceq^s \cap \preceq^{s\mathrm{op}}$	$\sqsubseteq^m \cap \sqsubseteq^s \cap \sqsubseteq^{s\mathrm{op}},\ \ \sqsubseteq^d \cap \sqsubseteq^s \cap \sqsubseteq^{s\mathrm{op}}$
$\preceq^d \cap \preceq^{d\mathrm{op}} \cap \preceq^s \cap \preceq^{s\mathrm{op}}$	$\sqsubseteq^m \cap \sqsubseteq^{m\mathrm{op}} \cap \sqsubseteq^s \cap \sqsubseteq^{s\mathrm{op}},\ \ \sqsubseteq^d \cap \sqsubseteq^{d\mathrm{op}} \cap \sqsubseteq^s \cap \sqsubseteq^{s\mathrm{op}}$
$\preceq^d$	$\sqsubseteq^m,\ \ \sqsubseteq^M,\ \ \sqsubseteq^d$
$\preceq^d \cap \preceq^{s\mathrm{op}}$	$\sqsubseteq^m \cap \sqsubseteq^{s\mathrm{op}},\ \ \sqsubseteq^M \cap \sqsubseteq^{s\mathrm{op}},\ \ \sqsubseteq^d \cap \sqsubseteq^{s\mathrm{op}}$

Table 1
The table of $\mathbf{CSPre}(\mathcal{D}, 1)$ and $\mathbf{Pre}(\mathcal{D})$ (we omit opposite preorders)

Proof. It is proved immediately from Proposition 6.3, 6.4, 6.5, and 6.7. □

The next lemma tells that $\mathbf{CSPre}(\mathcal{D}, 2)$ is enough to identify $\mathbf{Pre}(\mathcal{D})$.

Theorem 6.11 *We obtain* $\mathbf{Pre}(\mathcal{D}) \cong \mathbf{CSPre}(\mathcal{D}, 2)$.

Proof (Sketch). By Lemma 3.3 and [8, Lemma 3], it suffices to check $\sqsubseteq_2 \neq \sqsubseteq'_2$ whenever both $\sqsubseteq \neq \sqsubseteq'$ and $\sqsubseteq_1 = \sqsubseteq'_1$ hold. This is straightforward. □

For each $\preceq \in \mathbf{CSPre}(\mathcal{D}, 1)$, possible preorders on $\mathcal{D}$ whose evaluation at 1 equal $\preceq$ are show in Table 1.

In fact, $\mathbf{Pre}(\mathcal{D})$ is the opposite-closure of the collection of all preorders on $\mathcal{D}$ in the right column of Table 1. To check that the opposite-closure equals $\mathcal{C}_{\cap,\mathrm{op}}\{\sqsubseteq^r, \sqsubseteq^s, \sqsubseteq^d, \sqsubseteq^m, \sqsubseteq^M\}$, we remark $\mathrm{Eq}^{\mathcal{D}} = \sqsubseteq^r \cap \sqsubseteq^{r\mathrm{op}}$, $\sqsubseteq^M \cap \sqsubseteq^{M\mathrm{op}} = \sqsubseteq^d \cap \sqsubseteq^{d\mathrm{op}}$, $\sqsubseteq^s \cap \sqsubseteq^M = \sqsubseteq^s \cap \sqsubseteq^d$, $\sqsubseteq^r \not\trianglelefteq \sqsubseteq^s \cap \sqsubseteq^m$, and $\sqsubseteq^m \not\trianglelefteq \sqsubseteq^M \not\trianglelefteq \sqsubseteq^d$.

Thus, Table 1 shows that there are exactly 7 equivalence relations on $\mathcal{D}$: $\top^{\mathcal{D}}$, $\mathrm{Eq}^{\mathcal{D}}$, $\sqsubseteq^s \cap \sqsubseteq^{s\mathrm{op}}$, $\sqsubseteq^m \cap \sqsubseteq^{m\mathrm{op}}$, $\sqsubseteq^d \cap \sqsubseteq^{d\mathrm{op}}$, $\sqsubseteq^m \cap \sqsubseteq^{m\mathrm{op}} \cap \sqsubseteq^s \cap \sqsubseteq^{s\mathrm{op}}$, and $\sqsubseteq^d \cap \sqsubseteq^{d\mathrm{op}} \cap \sqsubseteq^s \cap \sqsubseteq^{s\mathrm{op}}$. There are exactly 9 partial orders on $\mathcal{D}$: $\mathrm{Eq}^{\mathcal{D}}$, $\sqsubseteq^r$, $\sqsubseteq^r \cap \sqsubseteq^{s\mathrm{op}}$, $\sqsubseteq^r \cap \sqsubseteq^{d\mathrm{op}}$, $\sqsubseteq^r \cap \sqsubseteq^{s\mathrm{op}} \cap \sqsubseteq^{d\mathrm{op}}$, and their opposite partial orders.

Remark 6.12 In the paper [13], Sokolova and Woracek proved that there are exactly 5 congruences on the convex algebra $[0,1] \cong \mathcal{D}1$. This fact corresponds to that there are exactly 5 equivalence relations in $\mathbf{CSPre}(\mathcal{D}, 1)$, namely $\mathrm{Eq}_{\mathcal{D}1}$, $\top_{\mathcal{D}1}$, $\preceq^s \cap \preceq^{s\mathrm{op}}$, $\preceq^d \cap \preceq^{d\mathrm{op}}$, and $\preceq^s \cap \preceq^{s\mathrm{op}} \cap \preceq^d \cap \preceq^{d\mathrm{op}}$.

Congruent and substitutive preorders on $\mathcal{D}X$ are equivalent to precongruences on the convex algebra $\mathcal{D}X$ over $|X|$-dementional vector space with an orthonormal basis $\{\, e_x \mid x \in X \,\}$. The precongruence needs to be closed under *linear operators* T such that $\forall x \in X.((\|Te_x\|_1 \leq 1) \wedge (0 \leq Te_x))$, where $\left\|\sum_{x\in X} p_x e_x\right\|_1 = \sum_{x\in X} |p_x|$ (1-norm). Thus generally speaking, there are more congruences on the convex algebra $\mathcal{D}X \subseteq [0,1]^X$ than equivalence relations in $\mathbf{CSPre}(\mathcal{D}, X)$ when $2 \leq |X|$. When $X \cong 1$, the closedness under operators is implied by the closedness under convex combinations, and hence congruences on $[0,1]$ equal equivalence relations in $\mathbf{CSPre}(\mathcal{D}, 1)$.

7 Coalgebraic Simulations between Markov Chains

Simulations between coalgebras are defined coalgebraically by using *relational liftings* of coalgebra functors. In this section, we focus on simulations between Markov chains (i.e. $\mathcal{D}$-coalgebras). We focus on the relational liftings of $\mathcal{D}$ that are constructed from preorders on $\mathcal{D}$ by the method in [5,7]. For a given preorder $\sqsubseteq \in \mathbf{Pre}(\mathcal{D})$, we construct the relational lifting $\overline{\mathcal{D}}^{(\sqsubseteq)}$ of $\mathcal{D}$ by

$$\overline{\mathcal{D}}^{(\sqsubseteq)}(R) = \sqsubseteq_X \circ \{\, (\mathcal{D}\pi_1(d), \mathcal{D}\pi_2(d)) \in \mathcal{D}X \times \mathcal{D}Y \mid d \in \mathcal{D}(R) \,\} \circ \sqsubseteq_Y$$

where $\pi_1 \colon R \to X$ and $\pi_2 \colon R \to Y$ are projections from a relation $R \subseteq X \times Y$.

We apply the preorders $\mathrm{Eq}^{\mathcal{D}}$, $\sqsubseteq^r$, $\sqsubseteq^s$, $\sqsubseteq^s \cap \sqsubseteq^{s\mathrm{op}}$, $\sqsubseteq^m$, $\sqsubseteq^M$, and $\sqsubseteq^d$ on $\mathcal{D}$ to the construction $\overline{\mathcal{D}}^{(-)}$. The first four cases are seen in earlier studies.

- $\overline{\mathcal{D}}^{(\mathrm{Eq}^{\mathcal{D}})}$-simulation, that is, $\mathcal{D}$-bisimulation in [1, Section 3] is a coalgebraic formulation of probabilistic bisimulation [9]. This fact is shown in [1].
- The study [3] shows that $\overline{\mathcal{D}}^{(\sqsubseteq^r)}$-simulations coincide Jonsson-Larsen simulations over Markov chains.
- It is easy to see that a relation R is a $\overline{\mathcal{D}}^{(\sqsubseteq^s)}$-simulation between Markov chains (X, ξ) and (Y, ξ') if and only if it is a simulation between two $\mathcal{P}$-colagebras

$(X, \mathrm{supp} \circ \xi)$ and $(Y, \mathrm{supp} \circ \xi')$ in the standard sense. We call $\overline{\mathcal{D}}^{(\sqsubseteq^s)}$-simulations *support-simulations*. See also [7, Example 4.5(4)].

- Analogous to $\overline{\mathcal{D}}^{(\sqsubseteq^s)}$-simulations, $\overline{\mathcal{D}}^{(\sqsubseteq^s \cap \sqsubseteq^{s\mathrm{op}})}$-simulations are obtained from bisimulation between two $\mathcal{P}$-colagebras. See also [7, Example 6.4]. We call $\overline{\mathcal{D}}^{(\sqsubseteq^s \cap \sqsubseteq^{s\mathrm{op}})}$-simulations support-bisimulations.

When we apply the the remaining three preorders $\sqsubseteq^m$, $\sqsubseteq^M$, and $\sqsubseteq^d$ on $\mathcal{D}$ to the construction $\overline{\mathcal{D}}^{(-)}$, we obtain the notion of probabilistic bisimulations, support-bisimulations, and reverse support-simulations *ignoring states with deadlocks* between Markov chains.

For two Markov chains (X, ξ) and (Y, ξ'), a relation $R \subseteq X \times Y$ is:

- a $\overline{\mathcal{D}}^{(\sqsubseteq^m)}$-simulation if and only if
$$(x, y) \in R \wedge \xi(x)[X] = 1 \implies (\xi(x), \xi'(y)) \in \overline{\mathcal{D}}^{(\mathrm{Eq}^{\mathcal{D}})}(R).$$
This is seen as a probabilistic bisimulation *ignoring states with deadlocks*.
- a $\overline{\mathcal{D}}^{(\sqsubseteq^M)}$-simulation if and only if
$$(x, y) \in R \wedge \xi(x)[X] = 1 \implies (\xi(x), \xi'(y)) \in \overline{\mathcal{D}}^{(\sqsubseteq^s \cap \sqsubseteq^{s\mathrm{op}})}(R).$$
This is seen as a support-bisimulation ignoring states with deadlocks.
- a $\overline{\mathcal{D}}^{(\sqsubseteq^d)}$-simulation if and only if
$$(x, y) \in R \wedge \xi(x)[X] = 1 \implies (\xi(x), \xi'(y)) \in \overline{\mathcal{D}}^{(\sqsubseteq^{s\mathrm{op}})}(R).$$
This is seen as a reverse support simulation ignoring states with deadlocks.

We give an example of $\overline{\mathcal{D}}^{(\sqsubseteq^m)}$-simulation. We consider two Markov chains (X, ξ) and (Y, ξ') and their start states $x \in X$ and $y \in Y$ as Fig. 6. The dashed arrows are a $\overline{\mathcal{D}}^{(\sqsubseteq^m)}$-simulation R between x and y. First, since the state x has a deadlock, the states x and y are assumed to be probabilistic bisimilar unconditionally. Next, since transitions started from the state x' has no deadlock, the state y' must be probabilistic bisimilar to the state x' in the sense of $\overline{\mathcal{D}}^{(\mathrm{Eq}^{\mathcal{D}})}$-(bi)similarity.

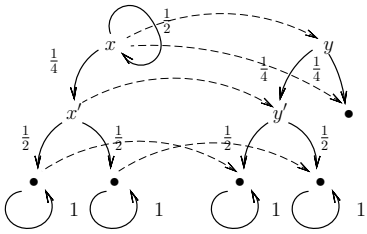

Fig. 6. A $\overline{\mathcal{D}}^{(\sqsubseteq^m)}$-simulation between Markov chains (X, ξ) and (Y, ξ')

8 Future Work

We have the following future work at this time:

- We expect to analyse preorders on other monads. For example, the convex module monad $\mathcal{CM}$ [6,14] that captures discrete probabilistic branching *combined with nondeterminism*.
- We expect to obtain preorders on the composite monad ST of monads S and T by using a distributive law $\delta\colon TS \Rightarrow ST$ from preorders on S and T.

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