Summary of thesis: Analysis of replica symmetry breaking in sparsely-connected spin glass models and non-equilibrium Brownian particles

Masahiko Ueda

In nature, there are many systems which contain random frustration. Typical examples of them are gene regulatory networks, neural networks, ecosystems, and human society. Although behaviors of elements of these systems are irregular, we can find that the systems sometimes exhibit stable ordered states in a macroscopic scale. Furthermore, stable pattern exhibited by the systems are sometimes diverse. It is a crucial step for studying complex systems to find universality of such systems.

In equilibrium statistical mechanics, randomly frustrated systems have extensively been studied. The representative examples are spin glass models. Particularly, in mean-field spin glass models, which can be studied theoretically, it has been known that states freeze into some irregular patterns at low temperatures. In addition, it has been known that the low temperature phase is characterized by a concept called replica symmetry breaking (RSB). RSB is a phenomenon that occurs by reflecting a rugged free energy landscape of the system, and indicates existence of several stable ordered phases. RSB is detected by a physical quantity called overlap, which expresses how much two independent identical systems (replicas) are similar to each other.

While the concept of RSB achieved great success in describing mean-field spin glass models, its utility for other glassy systems beyond them has not been known. This is mainly because of difficulty to handle these systems theoretically. Whether RSB is an effective concept for describing finite-dimensional spin glasses, structural glasses or non-equilibrium glasses is an interesting unsolved problem.

In this thesis, we focus on three problems. First, we develop a method to analyze an effective potential of overlap, called the Franz-Parisi potential, of sparsely-connected spin glass models in the replica method. There are two methods for analyzing sparse systems; the cavity method and the replica method. Since the cavity method relies on sparseness of random graphs, extension to loopy graphs is difficult. On the other hand, because the replica method can treat several models in a unified manner, when we consider extension to finite-dimensional models, results in sparse systems may be useful for them. We prove that the calculation results of the Franz-Parisi potential in terms of the replica method is equivalent to that by using the cavity method, similarly to the free energy of the models. We also prove that replica analysis of the Franz-Parisi potential under the replica symmetric ansatz is equivalent to the 1RSB cavity method with Parisi parameter x=1 for the free energy. In this proof, a characteristic structure between replicas, called ultrametric structure, plays a central role. Furthermore, by calculating the Franz-Parisi potential numerically in our method, we show the coexistence of the low-overlap phase and the high-overlap phase, which suggests that a mean-field picture is still useful for sparse systems.

Second, we develop an approximation method to calculate transition temperatures of sparsely-connected spin glass models in low computational cost. In sparse systems, since an order parameter is generally a distribution function of an effective field called a cavity field, we need to solve self-consistent equations of the distributions in order to calculate transition temperatures, which requires high computational cost. We obtain an approximation scheme to calculate the transition temperatures of spin glass models on regular random graphs with extremely high precision by analyzing the Franz-Parisi potential in an approximation which is exact in fully-connected models. In our method, we can calculate the transition temperatures only by solving a two-variable self-consistent equation.

Third, we study a non-equilibrium phenomenon where RSB plays an important role. Concretely, we consider a passive Brownian particle driven by a random velocity field, and study whether RSB in path ensemble occurs when we focus on trajectories of the particle. As a result, we numerically find that, in case of a velocity field driven by the noisy Burgers equation, an overlap between two independent particles becomes non-zero. Furthermore, by changing a boundary condition of the model, we theoretically prove that the model can be mapped exactly to directed polymers in random potentials, which exhibit RSB. In addition, by comparing the modified system with the original system, we conclude that the original system also exhibits RSB in the path ensemble. This result provides one example where the concept of RSB is useful even in non-equilibrium situation.

In sum, the thesis extends the range of application of replica method and the concept of RSB.