Performance Analysis and Sampled-Data Controller Synthesis for Bounded Persistent Disturbances

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Abstract

The disturbance rejection problem is one of the main issues in control system analysis and synthesis, and system norms are used to evaluate the effect of disturbances on the system outputs. Depending on the nature of the disturbance affecting the system and performance measures, one can define a number of different system norms. Among various system norms, the L_∞-induced norm of control systems is used to deal with the maximum amplitude of the regulated output for the worst persistent exogenous input with a unit magnitude. Because evaluating the maximum magnitude of the regulated output is very important in many control systems and bounded persistent disturbances such as steps and sinusoids are often encountered in control systems, the L_∞-induced norm control problem should be solved for performance evaluation and performance improvement. For the computation of the L_∞-induced norm of continuous-time linear time-invariant systems, we need to integrate the absolute value of the impulse response of the given system (i.e., we nee to compute the L_1 norm of the impulse response), where the response corresponds to the kernel function in the convolution formula for the input/output relation. Hence, the study associated with the treatment of the L_∞-induced norm has been called the L_1 problem. However, it is very difficult to compute this integral exactly or even approximately with an explicit upper bound and lower bound. Indeed, there have been no results on an exact computation of the L_∞-induced norm as well as its upper and lower bounds.

On the other hand, sampled-data systems arise in feedback control when continuous-time plants are to be controlled by discrete-time controllers, and they occur naturally in feedback control applications, such as process control, attitude control and so on. The study associated with sampled-data systems shows promise in providing deeper insight and improved performance in many applications since discrete-time controllers are so widespread in feedback control systems. Thus, the study associated with sampled-data systems plays important roles in practical senses and is one of the main issues in the analysis and synthesis of control systems. In particular, a number of studies of sampled-data systems relevant to the disturbance rejection problem have been addressed by many researchers. However, there have been no studies on giving an exact computation of the L_∞-induced norm as well as its

upper and lower bounds of sampled-data systems, and thus we pay attention to the studies associated with the L_∞-induced norm of sampled-data systems.

This thesis studies computing the L_∞-induced norms of continuous-time and sampled-data systems and develops two simple approaches named input approximation and kernel approximation for computing the L_∞-induced norms. They are two different approaches in terms of the viewpoint behind approximations but share a common technical feature that they employ a piecewise constant approximation or piecewise linear approximation scheme of functions.

This thesis first applies these two approaches to computing the L_∞-induced norm of stable continuous-time finite-dimensional linear time-invariant (FDLTI) systems. In the input approximation approach, the input of continuous-time systems is approximated by a piecewise constant or piecewise linear function and computation methods for an upper bound and lower bound of the L_∞-induced norm are given. In the kernel approximation approach, the kernel function in the convolution formula of continuous-time systems is approximated by a piecewise constant or piecewise linear function and computation methods for an upper bound and lower bound of the L_∞-induced norm are given. These approaches are introduced through the fast-lifting technique, by which a finite time interval $[0, h)$ with a sufficiently large h is divided into M subintervals with an equal width, and it is shown that the gap between the computed upper and lower bounds converges to 0 at the rate of $1/M$ in the piecewise constant approximation scheme and $1/M^2$ in the piecewise linear approximation scheme, under both the input approximation and kernel approximation approaches. The effectiveness and validity of the obtained theoretical results are demonstrated through numerical examples.

Stimulated by the success in the studies of the L_∞-induced norm analysis of continuous-time systems, this thesis next applies the two approximation approaches, i.e., input approximation and kernel approximation approaches, to the L_∞-induced norm analysis of sampled-data systems. These applications are also supported by the fast-lifting technique and the two approximation schemes, i.e., piecewise constant approximation and piecewise linear approximation schemes. Through these applications, an upper bound and lower bound of the L_∞-induced norm associated with sampled-data systems can be readily computed and it is shown that the gap between the computed upper and lower bounds converges to 0 at the rate of $1/M$ in the piecewise constant approximation scheme and $1/M^2$ in the piecewise linear approximation scheme, under both the input approximation and kernel approximation approaches. The effectiveness and validity of the developed theoretical results are demonstrated through numerical examples.

Furthermore, this thesis also deals with the L_1 optimal sampled-data controller synthesis problem, by which we mean the discrete-time controller synthesis minimizing the L_∞-induced

norm of sampled-data systems. To solve this problem, we develop two methods by using the ideas of the input approximation approach with the piecewise constant approximation and piecewise linear approximation schemes. In these methods, discretization procedures of the continuous-time generalized plant for the L_1 optimal sampled-data controller synthesis are derived. These procedures approximately convert the L_1 optimal sampled-data controller synthesis problem into the discrete-time l_1 optimal controller synthesis problem, where the latter problem can be solved with an existing method. By using the arguments of preadjoint operators, we give two important inequalities that form theoretical bases for the discretization procedures. More precisely, mathematical bases for the piecewise constant approximation and piecewise linear approximation schemes associated with the L_1 optimal sampled-data controller synthesis are shown through these inequalities together with convergence proofs again in the rate of $1/M$ and $1/M^2$, respectively. Effectiveness of the proposed methods is demonstrated through a numerical example.

Through the ideas of the input approximation and kernel approximation approaches, the readily computable upper and lower bounds of the L_∞-induced norm of continuous-time and sampled-data systems together with the associated convergence rates can be obtained. Furthermore, we provide methods for the L_1 optimal sampled-data controller synthesis, by which the L_∞-induced norm of sampled-data systems is minimized. We believe that this thesis offers innovative and fundamental methodologies associated with the L_∞-induced norm analysis and the L_1 optimal sampled-data controller synthesis. The theoretical results developed in this thesis will play quite important roles in dealing with the bounded persistent disturbances.

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Contents

Chapter 1

Introduction

This thesis is concerned with the problems associated with disturbance rejection control and sampled-data control systems, and this chapter reviews the background in these control problems. This chapter further discusses the contribution of this thesis over the existing results in the problems of disturbance rejection control and sampled-data control systems, and provides some mathematical preliminaries used in this thesis.

1.1 Background in Disturbance Rejection Control and Sampled-Data Control Systems

The background in the problems of disturbance rejection control and sampled-data control systems is reviewed in this section.

1.1.1 Background in Disturbance Rejection Control

The disturbance rejection control is very significant in control system analysis and synthesis. System norms are used to evaluate the effect of disturbances on the system outputs. Depending on the nature of the disturbance affecting the system and performance measures, one can define a number of different system norms. The system norms used in the disturbance rejection control problem are as follows.

The L_2 norm (or l_2 norm) can be used for evaluating the energy of continuous-time (or discrete-time) signals, and the L_2-induced norm of a continuous-time finite-dimensional linear time-invariant (FDLTI) system (or the l_2-induced norm of a discrete-time FDLTI system) corresponds to the H_∞ norm of the transfer matrix of the system, which is the supremum of the magnitude of the transfer function evaluated on the imaginary axis (or the unit circle in the discrete-time case). Hence, the study associated with the treatment of the L_2-induced (or l_2-induced) norm has been called the H_∞ problem. The H_∞ problem

was first investigated in [76], where the sensitivity reduction problem associated with single-input/single-output (SISO) continuous-time systems was dealt with in the measure of the H_∞ norm. Subsequently, there have been a number of studies on the continuous-time or discrete-time H_∞ problem since this system norm has been used as a typical measure in the sensitivity problem, robust control problem and so on. The output feedback design method for optimizing sensitivity to disturbances and robustness under plant perturbations relevant to SISO continuous-time systems was developed [77] in a weighted H_∞ norm. Furthermore, the determination of the optimal weighted sensitivity function and an upper bound on this norm relevant to SISO continuous-time systems were derived [31] by applying the theory of Sarason [64]. The methods for H_∞ optimal controller synthesis associated with multi-input/ multi-output (MIMO) continuous-time and discrete-time systems were obtained in [12],[32] by using the idea of the Wiener-Hopf approach [69],[74],[75], while the state-space approach to the MIMO continuous-time H_∞ optimal controller synthesis was proposed in [25]. A simple method for computing the H_∞ norm was derived in [7] by using the idea of bisection algorithm, while the linear matrix inequality (LMI) approach to computing the H_∞ norm was discussed in [8]. Nowadays, the LMI approach is one of the representative methods in the field of the H_∞ problem.

The H_2 norm can be used to evaluate the power of the output for a white noise input. The another interpretation of the H_2 norm is the L_2 norm (or the l_2 norm) of the output for unit impulse disturbances. The H_2 problem has originated from the problem of linear quadratic Gaussian (LQG) control [56], and the study of the LQG control problem was started with the study of Kalman Filters [44],[45]. The LQG control synthesis method was first introduced in [26], and much work has been reported on the LQG control problem since then. For example, a comprehensive analysis of the LQG control problem was given in [1], [57].

The reformulation and generalization of the LQG control problem as an H_2 norm problem is more recent, and the state-space approach to the MIMO continuous-time H_2 optimal controller synthesis was developed in [25]. Nowadays, the arguments in [25] are regarded as standard in the H_2 problem.

With these studies, the H_∞ and H_2 optimal controller synthesis problems have been well studied in a number of books such as [29],[57],[78],[79] and so on.

The L_∞ norm (or l_∞ norm) can be used for considering the maximum amplitude of signals, and the L_∞-induced norm of a continuous-time FDLTI system (or the l_∞-induced norm of a discrete-time FDLTI system) corresponds to the L_1 (or l_1) norm of the impulse response of the continuous-time (or discrete-time) system. Thus, the study associated with the treatment of the L_∞-induced (or l_∞-induced) norm has been named the L_1 (or l_1) problem. Along similar

lines to the H_∞ problem, the L_1 problem was first formulated in [70], where some special cases of the L_1 (and l_1) problem associated with SISO continuous-time and discrete-time systems were discussed. Subsequently, there have been a number of studies on the L_1 (or l_1) problem because evaluating the maximum amplitude of the output is very important in practical senses and this problem is pertinent to bounded persistent disturbances often encountered in control systems. Regarding a more general situation, the continuous-time L_1 problem of SISO continuous-time case was dealt with in [20], while the discrete-time l_1 problem of MIMO discrete-time case of square systems (i.e., the one-block problem) was dealt with in [19]. The central ideas for the solution of nonsquare (i.e., multiblock) problems were given in [21], including a method for constructing approximate suboptimal solutions iteratively. These ideas are based on the solution of a linear program representing a truncated version of the original problem. A similar method for solving the fixed input optimization problem was presented in [22]. A general treatment of the multiblock case was provided in [59], where an optimal solution is shown to exist under some assumptions. A method for computing lower bounds on the optimal norm was discussed in [23], while a rational approximation approach to the L_1 optimal controllers relevant to SISO continuous-time systems was given in [62]. Motivated by the lack of a solid understanding of the general discrete-time l_1 multiblock problem, a comprehensive treatment of the general l_1 multiblock problem was given in [24]. With these studies, the l_1 optimal control problems have been well studied in a number of books such as [18],[30] and so on. After that, a new method for solving the l_1 optimal control problem was presented in [47] by introducing the scaled-Q approach.

However, in contrast to the case of the discrete-time l_1 problem, there are few studies on the continuous-time L_1 problem. The L_1 problem associated with SISO continuous-time systems was dealt with in [20],[62] but there is no study on MIMO continuous-time systems. Thus, the author tackled the MIMO continuous-time L_1 analysis problem, by which the L_∞-induced norm computation of MIMO continuous-time systems is meant. As a result, the following results were obtained by the author. A computation method for the $L_\infty[0,h)$-induced norm of a compression operator, which corresponds to computing the L_∞-induced norm of a continuous-time system in a truncated fashion, was developed first in [49]. A more sophisticated argument to the compression operators was provided in [51] by using the idea of input approximation approach, and its extension to the L_∞-induced norm analysis of continuous-time systems with a new computation method named kernel approximation approach was studied first in [53]. Furthermore, a comparison between the input approximation approach and the kernel approximation approach in the L_∞-induced norm analysis of continuous-time systems was presented in [55]. These results relevant to the L_∞-induced norm analysis of continuous-time systems are discussed in this thesis.

1.1.2 Background in Sampled-Data Control Systems

Sampled-data systems arise in feedback control when continuous-time plants are to be controlled by discrete-time controllers, and they occur naturally in feedback control applications, such as process control, attitude control and so on. The study of sampled-data systems is very important in control system analysis and synthesis and shows promise in providing deeper insight and improved performance in many applications since discrete-time controllers are so widespread in feedback control systems and have several advantages over continuous-time controllers. For example, changing or tuning controllers requires just reprogramming the control algorithm in discrete-time controllers while hardware extension or reinstallation is need in continuous-time controllers. Thus, the study of discrete-time controller synthesis plays important roles in practical senses and is one of the main issues in control systems, and there are three basic ways to design discrete-time controllers. The first is to do a continuous-time synthesis and then a discrete-time implementation, the second is to discretize the plant and do a discrete-time controller synthesis, and the third is the so-called sampled-data controller synthesis. The advantages of the first way are that continuous-time controller synthesis is natural and one could normally expect recovery of the performance specifications as the sampling frequency increases. However, often in an industrial setting, the sampling frequency is not designable by the control engineer. The second way has the obvious disadvantage of ignoring intersample behavior. Furthermore, the sampling frequency must be selected *a priori* and adjusting its effect therefore is out of the synthesis loop. In connection with this, the study associated with sampled-data systems taking account of their intersample behavior (i.e., the third way) plays important roles in practical senses and is one of the main issues in control system analysis and synthesis.

Starting from the studies of sampled-data systems associated with the computation of the L_2-induced norm [13] and the input-output stability analysis [14], there have been a number of studies associated with sampled-data systems. Intersample behavior of sampled-data systems can be treated by using the ideas of the fast-sample/fast-hold (FSFH) approximation technique [46], the lifting technique [4],[68],[71], the FR operator approach [2] and so on. Frequency response gain of sampled-data systems could be computed through the FSFH approximation technique [73], the lifting technique [72] and the FR operator approach [2]. After that, a convenient idea of bisection algorithm for computing the frequency response gain of sampled-data systems [41] in both the lifting technique and the FR operator approach treatment was developed, while an idea of finite-rank approximation of compression operators in the Hilbert Schmidt norm was introduced in [37] for giving a simple method for computing upper and lower bounds of the frequency response gain of sampled-data systems. The

H_∞ sampled-data control problem was also dealt with in [42], and its solution procedures have been further developed through the FSFH approximation technique [46], the lifting technique [6],[40],[60],[61],[68], the FR operator approach [34] and so on [67]. Furthermore, computing the $L_2[0,h)$-induced norm of compression operators was developed in [27] as a fundamental study of the H_∞ sampled-data control problem. The study of the H_2 sampled-data control can also be classified by the type of the frameworks used in taking account of the intersample behavior, i.e., the lifting technique [5],[48],[60],[61], FR operator approach [33] and so on [15],[16],[39]. With these studies, the problems of sampled-data systems associated with the H_∞ and H_2 norms have been well studied in a number of books such as [17], [63] and so on.

Another very important framework to deal with sampled-data systems is the fast-lifting technique [38]. The fast-lifting technique has an integer parameter M as in the FSFH approximation technique, but it is used only to subdivide the sampling interval $[0,h)$ into M smaller pieces, while the FSFH approximation technique takes M equally spaced sampling points on the interval $[0,h)$; no information is hence lost as to signals on $[0,h)$ by applying fast-lifting. Through the fast-lifting technique, more sophisticated studies on the H_∞ sampled-data control problem have been developed as follows. A discretization procedure for the H_∞ sampled-data control problems was developed in [38] in a γ-independent fashion, where γ is the H_∞ performance level, when the direct feedthrough matrix D_{11} from the disturbance to the controlled output is assumed to be zero. This result was further extended in [36] for the case of $D_{11} \neq 0$. An idea of quasi-finite-rank approximation of compression operators on the $L_2[0,h)$-induced norm was developed in [35] to alleviate difficulties in γ-independent discretization for the H_∞ sampled-data control problem.

On the other hand, it should be noted, in contrast to the cases of the H_∞ and H_2 problems of sampled-data systems, that there are few studies on the L_1 problem of sampled-data systems. More precisely, the only approach to the L_1 problem of sampled-data systems has been to introduce a discrete-time system through the FSFH approximation technique as developed in [3],[28],[66], and it is shown that the l_∞-induced norm of the approximating discrete-time system converges to the L_∞-induced norm of the original sampled-data system as the FSFH approximation parameter M tends to infinity. However, these studies did not evaluate how close the l_∞-induced norm for a given M is to the exact value of the L_∞-induced norm. By noting that fast-lifting plays important roles in the H_∞ sampled-data control problems, the author developed a method for computing upper and lower bounds of the L_∞-induced norm of sampled-data systems by applying the idea of the fast-lifting technique. More specifically, the computation method developed in [49],[51] on compression operators has been used as a fundamental tool for the L_∞-induced norm computation of

sampled-data systems. In [50], readily computable upper and lower bounds of the L_∞-induced norm of sampled-data systems were derived by using the ideas of the fast-lifting technique and piecewise linear approximation scheme. Furthermore, a computation method for the L_∞-induced norm of sampled-data systems was presented in [54] by using the ideas of piecewise constant approximation and piecewise linear approximation schemes. The author also extended the arguments in [54] to the L_1 optimal controller synthesis of sampled-data systems. In connection with this, a mathematical basis for the piecewise linear approximation in the L_1 optimal controller synthesis problem of sampled-data systems was provided in [52]. These results relevant to the L_1 sampled-data control problem are discussed in this thesis.

1.2 Contribution of This Thesis

This section discusses the contribution of this thesis over the existing results in the L_∞-induced norm analysis of continuous-time and sampled-data systems and L_1 optimal controller synthesis in sampled-data systems.

1.2.1 L_∞-Induced Norm Analysis of Continuous-Time Systems

As mentioned in the above section, there have been a number of studies on the discrete-time l_∞-induced norm problem associated with MIMO FDLTI systems [18],[19],[21]–[24], [30],[47],[59],[70]. This is because evaluating the maximum amplitude of the output is very important in many control systems and this problem is also pertinent to dealing with bounded persistent disturbances such as steps and sinusoids, which are often encountered in control systems. However, in contrast to the case of the discrete-time l_∞-induced norm problem, there are few studies on the continuous-time L_∞-induced norm problem; the L_∞-induced norm problem associated with SISO FDLTI continuous-time systems was dealt with in [20], [62],[70] but there have been no studies on giving an exact computation method of the L_∞-induced norm as well as its upper and lower bounds associated with MIMO FDLTI continuous-time systems. This is because an accurate computation of the L_∞-induced norm associated with a MIMO FDLTI continuous-time system is very hard since we need to integrate the absolute value (i.e., we need to compute the L_1 norm) of the impulse response of the system, which corresponds to the kernel function in the convolution formula for its input/output relation.

With the difficulty of computing the above integral exactly in mind, this thesis provides two simple approaches named input approximation and kernel approximation for computing the L_∞-induced norm associated with a MIMO FDLTI system. Through these approxima-

tion approaches, the input or kernel function relevant to a MIMO continuous-time FDLTI system is approximated, and this leads to a simple method for computing the L_∞-induced norm of MIMO continuous-time FDLTI systems. More precisely, these two approximation approaches are supported by the application of fast-lifting which has the parameter M, and they share a common technical feature that they employ a piecewise constant approximation or piecewise linear approximation scheme of functions. One of the main contributions of this thesis is to show that the upper and lower bounds of the L_∞-induced norm associated with MIMO continuous-time FDLTI systems can be readily computed and the piecewise constant approximation and piecewise linear approximation schemes lead to gaps between the upper and lower bounds converging to 0 at the rate of $1/M$ and $1/M^2$, respectively, in both the input approximation and kernel approximation approaches.

1.2.2 L_∞-Induced Norm Analysis of Sampled-Data Systems

The L_∞-induced norm analysis of sampled-data systems has been dealt with in [3],[28], [66]. However, in contrast to the cases of the H_2 and H_∞ problems of sampled-data systems, no precise solution has been obtained even for the *analysis* of the L_∞-induced norm, for which only approximate methods have been provided. A drawback of these studies is that they are not pertinent to evaluating how close the approximately obtained l_∞-induced norm for a given M is to the exact value of the L_∞-induced norm.

As a significant advance over the existing results, this thesis provides readily computable upper and lower bounds of the L_∞-induced norm of sampled-data systems. More precisely, stimulated by the success in the L_∞-induced norm computation of continuous-time systems, we extend the input approximation and kernel approximation approaches, which involve the piecewise constant approximation and piecewise linear approximation schemes, to the computation of the L_∞-induced norm of sampled-data systems. Then, another main contribution of this thesis is to show that readily computable upper and lower bounds of the L_∞-induced norm of sample-data systems are obtained and the piecewise constant approximation and piecewise linear approximation schemes lead to gaps between the upper and lower bounds converging to 0 at the rate of $1/M$ and $1/M^2$, respectively, in both the input approximation and kernel approximation approaches.

1.2.3 L_1 Optimal Sampled-Data Controller Synthesis

As mentioned in the above subsection, we derive readily computable upper and lower bounds of the L_∞-induced norm of sampled-data systems by using the ideas of the input approximation and kernel approximation approaches. Unfortunately, however, these meth-

ods are restricted to analysis and cannot be used directly for synthesis. This is because they require to compute the $L_1[0, h/M)$ norms of kernel functions determined by the continuous-time system and the discrete-time controller and the structure of the way the controller parameters are involved in the kernel functions is complicated.

In this respect, this thesis also considers the L_1 optimal sampled-data controller synthesis problem, by which we mean the discrete-time controller synthesis minimizing the L_∞-induced norm of sampled-data systems. More precisely, we develop two methods in the input approximation approach with the piecewise constant approximation and piecewise linear approximation schemes. By applying the arguments of preadjoint operators, we provide two important inequalities that form theoretical bases for the piecewise constant approximation and piecewise linear approximation schemes to the L_1 optimal sampled-data controller synthesis problem. These inequalities show that the piecewise constant approximation and piecewise linear approximation schemes are in the convergence rate of $1/M$ and $1/M^2$, respectively, and the associated error bounds in the approximation schemes could be derived through these inequalities. In connection with these convergence rates, we further give two discretization procedures of continuous-time systems for the L_1 optimal sampled-data controller synthesis through the piecewise constant approximation and piecewise linear approximation treatment. The last main contribution of this thesis is thus the derivation of the theoretical bases for the piecewise constant approximation and piecewise linear approximation schemes in the L_1 optimal sampled-data controller synthesis with the developments of the discretization procedures. Furthermore, effectiveness of the developed methods is demonstrated through a numerical example.

1.3 Contents of This Thesis

In Chapter 2, we study computing the L_∞-induced norm of MIMO continuous-time FDLTI systems. More specifically, we develop the input approximation and kernel approximation approaches with application of the fast-lifting technique [38], which has parameter M. These two approximation approaches share a common technical feature that they employ a piecewise constant approximation or piecewise linear approximation scheme of functions. Through these approximation methods, this chapter shows that the input approximation and kernel approximation approaches derive readily computable upper and lower bounds of the L_∞-induced norm, and the gap between the upper and lower bounds in the piecewise constant approximation or piecewise linear approximation scheme converge to 0 at the rate of $1/M$ or $1/M^2$, respectively, in both the input approximation and kernel approximation approaches. Furthermore, this chapter investigates the relationship between such gaps in

the input approximation and kernel approximation approaches. The effectiveness of the developed computation methods is demonstrated through numerical examples.

In Chapter 3, we consider the problem of the L_∞-induced norm analysis of sampled-data systems. We first introduce operator theoretic approach to sampled-data systems by using the idea of the lifting technique [4],[6],[68],[71]. This approach derives the input and output operators describing the input/output relation of the sampled-data systems. In contrast to the case of continuous-time systems, we should apply adequate approximations to both the input and output operators in computing the L_∞-induced norm of sampled-data systems, even though the only input operator was treated approximately in continuous-time systems. With application of the fast-lifting technique, we develop the approximation methods for the output operator, by which the output of the sampled-data systems is approximated by a piecewise constant or piecewise linear function. We further apply the two approximation approaches used in Chapter 2, by which the L_∞-induced norm computation was carried out for continuous-time MIMO FDLTI systems, to the input operator, by noting that the structure of the input operator remains essentially the same as that in continuous-time FDLTI systems. Through these approximation methods, this chapter shows that an upper bound and lower bound of the L_∞-induced norm can be easily computed and the gaps between the computed upper and lower bounds converge to 0 as the fast-lifting parameter M tends to infinity. More precisely, the piecewise constant approximation and piecewise linear approximation schemes lead to the gaps between the upper and lower bounds converging to 0 at the rate of $1/M$ and $1/M^2$, respectively, in both the input approximation and kernel approximation approaches. Furthermore, a relationship between the gaps in the input approximation and kernel approximation approaches is discussed and the effectiveness of the resulting four types of computation methods is demonstrated through numerical examples.

In Chapter 4, we deal with the L_1 optimal controller synthesis problem of sampled-data systems. More precisely, in the lifted representation of sampled-data systems, we provide two discretization procedures of continuous-time systems for the L_1 optimal controller synthesis problem of sampled-data systems by using the ideas of piecewise constant approximation and piecewise linear approximation schemes under the input approximation approach. By applying the arguments of preadjoint operators, we provide two important inequalities that form theoretical bases for the piecewise constant approximation and piecewise linear approximation schemes to the L_1 optimal sampled-data controller synthesis problem. Through these inequalities, it is shown that the convergence rates for the L_1 optimal sampled-data controller synthesis in the piecewise constant approximation and piecewise linear approximation schemes are in $1/M$ and $1/M^2$, respectively, and the associated error bounds are derived. In connection with these convergence rates, we further give two discretization procedures of

continuous-time systems for the L_1 optimal sampled-data controller synthesis through the piecewise constant approximation and piecewise linear approximation treatment. The effectiveness of the developed discretization methods is examined through a numerical example.

In Chapter 5, we summarize the obtained results. We then give concluding remarks, with a possible direction of a future work for further developments in the rejection control problem of bounded persistent disturbances.

1.4 Notations and Mathematical Preliminaries

This section gives some mathematical preliminaries used in this thesis. Notations and the definitions of spaces used in this thesis are explained in detail. We particularly discuss the arguments of adjoint and preadjoint operators which play important roles in the L_1 optimal sampled-data controller synthesis.

The notations $\mathbb{R}^\nu_\infty$, $\mathbb{R}^\nu_1$ and $R(\cdot)$ are used to denote the Banach space of ν-dimensional real vectors equipped with vector ∞-norm, the Banach space of ν-dimensional real vectors equipped with vector 1-norm, and the range of an operator, respectively.

The dual space of a Banach space X, i.e., the space of all bounded linear functionals on X, is denoted by X^*.

Let X and Y be Banach spaces. For a linear operator $\mathbf{T} : X \to Y$, its adjoint [10],[11], [58] is denoted by $\mathbf{T}^* : Y^* \to X^*$, which by definition satisfies

$$\forall x \in X,\ \forall \phi \in Y^*, \quad \langle \mathbf{T}x, \phi \rangle = \langle x, \mathbf{T}^*\phi \rangle \tag{1.1}$$

where the notation $\langle y, \phi \rangle$ means the value of the linear functional ϕ at y. For the given Banach spaces X and Y, suppose that there exist unique Banach spaces, denoted by X_* and Y_*, such that their dual spaces $(X_*)^*$ and $(Y_*)^*$ coincide with X and Y, respectively. Then, if there exists an operator $\mathbf{T}_* : Y_* \to X_*$ such that $(\mathbf{T}_*)^* = \mathbf{T}$, then $\mathbf{T}_*$ is called the preadjoint [10], [11],[58] of $\mathbf{T} : X \to Y$; we can easily see that such an operator $\mathbf{T}_*$, if it exists, is unique. It is a fact that $\|\mathbf{T}_*\| = \|\mathbf{T}\|$, where $\|\mathbf{T}_*\|$ denotes the norm of $\mathbf{T}_*$ induced from the norms on Y_* and X_*, while $\|\mathbf{T}\|$ denotes the norm of $\mathbf{T} = (\mathbf{T}_*)^*$ induced from the (dual) norms on $(X_*)^* = X$ and $(Y_*)^* = Y$.

Regarding $(L_\infty[0,h))^\nu$, we sometimes drop ν and slightly abuse a term for simplicity, especially when we refer to the induced norm of an operator; for an operator $\mathbf{T} : X \to Y$ with X and Y being Banach spaces with norm $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively, we call $\|\mathbf{T}\| := \sup_{x \in X \setminus \{0\}} \|\mathbf{T}x\|_Y / \|x\|_X$ the $L_\infty[0,h)$-induced norm of $\mathbf{T}$ if either X or Y is $(L_\infty[0,h))^\nu$. A similar convention applies when $L_\infty[0,h)$ is replaced by a similar space.

The notation $\|\cdot\|$ is used to mean either the $L_\infty[0,h)$ norm of a vector function, i.e.,

$$\|f(\cdot)\| := \max_i \operatorname*{ess\,sup}_{0\le t<h} |f_i(t)| \tag{1.2}$$

(or that with h replaced by h/M or ∞), the $L_\infty[0,h)$-induced norm (or that with h/M or ∞ instead of h) of an operator, or the ∞-norm of a matrix or a vector, i.e., for $A \in \mathbb{R}_\infty^{n\times m}$,

$$\|A\| := \max_{i\in\{1,\cdots,n\}} \sum_{j=1}^{m} |a_{ij}| \tag{1.3}$$

whose distinction will be clear from the context. On the other hand, the notation $\|\cdot\|_1$ is used to mean either the $L_1[0,h)$ norm of a vector function, i.e.,

$$\|f(\cdot)\|_1 := \sum_i \int_0^h |f_i(t)|dt \tag{1.4}$$

(or that with h replaced by h/M or ∞), the $L_1[0,h)$-induced norm (or that with h/M or ∞ instead of h) of an operator, or the 1-norm of a matrix or a vector, i.e., for $A \in \mathbb{R}_1^{n\times m}$,

$$\|A\|_1 := \max_{j\in\{1,\cdots,m\}} \sum_{i=1}^{n} |a_{ij}| \tag{1.5}$$

whose distinction will also be clear from the context.

For a Banach space X, we identify the direct product $(X^m)^n$ with X^{mn} when we refer to the norm on the former. We also use the notation l_X to denote the space of all X-valued sequences, where X is a Banach space. For $\{\widehat{f}_k\}_{k=0}^\infty \in l_{(L_\infty[0,h))^\nu}$, we call $\|\{\widehat{f}_k\}\| := \sup_k \|\widehat{f}_k\|$ the $l^\infty_{L_\infty[0,h)}$ norm, and the induced norm $\|\mathbf{T}\| := \sup_{x\in X\setminus\{0\}} \|\mathbf{T}x\|_Y/\|x\|_X$ is sometimes called the $l^\infty_{L_\infty[0,h)}$ or $l^\infty_{L_\infty[0,h')}$-induced norm of $\mathbf{T}$ if $\mathbf{T}: X \to Y$ and either X or Y is $l_{(L_\infty[0,h))^\nu}$ or $l_{(L_\infty[0,h'))^\nu}$.

Throughout this thesis, $\mathcal{F}(G,H)$ denotes the so-called lower linear-fractional-transformation (LFT) given by $G_{11} + G_{12}H(I - G_{22}H)^{-1}G_{21}$ when $G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}$.

Chapter 2

L_∞-Induced Norm Analysis of Continuous-Time Systems

2.1 Introduction

The L_∞-induced norm of control systems is the peak magnitude of the output for the worst bounded persistent input with a unit peak magnitude. There have been a number of studies on the L_∞-induced norm problem associated with a linear time-invariant (LTI) system [20],[62],[70] and a positive system [9],[65] since evaluating the peak magnitude of the output is very important in many control systems. Because this norm corresponds to the L_1 norm of the impulse response of the system in the (strictly causal) finite-dimensional single-input/single-output (SISO) LTI case, the study associated with the treatment of the L_∞-induced norm has been called the L_1 problem. This problem is pertinent to dealing with bounded persistent disturbances such as steps and sinusoids, which are often encountered in control systems. Accurate computation of the L_∞-induced norm associated with an LTI system is very hard since we need to integrate the absolute value (i.e., we need to compute the L_1 norm) of the impulse response of the LTI system, which corresponds to the kernel function in the convolution formula for its input/output relation, and it is very difficult to compute this integral exactly. An exact computation of the integral could be done only when the relevant system is a *positive* finite-dimensional LTI system [43] (for which the impulse response is nonnegative and thus the operation of taking its absolute value may be eliminated, leading to an analytic formula for the integral), and there have been no studies on giving an exact computation of the L_∞-induced norm as well as its upper and lower bounds associated with (*not necessarily positive*) finite-dimensional LTI systems. This chapter studies to compute upper and lower bounds in such a way that these bounds can be made as close to each other as one desires.

In this chapter, we provide two simple approaches named input approximation and kernel

approximation for computing the L_∞-induced norm associated with a stable FDLTI system. They are two different approaches in terms of the viewpoint behind approximations but share a common technical feature that they employ a piecewise constant approximation or piecewise linear approximation scheme of functions. In these input and kernel approximation approaches, we first apply a truncation idea, by which the time interval $[0, \infty)$ is divided into $[0, h)$ and $[h, \infty)$ with a sufficiently large constant h. Then, the behavior of the system on the time interval $[0, h)$ is treated as accurately as possible while that on $[h, \infty)$ in a comparatively simple way. This is because the effect of the latter interval on the L_∞-induced norm is very small when h is large enough; this implies that evaluating the effect of the latter interval in a relatively rough way does not cause severe deterioration of the resulting upper and lower bounds for the induced norm, as long as the effect of the former interval is evaluated adequately. Such an accurate evaluation is achieved by first applying to the signals on the former interval the fast-lifting [38] treatment, which has an integer parameter M and simply divides (without applying sampling of signals) the time interval $[0, h)$ into M subintervals with an equal width (in the context of the present chapter, the role of fast-lifting is essentially the same as the conventional lifting [4],[6],[68],[71] in the studies of sampled-data systems and time-delay systems, except that the original interval $[0, h)$ is finite). With this fast-lifting treatment, the input as well as the kernel function associated with the convolution formula for FDLTI systems can be dealt with independently on each of the M subintervals. Fast-lifting plays a role in reducing the size of the intervals to be directly dealt with, and provides us with improved accuracy in the approximation of the input and kernel functions. We next develop the piecewise constant approximation and piecewise linear approximation schemes, by which the input and kernel functions associated with the convolution formula for FDLTI system are approximated by a piecewise constant or piecewise linear function; constant or linear approximation is applied to the input and kernel function on each of the M subintervals. Indeed, it is shown that the piecewise constant approximation and piecewise linear approximation schemes lead to approximation errors in the computation of the L_∞-induced norm converging to 0 at the rate of $1/M$ and $1/M^2$, respectively, in both the input approximation and kernel approximation approaches. To reveal the mutual connection between the input approximation and kernel approximation approaches, however, we further investigate the relationship between the error bounds in the input approximation and kernel approximation approaches and also give explicit upper and lower bounds for the L_∞-induced norms obtained by these two approaches. Finally, we demonstrate the effectiveness of the resulting four types of computation methods through numerical examples.

The organization of this chapter is as follows. In Section 2.2, we formulate the problem definition, i.e., computing the L_∞-induced norm of continuous-time FDLTI systems. We

develop the input approximation approach in Section 2.3. We further provide the kernel approximation approach in Section 2.4. In Section 2.5, we present the main result in this chapter and the guideline for taking the parameters used in computing the L_∞-induced norm. Comparison between the input approximation and kernel approximation approaches is given in Section 2.6. The effectiveness of the proposed computation methods is demonstrated through numerical examples in Section 2.7. In Section 2.8, we provide concluding remarks. Finally, the proofs of the lemmas given in this chapter and the rationale for taking the specific functions $f_0(\tau')$ and $f_1(\tau')$ given later are provided in Section 2.9.

2.2 Problem Formulation

This chapter is concerned with the stable continuous-time FDLTI system G shown in Figure 2.1. Suppose that G is described by

$$G : \begin{cases} \dfrac{dx}{dt} = Ax + Bw \\ z = Cx + Dw \end{cases} \tag{2.1}$$

where $x(t) \in \mathbb{R}^n_\infty$ is the state, $w(t) \in \mathbb{R}^{n_w}_\infty$ is the input and $z(t) \in \mathbb{R}^{n_z}_\infty$ is the output. It is well known that

$$z(t) = \int_0^t C\exp(A(t-\tau))Bw(\tau)d\tau + Dw(t) =: (\mathcal{G}_\infty w)(t) \quad (0 \le t < \infty) \tag{2.2}$$

where $\mathcal{G}_\infty$ is the operator from $(L_\infty)^{n_w}$ to $(L_\infty)^{n_z}$ associated with the input/output relation of the stable system (2.1). The L_∞-induced norm of the system (2.1) is given by

$$\sup_{\|w\|\le 1} \|(\mathcal{G}_\infty w)(\cdot)\| =: \|\mathcal{G}_\infty\| \tag{2.3}$$

where $\|\cdot\|$ on the left hand side denotes the L_∞ norm. The aim of this chapter is computing the L_∞-induced norm $\|\mathcal{G}_\infty\|$.

2.2.1 Truncation Idea

For simplicity, let us assume $D = 0$ for a while; we will return to the general case with $D \neq 0$ before we provide our final results. Then, by noting that $(\mathcal{G}_\infty w)(t)$ is a continuous

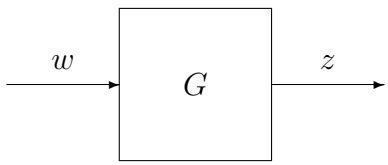


Figure 2.1: Continuous-time FDLTI System G.

function, the L_∞-induced norm $\|\mathcal{G}_\infty\|$ is described by

$$\|\mathcal{G}_\infty\| = \sup_{\|w\|\le 1} \sup_t \|(\mathcal{G}_\infty w)(t)\| = \sup_t \sup_{\|w\|\le 1} \|(\mathcal{G}_\infty w)(t)\| \tag{2.4}$$

Here, the L_∞-induced norm $\|\mathcal{G}_\infty\|$ can be rearranged as

$$\begin{aligned}
\|\mathcal{G}_\infty\| &= \sup_t \sup_{\|w\|\le 1} \|(\mathcal{G}_\infty w)(t)\| \\
&= \sup_t \sup_{\|w\|\le 1} \left\| \int_0^t C\exp(A(t-\tau))Bw(\tau)d\tau + Dw(t) \right\| \\
&= \sup_t \sup_{\|w\|\le 1} \left\| \int_0^t C\exp(A\theta)Bw(t-\theta)d\theta + Dw(t) \right\| \\
&= \sup_t \sup_{\|u\|\le 1} \left\| \int_0^t C\exp(A\theta)Bu(\theta)d\theta + Du(0) \right\| \\
&=: \sup_t \sup_{\|u\|\le 1} \|(\mathcal{G}u)(t)\| =: \|\mathcal{G}\|
\end{aligned} \tag{2.5}$$

where the equalities in the third and fourth lines are validated by letting $\theta := t - \tau$ and considering $u(\theta) := w(t-\theta)$ for $0 \le \theta \le t$. On the other hand, it readily follows from the property of L_∞ that

$$\sup_{\|u\|\le 1} \|(\mathcal{G}u)(T_1)\| \le \sup_{\|u\|\le 1} \|(\mathcal{G}u)(T_2)\| \tag{2.6}$$

whenever $0 \le T_1 < T_2$. This is because for every $u_1 \in L_\infty[0, T_1)$ such that $\|u_1\| \le 1$, the function u_2 defined as

$$u_2(t) := \begin{cases} u_1(t) & 0 \le t < T_1 \\ 0 & T_1 \le t < T_2 \end{cases} \tag{2.7}$$

belongs to $L_\infty[0, T_2)$, satisfies $\|u_2\| \le 1$ and $(\mathcal{G}u_1)(T_1) = (\mathcal{G}u_2)(T_2)$. By combining these arguments, the L_∞-induced norm $\|\mathcal{G}\|$ in (2.5) can be described by

$$\|\mathcal{G}\| = \lim_{t\to\infty} \sup_{\|u\|\le 1} \|(\mathcal{G}u)(t)\| \tag{2.8}$$

Hence, this chapter computes $\|\mathcal{G}_\infty\|$ by computing the L_∞-induced norm $\|\mathcal{G}\|$ instead because of some simplicities in the following arguments.

Remark 2.1 Even though we have been assuming for a while that $D = 0$ as mentioned above, note that $\mathcal{F}$ has been defined in (2.5) and (2.8) for a general D for later purposes.

To compute $\|\mathcal{G}\|$ when $D = 0$, we first introduce a truncation idea of $\mathcal{G}$. We thus take a sufficiently large h. Without loss of generality (see the limit in (2.8)), we then take t larger than h and decompose $\mathcal{G}$ into

$$(\mathcal{G}u)(t) = \mathcal{G}_h^- u + (\mathcal{G}_h^+ u)(t) \tag{2.9}$$

where

$$\mathcal{G}_h^- u := \int_0^h C\exp(A\theta)Bu(\theta)d\theta, \quad (\mathcal{G}_h^+ u)(t) := \int_h^t C\exp(A\theta)Bu(\theta)d\theta \tag{2.10}$$

Then, we have

$$\|\mathcal{G}_h^-\| - \|\mathcal{G}_h^+\| \leq \|\mathcal{G}\| \leq \|\mathcal{G}_h^-\| + \|\mathcal{G}_h^+\| \tag{2.11}$$

where

$$\|\mathcal{G}_h^-\| := \sup_{\|u\|\leq 1} \|\mathcal{G}_h^- u\|, \quad \|\mathcal{G}_h^+\| := \lim_{t\to\infty} \sup_{\|u\|\leq 1} \|(\mathcal{G}_h^+ u)(t)\| \tag{2.12}$$

It will be explained in Section 2.5 that $\|\mathcal{G}_h^+\|$ has an upper bound proportional to $\|C\exp(Ah)\|$ and the latter norm becomes arbitrarily small by taking h sufficiently large by the stability assumption of (2.1). Hence, our approach to computing $\|\mathcal{G}\|$ uses (2.11), in which $\|\mathcal{G}_h^-\|$ is computed as accurately as possible while the computation of $\|\mathcal{G}_h^+\|$ is treated in a comparatively simple way; we aim at computing upper and lower bounds of the L_∞-induced norm $\|\mathcal{G}\|$ through approximations of $\mathcal{G}_h^-$ and an upper bound computation of $\|\mathcal{G}_h^+\|$. The choice of h (as well as other parameters to be introduced) will be discussed in Section 2.5.

2.2.2 Fast-Lifting Treatment of $\mathcal{G}_h^-$

In this subsection, we suppose that h is given and aim at computing upper and lower bounds of $\|\mathcal{G}_h^-\|$. This is because a closed-form expression for this norm (can readily be obtained but) requires us to compute the integral of the absolute value of each entry of the matrix function $C\exp(A\theta)B$. Since it is very hard to perform such computations exactly, we consider computing the norm approximately but in such a way that its upper and lower bounds are available. To achieve this goal, we introduce input approximation or kernel[1] approximation approach, where the former is related to $u(\theta)$ while the latter to $C\exp(A\theta)B$. They are two different approaches in terms of the viewpoint behind approximations but share a common technical feature that they use either a piecewise constant approximation or

[1]Even though this term should make sense only such a part of (2.5) relevant to w is referred to, we retain this term with a slight abuse of terminology even when we view such a part of (2.5) relevant to u.

piecewise linear approximation scheme of (either the input or kernel) functions. Furthermore, the associated approximation errors converge to 0 at the rate of $1/M$ and $1/M^2$ in the piecewise constant approximation and piecewise linear approximation schemes, respectively. Here, M is the parameter of fast-lifting [38] applied to subdivide the interval $[0, h)$ into M subintervals with an equal width, as a preliminary step to develop such approximation schemes.

To describe the details of the approximate computation methods for $\|\mathcal{G}_h^-\|$, we first review the fast-lifting technique [38] (which in the context of the present chapter is nothing but lifting [4],[6],[68],[71] applied to signals with finite duration). As shown in Figure 2.2, for $M \in \mathbb{N}$ and $h' := h/M$, fast-lifting is defined as the mapping from $f \in (L_\infty[0, h))^\nu$ to $\check{f} := [(f^{(1)})^T \ \cdots \ (f^{(M)})^T]^T \in (L_\infty[0, h'))^{M\nu}$, and is denoted by $\check{f} := \mathbf{L}_M f$, where

$$f^{(i)}(\theta') := f((i-1)h' + \theta') \quad (0 \le \theta' < h') \tag{2.13}$$

It is easy to see from the fast-lifting treatment of $u(\theta)$ $(0 \le \theta < h)$ that the operation of $\mathcal{G}_h^-$ on u is described by

$$\mathcal{G}_h^- u = \sum_{i=1}^{M} C(A'_d)^{i-1} \mathbf{B}' u^{(i)} \tag{2.14}$$

where

$$\mathbf{B}' u^{(i)} := \int_0^{h'} \exp(A\theta') B u^{(i)}(\theta') d\theta', \quad A'_d := \exp(Ah') \tag{2.15}$$

Note that right hand side of (2.14) corresponds to the expanded representation of $\mathcal{G}_h^- \mathbf{L}_M^{-1} \check{u}$, where $\check{u} = \mathbf{L}_M u = [(u^{(1)})^T \ \cdots \ (u^{(M)})^T]^T$.

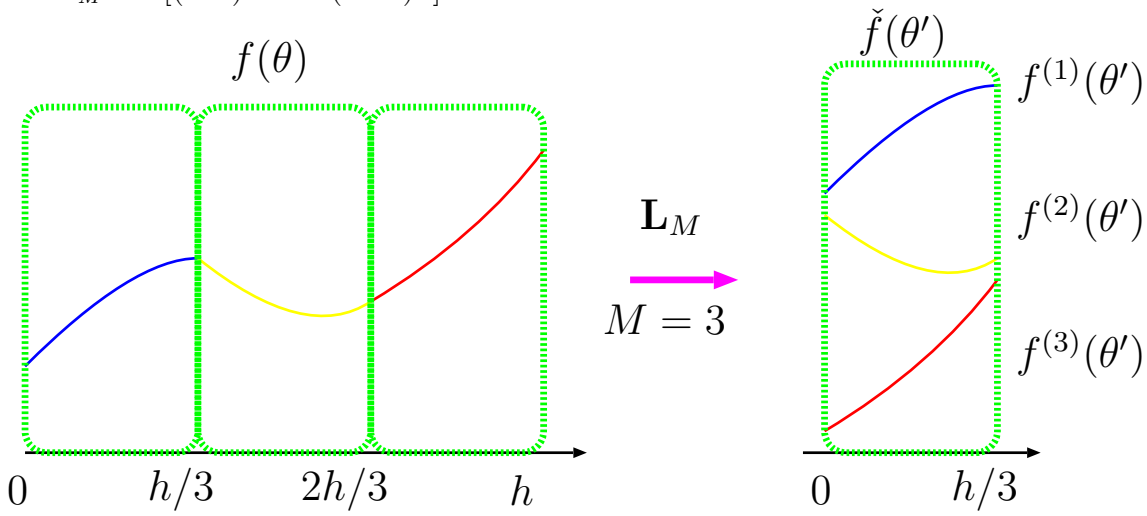


Figure 2.2: Fast-lifting $\mathbf{L}_M$.

It readily follows that

$$\|\mathcal{G}_h^-\| = \|\mathcal{G}_h^- \mathbf{L}_M^{-1}\| \tag{2.16}$$

where $\|\cdot\|$ on the right hand side denotes the induced norm from $(L_\infty[0,h'))^{Mn_w}$ to $\mathbb{R}_\infty^{n_z}$. Regarding the right hand side, it follows from (2.14) that the operator $\mathcal{G}_h^- \mathbf{L}_M^{-1}$ is described by

$$\mathcal{G}_h^- \mathbf{L}_M^{-1} = C'_{dM}\overline{\mathbf{B}'} \tag{2.17}$$

where

$$C'_{dM} = \begin{bmatrix} C & CA'_d & \cdots & C(A'_d)^{M-1} \end{bmatrix} \tag{2.18}$$

and $\overline{(\cdot)}$ denotes $\mathrm{diag}[(\cdot),\cdots,(\cdot)]$ consisting of M copies of $(\cdot)$.

As mentioned before, it is difficult to compute $\|\mathcal{G}_h^-\|$ exactly since computing the integral of the absolute value of each entry of the matrix function $C\exp(A\theta)B$ is very hard. We thus aim at its approximate computation, for which the above application of fast-lifting is helpful when we are to compute $\|\mathcal{G}_h^-\|$ by computing $\|\mathcal{G}_h^- \mathbf{L}_M^{-1}\| = \|C'_{dM}\overline{\mathbf{B}'}\|$ instead. This is because the input function and kernel function $C\exp(A\theta')B,\ 0 \le \theta \le h'$ associated with the operator $\mathbf{B}'$ are defined on a smaller interval than the interval $[0,h)$ on which $\mathcal{G}_h^-$ is defined. This provides us with a better chance for more accurate approximation. In particular, we aim at computing upper and lower bounds of $\|\mathcal{G}_h^-\|$ through the input approximation or kernel approximation approach.

2.3 Input Approximation Approach

In this section, we provide the first method for computing the L_∞-induced norm of continuous-time FDLTI systems, i.e., the input approximation approach. Through the input approximation approach, we readily compute the upper and lower bounds of $\|\mathcal{G}_h^-\|$. More precisely, constant and linear approximations to the input of $\mathbf{B}'$ (which by (2.17) lead to piecewise constant and piecewise linear approximations to the input of $\mathcal{G}_h^-$) are introduced for computing $\|\mathcal{G}_h^-\|$, as well as the associated convergence rates in M.

2.3.1 Piecewise Constant Approximation Scheme

We introduce the averaging operator $\mathbf{J}'_0 : (L_\infty[0,h'))^{n_w} \to (L_\infty[0,h'))^{n_w}$ defined by

$$(\mathbf{J}'_0 u)(\theta') = \frac{1}{h'}\int_0^{h'} u(\tau')d\tau' \quad (0 \le \theta' < h') \tag{2.19}$$

and the operator $\mathbf{B}'_{\mathrm{ic0}} : (L_\infty[0,h))^{n_w} \to \mathbb{R}^n_\infty$ defined as $\mathbf{B}'_{\mathrm{ic0}} := \mathbf{B}'\mathbf{J}'_0$, where the subscripts i and c stand for input approximation and continuous-time systems, respectively. In other words, introducing the operator $\mathbf{B}'_{\mathrm{ic0}}$ corresponds to restricting the input of $\mathbf{B}'$ to constant functions and that $\mathbf{B}'_{\mathrm{ic0}}u = \mathbf{B}'u$ whenever u is a constant function.

We next consider the operator $\mathcal{G}^-_{hM\mathrm{i}0}$ obtained by replacing $\mathbf{B}'$ with $\mathbf{B}'_{\mathrm{ic0}}$ in (2.17):

$$\mathcal{G}^-_{hM\mathrm{i}0} = C'_{dM}\overline{\mathbf{B}'_{\mathrm{ic0}}} \tag{2.20}$$

It is easy to see that $\mathcal{G}^-_{hM\mathrm{i}0}$ is the fast-lifted counterpart to the piecewise constant approximation of $\mathcal{G}^-_h$ (under the input approximation approach). This chapter shows that $\|\mathcal{G}^-_{hM\mathrm{i}0}\|$ can be computed exactly and tends to $\|\mathcal{G}^-_h\|$ as $M \to \infty$. The following two lemmas play significant roles in establishing the above facts and the associated convergence rate.

Remark 2.2 The treatment of D has been recovered in the second lemma. The same argument is repeatedly applied to the arguments of Lemmas 2.4, 2.6 and 2.8 given later.

Lemma 2.1 The following inequality holds:

$$\|\mathbf{B}' - \mathbf{B}'_{\mathrm{ic0}}\| \le \frac{h^2}{M^2}\|A\| \cdot \|B\| e^{\|A\|h/M} \tag{2.21}$$

Lemma 2.2 Let $B'_{0\mathrm{c}d}$ be the matrix defined as

$$B'_{0\mathrm{c}d} := \int_0^{h'} \exp(A\theta')B d\theta' \tag{2.22}$$

Then, $\|\mathcal{G}^-_{hM\mathrm{i}0}\|$ coincides with the ∞-norm of the finite-dimensional matrix $G^-_{hM\mathrm{i}0}$ given by

$$G^-_{hM\mathrm{i}0} = \begin{bmatrix} CB'_{0\mathrm{c}d} & \cdots & C(A'_d)^{M-1}B'_{0\mathrm{c}d} & D \end{bmatrix} \tag{2.23}$$

The proofs of these lemmas are deferred to the appendix of this chapter, i.e., Subsection 2.9.1 since they are quite technical. From Lemmas 2.1 and 2.2, we can readily obtain the following result.

Theorem 2.1 The inequality

$$\|G^-_{hM\mathrm{i}0}\| - \frac{K_{M\mathrm{ic}0}}{M} \le \|\mathcal{G}^-_h\| \le \|G^-_{hM\mathrm{i}0}\| + \frac{K_{M\mathrm{ic}0}}{M} \tag{2.24}$$

holds with $K_{M\mathrm{ic}0}$ given by

$$K_{M\mathrm{ic}0} := \frac{h^2}{M}\|C'_{dM}\| \cdot \|A\| \cdot \|B\| e^{\|A\|h/M} \tag{2.25}$$

Furthermore, $K_{M\mathrm{ic}0}$ has the following uniform upper bound with respect to M:

$$K^U_{\mathrm{ic}0} := h^2\|C\| \cdot \|A\| \cdot \|B\| e^{\|A\|h} \tag{2.26}$$

Remark 2.3 By noting that the piecewise constant approximation is norm-contractive in the input approximation approach, we can show that the lower bound of $\|\mathcal{G}_h^-\|$ in (2.24) can in fact be replaced by $\|G_{hM\mathrm{i}0}^-\|$, i.e., the following inequality holds:

$$\|G_{hM\mathrm{i}0}^-\| \le \|\mathcal{G}_h^-\| \le \|G_{hM\mathrm{i}0}^-\| + \frac{K_{M\mathrm{i}0}}{M} \tag{2.27}$$

Remark 2.4 The second assertion of Theorem 2.1 can be proved easily if we note from (2.18) that

$$\|C'_{dM}\| e^{\|A\|h/M} \le M\|C\| e^{\|A\|h} \tag{2.28}$$

The same arguments are repeatedly applied to the arguments associated with K_{ic1}^U, K_{kc0}^U and K_{kc1}^U in Theorems 2.2, 2.3 and 2.4, respectively.

Theorem 2.1 implies that an upper bound and a lower bound of $\|\mathcal{G}_h^-\|$ can be computed through matrix ∞-norm computations, and as the fast-lifting parameter M becomes larger, the gap between the upper and lower bounds tends to 0 at no slower convergence rate than $1/M$.

2.3.2 Piecewise Linear Approximation Scheme

We next introduce the 'linearizing' operator $\mathbf{J}'_1 : (L_\infty[0,h'))^{n_w} \to (L_\infty[0,h'))^{n_w}$ given by

$$(\mathbf{J}'_1 u)(\theta') = \int_0^{h'} f_0(\tau')u(\tau')d\tau' + \theta' \int_0^{h'} f_1(\tau')u(\tau')d\tau' \tag{2.29}$$

where the scalar-valued functions $f_0(\tau')$ and $f_1(\tau')$ are defined as

$$f_0(\tau') = -\frac{6}{(h')^2}\tau' + \frac{4}{h'}, \quad f_1(\tau') = \frac{12}{(h')^3}\tau' - \frac{6}{(h')^2} \tag{2.30}$$

The rationale for taking such specific functions will be precisely discussed in the appendix, Subsection 2.9.3, but we would like remark that among important properties of $\mathbf{J}'_1$ is that $\mathbf{J}'_1 u = u$ whenever u is a linear function. Let us further introduce the operator $\mathbf{B}'_{\mathrm{ic1}} := \mathbf{B}'\mathbf{J}'_1$. Introducing this operator is equivalent to restricting the input of $\mathbf{B}'$ to linear functions, and $\mathbf{B}'_{\mathrm{ic1}} u = \mathbf{B}' u$ whenever u is a linear function.

We next consider the operator $\mathcal{G}_{hM\mathrm{i}1}^-$ obtained by replacing $\mathbf{B}'$ with $\mathbf{B}'_{\mathrm{ic1}}$ in (2.17):

$$\mathcal{G}_{hM\mathrm{i}1}^- = C'_{dM}\overline{\mathbf{B}'_{\mathrm{ic1}}} \tag{2.31}$$

It is easy to see that $\mathcal{G}^-_{hMi1}$ is the fast-lifted counterpart to the piecewise linear approximation of $\mathcal{G}^-_h$ (under the input approximation approach). In the following, we show that $\|\mathcal{G}^-_{hMi1}\|$ can be computed exactly and converges to $\|\mathcal{G}^-_h\|$ as $M \to \infty$. The following two lemmas are important in establishing the above facts together with the associated convergence rate.

Lemma 2.3 The following inequality holds:

$$\|\mathbf{B}' - \mathbf{B}'_{\mathrm{ic1}}\| \leq \frac{h^3}{2M^3}\|A\|^2 \cdot \|B\| e^{\|A\|h/M} \tag{2.32}$$

Lemma 2.4 Let $T_{j\mathrm{ic}}$ $(j = 1, \cdots, M)$ be the matrix consisting of the $L_1[0, h')$ norm of each entry of the matrix linear function

$$S_{j\mathrm{c}0} + S_{j\mathrm{c}1}\theta' := C(A'_d)^{j-1}(Y_{\mathrm{c}0} + Y_{\mathrm{c}1}\theta') \tag{2.33}$$

where the matrices $Y_{\mathrm{c}0}$ and $Y_{\mathrm{c}1}$ are defined as

$$Y_{\mathrm{c}0} := -\frac{6}{(h')^2}B'_{1\mathrm{c}d} + \frac{4}{h'}B'_{0\mathrm{c}d}, \quad Y_{\mathrm{c}1} := \frac{12}{(h')^3}B'_{1\mathrm{c}d} - \frac{6}{(h')^2}B'_{0\mathrm{c}d} \tag{2.34}$$

through the matrices $B'_{0\mathrm{c}d}$ and

$$B'_{1\mathrm{c}d} := \int_0^{h'} \exp(A\theta')\theta' B d\theta' \tag{2.35}$$

Then, $\|\mathcal{G}^-_{hMi1}\|$ coincides with the ∞-norm of the finite-dimensional matrix G^-_{hMi1} given by

$$G^-_{hMi1} := \begin{bmatrix} T_{1\mathrm{ic}} & \cdots & T_{M\mathrm{ic}} & D \end{bmatrix} \tag{2.36}$$

The proofs of these lemmas are given in the appendix of this chapter, Subsection 2.9.2. From Lemmas 2.3 and 2.4, we can easily obtain the following result.

Theorem 2.2 The inequality

$$\|G^-_{hMi1}\| - \frac{K_{M\mathrm{ic}1}}{M^2} \leq \|\mathcal{G}^-_h\| \leq \|G^-_{hMi1}\| + \frac{K_{M\mathrm{ic}1}}{M^2} \tag{2.37}$$

holds with $K_{M\mathrm{ic}1}$ given by

$$K_{M\mathrm{ic}1} := \frac{h^3}{2M}\|C'_{dM}\| \cdot \|A\|^2 \cdot \|B\| e^{\|A\|h/M} \tag{2.38}$$

Furthermore, $K_{M\mathrm{ic}1}$ has the following uniform upper bound with respect to M:

$$K^U_{\mathrm{ic}1} := \frac{h^3}{2}\|C\| \cdot \|A\|^2\|B\| e^{\|A\|h} \tag{2.39}$$

Theorem 2.2 implies that an upper bound and a lower bound of $\|\mathcal{G}_h^-\|$ can be computed through the matrix ∞-norm $\|G_{hM\mathrm{i}1}^-\|$, and as the fast-lifting parameter M becomes larger, the gap between the upper and lower bounds tends to 0 at no slower convergence rate than $1/M^2$.

Remark 2.5 The idea of the input approximation approach developed in this chapter could be used in developing a computation method for the L_∞-induced norm of sampled-data systems. The details are pursued in Chapter 3.

2.4 Kernel Approximation Approach

In this section, we provide the second method for computing the L_∞-induced norm of continuous-time FDLTI systems, i.e., the kernel approximation approach. Through the kernel approximation approach, we can easily compute the upper and lower bounds of $\|\mathcal{G}_h^-\|$. In other words, we apply piecewise constant and piecewise linear approximations to the kernel function $C\exp(A\theta)B$ (or more precisely, constant and linear approximations of the kernel function $\exp(A\theta')B,\ 0 \le \theta' < h'$ of $\mathbf{B}'$) and show the associated convergence rates in M.

2.4.1 Piecewise Constant Approximation Scheme

We introduce the operator $\mathbf{B}'_{\mathrm{kc0}} : (L_\infty[0,h'))^{n_w} \to \mathbb{R}^n_\infty$ defined as

$$\mathbf{B}'_{\mathrm{kc0}}u := \int_0^{h'} Bu(\theta')d\theta' \tag{2.40}$$

where the subscript k stands for kernel approximation. Introducing the operator $\mathbf{B}'_{\mathrm{kc0}}$ corresponds to the zero-order approximation of the kernel function $\exp(A\theta')B = \sum_{i=0}^{\infty} \frac{(A\theta')^i}{i!}B$ of the operator $\mathbf{B}'$.

We next consider the operator $\mathcal{G}_{hM\mathrm{k0}}^-$ obtained by replacing $\mathbf{B}'$ with $\mathbf{B}'_{\mathrm{kc0}}$ in (2.17):

$$\mathcal{G}_{hM\mathrm{k0}}^- := C'_{dM}\overline{\mathbf{B}'_{\mathrm{kc0}}} \tag{2.41}$$

It is easy to see that $\mathcal{G}_{hM\mathrm{k0}}^-$ is the fast-lifted counterpart to the piecewise constant approximation of $\mathcal{G}_h^-$ (under the kernel approximation approach). This chapter shows that $\|\mathcal{G}_{hM\mathrm{k0}}^-\|$ can be computed exactly and tends to $\|\mathcal{G}_h^-\|$ as $M \to \infty$. The following two lemmas play important roles in establishing the above facts and the associated convergence rate.

Lemma 2.5 The following inequality holds:

$$\|\mathbf{B}' - \mathbf{B}'_{\mathrm{kc0}}\| \leq \frac{h^2}{2M^2}\|A\| \cdot \|B\| e^{\|A\|h/M} \tag{2.42}$$

Lemma 2.6 $\|\mathcal{G}^-_{hM\mathrm{k0}}\|$ coincides with the ∞-norm of the finite-dimensional matrix $G^-_{hM\mathrm{k0}}$ given by

$$G^-_{hM\mathrm{k0}} := \begin{bmatrix} CBh' & \cdots & C(A'_d)^{M-1}Bh' & D \end{bmatrix} \tag{2.43}$$

The proofs of these lemmas are omitted because they are essentially the same as those of the input approximation approach. From Lemmas 2.5 and 2.6, we can readily obtain the following result.

Theorem 2.3 The inequality

$$\|G^-_{hM\mathrm{k0}}\| - \frac{K_{M\mathrm{kc0}}}{M} \leq \|\mathcal{G}^-_h\| \leq \|G^-_{hM\mathrm{k0}}\| + \frac{K_{M\mathrm{kc0}}}{M} \tag{2.44}$$

holds with $K_{M\mathrm{kc0}}$ defined as

$$K_{M\mathrm{kc0}} := \frac{h^2}{2M}\|C'_{dM}\| \cdot \|A\| \cdot \|B\| e^{\|A\|h/M} \tag{2.45}$$

Furthermore, $K_{M\mathrm{kc0}}$ has a uniform upper bound with respect to M given by

$$K^U_{\mathrm{kc0}} := \frac{h^2}{2}\|C\| \cdot \|A\| \cdot \|B\| e^{\|A\|h} \tag{2.46}$$

2.4.2 Piecewise Linear Approximation Scheme

We introduce the operator $\mathbf{B}'_{\mathrm{kc1}} : (L_\infty[0, h'))^{n_w} \to \mathbb{R}^n_\infty$ defined as

$$\mathbf{B}'_{\mathrm{kc1}}u := \int_0^{h'} (I + A\theta')Bu(\theta')d\theta' \tag{2.47}$$

Introducing the operator $\mathbf{B}'_{\mathrm{kc1}}$ is equivalent to the first-order approximation of the kernel function $\exp(A\theta')B = \sum_{i=0}^{\infty} \frac{(A\theta')^i}{i!}B$ of the operator $\mathbf{B}'$.

We next consider the operator $\mathcal{G}^-_{hM\mathrm{k1}}$ obtained by replacing $\mathbf{B}'$ with $\mathbf{B}'_{\mathrm{kc1}}$ in (2.17):

$$\mathcal{G}^-_{hM\mathrm{k1}} = C'_{dM}\overline{\mathbf{B}'_{\mathrm{kc1}}} \tag{2.48}$$

It is easy to see that $\mathcal{G}^-_{hM\mathrm{k1}}$ is the fast-lifted counterpart to the piecewise linear approximation of $\mathcal{G}^-_h$ (under the kernel approximation approach). In the following, we show that $\|\mathcal{G}^-_{hM\mathrm{k1}}\|$ can be computed exactly and converges to $\|\mathcal{G}^-_h\|$ as $M \to \infty$. The following two lemmas are significant in establishing the above facts together with the associated convergence rate.

Lemma 2.7 The following inequality holds:

$$\|\mathbf{B}' - \mathbf{B}'_{\mathrm{kc1}}\| \leq \frac{h^3}{6M^3}\|A\|^2 \cdot \|B\| e^{\|A\|h/M} \tag{2.49}$$

Lemma 2.8 Let $T_{j\mathrm{kc}}$ $(j = 1, \cdots, M)$ be the matrix consisting of the $L_1[0, h')$ norm of each entry of the matrix linear function $C(A'_d)^{j-1}(I + A\theta')B$ involved in (2.48). Then, $\|\mathcal{G}^-_{hM\mathrm{k1}}\|$ coincides with the ∞-norm of the finite-dimensional matrix $G^-_{hM\mathrm{k1}}$ given by

$$G^-_{hM\mathrm{k1}} := \begin{bmatrix} T_{1\mathrm{kc}} & \cdots & T_{M\mathrm{kc}} & D \end{bmatrix} \tag{2.50}$$

The proofs of these lemmas are also omitted since they are essentially the same as those of the input approximation approach. From Lemmas 2.7 and 2.8, we can readily obtain the following theorem.

Theorem 2.4 The inequality

$$\|G^-_{hM\mathrm{k1}}\| - \frac{K_{M\mathrm{kc1}}}{M^2} \leq \|\mathcal{G}^-_h\| \leq \|G^-_{hM\mathrm{k1}}\| + \frac{K_{M\mathrm{kc1}}}{M^2} \tag{2.51}$$

holds with $K_{M\mathrm{kc1}}$ defined as

$$K_{M\mathrm{kc1}} := \frac{h^3}{6M}\|C'_{dM}\| \cdot \|A\|^2 \cdot \|B\| e^{\|A\|h/M} \tag{2.52}$$

Furthermore, $K_{M\mathrm{kc1}}$ has a uniform upper bound with respect to M given by

$$K^U_{\mathrm{kc1}} := \frac{h^3}{6}\|C\| \cdot \|A\|^2 \cdot \|B\| e^{\|A\|h} \tag{2.53}$$

Remark 2.6 It is expected that the idea of the kernel approximation approach developed in this chapter may be used in the computation of the L_∞-induced norm of sampled-data systems. Such a computation method is also expected to lead to an improved gap between its upper and lower bounds than the computation method through the input approximation approach. The details are pursued in Chapter 3.

2.5 Computation of the L_∞-Induced Norm and Guideline for Taking Parameters

This section is dedicated to a computation method for an upper bound of $\|\mathcal{G}^+_h\|$, which together with the arguments in the preceding sections leads to methods for computing upper and lower bounds of the L_∞-induced norm $\|\mathcal{G}\|$ of the FDLTI system (2.1). Theses bounds are ensured to converge to each other as the parameters h and M tends to ∞.

2.5.1 Computing Upper Bound of $\|\mathcal{G}_h^+\|$

We first note from (2.12) (with t replaced by $t+h$) and (2.10) that

$$\|\mathcal{G}_h^+\| \leq \lim_{t\to\infty} \sup_{\|u\|\leq 1} \left\| \int_0^t \exp(A\theta)Bu(\theta+h)d\theta \right\| \cdot \|C\exp(Ah)\| \tag{2.54}$$

If we take $q>0$ such that $\|\exp(Aq)\|<1$, it readily follows that

$$\begin{aligned}
&\lim_{t\to\infty} \sup_{\|u\|\leq 1} \left\| \int_0^t \exp(A\theta)Bu(\theta+h)d\theta \right\| \\
&\leq (1+\|\exp(Aq)\|+\|\exp(2Aq)\|+\cdots)\cdot \sup_{\|v\|\leq 1} \left\| \int_0^q \exp(A\theta)Bv(\theta)d\theta \right\| \\
&\leq \frac{1}{1-\|\exp(Aq)\|}\cdot qe^{\|A\|q}\|B\| \qquad (2.55)
\end{aligned}$$

Summarizing (2.54) and (2.55), we can obtain the following result.

Proposition 2.1 If we take $q>0$ such that $\|\exp(Aq)\|<1$, then

$$\|\mathcal{G}_h^+\| \leq \frac{qe^{\|A\|q}\|B\|}{1-\|\exp(Aq)\|}\|C\exp(Ah)\| =: K_{hq} \tag{2.56}$$

and K_{hq} converges to 0 regardless of q as $h\to\infty$.

2.5.2 Main Results in the Computation of the L_∞-Induced Norm of Continuous-Time FDLTI Systems

In this subsection, we give the main results in the computation of the L_∞-induced norm of continuous-time FDLTI systems. Combining Theorems 2.1–2.4, Proposition 2.1 and Remark 2.3 together with (2.11), we are led to the following main results.

Theorem 2.5 If we take $q>0$ such that $\|\exp(Aq)\|<1$, then

$$\|G_{hM\mathrm{i}0}^-\| - K_{hq} \leq \|\mathcal{G}\| \leq \|G_{hM\mathrm{i}0}^-\| + \frac{K_{M\mathrm{ic}0}}{M} + K_{hq} \tag{2.57}$$

$$\|G_{hM\mathrm{i}1}^-\| - \frac{K_{M\mathrm{ic}1}}{M^2} - K_{hq} \leq \|\mathcal{G}\| \leq \|G_{hM\mathrm{i}1}^-\| + \frac{K_{M\mathrm{ic}1}}{M^2} + K_{hq} \tag{2.58}$$

$$\|G_{hM\mathrm{k}0}^-\| - \frac{K_{M\mathrm{kc}0}}{M} - K_{hq} \leq \|\mathcal{G}\| \leq \|G_{hM\mathrm{k}0}^-\| + \frac{K_{M\mathrm{kc}0}}{M} + K_{hq} \tag{2.59}$$

$$\|G_{hM\mathrm{k}1}^-\| - \frac{K_{M\mathrm{kc}1}}{M^2} - K_{hq} \leq \|\mathcal{G}\| \leq \|G_{hM\mathrm{k}1}^-\| + \frac{K_{M\mathrm{kc}1}}{M^2} + K_{hq} \tag{2.60}$$

Furthermore, $K_{M\mathrm{ic0}}$, $K_{M\mathrm{ic1}}$, $K_{M\mathrm{kc0}}$ and $K_{M\mathrm{kc1}}$ have uniform upper bounds K^U_{ic0}, K^U_{ic1}, K^U_{kc0} and K^U_{kc1} defined as (2.26), (2.39), (2.46) and (2.53), respectively. Thus, the error bounds $K_{M\mathrm{i0}}/M$, $K_{M\mathrm{i1}}/M^2$, $K_{M\mathrm{k0}}/M$ and $K_{M\mathrm{k1}}/M^2$ in (2.57)–(2.60) converge to 0 as $M \to \infty$, while K_{hq} converges to 0 regardless of q as $h \to \infty$.

It should be noted in Theorem 2.5 that the uniform upper bounds K^U_{ic0}, K^U_{ic1}, K^U_{kc0} and K^U_{kc1} given in (2.26), (2.39), (2.46) and (2.53), respectively, depend on h, and increase as h is increased to reduce K_{hq}. However, K_{hq} is bounded from above in the exponential order $e^{\sigma h}$ in h regardless of q, where $\sigma < 0$ is the maximum real part of the eigenvalues of A. It is hence expected that K_{hq} can be made small with a modest h and thus we can keep the uniform upper bounds K^U_{ic0}, K^U_{ic1}, K^U_{kc0} and K^U_{kc1} modest.

2.5.3 Guideline for Taking Parameters

Regarding a guideline for taking the parameters h, M and q, we can summarize the above arguments as follows. It may be reasonable to take a relatively small q as long as $\|\exp(Aq)\| < 1$; this is to avoid undue increase of K_{hq}, or in particular $e^{\|A\|q}$. Once q is fixed, the next step would be to take an h such that K_{hq} is as small as we wish; this is always possible by taking h sufficiently large. Once h is also fixed, the uniform upper bounds K^U_{ic0}, K^U_{ic1}, K^U_{kc0} and K^U_{kc1} in (2.26), (2.39), (2.46) and (2.53), respectively, are determined, and thus the last step would be taking an M such that K^U_{ic0}/M, K^U_{ic1}/M^2, K^U_{kc0}/M and K^U_{kc1}/M^2 are as small as we wish. It is obvious that following this kind of guideline leads to computation methods for the L_∞-induced norm of the FDLTI system (2.1) to any degree of accuracy.

2.6 Comparison between the Input and Kernel Approximation Approaches

In this section, we compare effectiveness of the input approximation and kernel approximation approaches in the treatment of $\mathcal{G}_h^-$. We see that $K_{M\mathrm{kc0}}$ and $K_{M\mathrm{kc1}}$ relevant to the approximation errors in the kernel approximation approach developed in this chapter are smaller than $K_{M\mathrm{ic0}}$ and $K_{M\mathrm{ic1}}$, respectively, relevant to those for the input approximation approach. More precisely, we can see from (2.25) and (2.45) that $K_{M\mathrm{kc0}} = K_{M\mathrm{ic0}}/2$ for the piecewise constant approximation scheme, while (2.38) and (2.52) implies that $K_{M\mathrm{kc1}} = K_{M\mathrm{ic1}}/3$ for the piecewise linear approximation scheme. If we note that the treatment of the truncated part $\mathcal{G}_h^+$ discussed in the preceding section is common for all the four methods discussed in this section, the following interpretations of these two relations will be justified.

The former relation implies that the gap between the upper and lower bounds in (2.57) and that in (2.59) coincide with each other. This could be interpreted as implying that the overall ability is the same for the input approximation and kernel approximation approaches as far as the piecewise constant approximation scheme is taken. For the piecewise linear approximation scheme, on the other hand, the latter relation implies that the gap between the upper and lower bounds in (2.60) for the kernel approximation approach is one third of that in (2.58) for the input approximation approach. Meanwhile, it will be (numerically) demonstrated in the following section that the piecewise linear approximation scheme is superior to the piecewise constant approximation scheme in the computation of $\|\mathcal{G}\|$ under both the input approximation and kernel approximation approaches. Summarizing these observation clearly indicates an advantage of the method with combined use of the piecewise linear approximation scheme and the kernel approximation approach over the other three methods.

2.7 Numerical Examples

In this section, we study numerical examples and examine effectiveness of the computation methods discussed in this chapter.

Let us first consider the stable SISO FDLTI oscillatory system

$$A = \begin{bmatrix} 0 & -2 \\ 2 & -2 \end{bmatrix}, \ B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \ C = \begin{bmatrix} 1 & 1 \end{bmatrix}, \ D = 1 \tag{2.61}$$

We compute estimates of its L_∞-induced norm, or equivalently $\|\mathcal{G}\|$, by taking the fast-lifting parameter M ranging from 500 to 5000 on the condition that $h = 25$ and $q = 2$ following the guideline in Subsection 2.5.3, which leads to $K_{hq} = 2.26 \times 10^{-7}$. The results for the upper and lower bounds of $\|\mathcal{G}\|$ obtained by Theorem 2.5 and the computation times under the piecewise constant approximation scheme are shown in Table 2.1, while those with the piecewise linear approximation scheme are shown in Table 2.2. We are mainly interested in the comparison between the input approximation approach and the kernel approximation approach developed in this chapter. Hence, these (and the following) tables consist of Case (a) for the input approximation approach and Case (b) for the kernel approximation approach.

Table 2.1: Results with piecewise constant approximation scheme in SISO example.

Case (a): Input approximation approach

M	500	1000	2000	5000
$\|G^-_{hM\mathrm{i}0}\| + \dfrac{K_{M\mathrm{ic}0}}{M} + K_{hq}$	3.703542	3.361762	3.215702	3.135198
$\|G_{hM\mathrm{i}0}\| - K_{hq}$	3.084104	3.084248	3.084370	3.084370
time (sec)	0.015281	0.030330	0.036423	0.079816

Case (b): Kernel approximation approach

M	500	1000	2000	5000
$\|G^-_{hM\mathrm{k}0}\| + \dfrac{K_{M\mathrm{kc}0}}{M} + K_{hq}$	3.392950	3.222860	3.149950	3.109775
$\|G^-_{hM\mathrm{k}0}\| - \dfrac{K_{M\mathrm{kc}0}}{M} - K_{hq}$	2.773512	2.945347	3.018618	3.058947
time (sec)	0.015093	0.024384	0.036143	0.078988

Table 2.2: Results with piecewise linear approximation scheme in SISO example.

Case (a): Input approximation approach

M	500	1000	2000	5000
$\|G^-_{hM\mathrm{i}1}\| + \dfrac{K_{M\mathrm{ic}1}}{M^2} + K_{hq}$	3.146314	3.098249	3.087656	3.084882
$\|G^-_{hM\mathrm{i}1}\| - \dfrac{K_{M\mathrm{ic}1}}{M^2} - K_{hq}$	3.022426	3.070497	3.081089	3.083865
time (sec)	0.020353	0.035159	0.049186	0.120428

Case (b): Kernel approximation approach

M	500	1000	2000	5000
$\|G^-_{hM\mathrm{k}1}\| + \dfrac{K_{M\mathrm{kc}1}}{M^2} + K_{hq}$	3.108609	3.089882	3.085686	3.084578
$\|G^-_{hM\mathrm{k}1}\| - \dfrac{K_{M\mathrm{kc}1}}{M^2} - K_{hq}$	3.067312	3.080631	3.083497	3.084238
time (sec)	0.019011	0.032270	0.038521	0.119789

We next consider the stable MIMO FDLTI oscillatory system

$$A = \begin{bmatrix} -1 & 0 & 2 & 2 \\ 1 & -1 & 2 & 3 \\ 0 & -2 & -2 & 0 \\ 1 & -1 & -1 & -2 \end{bmatrix}, \; B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 2 & 0 \\ 1 & -1 \end{bmatrix}, \; C = \begin{bmatrix} 1 & 1 & 0 & -1 \\ 2 & 1 & -1 & 1 \end{bmatrix}, \; D = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} \tag{2.62}$$

We compute the upper and lower bounds of its L_∞-induced norm by taking the fast-lifting parameter M ranging from 500 to 5000 on the condition that $h = 25$ and $q = 2$, which leads to $K_{hq} = 2.65 \times 10^{-8}$. The results are shown in Tables 2.3 and 2.4.

We can see from these tables that the error bounds for the computation of $\|\mathcal{G}\|$ (i.e.,

Table 2.3: Results with piecewise constant approximation scheme in MIMO example.

Case (a): Input approximation approach

M	500	1000	2000	5000
$\|G^-_{hM\mathrm{i}0}\| + \dfrac{K_{M\mathrm{ic}0}}{M} + K_{hq}$	18.548486	13.828079	11.996866	11.041614
$\|G_{hM\mathrm{i}0}\| - K_{hq}$	10.458708	10.458788	10.459380	10.459432
time (sec)	0.017563	0.033310	0.048476	0.094044

Case (b): Kernel approximation approach

M	500	1000	2000	5000
$\|G^-_{hM\mathrm{k}0}\| + \dfrac{K_{M\mathrm{kc}0}}{M} + K_{hq}$	14.576154	12.181119	11.246745	10.758005
$\|G^-_{hM\mathrm{k}0}\| - \dfrac{K_{M\mathrm{kc}0}}{M} - K_{hq}$	6.486376	8.811828	9.709259	10.175823
time (sec)	0.015604	0.026847	0.040918	0.090371

Table 2.4: Results with piecewise linear approximation scheme in MIMO example.

Case (a): Input approximation approach

M	500	1000	2000	5000
$\|G^-_{hM\mathrm{i}1}\| + \dfrac{K_{M\mathrm{ic}1}}{M^2} + K_{hq}$	11.875157	10.754256	10.526707	10.469630
$\|G^-_{hM\mathrm{i}1}\| - \dfrac{K_{M\mathrm{ic}1}}{M^2} - K_{hq}$	9.043735	10.164630	10.392177	10.449254
time (sec)	0.025248	0.038933	0.064876	0.134318

Case (b): Kernel approximation approach

M	500	1000	2000	5000
$\|G^-_{hM\mathrm{k}1}\| + \dfrac{K_{M\mathrm{kc}1}}{M^2} + K_{hq}$	10.943846	10.560772	10.482617	10.462958
$\|G^-_{hM\mathrm{k}1}\| - \dfrac{K_{M\mathrm{kc}1}}{M^2} - K_{hq}$	10.000038	10.364230	10.437774	10.456166
time (sec)	0.022003	0.038181	0.061540	0.133024

the gaps between the upper and lower bounds) decrease by taking the fast-lifting parameter M larger for all estimates. Hence, all the four approximation methods discussed in this chapter can be validated as methods for computing the L_∞-induced norm. A more important concern in this chapter, however, lies in the effectiveness comparison between (a) the input approximation approach and (b) the kernel approximation approach. In this respect, we had an earlier discussion in Section 2.6, which implies that, under the piecewise constant approximation scheme, the kernel approximation approach can provide no advantage over the input approximation approach in reducing the gap between the computed upper and lower bounds. As seen from Tables 2.1 and 2.3, the convergence of this gap (common for the input approximation and kernel approximation approaches) is not fast with respect to M.

This suggests us to use the piecewise linear approximation scheme instead, which exhibits much faster convergence as seen from Tables 2.2 and 2.4. We can further observe from these tables that once we switch to the piecewise linear approximation scheme, an advantage of the kernel approximation approach over the input approximation approach is prominent. This is because the range between the upper and lower bounds obtained by the kernel approximation approach is always contained in (and thus less conservative than) that by the input approximation approach for the same M. Furthermore, the computation times in the kernel approximation approach are slightly smaller than those in the input approximation approach under the same parameter M. As an overall evaluation, the kernel approximation approach with the piecewise linear approximation scheme exhibits the smallest range for the L_∞-induced norm estimates with relatively short computation times among the four methods discussed in this chapter.

2.8 Concluding Remarks

In this chapter, we tackled a difficult problem of accurately computing the L_∞-induced norm associated with a stable FDLTI system, which is very important in many control systems. To this problem, we applied a truncation idea with a sufficiently large h, which mostly reduces the problem to the induced-norm computation of $\mathcal{F}_h^-$ defined on the time interval $[0, h)$ (describing the input/output relation of the FDLTI system on that interval). We first introduced the input approximation approach to the L_∞-induced norm computation associated with continuous-time FDLTI systems based on the fast-lifting treatment. We next developed the kernel approximation approach to the L_∞-induced norm computation, which is also based on fast-lifting. In these two approximation approaches, we applied two schemes in approximating input and kernel functions, i.e., the piecewise constant approximation scheme and the piecewise linear approximation scheme. It was then shown that the approximation errors in our two approaches converge to 0 at the rates of $1/M$ and $1/M^2$ in the piecewise constant approximation and piecewise linear approximation schemes, respectively, as the fast-lifting parameter M tends to infinity. Even though these convergence rates are qualitatively the same in the two approximation approaches, our detailed analysis showed that the approximation errors through the kernel approximation approach are smaller than those through the input approximation approach, when the piecewise linear approximation scheme is used. We then gave a method for evaluating the effect on the truncated interval $[h, \infty)$, and this was used commonly in both the input approximation and kernel approximation approaches. Through this evaluation together with the input approximation and kernel approximation approaches, we can compute the L_∞-induced norm of FDLTI systems

to any degree of accuracy. Finally, we examined effectiveness of the two approximation approaches through numerical studies and confirmed that the kernel approximation approach works more effectively than the input approximation approach, not only in accuracy but also in computation times, especially when the piecewise linear approximation scheme is taken.

We remark that the input approximation and kernel approximation approaches developed in this chapter can be extended to the computation of the L_∞-induced norm of sampled-data systems (i.e., the L_1 analysis of sampled-data systems), and such extensions will be discussed in Chapter 3.

We further note that constructing the jth-order approximants $\mathbf{B}'_{\mathrm{ic}j}$ and $\mathbf{B}'_{\mathrm{kc}j}$ to $\mathbf{B}'$ (with desired properties from the jth-order approximation viewpoint) could be carried out even for $j \geq 2$ by following the same line of arguments as in the appendix, i.e., Subsection 2.9.3 and Section 2.4, respectively. However, the overall performance improvement by taking $j \geq 2$ may not be definite since it would take a longer time to compute the $L_1[0, h')$ norms of jth-order polynomials when $j \geq 2$. This is in sharp contrast with the present chapter dealing only with $j = 0$ and $j = 1$ (i.e., constant and linear functions) and might govern the overall performance as the fast-lifting parameter M becomes larger. Analyzing such an aspect and developing an effective computation method exploiting a jth-order approximation idea for $j \geq 2$ may be an interesting future topic.

Finally, we remark that for the class of positive finite-dimensional LTI systems [43], the L_∞-induced norm computation reduces to the finite-dimensional matrix ∞-norm computation $\|D - CA^{-1}B\|$ as shown in [9] (for essentially the same reason as that stated in Section 2.1). More interestingly, this explicit result has been extended to positive LTI systems with distributed delays in [65], where an 'equivalent' *finite-dimensional* LTI positive system has been clarified that possesses the same L_∞-induced norm as the original system with distributed delays. This might suggest that computing upper and lower bounds of the L_∞-induced norm of (not necessarily positive) LTI systems with (distributed) delays may still be a tractable problem. If the L_∞-induced norm of (not necessarily positive) LTI systems with (distributed) delays can also be reduced to the L_∞-induced norm associated with 'equivalent' LTI systems without delays, both the input approximation method and kernel approximation method discussed in the present chapter can immediately be applied to the latter systems, but it is nontrivial whether such equivalent systems do exist generically. This may be an interesting topic to study. On the other hand, aiming at direct extension of the kernel approximation method in the present chapter to the distributed delay systems, in which the kernel functions associated with distributed delays are approximated, could also be a (quite nontrivial but) interesting future topic.

2.9 Appendix

This section is concerned with the proofs of the lemmas presented in this chapter. They are based on the Taylor expansion of the matrix exponential of $A\theta'$ (or Ah'), and the proofs of lemmas in the kernel approximation approach proceed in essentially the same way as those of lemmas in the input approximation approach. Hence, only the proofs of the lemmas in the input approximation approach are given. Furthermore, this section is devoted to explaining the rationale for taking the specific functions $f_0(\tau')$ and $f_1(\tau')$ given in (2.30).

2.9.1 Proofs of Lemmas 2.1 and 2.2

Proof of Lemma 2.1:

By considering the Taylor expansion of $exp(A\theta')$, it readily follows that

$$(\mathbf{B}' - \mathbf{B}'_{\mathrm{ic0}})u = \int_0^{h'} \left\{ \exp(A\theta')B - \frac{1}{h'}B'_{\mathrm{0c}d} \right\} u(\theta')d\theta' \tag{2.63}$$

Because

$$\exp(A\theta')B - \frac{1}{h'}B'_{\mathrm{0c}d} = \sum_{i=1}^{\infty} \left\{ \frac{(A)^i(\theta')^i}{i!} - \frac{(A)^i(h')^i}{(i+1)!} \right\} B \tag{2.64}$$

we can confirm that the following inequality is established.

$$\|\mathbf{B}' - \mathbf{B}'_{\mathrm{ic0}}\| \leq \|B\| \int_0^{h'} \sum_{i=1}^{\infty} \left\{ \frac{\|A\|^i(\theta')^i}{i!} + \frac{\|A\|^i(h')^i}{(i+1)!} \right\} d\theta' \leq (h')^2\|A\| \cdot \|B\| e^{\|A\|h'} \tag{2.65}$$

Thus Lemma 2.1 is proved.

Proof of Lemma 2.2:

By noting that the input of $\mathbf{B}'_{\mathrm{ic0}}$ may be confined to constant vector functions, we can easily see that, for a n_w-dimensional vector w_d, the relation

$$\{\mathbf{B}'_{\mathrm{ic0}}w \mid \|w\| \leq 1\} = \{B'_{\mathrm{0c}d}w_d \mid \|w_d\| \leq 1\} \tag{2.66}$$

holds. We hence may replace $C'_{dM}\overline{\mathbf{B}'_{\mathrm{ic0}}}$ with $C'_{dM}\overline{B'_{\mathrm{0c}d}}$ by identifying constant vector functions with constant vectors. Thus, the Lemma 2.2 is proved.

2.9.2 Proofs of Lemmas 2.3 and 2.4

Proof of Lemma 2.3:

Since f_0 and f_1 are scalar valued functions, it follows from the Taylor expansion of $\exp(A\theta')$ that

$$(\mathbf{B}' - \mathbf{B}'_{\mathrm{ic1}})u = \int_0^{h'} \{\exp(A\theta')B - B'_{0\mathrm{cd}} f_0(\theta') - B'_{1\mathrm{cd}} f_1(\theta')\} u(\theta') d\theta' \tag{2.67}$$

By the definition of f_0 and f_1, we can show that

$$\begin{aligned}
&\exp(A\theta')B - B'_{0\mathrm{cd}} f_0(\theta') - B'_{1\mathrm{cd}} f_1(\theta') \\
=&\sum_{j=2}^{\infty} \left\{ \frac{(A)^j(\theta')^j}{j!} - \left(\frac{6j}{(j+2)!}(A)^j(h')^{j-1} \right) \theta' \right\} B + \sum_{j=2}^{\infty} \frac{2(j-1)}{(j+2)!}(A)^j(h')^j B \\
=:&L_{A\mathrm{c}}(\theta')B
\end{aligned} \tag{2.68}$$

Here, because we can establish the inequalities

$$\begin{aligned}
&\int_0^{h'} \|L_{A\mathrm{c}}(\theta')\| d\theta' \\
\leq &\int_0^{h'} \sum_{j=2}^{\infty} \left\{ \frac{\|A\|^j(\theta')^j}{j!} + \left(\frac{6j}{(j+2)!}\|A\|^j(h')^{j-1} \right) \theta' \right\} d\theta' + \int_0^{h'} \sum_{j=2}^{\infty} \frac{2(j-1)}{(j+2)!}\|A\|^j(h')^j d\theta' \\
\leq &\frac{1}{2}(h')^3\|A\|^2 e^{\|A\|h'}
\end{aligned} \tag{2.69}$$

we have

$$\|\mathbf{B}' - \mathbf{B}'_{\mathrm{ic1}}\| \leq \frac{(h')^3}{2}\|A\|^2 \cdot \|B\| e^{\|A\|h'} \tag{2.70}$$

This completes the proof.

Proof of Lemma 2.4:

We next consider the computation of $\|\mathcal{G}^-_{hM11}\|$ (assuming that $D = 0$). This norm is described by

$$\|\mathcal{G}^-_{hM\mathrm{i}1}\| = \sup_{\|\check{u}\| \leq 1} \|\mathcal{G}^-_{hM\mathrm{i}1}\check{u}\| \tag{2.71}$$

where

$$\mathcal{G}^-_{hM\mathrm{i}1}\check{u} = \sum_{j=1}^{M} C(A'_d)^{j-1}\mathbf{B}'_{\mathrm{ic1}}u^{(j)}, \quad [(u^{(1)})^T, \cdots, (u^{(M)})^T] := \check{u} \tag{2.72}$$

By the definition of $\mathbf{B}'_{\mathrm{ic1}}$, we can see that

$$
\begin{aligned}
C(A'_d)^{j-1}\mathbf{B}'_{\mathrm{ic1}}u^{(j)} &= \int_0^{h'} C(A'_d)^{j-1}\{f_0(\theta')B'_{0\mathrm{c}d} + f_1(\theta')B'_{1\mathrm{c}d}\}u^{(j)}(\theta')d\theta' \\
&= \int_0^{h'} C(A'_d)^{j-1}\left\{\left(-\frac{6}{(h')^2}\theta' + \frac{4}{h'}\right)B'_{0\mathrm{c}d} + \left(\frac{12}{(h')^3}\theta' - \frac{6}{(h')^2}\right)B'_{0\mathrm{c}d}\right\}u^{(j)}(\theta')d\theta' \\
&= \int_0^{h'} C(A'_d)^{j-1}(Y_{\mathrm{c}0} + Y_{\mathrm{c}1}\theta')u^{(j)}(\theta')d\theta' \qquad (2.73)
\end{aligned}
$$

Note that the integrand involves the function used in defining $T_{j\mathrm{ic}}$. Hence, by the property of $L_\infty[0,h')$ and the definition of $G^-_{hM\mathrm{i}1}$, it follows that $\|\mathcal{G}^-_{hM\mathrm{i}1}\|$ coincides with the ∞-norm of the finite-dimensional matrix $G^-_{hM\mathrm{i}1}$ given by (2.36) with D removed. Then, the assertion of Lemma 2.4 for the case $D \neq 0$ follows immediately again by the property of $L_\infty[0,h')$.

2.9.3 The Rationale behind Our Specific Choice of $f_0(\tau')$ and $f_1(\tau')$

This subsection is devoted to explaining the rationale behind our specific choice of $f_0(\tau')$ and $f_1(\tau')$ given in (2.30). The aim of introducing the functions $f_0(\tau')$ and $f_1(\tau')$ is computing $\|\mathcal{G}^-_h\|$ in the order of $1/M^2$. To this end, we first note that

$$
\mathcal{G}^-_h\mathbf{L}_M^{-1} - \mathcal{G}^-_{hM\mathrm{i}1} = \begin{bmatrix} C(\mathbf{B}' - \mathbf{B}'_{\mathrm{ic1}}) & \cdots & C(A'_d)^{M-1}(\mathbf{B}' - \mathbf{B}'_{\mathrm{ic1}}) \end{bmatrix} \qquad (2.74)
$$

By noting (2.74), we aim at finding scalar-valued linear functions $f_0(\tau')$ and $f_1(\tau')$ such that $\|\mathbf{B}' - \mathbf{B}'_{\mathrm{ic1}}\|$ converges to 0 in the order of $1/M^3$. Taking such $f_0(\tau')$ and $f_1(\tau')$ makes $\|\mathcal{G}^-_h\mathbf{L}_M^{-1} - \mathcal{G}^-_{hM\mathrm{i}1}\|$ converge to 0 in the order of $1/M^2$. If we note (2.67), it suffices to take $f_0(\tau')$ and $f_1(\tau')$ such that

$$
\sup_{\|u\|\le 1}\left\|\int_0^{h'}\{\exp(A\theta')B - B'_{0\mathrm{c}d}f_0(\theta') - B'_{1\mathrm{c}d}f_1(\theta')\}u(\theta')d\theta'\right\| \qquad (2.75)
$$

converges to 0 in the order of $1/M^3$. It is easy to see that the above condition is satisfied if

$$
\int_0^{h'}\|\{\exp(A\theta')B - B'_{0\mathrm{c}d}f_0(\theta') - B'_{1\mathrm{c}d}f_1(\theta')\}\|\,d\theta' \qquad (2.76)
$$

is bounded by the order of $1/M^3$. It follows readily from the Taylor expansion of $\exp(A\theta')$ that

$$
\begin{aligned}
&\exp(A\theta') - B'_{0cd} f_0(\theta') - B'_{1cd} f_1(\theta') \\
&= \left\{ \sum_{i=0}^{\infty} \frac{(A)^i (\theta')^i}{i!} - f_0(\theta') \sum_{i=0}^{\infty} \frac{(A)^i (h')^{i+1}}{(i+1)!} - f_1(\tau') \sum_{i=0}^{\infty} \frac{(A)^i (h')^{i+2}}{i!(i+2)} \right\} B \\
&= \left\{ (I + A\theta') - \left(Ih' + \frac{A}{2}(h')^2 \right) f_0(\theta') - \left(\frac{I}{2}(h')^2 + \frac{A}{3}(h')^3 \right) f_1(\theta') \right\} B \\
&+ \left\{ \sum_{i=2}^{\infty} \frac{(A)^i (\theta')^i}{i!} - f_0(\theta') \sum_{i=2}^{\infty} \frac{(A)^i (h')^{i+1}}{(i+1)!} - f_1(\theta') \sum_{i=2}^{\infty} \frac{(A)^i (h')^{i+2}}{i!(i+2)} \right\} B
\end{aligned}
\tag{2.77}
$$

where the integral from $\theta' = 0$ to $\theta' = h'$ of the norm of the second term can obviously be bounded by the order of $1/M^3$ regardless of the linear functions $f_0(\theta')$ and $f_1(\theta')$ such that $\int_0^{h'} |f_i(\theta')| d\theta'$ is proportional to $(h')^{-i}$ $(i = 0, 1)$. Hence, let us note the first term and require these linear functions to further satisfy that

$$
\int_0^{h'} \left\| (I + A\theta') - \left(Ih' + \frac{A}{2}(h')^2 \right) f_0(\theta') - \left(\frac{I}{2}(h')^2 + \frac{A}{3}(h')^3 \right) f_1(\theta') \right\| d\theta' \tag{2.78}
$$

is bounded by the order of $1/M^3$. An easy way to satisfy these conditions is to assume that $f_0(\theta') = \frac{f_{01}}{(h')^2}\theta' + \frac{f_{00}}{h'}$ and $f_1(\theta') = \frac{f_{11}}{(h')^3}\theta' + \frac{f_{10}}{(h')^2}$ (with $f_{01}, f_{00}, f_{11}, f_{10}$ independent of h') and seek for these coefficients such that the function in the integrand of (2.78) is identically zero (with respect A as well as τ'). This treatment leads to the unique solutions for $f_0(\theta')$ and $f_1(\theta')$ given by (2.30).

Remark 2.7 The use of $\mathbf{J}'_0$ in the piecewise constant approximation method was motivated by the study in [3], but a similar argument can actually show that this operator follows as a unique solution in that method.

Chapter 3

L_∞-Induced Norm Analysis of Sampled-Data Systems

3.1 Introduction

There have been a number of studies associated with sampled-data systems [2]–[6],[14]–[16],[28],[33],[36],[38],[41],[46],[60],[61],[66]–[68],[71],[73] taking account of their intersample behavior by using the lifting technique [4],[6],[68],[71], the fast-sample/fast-hold (FSFH) approximation approach [46],[73] or the FR-operator approach [2],[33], and so on. These studies can also be classified by the type of system norms dealt with, where the typical studies for sampled-data systems are the H_∞ problem [6],[34],[36],[38],[60],[61],[67],[68] and the H_2 problem [5],[15],[33],[60],[61]. These norms play important roles in the analysis and synthesis for sampled-data systems relevant to practical control problems. However, they cannot be used for dealing with the problems of bounded persistent disturbances, such as steps and sinusoids, which are often encountered in control systems. In this regard, we need to consider the L_∞-induced norm of sampled-data systems, and this problem has been named the L_1 problem of sampled-data systems [3],[28],[66]. In [3],[28],[66], a sampled-data system is approximated by a discrete-time system through the FSFH approximation technique, and it is shown that the l_∞-induced norm of the approximating discrete-time system converges to the L_∞-induced norm of the original sampled-data system as the FSFH approximation parameter M tends to infinity. However, these studies do not evaluate how close the l_∞-induced norm for a given M is to the exact value of the L_∞-induced norm. In other words, no upper and lower bounds have been derived for the L_∞-induced norm through the FSFH approximation approach.

As a significant advance over the existing result, this chapter derives two approaches for computing the upper and lower bounds of the L_∞-induced norm of sampled-data systems. By using the idea of lifting [4],[6],[68],[71], we first derive input and output operators de-

scribing the input/output relation of sampled-data systems. In contrasted to the case of continuous-time systems, both the input and output operators should be approximated in computing the L_∞-induced norm of sampled-data systems. Thus, we develop two approximation methods for the output operators, by which the output of sampled-data systems is approximated by a piecewise constant or piecewise linear function. Furthermore, by noting that the structure of the input operator is essentially equivalent to that of the convolution formula of continuous-time FDLTI systems, we extend the two approximation approaches used in continuous-time systems to approximating the input operator. In other words, the input operator of sampled-data systems is approximated through the idea of input approximation or kernel approximation approach. This is stimulated by the success of employing these ideas in Chapter 2 in computing the L_∞-induced norm of continuous-time FDLTI systems. The fast-lifting technique introduced in Chapter 2 also plays an important role in introducing both approximation approaches, which divides the sampling interval $[0, h)$ into M subintervals with an equal width (without applying sampling of signals). Two approximation approaches involve piecewise constant approximation and piecewise linear approximation schemes, by which the input of the sampled-data system or the kernel function of the input operator is approximated by a piecewise constant or piecewise linear function. Even though the central part of the method with the input approximation approach and the piecewise constant approximation scheme essentially coincides with a conventional method via FSFH approximation after all, we show that our new arguments supported by the application of fast-lifting not only successfully allow us to develop the extended piecewise linear approximation scheme but also lead to upper and lower bounds of the L_∞-induced norm, whose gap converges to 0 at the rate of $1/M$ in the piecewise constant approximation and $1/M^2$ in the extended (i.e., piecewise linear) approximation scheme, under both the input approximation and kernel approximation approaches. Furthermore, we examine effectiveness of these methods through numerical studies, and we show that the kernel approximation approach with the piecewise linear approximation scheme achieves the best efficiency in the four computation methods.

The organization of this chapter is as follows. In Section 3.2, we first introduce the lifting approach to sampled-data systems and formulate the problem definition, i.e., L_∞-induced norm analysis of sampled-data systems. We next develop the input approximation approach in Section 3.3. In Section 3.4, we further give the kernel approximation approach. The main results in this chapter together with the guideline for taking approximation parameters are given in Section 3.5. We then do comparison between the input approximation and kernel approximation approaches in Section 3.6. In Section 3.7, the effectiveness of the proposed methods is demonstrated through numerical examples. We give concluding remarks

in Section 3.8. Finally, the proofs of the lemmas given in this chapter and the derivation of Theorem 3.2, which is concerned with the L_∞-induced norm computation through the input approximation approach with the piecewise linear approximation scheme, are provided in Section 3.9.

3.2 Problem Formulation

In this section, we first introduce sampled-data systems and discuss the L_∞-induced norm of the sampled-data systems. We then derive the lifted representation of sampled-data systems [4],[6],[68],[71] and describe the L_∞-induced norm in such representation.

3.2.1 Lifted Representation of Sampled-Data Systems

This chapter is concerned with the sampled-data system Σ_{SD} shown in Figure 3.1, where P denotes the continuous-time finite-dimensional linear time-invariant (FDLTI) system, while Ψ, $\mathcal{H}$ and $\mathcal{S}$ denote the discrete-time LTI controller, the zero-order hold and the ideal sampler, respectively, operating with sampling period h in a synchronous fashion. Solid lines and dashed lines in Fig. 3.1 are used to represent continuous-time signals and discrete-time signals, respectively. Suppose that P and Ψ are described respectively by

$$P : \begin{cases} \dfrac{dx}{dt} = Ax + B_1 w + B_2 u \\ z = C_1 x + D_{11} w + D_{12} u \\ y = C_2 x \end{cases} \tag{3.1}$$

$$\Psi : \begin{cases} \psi_{k+1} = A_\Psi \psi_k + B_\Psi y_k \\ u_k = C_\Psi \psi_k + D_\Psi y_k \end{cases} \tag{3.2}$$

where $x(t) \in \mathbb{R}^n_\infty,\ w(t) \in \mathbb{R}^{n_w}_\infty,\ u(t) \in \mathbb{R}^{n_u}_\infty,\ z(t) \in \mathbb{R}^{n_z}_\infty,\ y(t) \in \mathbb{R}^{n_y}_\infty,\ \psi_k \in \mathbb{R}^{n_\Psi}_\infty,\ y_k = y(kh)$ and $u(t) = u_k$ $(kh \le t < (k+1)h)$. The assumptions that $D_{21} = 0$ and $D_{22} = 0$ are made

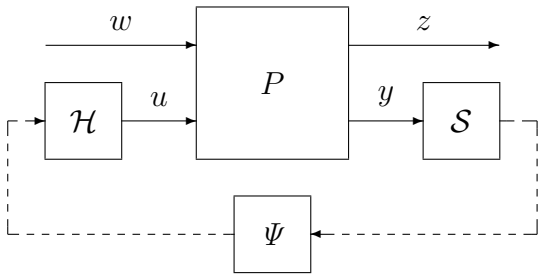


Figure 3.1: Sampled-data system Σ_{SD}.

to guarantee that the sampler operates on continuous signals and the map from u to y is strictly causal, respectively.

The L_∞-induced norm of the sampled-data system Σ_{SD} denoted by $\|\Sigma_{\mathrm{SD}}\|$, is defined as

$$\|\Sigma_{\mathrm{SD}}\| := \sup_{\|w\|\leq 1} \|z\| \tag{3.3}$$

We next introduce the lifting technique [4],[6],[68],[71]. Given $f(\cdot) \in (L_\infty)^\nu$, its lifting $\{\widehat{f}_k\}_{k=0}^\infty$ with $\widehat{f}_k(\cdot) \in (L_\infty[0,h))^\nu$ $(k \in \mathbb{N}_0)$ (with sampling period h) is defined as follows:

$$\widehat{f}_k(\theta) = f(kh+\theta) \quad (0 \leq \theta < h) \tag{3.4}$$

As shown in Figure 3.2, the lifting can be visualized as taking a continuous signal and dividing it up into a sequence of pieces each corresponding to the function over an interval of length h. Let us denote the lifting by $\mathbf{W}_h : (L_\infty)^\nu \mapsto l_{(L_\infty[0,h))^\nu}$. Then, we easily see that $\mathbf{W}_h$ is a liner isomorphism and is an isometry, i.e., it preserves norm.

By applying lifting to w and z, the lifted representation of the sampled-data system Σ_{SD} is described by

$$\begin{cases} \xi_{k+1} &= \mathcal{A}\xi_k + \mathcal{B}\widehat{w}_k \\ \widehat{z}_k &= \mathcal{C}\xi_k + \mathcal{D}\widehat{w}_k \end{cases} \tag{3.5}$$

with $\xi_k := [x_k^T \ \psi_k^T]^T$ $(x_k := x(kh))$, the matrix $\mathcal{A}$ and the operators $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ defined as

$$\mathcal{A} = \begin{bmatrix} A_d + B_{2d}D_\Psi C_{2d} & B_{2d}C_\Psi \\ B_\psi C_{2d} & A_\Psi \end{bmatrix} : \mathbb{R}_\infty^{n+n_\Psi} \to \mathbb{R}_\infty^{n+n_\Psi} \tag{3.6}$$

$$\mathcal{B} = J_\Sigma \mathbf{B}_1 : (L_\infty[0,h))^{n_w} \to \mathbb{R}_\infty^{n+n_\Psi} \tag{3.7}$$

$$\mathcal{C} = \mathbf{M}_1 C_\Sigma : \mathbb{R}_\infty^{n+n_\Psi} \to (L_\infty[0,h))^{n_z} \tag{3.8}$$

$$\mathcal{D} = \mathbf{D}_{11} : (L_\infty[0,h))^{n_w} \to (L_\infty[0,h))^{n_z} \tag{3.9}$$

where

$$A_d := \exp(Ah), \quad B_{2d} := \int_0^h \exp(A\theta)B_2 d\theta, \quad C_{2d} := C_2 \tag{3.10}$$

$$J_\Sigma = \begin{bmatrix} I \\ 0 \end{bmatrix} \in \mathbb{R}_\infty^{(n+n_\Psi)\times n}, \quad C_\Sigma := \begin{bmatrix} I & 0 \\ D_\Psi C_{2d} & C_\Psi \end{bmatrix} \tag{3.11}$$

$$\mathbf{B}_1 w = \int_0^h \exp(A(h-\theta))B_1 w(\theta) d\theta \tag{3.12}$$

$$\left(\mathbf{M}_1 \begin{bmatrix} x \\ u \end{bmatrix}\right)(\theta) = C_0 \exp(A_2\theta) \begin{bmatrix} x \\ u \end{bmatrix} \tag{3.13}$$

$$A_2 := \begin{bmatrix} A & B_2 \\ 0 & 0 \end{bmatrix}, \quad C_0 := \begin{bmatrix} C_1 & D_{12} \end{bmatrix} \tag{3.14}$$

$$(\mathbf{D}_{11} w)(\theta) = \int_0^\theta C_1 \exp(A(\theta-\tau))B_1 w(\tau) d\tau + D_{11} w(\theta) \tag{3.15}$$

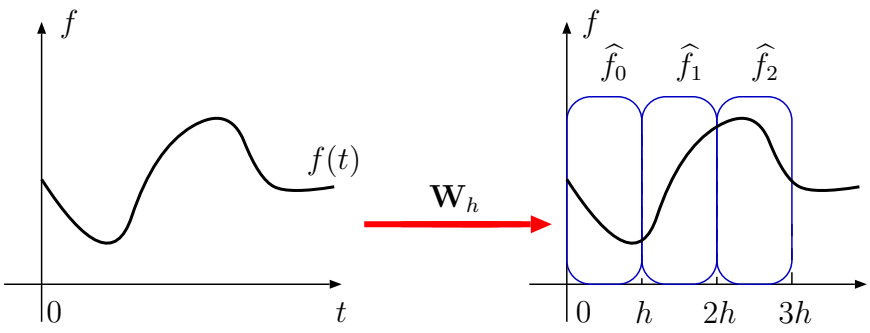


Figure 3.2: Lifting technique.

In the following, we denote the lifted representation of the sampled-data system Σ_{SD} in (3.5) by $\widehat{\Sigma}_{\mathrm{SD}}$. By using the properties of L_∞ and $L_\infty[0,h)$, the L_∞-induced norm of the lifted sampled-data system $\widehat{\Sigma}_{\mathrm{SD}}$ denoted by $\|\widehat{\Sigma}_{\mathrm{SD}}\|$, can be described by

$$\|\widehat{\Sigma}_{\mathrm{SD}}\| = \sup_{\|\{\widehat{w}_i\}_{i=0}^{\infty}\| \leq 1} \sup_{k \in \mathbb{N}_0} \|\widehat{z}_k\| \tag{3.16}$$

where $\|\{\widehat{w}_i\}_{i=0}^{\infty}\| \leq 1$ of the right hand side of (3.16) implies that

$$\sup_{i \in \mathbb{N}_0} \|\widehat{w}_i\| \leq 1 \tag{3.17}$$

As mentioned above, because the lifting $\mathbf{W}_h$ is norm preserving, it readily follows that

$$\|\Sigma_{\mathrm{SD}}\| = \|\widehat{\Sigma}_{\mathrm{SD}}\| \tag{3.18}$$

Thus, in the following, we call $\|\widehat{\Sigma}_{\mathrm{SD}}\|$ also the L_∞-induced norm of the sampled-data system Σ_{SD}. We further assume that the sampled-data system Σ_{SD} is internally asymptotically stable, i.e., $\mathcal{A}$ has all its eigenvalues in the open unit disc. This is necessary (and sufficient by the stability of Σ_{SD}) for the L_∞-induced norm of the sampled-data system Σ_{SD} to be bounded/well-defined.

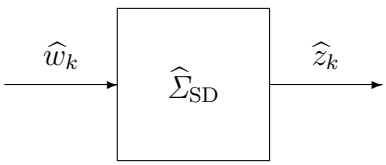


Figure 3.3: Lifted representation of the sampled-data system Σ_{SD}.

3.2.2 Toeplitz Structure of Input /Output Relation and Truncation Idea

In this subsection, we give preliminaries for the arguments in this chapter, i.e., the Toeplitz structure of the input/output relation of $\widehat{\Sigma}_{\mathrm{SD}}$ and its truncation idea.

To compute the L_∞-induced norm $\|\widehat{\Sigma}_{\mathrm{SD}}\|$, we first note (3.5) and describe the relation between $\widehat{w}_k$ and $\widehat{z}_k$ ($k \in \mathbb{N}_0$) as follows:

$$\begin{bmatrix} \widehat{z}_0 \\ \widehat{z}_1 \\ \widehat{z}_2 \\ \widehat{z}_3 \\ \vdots \end{bmatrix} = \begin{bmatrix} \mathcal{D} & 0 & \cdots & & & \\ \mathcal{CB} & \mathcal{D} & 0 & \cdots & & \\ \mathcal{CAB} & \mathcal{CB} & \mathcal{D} & 0 & \cdots & \\ \mathcal{CA}^2\mathcal{B} & \mathcal{CAB} & \mathcal{CB} & \mathcal{D} & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \begin{bmatrix} \widehat{w}_0 \\ \widehat{w}_1 \\ \widehat{w}_2 \\ \widehat{w}_3 \\ \vdots \end{bmatrix} \tag{3.19}$$

Since the above operator has a Toeplitz structure and since $\|f\| = \sup_{k \in \mathbb{N}_0} \|\widehat{f}_k\|$ for $f \in L_\infty$, it follows readily from the properties of L_∞ that the L_∞-induced norm $\|\widehat{\Sigma}_{\mathrm{SD}}\|$ coincides with the $L_\infty[0,h)$-induced norm of

$$\mathcal{P} := \begin{bmatrix} \mathcal{D} & \mathcal{CB} & \mathcal{CAB} & \mathcal{CA}^2\mathcal{B} & \cdots \end{bmatrix} \tag{3.20}$$

Remark 3.1 Essentially the same assertion can be found in [66], but noting the Toeplitz structure leads to a concise statement (as above) as well as an obvious proof of the assertion.

Remark 3.2 Implicitly assumed in (3.5) (and thus (3.20)) is the assumption that $t = 0$ is a sampling instant. One might argue that if an intersample instant is taken as $t = 0$, the corresponding L_∞-induced norm might become different from the present one. Since the input-output mapping of Σ_{SD} between w and z is h-periodic, however, this is not the case as an immediate property of an induced norm (as in the H_∞ or L_2-induced norm).

It is, however, still difficult to compute $\|\mathcal{P}\|$ since $\mathcal{P}$ consists of an infinite number of columns. To alleviate this difficulty, we take an $N \in \mathbb{N}$, decompose $\mathcal{P}$ into

$$\mathcal{P} = \mathcal{P}_N^- + \mathcal{P}_N^+ \tag{3.21}$$

$$\mathcal{P}_N^- := \begin{bmatrix} \mathcal{D} & \mathcal{CB} & \cdots & \mathcal{CA}^N\mathcal{B} & 0 & 0 & \cdots \end{bmatrix} \tag{3.22}$$

$$\mathcal{P}_N^+ := \begin{bmatrix} 0 & \cdots & 0 & \mathcal{CA}^{N+1}\mathcal{B} & \mathcal{CA}^{N+2}\mathcal{B} & \cdots \end{bmatrix} \tag{3.23}$$

and compute the $L_\infty[0,h)$-induced norm $\|\mathcal{P}_N^-\|$ as accurately as possible while the computation of $\|\mathcal{P}_N^+\|$ is treated in a comparatively simple way (because this norm is expected to be small when N is large enough); we aim at computing upper and lower bounds of $\|\mathcal{P}\|$ through approximation of $\mathcal{P}_N^-$ and computing an upper bound of $\|\mathcal{P}_N^+\|$. The choice of N (as well as other parameters to be introduced) will be discussed in Subsection 3.5.3.

3.2.3 Fast-Lifting Treatment of $\mathcal{P}_N^-$

As a key idea in the computation of the L_∞-induced norm of sampled-data systems within any prescribed error bound, we next apply the fast-lifting technique introduced in Chapter 2. Approximations are applied later in the following section on the top of the fast-lifting treatment, by which signals on $[0, h/M)$ are constrained to constant functions or linear functions. Piecewise constant approximation or piecewise linear approximation scheme of signals on $[0, h)$ can be achieved easily in such a way.

It readily follows from the norm-preserving property of $\mathbf{L}_M$ that

$$\|\mathcal{P}_N^-\| = \left\| \begin{bmatrix} \mathbf{L}_M \mathcal{D} \mathbf{L}_M^{-1} & \cdots & \mathbf{L}_M \mathcal{C} \mathcal{A}^N \mathcal{B} \mathbf{L}_M^{-1} \end{bmatrix} \right\| \tag{3.24}$$

To facilitate the treatment of the right-hand side, we introduce $\mathbf{D}_{11}', \mathbf{B}_1'$ and $\mathbf{M}_1'$ defined as $\mathbf{D}_{11}$, $\mathbf{B}_1$ and $\mathbf{M}_1$, respectively, with the horizon $[0, h)$ replaced by $[0, h')$ $(= [0, h/M))$, and also introduce the matrices

$$A_d' := \exp(Ah'), \quad A_{2d}' := \exp(A_2 h'), \quad J := \begin{bmatrix} I \\ 0 \end{bmatrix} \in \mathbb{R}_\infty^{(n+n_u)\times n} \tag{3.25}$$

Then, it is easy to see that $\mathbf{L}_M \mathcal{D} \mathbf{L}_M^{-1}$ and $\mathbf{L}_M \mathcal{C} \mathcal{A}^j \mathcal{B} \mathbf{L}_M^{-1}$ $(j = 0, \cdots, N)$ in (3.24) are described respectively by

$$\mathbf{L}_M \mathcal{D} \mathbf{L}_M^{-1} = \overline{\mathbf{M}_1'} \Delta_M^0 \overline{\mathbf{B}_1'} + \overline{\mathbf{D}_{11}'} \tag{3.26}$$

$$\mathbf{L}_M \mathcal{C} \mathcal{A}^j \mathcal{B} \mathbf{L}_M^{-1} = \overline{\mathbf{M}_1'} A_{2dM}' C_\Sigma \mathcal{A}^j J_\Sigma A_{dM}' \overline{\mathbf{B}_1'} \tag{3.27}$$

where

$$A_{dM}' := \begin{bmatrix} (A_d')^{M-1} & \cdots I \end{bmatrix}, \quad A_{2dM}' := \begin{bmatrix} I \\ \vdots \\ (A_{2d}')^{M-1} \end{bmatrix} \tag{3.28}$$

$$\Delta_M^0 := \begin{bmatrix} 0 & 0 & \cdots & 0 \\ J & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ (A_{2d}')^{M-2} J & \cdots & J & 0 \end{bmatrix} \tag{3.29}$$

and $\overline{(\cdot)}$ denotes diag$[(\cdot), \cdots, (\cdot)]$ consisting of M copies of $(\cdot)$. Hence, the operator matrix on the right hand side of (3.24) admits the representation

$$\mathcal{P}_{NM}^- = \begin{bmatrix} \overline{\mathbf{M}_1'} \Delta_M^0 \overline{\mathbf{B}_1'} + \overline{\mathbf{D}_{11}'} & \overline{\mathbf{M}_1'} \mathcal{A}_{M0} \overline{\mathbf{B}_1'} & \cdots & \overline{\mathbf{M}_1'} \mathcal{A}_{MN} \overline{\mathbf{B}_1'} \end{bmatrix} \tag{3.30}$$

where

$$\mathcal{A}_{Mj} := A_{2dM}' C_\Sigma \mathcal{A}^j J_\Sigma A_{dM}' \quad (j = 0, \cdots, N) \tag{3.31}$$

Remark 3.3 The structure of the operator $\mathbf{B}_1'$ is essentially equivalent to that of the operator $\mathbf{B}'$ discussed in Chapter 2. We hence readily extend the input approximation and kernel approximation approaches given in Chapter 2 to the approximation of the operator $\mathbf{B}_1'$ associated with sampled-data systems, in Sections 3.3 and 3.4, respectively.

3.3 Input Approximation Approach

This section gives computation methods for the L_∞-induced norm of sampled-data systems by using the idea of the input approximation approach discussed in Chapter 2. This approach is developed in the fast-lifted representation $\mathcal{P}_{NM}^-$ and leads to piecewise constant approximation or piecewise linear approximation scheme of the operators $\mathbf{B}_1$, $\mathbf{M}_1$ and $\mathbf{D}_{11}$ (i.e., constant or linear approximation of the operators $\mathbf{B}'$, $\mathbf{M}_1'$ and $\mathbf{D}_{11}'$ involved in $\mathcal{P}_{NM}^-$).

3.3.1 Piecewise Constant Approximation Scheme

In this subsection, we suppose that N is given and aim at computing upper and lower bounds of $\|\mathcal{P}_N^-\|(=\|\mathcal{P}_{NM}^-\|)$ through piecewise constant approximation scheme of $\mathcal{P}_N^-$.

In piecewise constant approximation scheme, a central role is played by the 'averaging' operator $\mathbf{J}_0'$ defined as (2.19). We introduce the operator $\mathbf{B}_{\mathrm{i}0}' := \mathbf{B}_1'\mathbf{J}_0'$, i.e.,

$$\mathbf{B}_{\mathrm{i}0}' w = \int_0^{h'} \exp(A(h'-\theta'))B_1 \cdot (\mathbf{J}_0' w)(\theta')d\theta' \tag{3.32}$$

which corresponds to restricting the input of $\mathbf{B}_1'$ to constant functions. Obviously, $\mathbf{B}_{\mathrm{i}0}' w = \mathbf{B}_1' w$ whenever w is a constant function. On the other hand, we further introduce the operators $\mathbf{M}_{\mathrm{a}0}'$ and $\mathbf{D}_{\mathrm{a}0}'$ defined respectively by

$$\left(\mathbf{M}_{\mathrm{a}0}' \begin{bmatrix} x \\ u \end{bmatrix}\right)(\theta') = C_0 \begin{bmatrix} x \\ u \end{bmatrix} \quad (0 \le \theta' < h') \tag{3.33}$$

$$(\mathbf{D}_{\mathrm{a}0}' w)(\theta') = D_{11} w(\theta') \quad (0 \le \theta' < h') \tag{3.34}$$

Remark 3.4 The above two operators $\mathbf{M}_{\mathrm{a}0}'$ and $\mathbf{D}_{\mathrm{a}0}'$ will also be used in the kernel approximation approach with the piecewise constant approximation scheme in Subsection 3.4.1.

Introducing the operator $\mathbf{M}_{\mathrm{a}0}'$ corresponds to a zero-order approximation of the Taylor expansion of the output $= C_0 \sum_{i=0}^{\infty} \frac{(A_2\theta')^i}{i!} \begin{bmatrix} x \\ u \end{bmatrix}$ of $\mathbf{M}_1'$. The operator $\mathbf{D}_{\mathrm{a}0}'$ means the operator of multiplication by the matrix D_{11}.

We are in a position to introduce the constant approximation $\mathcal{P}^-_{NM\mathrm{i}0}$ for $\mathcal{P}^-_{NM}$, by which we mean to replace $\mathbf{B}'_1$, $\mathbf{M}'_1$ and $\mathbf{D}'_{11}$ in (3.30) with $\mathbf{B}'_{\mathrm{i}0}$, $\mathbf{M}'_{\mathrm{a}0}$ and $\mathbf{D}'_{\mathrm{a}0}$, respectively:

$$\mathcal{P}^-_{NM\mathrm{i}0} = \left[\overline{\mathbf{M}'_{\mathrm{a}0}}\Delta^0_M\overline{\mathbf{B}'_{\mathrm{i}0}} + \overline{\mathbf{D}'_{\mathrm{a}0}} \quad \overline{\mathbf{M}'_{\mathrm{a}0}}\mathcal{A}_{M0}\overline{\mathbf{B}'_{\mathrm{i}0}} \quad \cdots \quad \overline{\mathbf{M}'_{\mathrm{a}0}}\mathcal{A}_{MN}\overline{\mathbf{B}'_{\mathrm{i}0}}\right] \tag{3.35}$$

This corresponds to piecewise constant approximation of $\mathcal{P}^-_N$. This subsection shows that $\|\mathcal{P}^-_{NM\mathrm{i}0}\|$ can be computed exactly and converges, as $M \to \infty$, to $\|\mathcal{P}^-_N\|$ at the rate of $1/M$. To establish a more precise assertion relevant to upper and lower bounds of $\|\mathcal{P}^-_N\|$, the following three lemmas are important.

Lemma 3.1 The following inequality holds.

$$\|\mathbf{B}'_1 - \mathbf{B}'_{\mathrm{i}0}\| \le \frac{h^2}{M^2}\|A\| \cdot \|A'_d B_1\| e^{\|A\|h/M} \tag{3.36}$$

Lemma 3.2 The following inequality holds.

$$\|\mathbf{M}'_1 - \mathbf{M}'_{\mathrm{a}0}\| \le \frac{h}{M}\|C_0 A_2\| e^{\|A_2\|h/M} \tag{3.37}$$

Lemma 3.3 The inequality

$$\|(\overline{\mathbf{M}'_1}\Delta^0_M\overline{\mathbf{B}'_1} + \overline{\mathbf{D}'_{11}}) - (\overline{\mathbf{M}'_{\mathrm{a}0}}\Delta^0_M\overline{\mathbf{B}'_{\mathrm{i}0}} + \overline{\mathbf{D}'_{\mathrm{a}0}})\| \le \frac{K_{M\mathcal{D}\mathrm{i}0}}{M} \tag{3.38}$$

holds, where

$$\begin{aligned} K_{M\mathcal{D}\mathrm{i}0} := h\|C_1\| \cdot \|B_1\| e^{\|A\|h/M} &+ \frac{h^2}{M}\|A\| \cdot \|B_1\| e^{\|A\|h/M} \\ &\cdot \sum_{k=0}^{M-2} \left\{\|C_1(A'_d)^{k+1}\| + \|C_1(A'_d)^k\| e^{\|A\|h/M}\right\} \end{aligned} \tag{3.39}$$

Furthermore, $K_{M\mathcal{D}\mathrm{i}0}$ has a uniform upper bound with respect to M given by

$$K^U_{\mathcal{D}\mathrm{i}0} := h\|C_1\| \cdot \|B_1\| e^{\|A\|h} + 2h^2\|C_1\| \cdot \|A\| \cdot \|B_1\| \cdot e^{\|A\|h} \tag{3.40}$$

The proof of Lemma 3.1 is omitted because it is essentially the same as that of Lemma 2.1 in Chapter 2, while the proofs of Lemmas 3.2 and 3.3 are given in Subsection 3.9.1. From Lemmas 3.1 and 3.2, we can obtain the following result.

Proposition 3.1 The inequality

$$\|\overline{\mathbf{M}'_1}\mathcal{A}_{Mj}\overline{\mathbf{B}'_1} - \overline{\mathbf{M}'_{\mathrm{a}0}}\mathcal{A}_{Mj}\overline{\mathbf{B}'_{\mathrm{i}0}}\| \leq \frac{K_{Mj\mathrm{i}0}}{M} \tag{3.41}$$

holds for $j = 0, \cdots, N$, where

$$K_{Mj\mathrm{i}0} := e^{\|A\|h/M} \cdot \|\mathcal{A}_{Mj}\| \cdot \frac{h^2}{M} \cdot \{\|C_0A_2\| \cdot \|B_1\|e^{\|A_2\|h/M} + \|C_0\| \cdot \|A\| \cdot \|A'_dB_1\|\} \tag{3.42}$$

Furthermore, $K_{Mj\mathrm{i}0}$ has a uniform upper bound with respect to M and j given by

$$K^U_{\mathcal{CAB}\mathrm{i}0} := h^2e^{\|A\|h} \cdot \|B_1\| \cdot K_* \cdot \{\|C_0A_2\| \cdot e^{\|A_2\|h} + \|C_0\| \cdot \|A\| \cdot e^{\|A\|h}\} \tag{3.43}$$

where

$$K_* := \max_{i\in\mathbb{N}_0} \|\mathcal{A}^i\| \cdot e^{(\|A\|+\|A_2\|)h} \cdot \|C_\Sigma\| \tag{3.44}$$

Remark 3.5 $\max_{i\in\mathbb{N}_0}\|\mathcal{A}^i\|$ exists since $\mathcal{A}^i \to 0$ as $i \to \infty$ by the stability assumption of Σ_{SD}.

Proof. By applying triangle inequality to (3.41), we readily have the following inequality.

$$\begin{aligned}\|\overline{\mathbf{M}'_1}\mathcal{A}_{Mj}\overline{\mathbf{B}'_1} - \overline{\mathbf{M}'_{\mathrm{a}0}}\mathcal{A}_{Mj}\overline{\mathbf{B}'_{\mathrm{i}0}}\| &\leq \|(\overline{\mathbf{M}'_1} - \overline{\mathbf{M}'_{\mathrm{a}0}})\mathcal{A}_{Mj}\overline{\mathbf{B}'_1}\| + \|\overline{\mathbf{M}'_{\mathrm{a}0}}\mathcal{A}_{Mj}(\overline{\mathbf{B}'_1} - \overline{\mathbf{B}'_{\mathrm{i}0}})\| \\ &\leq \|(\overline{\mathbf{M}'_1} - \overline{\mathbf{M}'_{\mathrm{a}0}})\| \cdot \|\mathcal{A}_{Mj}\| \cdot \|\overline{\mathbf{B}'_1}\| + \|\overline{\mathbf{M}'_{\mathrm{a}0}}\| \cdot \|\mathcal{A}_{Mj}\| \cdot \|\overline{\mathbf{B}'_1} - \overline{\mathbf{B}'_{\mathrm{i}0}}\|\end{aligned} \tag{3.45}$$

By noting that

$$\|\mathbf{B}'_1\| \leq \frac{h}{M}\|B_1\|e^{\|A\|h/M} \tag{3.46}$$

$$\|\mathbf{M}'_{\mathrm{a}0}\| \leq \|C_0\| \tag{3.47}$$

with (3.36) and (3.37), it readily follows that

$$\|\overline{\mathbf{M}'_1}\mathcal{A}_{Mj}\overline{\mathbf{B}'_1} - \overline{\mathbf{M}'_{\mathrm{a}0}}\mathcal{A}_{Mj}\overline{\mathbf{B}'_{\mathrm{i}0}}\| \leq \frac{K_{Mj\mathrm{i}0}}{M} \tag{3.48}$$

The second assertion can be proved easily if we note that

$$\|A'_d\| \leq e^{\|A\|h} \tag{3.49}$$

$$\|\mathcal{A}_{Mj}\| \leq Me^{(\|A\|+\|A_2\|)h}\|C_\Sigma\| \cdot \max_{i\in\mathbb{N}_0}\|\mathcal{A}^i\| \tag{3.50}$$

This completes the proof. Q.E.D.

Remark 3.6 Because of the structure of Δ_M^0, the left-hand side of (3.38) is actually independent of B_2 and D_{12} involved in (3.14). Similarly for Lemmas 3.6, 3.8 and 3.10 given later.

Remark 3.7 Even though the arguments in Lemmas 3.1 and 3.2 could be employed in establishing the assertion of (3.38) with changing $K_{M\mathcal{D}\mathrm{i}0}$ slightly, Lemma 3.3 gives us a less conservative error bound for approximating $\mathcal{D}$.

Proposition 3.1 and Lemma 3.3 readily lead to the following result.

Proposition 3.2 The inequality

$$\|\mathcal{P}_{NM}^{-} - \mathcal{P}_{NM\mathrm{i}0}^{-}\| \leq \frac{K_{M\mathrm{i}0}}{M} \tag{3.51}$$

holds, where

$$K_{M\mathrm{i}0} := K_{M\mathcal{D}\mathrm{i}0} + \sum_{j=0}^{N} K_{Mj\mathrm{i}0} \tag{3.52}$$

In addition, $K_{M\mathrm{i}0}$ has a uniform upper bound with respect to M given by

$$K_{\mathrm{i}0}^{U} := K_{\mathcal{D}\mathrm{i}0}^{U} + (N+1) \cdot K_{\mathcal{CAB}\mathrm{i}0}^{U} \tag{3.53}$$

Remark 3.8 By noting that the piecewise constant approximation is norm-contractive, we can show that the lower bound of $\|\mathcal{P}_{NM}^{-}\|$ in (3.51) can in fact be replaced by $\|\mathcal{P}_{NM\mathrm{i}0}^{-}\|$, i.e., the following inequality holds:

$$\|\mathcal{P}_{NM\mathrm{i}0}^{-}\| \leq \|\mathcal{P}_{N}^{-}\| \leq \|\mathcal{P}_{NM\mathrm{i}0}^{-}\| + \frac{K_{M\mathrm{i}0}}{M} \tag{3.54}$$

To evaluate $\|\mathcal{P}_{N}^{-}\| = \|\mathcal{P}_{NM}^{-}\|$ through (3.51) and the triangle inequality, we next provide a method for (exactly) computing $\|\mathcal{P}_{NM\mathrm{i}0}^{-}\|$. To facilitate the arguments, let us first suppose that $D_{11} = 0$ (so that $\mathbf{D}_{\mathrm{a}0}' = 0$). Since $\|w\| \geq \|\mathbf{J}_0' w\|$ whenever $w \in (L_\infty[0,h))^{n_w}$ and since $\mathbf{J}_0' w$ is a constant function, it follows readily from (3.35) that the input of $\mathcal{P}_{NM\mathrm{i}0}^{-}$ may always be assumed to be a constant function when we evaluate $\|\mathcal{P}_{NM\mathrm{i}0}^{-}\|$. By (3.33), the output of $\mathcal{P}_{NM\mathrm{i}0}^{-}$ is also a constant function determined by the matrix C_0. Hence, $\|\mathcal{P}_{NM\mathrm{i}0}^{-}\|$ coincides with the ∞-norm of the matrix obtained by replacing the operators $\mathbf{B}_{\mathrm{i}0}'$ and $\mathbf{M}_{\mathrm{a}0}'$ with B_{0d}'

and C_0, respectively, where B'_{0d} is the matrix representing an 'equivalent operation' in (3.32) for constant functions w:

$$B'_{0d} := \int_0^{h'} \exp(A(h' - \theta'))B_1 d\theta' \tag{3.55}$$

Combining the above arguments leads to the following prelude to the first main result in this chapter.

Theorem 3.1 The inequality

$$\|P^-_{NMi0}\| \leq \|\mathcal{P}^-_N\| \leq \|P^-_{NMi0}\| + \frac{K_{Mi0}}{M} \tag{3.56}$$

holds, where

$$P^-_{NMi0} := \begin{bmatrix} \overline{D_{11}} & \overline{C_0}\Delta^0_M\overline{B'_{0d}} & \overline{C_0}\mathcal{A}_{M0}\overline{B'_{0d}} & \cdots & \overline{C_0}\mathcal{A}_{MN}\overline{B'_{0d}} \end{bmatrix} \tag{3.57}$$

Remark 3.9 The above arguments under the assumption $D_{11} = 0$ immediately lead to (3.57) without the extra entry $\overline{D_{11}}$, but it is not hard to see that dealing with $D_{11} \neq 0$ and thus the corresponding multiplication operator $\overline{\mathbf{D}'_{\mathrm{a}0}}$ in (3.35) simply leads to introducing this extra entry by the property of $L_\infty[0, h')$; the treatment of D_{11} is essentially the same as that in [66]. Similarly for Theorems 3.2, 3.3 and 3.4

We can summarize the arguments in this subsection as follows: Computing the approximate value $\|\mathcal{P}^-_N\|$ for the L_∞-induced norm can be achieved by piecewise constant approximation through the fast-lifted treatment, its upper and lower bounds can be computed exactly through matrix manipulations, and the gap between these bounds tends to 0 at the rate of $1/M$ (since K_{Mi0} has a uniform upper bound $K^U_{\mathrm{i}0}$ given in (3.53)).

Remark 3.10 We would like to note that although the use of P^-_{NMi0} (and thus the central part of the computation method in this subsection) has something in common with [3],[28], [66] (and would essentially recover the computations in these studies if we were to consider only the limit of $\|P^-_{NMi0}\|$ for $N \to \infty$), the overall method with piecewise constant approximation here is completely different from that in these existing studies. This is because the present paper provides readily computable upper and lower bounds of the L_∞-induced norm (aside from the extension to piecewise linear approximation discussed in the following subsection), while the existing studies only show the convergence rate without providing any readily computable upper and lower bounds.

3.3.2 Piecewise Linear Approximation Scheme

This subsection considers computing upper and lower bounds of $\|\mathcal{P}_N^-\|$ through piecewise linear approximation scheme of $\mathcal{P}_N^-$.

A key idea in this direction is to use the 'linearizing' operator $\mathbf{J}_1'$ defined as (2.29). This specific operator satisfies $\mathbf{J}_1' w = w$ for any linear function w (among other technically important properties). The rationale for taking such specific structure of $\mathbf{J}_1'$ in sampled-data problem is essentially the same as that of continuous-time problem given in Subsection 2.9.3. By using the operator $\mathbf{J}_1'$, we introduce the operator $\mathbf{B}_{\mathrm{i}1}' := \mathbf{B}_1'\mathbf{J}_1'$, i.e.,

$$\mathbf{B}_{\mathrm{i}1}' w = \int_0^{h'} \exp(A(h'-\theta'))B_1 \cdot (\mathbf{J}_1' w)(\theta') d\theta' \tag{3.58}$$

This is equivalent to restricting the input of $\mathbf{B}_1'$ to linear functions. We further introduce the operators $\mathbf{M}_{\mathrm{a}1}'$ and $\mathbf{D}_{\mathrm{a}1}'$ defined respectively by

$$\left(\mathbf{M}_{\mathrm{a}1}' \begin{bmatrix} x \\ u \end{bmatrix}\right)(\theta') = C_0(I + A_2\theta') \begin{bmatrix} x \\ u \end{bmatrix} \quad (0 \le \theta' < h') \tag{3.59}$$

$$(\mathbf{D}_{\mathrm{a}1}' w)(\theta') = C_1 B_1 \int_0^{\theta'} w(\tau') d\tau' + D_{11} w(\theta') \quad (0 \le \theta' < h') \tag{3.60}$$

Remark 3.11 The above two operators $\mathbf{M}_{\mathrm{a}1}'$ and $\mathbf{D}_{\mathrm{a}1}'$ will also be used in the kernel approximation approach with the piecewise linear approximation scheme in Subsection 3.4.2.

Introducing $\mathbf{M}_{\mathrm{a}1}'$ corresponds to the first-order approximation of the Taylor expansion of the output $C_0 \sum_{i=0}^{\infty} \frac{(A_2\theta')^i}{i!} \begin{bmatrix} x \\ u \end{bmatrix}$ of $\mathbf{M}_1'$ and thus its output is also a linear function. $\mathbf{D}_{\mathrm{a}1}'$ is introduced to approximate $\mathbf{D}_{11}'$, and its specific structure plays important roles in establishing Lemma 3.6 given later. We further note that, unlike $\mathbf{B}_{\mathrm{i}1}'$, introducing $\mathbf{D}_{\mathrm{a}1}'$ is not equivalent to restricting the input of $\mathbf{D}_{11}'$ to linear functions even when $D_{11} = 0$. It may be quite interesting to note that the compact operator $\mathbf{D}_{11}' - D_{11}$ is approximated by the infinite-rank but rather amenable integral operator $\mathbf{D}_{\mathrm{a}1}' - D_{11} = C_1 B_1 \int_0^{\theta'} \cdot \, d\tau$.

We are in a position to introduce the linear approximation $\mathcal{P}_{NM\mathrm{i}1}^-$ for $\mathcal{P}_{NM}^-$, by which we mean to replace $\mathbf{B}_1'$, $\mathbf{M}_1'$ and $\mathbf{D}_{11}'$ in (3.30) with $\mathbf{B}_{\mathrm{i}1}'$, $\mathbf{M}_{\mathrm{a}1}'$ and $\mathbf{D}_{\mathrm{a}1}'$, respectively:

$$\mathcal{P}_{NM\mathrm{i}1}^- = \left[\overline{\mathbf{M}_{\mathrm{a}1}'}\Delta_M^0\overline{\mathbf{B}_{\mathrm{i}1}'} + \overline{\mathbf{D}_{\mathrm{a}1}'} \quad \overline{\mathbf{M}_{\mathrm{a}1}'}\mathcal{A}_{M0}\overline{\mathbf{B}_{\mathrm{i}1}'} \quad \cdots \quad \overline{\mathbf{M}_{\mathrm{a}1}'}\mathcal{A}_{MN}\overline{\mathbf{B}_{\mathrm{i}1}'}\right] \tag{3.61}$$

This in turn defines piecewise linear approximation of $\mathcal{P}_N^-$. This subsection shows that $\|\mathcal{P}_{NM\mathrm{i}1}^-\|$ can be computed exactly and converges to $\|\mathcal{P}_N^-\|$ at the rate of $1/M^2$. The following three lemmas are important in establishing a more precise assertion.

Lemma 3.4 The following inequality holds.

$$\|\mathbf{B}_1' - \mathbf{B}_{\mathrm{i}1}'\| \leq \frac{h^3}{2M^3}\|A\|^2 \cdot \|A_d' B_1\| e^{\|A\|h/M} \tag{3.62}$$

Lemma 3.5 The following inequality holds.

$$\|\mathbf{M}_1' - \mathbf{M}_{\mathrm{a}1}'\| \leq \frac{h^2}{2M^2}\|C_0 A_2^2\| e^{\|A_2\|h/M} \tag{3.63}$$

Lemma 3.6 The inequality

$$\|(\overline{\mathbf{M}_1'}\Delta_M^0\overline{\mathbf{B}_1'} + \overline{\mathbf{D}_{11}'}) - (\overline{\mathbf{M}_{\mathrm{a}1}'}\Delta_M^0\overline{\mathbf{B}_{\mathrm{i}1}'} + \overline{\mathbf{D}_{\mathrm{a}1}'})\| \leq \frac{K_{M\mathcal{D}\mathrm{i}1}}{M^2} \tag{3.64}$$

holds, where

$$\begin{aligned} K_{M\mathcal{D}\mathrm{i}1} := \frac{1}{2}\|C_1\| \cdot \|A\| \cdot \|B_1\| h^2 e^{\|A\|h/M} + \frac{1}{2}\|A\|^2 \cdot \|B_1\| e^{\|A\|h/M}\frac{h^3}{M} \\ \cdot \sum_{k=0}^{M-2}\left\{\left\|C_1(A_d')^k\right\| e^{\|A\|h/M} + \max_{0\leq\theta'<h'}\left\|C_1(I+A\theta')(A_d')^{k+1}\right\|\right\} \end{aligned} \tag{3.65}$$

Furthermore, $K_{M\mathcal{D}\mathrm{i}1}$ has a uniform upper bound with respect to M given by

$$K_{\mathcal{D}\mathrm{i}1}^U := \frac{1}{2}\|C_1\| \cdot \|A\| \cdot \|B_1\| h^2 e^{\|A\|h} + \frac{1}{2}\|C_1\| \cdot \|A\|^2 \cdot \|B_1\| \cdot h^3 e^{\|A\|h}(2 + \|A\|h) \tag{3.66}$$

The proof of Lemma 3.4 is omitted since it is essentially the same as that of Lemma 2.3 in Chapter 2, while the proofs of Lemmas 3.5 and 3.6 are given in Subsection 3.9.1. From Lemmas 3.4 and 3.5, we easily have the following result.

Proposition 3.3 The inequality

$$\|\overline{\mathbf{M}_1'}\mathcal{A}_{Mj}\overline{\mathbf{B}_1'} - \overline{\mathbf{M}_{\mathrm{a}1}'}\mathcal{A}_{Mj}\overline{\mathbf{B}_{\mathrm{i}1}'}\| \leq \frac{K_{Mj11}}{M^2} \tag{3.67}$$

holds for $j = 0, \cdots, N$, where

$$\begin{aligned} K_{Mj11} = \frac{1}{2}e^{\|A\|h/M}\|\mathcal{A}_{Mj}\|\frac{h^3}{M} \cdot \Big\{ \max_{\theta'\in\{0,h'\}} \|C_0\,(I + A_2\theta')\|\,\|A\|^2 \cdot \|A_d' B_1\| \\ + \left\|C_0 A_2^2\right\| e^{\|A_2\|h/M}\|B_1\|\Big\} \end{aligned} \tag{3.68}$$

Furthermore, $K_{Mj\mathrm{i}1}$ has a uniform upper bound with respect to M and j defined as

$$K_{\mathcal{CAB}\mathrm{i}1}^U := \frac{1}{2}h^3 e^{\|A\|h}\|B_1\|K_* \cdot \left\{ (\|C_0\| + \|C_0 A_2\|\,h)\,\|A\|^2 e^{\|A\|h} + \left\|C_0 A_2^2\right\| e^{\|A_2\|h}\right\} \tag{3.69}$$

where K_* is given by (3.44).

Proof. Applying triangular inequality to (3.67) leads to

$$\begin{aligned}&\|\overline{\mathbf{M}'_1}\mathcal{A}_{Mj}\overline{\mathbf{B}'_1} - \overline{\mathbf{M}'_{\mathrm{a}1}}\mathcal{A}_{Mj}\overline{\mathbf{B}'_{\mathrm{i}1}}\| \leq \|(\overline{\mathbf{M}'_1} - \overline{\mathbf{M}'_{\mathrm{a}1}})\mathcal{A}_{Mj}\overline{\mathbf{B}'_1}\| + \|\overline{\mathbf{M}'_{\mathrm{a}1}}\mathcal{A}_{Mj}(\overline{\mathbf{B}'_1} - \overline{\mathbf{B}'_{\mathrm{i}1}})\| \\ \leq &\|(\overline{\mathbf{M}'_1} - \overline{\mathbf{M}'_{\mathrm{a}1}})\| \cdot \|\mathcal{A}_{Mj}\| \cdot \|\overline{\mathbf{B}'_1}\| + \|\overline{\mathbf{M}'_{\mathrm{a}1}}\| \cdot \|\mathcal{A}_{Mj}\| \cdot \|\overline{\mathbf{B}'_1} - \overline{\mathbf{B}'_{\mathrm{i}1}}\| \end{aligned} \tag{3.70}$$

By noting that

$$\|\mathbf{M}'_{\mathrm{a}1}\| = \max_{\theta' \in \{0,h'\}} \|C_0(I + A_2\theta')\| \tag{3.71}$$

with (3.46), (3.62) and (3.63), we easily have (3.67). The second assertion could be proved readily if we note (3.49), (3.50) and

$$\max_{\theta' \in \{0,h'\}} \|C_0(I + A_2\theta')\| \leq \|C_0\| + \|C_0A_2\|h \tag{3.72}$$

This completes the proof. Q.E.D.

From Proposition 3.3 and Lemma 3.6, we readily obtain the following result.

Proposition 3.4 The inequality

$$\|\mathcal{P}^-_{NM} - \mathcal{P}^-_{NM1}\| \leq \frac{K_{M\mathrm{i}1}}{M^2} \tag{3.73}$$

holds, where

$$K_{M\mathrm{i}1} := K_{M\mathcal{D}\mathrm{i}1} + \sum_{j=0}^{N} K_{Mj\mathrm{i}1} \tag{3.74}$$

In addition, $K_{M\mathrm{i}1}$ has a uniform upper bound with respect to M given by

$$K^U_{\mathrm{i}1} := K^U_{\mathcal{D}\mathrm{i}1} + (N+1) \cdot K^U_{\mathcal{CAB}\mathrm{i}1} \tag{3.75}$$

With an application of the triangle inequality to (3.73) in mind, we now turn to giving a method for (exactly) computing the $L_\infty[0, h')$-induced norm

$$\|\mathcal{P}^-_{NM\mathrm{i}1}\| = \sup_{\|w\| \leq 1} \|(\mathcal{P}^-_{NM\mathrm{i}1}w)(\cdot)\| \tag{3.76}$$

We note on the right hand side of (3.76) that

$$(\mathcal{P}^-_{NM\mathrm{i}1}w)(\theta') = \sum_{j=0}^{N} \left(\overline{\mathbf{M}'_{\mathrm{a}1}}\mathcal{A}_{Mj}\overline{\mathbf{B}'_{\mathrm{i}1}}w_{j+1}\right)(\theta') + \left((\overline{\mathbf{M}'_{\mathrm{a}1}}\Delta^0_M\overline{\mathbf{B}'_{\mathrm{i}1}} + \overline{\mathbf{D}'_{\mathrm{a}1}})w_0\right)(\theta') \tag{3.77}$$

where $w =: [w_0^T, \cdots, w_{N+1}^T]^T$.

We first consider the matrix function $\left((\overline{\mathbf{M}'_{\mathrm{a1}}}\Delta_M^0\overline{\mathbf{B}'_{\mathrm{i1}}} + \overline{\mathbf{D}'_{\mathrm{a1}}})w_0\right)(\theta')$ in (3.77). Let us further introduce the partitioned notation $w_j =: [(w_j^{(1)})^T, \cdots, (w_j^{(M)})^T]^T$ $(j = 0, \cdots, N+1)$ by noting that w_j is in fact a fast-lifting representation of a signal on $[0, h)$. Then, for every $p \in \{1, \cdots, M\}$, $w_0^{(p)}$ appears only on the ith block row in $\overline{\mathbf{D}'_{\mathrm{a1}}}w_0$ while it appears only on the kth block rows in $\overline{\mathbf{M}'_{\mathrm{a1}}}\Delta_M^0\overline{\mathbf{B}'_{\mathrm{i1}}}w_0$ with $k > p$; this is because of the strict block lower triangular structure of Δ_M^0. We further note in (3.76) that

$$\|(\mathcal{P}^-_{NM\mathrm{i}1}w)(\cdot)\| = \max_{p\in\{1,\cdots,M\}} \|(\mathcal{P}^-_{NM\mathrm{i}1}w)_p(\cdot)\| \tag{3.78}$$

where $(\cdot)_p$ denotes the pth block row of $(\cdot)$. This implies that the block rows mentioned above can be handled one by one, and thus when $(\overline{\mathbf{M}'_{\mathrm{a1}}}\Delta_M^0\overline{\mathbf{B}'_{\mathrm{i1}}} + \overline{\mathbf{D}'_{\mathrm{a1}}})w_0$ is expanded into $\overline{\mathbf{D}'_{\mathrm{a1}}}w_0$ and $\overline{\mathbf{M}'_{\mathrm{a1}}}\Delta_M^0\overline{\mathbf{B}'_{\mathrm{i1}}}w_0$, the input w_0 in the first term may be handled independently of that in the second term (i.e., they may be regarded to be different functions), as long as we further take $\sup\limits_{\|w\|\le 1}$ as in (3.76). This is equivalent to saying that $\mathcal{P}^-_{NM\mathrm{i}1}$ may be redefined as

$$\mathcal{P}^-_{NM\mathrm{i}1} = \left[\overline{\mathbf{D}'_{\mathrm{a1}}} \quad \overline{\mathbf{M}'_{\mathrm{a1}}}\Delta_M^0\overline{\mathbf{B}'_{\mathrm{i1}}} \quad \overline{\mathbf{M}'_{\mathrm{a1}}}\mathcal{A}_{M0}\overline{\mathbf{B}'_{\mathrm{i1}}} \quad \cdots \quad \overline{\mathbf{M}'_{\mathrm{a1}}}\mathcal{A}_{MN}\overline{\mathbf{B}'_{\mathrm{i1}}}\right] \tag{3.79}$$

without changing $\|\mathcal{P}^-_{NM\mathrm{i}1}\|$. Noting the definition of $\mathbf{D}'_{\mathrm{a1}}$ in (3.60), let us further introduce the integral operator $\mathbf{D}'_{\mathrm{a10}} := \mathbf{D}'_{\mathrm{a1}} - D_{11}$. Then, it follows again from the property of $L_\infty[0, h')$ that $\mathcal{P}^-_{NM\mathrm{i}1}$ may be redefined further, without changing its norm, as

$$\mathcal{P}^-_{NM\mathrm{i}1} = \left[\overline{D_{11}} \quad \overline{\mathbf{D}'_{\mathrm{a10}}} \quad \overline{\mathbf{M}'_{\mathrm{a1}}}\Delta_M^0\overline{\mathbf{B}'_{\mathrm{i1}}} \quad \overline{\mathbf{M}'_{\mathrm{a1}}}\mathcal{A}_{M0}\overline{\mathbf{B}'_{\mathrm{i1}}} \quad \cdots \quad \overline{\mathbf{M}'_{\mathrm{a1}}}\mathcal{A}_{MN}\overline{\mathbf{B}'_{\mathrm{i1}}}\right] \tag{3.80}$$

Throughout the rest of this chapter, we mean (3.80) by $\mathcal{P}^-_{NM\mathrm{i}1}$.

Summarizing the above arguments, we may replace the function $((\overline{\mathbf{M}'_{\mathrm{a1}}}\Delta_M^0\overline{\mathbf{B}'_{\mathrm{i1}}} + \overline{\mathbf{D}'_{\mathrm{a1}}})w_0)(\theta')$ in (3.77) by $\overline{D_{11}}w_{00} + \overline{\mathbf{D}'_{\mathrm{a10}}}w_{01} + \overline{\mathbf{M}'_{\mathrm{a1}}}\Delta_M^0\overline{\mathbf{B}'_{\mathrm{i1}}}w_{02}$, where $\|w_{0i}\| \le 1$ $(i = 0, 1, 2)$, as long as we consider evaluating $\|\mathcal{P}^-_{NM\mathrm{i}1}\|$. Note that the third term is a linear function by the definition of $\mathbf{M}'_{\mathrm{a1}}$, as well as all the terms in (3.77) except the last. For simplicity, let us suppose $D_{11} = 0$ for a while, even though we will eventually deal with the case of $D_{11} \neq 0$. Then, all the terms of (3.77) are linear functions except $\overline{\mathbf{D}'_{\mathrm{a10}}}w_{01}$, where $\mathbf{D}'_{\mathrm{a10}}$ is simply an integral operator. By noting the signs of each elements in each rows of the matrix C_1B_1 involved in $\mathbf{D}'_{\mathrm{a10}}$, we could choose the each entry of w_{01} as 1 or -1, and this procedure leads to that the output of the operator $\mathbf{D}'_{\mathrm{a10}}$ becomes a linear function with its slope to be maximized or minimized. Then, the output of $\mathcal{P}^-_{NM\mathrm{i}1}$ for an arbitrary w is always contained in the sector determined by the maximum and minimum slopes. Here, the $L_\infty[0, h')$ norm of any function in the sector is bounded by the maximum of the following two values: one is the absolute

value of the sector at $\theta' = 0$ and the other is the maximum of the two absolute values of the sector at h'. Thus, (3.76) reduces to

$$\|\mathcal{P}^-_{NMi1}\| = \sup_{\|w\|\le 1} \max_{\theta'=0,h'} \|(\mathcal{P}^-_{NMi1}w)(\theta')\| = \sup_{\|w\|\le 1} \left\| \begin{bmatrix} (\mathcal{P}^-_{NMi1}w)(0) \\ (\mathcal{P}^-_{NMi1}w)(h') \end{bmatrix} \right\| \tag{3.81}$$

where w is redefined as $[w_{00}^T, w_{01}^T, w_{02}^T, w_1^T, \cdots, w_{N+1}^T]^T$, and $(\mathcal{P}^-_{NMi1}w)(h')$ is defined by continuity of a linear function.

By using the matrices $V_{pi}^{[0]}$, $V_{pi}^{[h']}$, $T_{jpi}^{[0]}$ and $T_{jpi}^{[h']}$ $(p = 1, \cdots, M; j = 0, \cdots, N)$ given in Subsection 3.9.2, we can obtain the following prelude to the second main result. In particular, it gives an exact computation method for the $L_\infty[0, h')$-induced norm $\|\mathcal{P}^-_{NMi1}\|$ given by (3.81). See Subsection 3.9.2 for the arguments leading to this result.

Theorem 3.2 The inequality

$$\|\mathcal{P}^-_{NMi1}\| - \frac{K_{Mi1}}{M^2} \le \|\mathcal{P}^-_N\| \le \|\mathcal{P}^-_{NMi1}\| + \frac{K_{Mi1}}{M^2} \tag{3.82}$$

holds. Furthermore, $\|\mathcal{P}^-_{NMi1}\|$ coincides with the ∞-norm of the finite-dimensional matrix

$$P^-_{NMi1} := \begin{bmatrix} U_{\mathrm{i}}^{[0]} & T_{0\mathrm{i}}^{[0]} & \cdots & T_{N\mathrm{i}}^{[0]} \\ U_{\mathrm{i}}^{[h']} & T_{0\mathrm{i}}^{[h']} & \cdots & T_{N\mathrm{i}}^{[h']} \end{bmatrix} \tag{3.83}$$

where

$$U_{\mathrm{i}}^{[0]} := \begin{bmatrix} \overline{D_{11}} & 0 & V_{\mathrm{i}}^{[0]} \end{bmatrix} \tag{3.84}$$

$$U_{\mathrm{i}}^{[h']} := \begin{bmatrix} \overline{D_{11}} & \overline{C_1 B_1} h' & V_{\mathrm{i}}^{[h']} \end{bmatrix} \tag{3.85}$$

$$T_{j\mathrm{i}}^{[0]} := \begin{bmatrix} T_{j1\mathrm{i}}^{[0]} & \cdots & T_{jM\mathrm{i}}^{[0]} \end{bmatrix} \quad (j = 0, \cdots, N) \tag{3.86}$$

$$T_{j\mathrm{i}}^{[h']} := \begin{bmatrix} T_{j1\mathrm{i}}^{[h']} & \cdots & T_{jM\mathrm{i}}^{[h']} \end{bmatrix} \quad (j = 0, \cdots, N) \tag{3.87}$$

with $V_{\mathrm{i}}^{[0]} := [V_{1\mathrm{i}}^{[0]} \ \cdots \ V_{M\mathrm{i}}^{[0]}]$ and $V_{\mathrm{i}}^{[h']} := [V_{1\mathrm{i}}^{[h']} \ \cdots \ V_{M\mathrm{i}}^{[h']}]$.

Remark 3.12 In (3.83) (or more precisely in (3.84) and (3.85)), we have recovered the general case with $D_{11} \neq 0$, which can be validated as in Remark 3.9.

To summarize, we have shown in this subsection that similar arguments to the preceding subsection can be developed by piecewise linear approximation through the fast-lifted treatment, in which the gap between the upper and lower bounds of $\|\mathcal{P}^-_N\|$ tends to 0 at the rate of $1/M^2$.

3.4 Kernel Approximation Approach

This section provides methods for computing the L_∞-induced norm of sampled-data systems by using the idea of the kernel approximation approach discussed in Chapter 2. This approach also employs the fast-lifted representation $\mathcal{P}^-_{NM}$ and leads to piecewise constant approximation or piecewise linear approximation scheme of the operators $\mathbf{B}_1$, $\mathbf{M}_1$ and $\mathbf{D}_{11}$ (i.e., constant or linear approximation of the operators $\mathbf{B}'_1$, $\mathbf{M}'_1$ and $\mathbf{D}'_{11}$ involved in $\mathcal{P}^-_{NM}$). More precisely, the approximation operators $\mathbf{M}'_{\mathrm{a}0}$, $\mathbf{D}'_{\mathrm{a}0}$, $\mathbf{M}'_{\mathrm{a}1}$ and $\mathbf{D}'_{\mathrm{a}1}$ are also used in this section, and the only difference between the kernel approximation approach and the input approximation approach is the way the operator $\mathbf{B}'_1$ is approximated. This is because the operators $\mathbf{M}'_1$ and $\mathbf{D}'_{11}$ are comparatively simple and thus there seem to be little variations for their reasonable approximations. Hence, the main contribution of the present section over the results in Section 3.3 is a new approximation approach of the operator $\mathbf{B}'_1$. As it turns out, the new method for approximating the operator $\mathbf{B}'_1$ in this section leads to improved upper and lower bounds for the L_∞-induced norm of sampled-data systems.

3.4.1 Piecewise Constant Approximation Scheme

In this subsection, we suppose that N is given and aim at computing upper and lower bounds of $\|\mathcal{P}^-_N\|$ through piecewise constant approximation scheme of $\mathcal{P}^-_N$.

The main idea in this subsection is approximation of the kernel function $\exp(A(h'-\theta'))B_1$ of the operator $\mathbf{B}'_1$. In other words, we introduce operator $\mathbf{B}'_{\mathrm{k}0}$ described by

$$\mathbf{B}'_{\mathrm{k}0}w = \int_0^{h'} A'_d B_1 w(\theta')d\theta' \tag{3.88}$$

This approximation corresponds to the zero-order approximation of the kernel function $\exp(A(h'-\theta'))B_1 = A'_d \sum_{i=0}^{\infty} \frac{(-A\theta')^i}{i!} B_1$ of the operator $\mathbf{B}'_1$. Furthermore, we consider the operators $\mathbf{M}'_{\mathrm{a}0}$ and $\mathbf{D}'_{\mathrm{a}0}$ defined as (3.33) and (3.34), respectively.

We are in a position to introduce the constant approximation $\mathcal{P}^-_{NM\mathrm{k}0}$ for $\mathcal{P}^-_{NM}$, by which we mean to replace $\mathbf{B}'_1$, $\mathbf{M}'_1$ and $\mathbf{D}'_{11}$ in (3.30) with $\mathbf{B}'_{\mathrm{k}0}$, $\mathbf{M}'_{\mathrm{a}0}$ and $\mathbf{D}'_{\mathrm{a}0}$, respectively:

$$\mathcal{P}^-_{NM\mathrm{k}0} = \left[\overline{\mathbf{M}'_{\mathrm{a}0}}\Delta^0_M\overline{\mathbf{B}'_{\mathrm{k}0}} + \overline{\mathbf{D}'_{\mathrm{a}0}} \quad \overline{\mathbf{M}'_{\mathrm{a}0}}\mathcal{A}_{M0}\overline{\mathbf{B}'_{\mathrm{k}0}} \quad \cdots \quad \overline{\mathbf{M}'_{\mathrm{a}0}}\mathcal{A}_{MN}\overline{\mathbf{B}'_{\mathrm{k}0}}\right] \tag{3.89}$$

This corresponds to piecewise constant approximation of $\mathcal{P}^-_N$. In this subsection, we show that $\|\mathcal{P}^-_{NM\mathrm{k}0}\|$ can be computed exactly and converges, as $M \to \infty$, to $\|\mathcal{P}^-_N\|$ at the rate of $1/M$. To establish a more precise assertion associated with upper and lower bounds of $\|\mathcal{P}^-_N\|$, the following two lemmas are significant.

Lemma 3.7 The following inequality holds.

$$\|\mathbf{B}_1' - \mathbf{B}_{\mathrm{k0}}'\| \leq \frac{h^2}{2M^2}\|A\| \cdot \|A_d' B_1\| e^{\|A\|h/M} \tag{3.90}$$

Lemma 3.8 The inequality

$$\|(\overline{\mathbf{M}_1'}\Delta_M^0\overline{\mathbf{B}_1'} + \overline{\mathbf{D}_{11}'}) - (\overline{\mathbf{M}_{\mathrm{a0}}'}\Delta_M^0\overline{\mathbf{B}_{\mathrm{k0}}'} + \overline{\mathbf{D}_{\mathrm{a0}}'})\| \leq \frac{K_{M\mathcal{D}\mathrm{k0}}}{M} \tag{3.91}$$

holds, where

$$K_{M\mathcal{D}\mathrm{k0}} := h\|C_1\| \cdot \|B_1\| e^{\|A\|h/M} + \frac{h^2}{M}\|A\| \cdot \|B_1\| e^{\|A\|h/M} \cdot \sum_{k=0}^{M-2}\left(\frac{1}{2}\|C_1(A_d')^{k+1}\| + \|C_1(A_d')^k\| e^{\|A\|h/M}\right) \tag{3.92}$$

Furthermore, $K_{M\mathcal{D}\mathrm{k0}}$ has a uniform upper bound with respect to M given by

$$K_{\mathcal{D}\mathrm{k0}}^U := h\|C_1\| \cdot \|B_1\| e^{\|A\|h} + \frac{3h^2}{2}\|C_1\| \cdot \|A\| \cdot \|B_1\| e^{\|A\|h} \tag{3.93}$$

Remark 3.13 From the comparisons between Lemmas 3.1 and 3.7 with Lemmas 3.3 and 3.8, we can easily see that the kernel approximation approach leads to smaller error bounds than those in the input approximation approach under the piecewise constant approximation scheme. This is very important in deriving improved upper and lower bounds of the L_∞-induced norm of sampled-data systems.

The proofs of these lemmas are omitted because they are essentially the same as those of the input approximation approach. From Lemmas 3.2 and 3.7, we readily have the following result.

Proposition 3.5 The inequality

$$\|\overline{\mathbf{M}_1'}\mathcal{A}_{Mj}\overline{\mathbf{B}_1'} - \overline{\mathbf{M}_{\mathrm{a0}}'}\mathcal{A}_{Mj}\overline{\mathbf{B}_{\mathrm{k0}}'}\| \leq \frac{K_{Mj\mathrm{k0}}}{M} \tag{3.94}$$

holds for $j = 0, \cdots, N$, where

$$K_{Mj\mathrm{k0}} := e^{\|A\|h/M}\|\mathcal{A}_{Mj}\|\frac{h^2}{M} \cdot \left\{\|C_0A_2\| \cdot \|B_1\| e^{\|A_2\|h/M} + \frac{1}{2}\|C_0\| \cdot \|A\| \cdot \|A_d'B_1\|\right\} \tag{3.95}$$

Furthermore, $K_{Mj\mathrm{k0}}$ has a uniform upper bound with respect to M and j given by

$$K_{\mathcal{CAB}\mathrm{k0}}^U := h^2 e^{\|A\|h} \cdot \|B_1\| \cdot K_* \cdot \left\{\|C_0A_2\| e^{\|A_2\|h} + \frac{1}{2}\|C_0\| \cdot \|A\| e^{\|A\|h}\right\} \tag{3.96}$$

where K_* is given by (3.44).

The proof of Proposition 3.5 is omitted because it is essentially the same as that of Proposition 3.2. From Proposition 3.5 and Lemma 3.8, we easily have the following result.

Proposition 3.6 The inequality

$$\|\mathcal{P}^-_{NM} - \mathcal{P}^-_{NM\mathrm{k}0}\| \leq \frac{K_{M\mathrm{k}0}}{M} \tag{3.97}$$

holds, where

$$K_{M\mathrm{k}0} := K_{M\mathcal{D}\mathrm{k}0} + \sum_{j=1}^{N} K_{Mj\mathrm{k}0} \tag{3.98}$$

In addition, $K_{M\mathrm{k}0}$ has a uniform upper bound with respect to M given by

$$K^U_{\mathrm{k}0} := K^U_{\mathcal{D}\mathrm{k}0} + (N+1) \cdot K^U_{\mathcal{CAB}\mathrm{k}0} \tag{3.99}$$

To evaluate $\|\mathcal{P}^-_N\| = \|\mathcal{P}^-_{NM}\|$ through (3.97) and the triangle inequality, we give a method for exactly computing $\|\mathcal{P}_{NM\mathrm{k}0}\|$. To facilitate the arguments, we first suppose that $D_{11} = 0$ (so that $\mathbf{D}'_{\mathrm{a}0} = 0$) for a while, even though we will eventually deal with the case of $D_{11} \neq 0$. It readily follows from (3.89) that the output of $\mathcal{P}^-_{NM\mathrm{k}0}$ is a constant function determined by the matrix C_0. Furthermore, the input of $\mathcal{P}^-_{NM\mathrm{k}0}$ may always be assumed to be a constant function when we evaluate $\|\mathcal{P}_{NM\mathrm{k}0}\|$. This is because we can see easily from (3.88) that the following relation holds for a constant function $w_d \in (L_\infty[0, h')^{n_w}$.

$$\{\mathbf{B}'_{\mathrm{k}0} w_d \mid \|w_d\| \leq 1\} = \{\mathbf{B}'_{\mathrm{k}0} w \mid \|w\| \leq 1\} \tag{3.100}$$

Hence, $\|\mathcal{P}^-_{NM\mathrm{k}0}\|$ coincides with the ∞-norm of the matrix obtained by replacing the operators $\mathbf{B}'_{\mathrm{k}0}$ and $\mathbf{M}'_{\mathrm{a}0}$ with $A'_d B_1 h'$ and C_0, respectively. Combining the above arguments leads to the following result.

Theorem 3.3 The inequality

$$\|P^-_{NM\mathrm{k}0}\| - \frac{K_{M\mathrm{k}0}}{M} \leq \|\mathcal{P}^-_N\| \leq \|P^-_{NM\mathrm{k}0}\| + \frac{K_{M\mathrm{k}0}}{M} \tag{3.101}$$

holds, where

$$P^-_{NM\mathrm{k}0} := \begin{bmatrix} \overline{D_{11}} & \overline{C_0}\Delta^0_M \overline{A'_d B_1 h'} & \overline{C_0}\mathcal{A}_{M0}\overline{A'_d B_1 h'} & \cdots & \overline{C_0}\mathcal{A}_{MN}\overline{A'_d B_1 h'} \end{bmatrix} \tag{3.102}$$

Remark 3.14 As mentioned in Remark 3.9, we have recovered in (3.102) the general case with $D_{11} \neq 0$.

To summarize, we have studied computing the approximate value $\|\mathcal{P}_N^-\|$ for the L_∞-induced norm by piecewise constant approximation through the fast-lifted treatment, in which its upper and lower bounds can be computed exactly through matrix manipulations, and the gap between these bounds tends to 0 at the rate of $1/M$ (because $K_{M\mathrm{k}0}$ has a uniform upper bound $K_{\mathrm{k}0}^U$ given in (3.99)) .

3.4.2 Piecewise Linear Approximation Scheme

This subsection studies computing upper and lower bounds of $\|\mathcal{P}_N^-\|$ through piecewise linear approximation scheme of $\mathcal{P}_N^-$.

We introduce the operator $\mathbf{B}'_{\mathrm{k}1} : (L_\infty[0,h))^{n_w} \to \mathbb{R}^n_\infty$ defined as

$$\mathbf{B}'_{\mathrm{k}1} w = \int_0^{h'} A'_d(I - A\theta')B_1 w(\theta')d\theta' \tag{3.103}$$

Introducing the operator $\mathbf{B}'_{\mathrm{k}1}$ corresponds to the first-order approximation of the kernel function $A'_d \sum_{i=0}^{\infty} \frac{(-A\theta')^i}{i!} B_1$ of the operator $\mathbf{B}'_1$.

We next consider the operator $\mathcal{P}^-_{NM\mathrm{k}1}$ obtained by replacing $\mathbf{B}'_1$, $\mathbf{M}'_1$ and $\mathbf{D}'_{11}$ with $\mathbf{B}'_{\mathrm{k}1}$, $\mathbf{M}'_{\mathrm{a}1}$ and $\mathbf{D}'_{\mathrm{a}1}$, respectively, in (3.30). In other words, we define

$$\mathcal{P}^-_{NM\mathrm{k}1} = \left[\overline{\mathbf{M}'_{\mathrm{a}1}}\Delta^0_M\overline{\mathbf{B}'_{\mathrm{k}1}} + \overline{\mathbf{D}'_{\mathrm{a}1}} \quad \overline{\mathbf{M}'_{\mathrm{a}1}}\mathcal{A}_{M0}\overline{\mathbf{B}'_{\mathrm{k}1}} \quad \cdots \quad \overline{\mathbf{M}'_{\mathrm{a}1}}\mathcal{A}_{MN}\overline{\mathbf{B}'_{\mathrm{k}1}}\right] \tag{3.104}$$

This subsection shows that $\|\mathcal{P}^-_{NM\mathrm{k}1}\|$ can be computed exactly and converges to $\|\mathcal{P}_N^-\|$ at the rate of $1/M^2$. The following two lemmas are important in establishing a more precise assertion.

Lemma 3.9 The following inequality holds.

$$\|\mathbf{B}'_1 - \mathbf{B}'_{\mathrm{k}1}\| \le \frac{h^3}{6M^3}\|A\|^2 \cdot \|A'_d B_1\| e^{\|A\|h/M} \tag{3.105}$$

Lemma 3.10 The inequality

$$\|\overline{\mathbf{M}'_1}\Delta^0_M\overline{\mathbf{B}'_1} + \overline{\mathbf{D}'_{11}} - (\overline{\mathbf{M}'_{\mathrm{a}1}}\Delta^0_M\overline{\mathbf{B}'_{\mathrm{k}1}} + \overline{\mathbf{D}'_{\mathrm{a}1}})\| \le \frac{K_{M\mathcal{D}\mathrm{k}1}}{M^2} \tag{3.106}$$

holds with $K_{M\mathcal{D}\mathrm{k}1}$ defined as

$$\begin{aligned} K_{M\mathcal{D}\mathrm{k}1} := \frac{1}{2}\|C_1\| \cdot \|A\| \cdot \|B_1\| h^2 e^{\|A\|h/M} + \frac{1}{2}\|A\|^2 \cdot \|B_1\| e^{\|A\|h/M}\frac{h^3}{M} \\ \cdot \sum_{i=0}^{M-2}\left\{\|C_1(A'_d)^i\| e^{\|A\|h/M} + \frac{1}{3}\max_{\theta'\in\{0,h'\}}\|C_1(I + A\theta')(A'_d)^{i+1}\|\right\} \end{aligned} \tag{3.107}$$

Furthermore, $K_{M\mathcal{D}\mathrm{k1}}$ has a uniform upper bound with respect to M given by

$$K^U_{\mathcal{D}\mathrm{k1}} := \frac{1}{2}\|C_1\| \cdot \|A\| \cdot \|B_1\| h^2 e^{\|A\|h} + \frac{1}{2}\|C_1\| \cdot \|A\|^2 \cdot \|B_1\| \cdot h^3 e^{\|A\|h} \left(\frac{4 + \|A\|h}{3}\right) \tag{3.108}$$

Remark 3.15 From the comparisons between Lemmas 3.4 and 3.9 with Lemmas 3.6 and 3.10, it readily follows that the kernel approximation approach leads to smaller error bounds than those in the input approximation approach under the piecewise linear approximation scheme.

The proofs of these lemmas are omitted. From Lemmas 3.5 and 3.9, we easily have the following result.

Proposition 3.7 The inequality

$$\|\overline{\mathbf{M}'_1}\mathcal{A}_{Mj}\overline{\mathbf{B}'_1} - \overline{\mathbf{M}'_{\mathrm{a}1}}\mathcal{A}_{Mj}\overline{\mathbf{B}'_{\mathrm{k}1}}\| \leq \frac{K_{Mj\mathrm{k}1}}{M^2} \tag{3.109}$$

holds for $j = 0, \cdots, N$, where

$$K_{Mj\mathrm{k}1} = \frac{1}{2}e^{\|A\|h/M}\|\mathcal{A}_{Mj}\|\frac{h^3}{M} \cdot \left\{\frac{1}{3}\max_{\theta' \in \{0,h'\}} \|C_0(I + A_2\theta')\| \cdot \|A\|^2 \cdot \|A'_d B_1\| + \|C_0 A_2^2\| e^{\|A_2\|h/M}\|B_1\|\right\} \tag{3.110}$$

Furthermore, $K_{Mj\mathrm{k}1}$ has a uniform upper bound with respect to M and j defined as

$$K^U_{\mathcal{CAB}\mathrm{k}1} := \frac{1}{2}h^3 e^{\|A\|h}\|B_1\|K_* \cdot \left\{\frac{1}{3}\left(\|C_0\| + \|C_0 A_2\|h\right)\|A\|^2 e^{\|A\|h} + \|C_0 A_2^2\| e^{\|A_2\|h}\right\} \tag{3.111}$$

The proof of this proposition is omitted. From Proposition 3.7 and Lemma 3.10, we readily have the following result.

Proposition 3.8 The inequality

$$\|\mathcal{P}^-_{NM} - \mathcal{P}^-_{NM\mathrm{k}1}\| \leq \frac{K_{M\mathrm{k}1}}{M^2} \tag{3.112}$$

holds, where

$$K_{M\mathrm{k}1} := K_{M\mathcal{D}\mathrm{k}1} + \sum_{j=1}^{N} K_{Mj\mathrm{k}1} \tag{3.113}$$

In addition, $K_{M\mathrm{k}1}$ has a uniform upper bound with respect to M given by

$$K^U_{\mathrm{k}1} := K^U_{\mathcal{D}\mathrm{k}1} + (N+1)K^U_{\mathcal{CAB}\mathrm{k}1} \tag{3.114}$$

We next give a method for exactly computing $\|\mathcal{P}^-_{NM\mathrm{k}1}\|$. By essentially the same arguments to those in Subsection 3.3.2, which deal with the computation method for $\|\mathcal{P}^-_{NM\mathrm{i}1}\|$, we can have the following result.

Proposition 3.9 Let $V^{[0]}_{\mathrm{k}}$ be the matrix consisting of the $L_1[0,h')$ norm of each entry of the matrix linear function $\overline{C_0}\Delta^0_M\overline{A'_d(I-A\theta')B_1}$, while let $V^{[h']}_{\mathrm{k}}$ be the matrix constructed in the same way from $\overline{C_0(I+A_2h')}\Delta^0_M\overline{A'_d(I-A\theta')B_1}$. Furthermore, let $T^{[0]}_{j\mathrm{k}}$ $(j=0,\cdots,N)$ be the matrix consisting of the $L_1[0,h')$ norm of each entry of the matrix linear function $\overline{C_0}\mathcal{A}_{Mj}\overline{A'_d(I-A\theta')B_1}$, while let $T^{[h']}_{j\mathrm{k}}$ $(j=0,\cdots,N)$ be the matrix constructed in the same way from $\overline{C_0(I+A_2h')}\mathcal{A}_{Mj}\overline{A'_d(I-A\theta')B_1}$. Then, $\|\mathcal{P}^-_{NM\mathrm{k}1}\|$ coincides with the ∞-norm of the finite-dimensional matrix

$$P^-_{NM\mathrm{k}1} := \begin{bmatrix} U^{[0]}_{\mathrm{k}} & T^{[0]}_{0\mathrm{k}} & \cdots & T^{[0]}_{N\mathrm{k}} \\ U^{[h']}_{\mathrm{k}} & T^{[h']}_{0\mathrm{k}} & \cdots & T^{[h']}_{N\mathrm{k}} \end{bmatrix} \tag{3.115}$$

where

$$U^{[0]}_{\mathrm{k}} := \begin{bmatrix} \overline{D_{11}} & 0 & V^{[0]}_{\mathrm{k}} \end{bmatrix} \tag{3.116}$$

$$U^{[h']}_{\mathrm{k}} := \begin{bmatrix} \overline{D_{11}} & \overline{C_1B_1h'} & V^{[h']}_{\mathrm{k}} \end{bmatrix} \tag{3.117}$$

Combining Propositions 3.8 and 3.9 lead to the following result.

Theorem 3.4 The following inequality holds:

$$\|P^-_{NM\mathrm{k}1}\| - \frac{K_{M\mathrm{k}1}}{M^2} \le \|\mathcal{P}^-_N\| \le \|P^-_{NM\mathrm{k}1}\| + \frac{K_{M\mathrm{k}1}}{M^2} \tag{3.118}$$

This theorem implies that upper and lower bounds of $\|\mathcal{P}^-_N\|$ can be obtained through $\|P^-_{NM\mathrm{k}1}\|$ together with $K_{M\mathrm{k}1}/M^2$, and by taking the fast-lifting parameter M larger, the gap between those upper and lower bounds converges to 0 at no slower convergence rate than $1/M^2$ (because $K_{M\mathrm{k}1}$ has a uniform upper bound $K^U_{\mathrm{k}1}$).

3.5 Computation of the L_∞-Induced Norm and Guideline for Taking Approximation Parameters

This section gives a computation method for an upper bound of $\|\mathcal{P}^+_N\|$, which together with the arguments in the preceding sections leads to methods for computing upper and lower bounds of the L_∞-induced norm $\|\mathcal{P}\|$ of the sampled-data system Σ_{SD}. These bounds are ensured to converge to each other as the parameters M and N tend to ∞.

3.5.1 Computing Upper Bound of $\|\mathcal{P}_N^+\|$

In this subsection, we give a computation method for an upper bound of $\|\mathcal{P}_N^+\|$. We first note that

$$\|\mathcal{P}_N^+\| = \left\| \mathcal{C} \begin{bmatrix} \mathcal{A}^{N+1} & \mathcal{A}^{N+2} & \cdots \end{bmatrix} \begin{bmatrix} \mathcal{B} & & \\ & \mathcal{B} & \\ & & \ddots \end{bmatrix} \right\| \leq \|\mathcal{C}\| \cdot \|\mathcal{B}\| \cdot \| \begin{bmatrix} \mathcal{A}^{N+1} & \mathcal{A}^{N+2} & \cdots \end{bmatrix} \| \tag{3.119}$$

We take a constant $L \in \mathbb{N}$ such that $\|\mathcal{A}^L\| < 1$, which does exist by the stability assumption of Σ_{SD}. Then, if we note that introducing

$$\mathcal{A}_{NL} := \begin{bmatrix} \mathcal{A}^{N+1} & \mathcal{A}^{N+2} & \cdots & \mathcal{A}^{N+L} \end{bmatrix} \tag{3.120}$$

leads to

$$\begin{bmatrix} \mathcal{A}^{N+1} & \mathcal{A}^{N+2} & \cdots \end{bmatrix} = \begin{bmatrix} I & \mathcal{A}^L & \cdots \end{bmatrix} \begin{bmatrix} \mathcal{A}_{NL} & & \\ & \mathcal{A}_{NL} & \\ & & \ddots \end{bmatrix} \tag{3.121}$$

it readily follows that

$$\left\| \begin{bmatrix} \mathcal{A}^{N+1} & \mathcal{A}^{N+2} & \cdots \end{bmatrix} \right\| \leq \frac{\|\mathcal{A}_{NL}\|}{1 - \|\mathcal{A}^L\|} \tag{3.122}$$

Summarizing (3.119) and (3.122), we obtain the following result.

Proposition 3.10 If $\|\mathcal{A}^L\| < 1$, then

$$\|\mathcal{P}_N^+\| \leq \frac{\|\mathcal{A}_{NL}\|}{1 - \|\mathcal{A}^L\|} \|C_0\| e^{\|A_2\|h} \|C_\Sigma\| h e^{\|A\|h} \|B_1\| =: K_{NL} \tag{3.123}$$

and K_{NL} converges to 0 regardless of L as $N \to \infty$.

Proof. To derive (3.123), it suffices to show that

$$\|\mathcal{C}\| \leq \|C_0\| e^{\|A_2\|h} \|C_\Sigma\| \tag{3.124}$$

$$\|\mathcal{B}\| \leq h e^{\|A\|h} \|B_1\| \tag{3.125}$$

but these inequalities follow readily from the definition of $\mathcal{B}$ and $\mathcal{C}$. The last assertion is immediate from the fact that $\|\mathcal{A}_{NL}\| \to 0$ as $N \to \infty$. Q.E.D.

3.5.2 Main Results in the Computation of the L_∞-Induced Norm of Sampled-Data Systems

In this subsection, we give the main results on computing the L_∞-induced norm of sampled-data systems. Combining Theorems 3.1–3.4 and Proposition 3.10 together with (3.21), we are led to the following results.

Theorem 3.5 If $\|\mathcal{A}^L\| < 1$, then

$$\|P^-_{NM\mathrm{i}0}\| - K_{NL} \leq \|\mathcal{P}\| \leq \|P_{NM\mathrm{i}0}\| + \frac{K_{M\mathrm{i}0}}{M} + K_{NL} \tag{3.126}$$

$$\|P_{NM\mathrm{i}1}\| - \frac{K_{M\mathrm{i}1}}{M^2} - K_{NL} \leq \|\mathcal{P}\| \leq \|P_{NM\mathrm{i}1}\| + \frac{K_{M\mathrm{i}1}}{M^2} + K_{ML} \tag{3.127}$$

$$\|P_{NM\mathrm{k}0}\| - \frac{K_{M\mathrm{k}0}}{M} - K_{NL} \leq \|\mathcal{P}\| \leq \|P_{NM\mathrm{k}0}\| + \frac{K_{M\mathrm{k}0}}{M} + K_{NL} \tag{3.128}$$

$$\|P_{NM\mathrm{k}1}\| - \frac{K_{M\mathrm{k}1}}{M^2} - K_{NL} \leq \|\mathcal{P}\| \leq \|P_{NM\mathrm{k}1}\| + \frac{K_{M\mathrm{k}1}}{M^2} + K_{NL} \tag{3.129}$$

Furthermore, $K_{M\mathrm{i}0}$, $K_{M\mathrm{i}1}$, $K_{M\mathrm{k}0}$ and $K_{M\mathrm{k}1}$ have uniform upper bounds $K^U_{\mathrm{i}0}$, $K^U_{\mathrm{i}1}$, $K^U_{\mathrm{k}0}$ and $K^U_{\mathrm{k}1}$ defined as (3.53), (3.75), (3.99) and (3.114), respectively. Thus, the error bounds $K_{M\mathrm{i}0}/M$, $K_{M\mathrm{i}1}/M^2$, $K_{M\mathrm{k}0}/M$ and $K_{M\mathrm{k}1}/M^2$ in (3.126)–(3.129) converge to 0 as $M \to \infty$, while K_{NL} converges to 0 regardless of L as $N \to \infty$.

3.5.3 Guideline for Taking Parameters

It should be noted in (3.126)–(3.129) that the uniform upper bounds $K^U_{\mathrm{i}0}$ of $K_{M\mathrm{i}0}$, $K^U_{\mathrm{i}1}$ of $K_{M\mathrm{i}1}$, $K^U_{\mathrm{k}0}$ of $K_{M\mathrm{k}0}$ and $K^U_{\mathrm{k}1}$ of $K_{M\mathrm{k}1}$ given in (3.53), (3.75), (3.99) and (3.114), respectively, depend on N, and increase as N is increased to reduce K_{NL}. However, K_{NL} is bounded from above in the exponential order ρ^N in N regardless of L, for any $\rho < 1$ larger than the spectral radius of $\mathcal{A}$ and thus should reduce relatively fast with respect to N. Hence, it is expected that we can keep the uniform upper bounds $K^U_{\mathrm{i}0}$, $K^U_{\mathrm{i}1}$, $K^U_{\mathrm{k}0}$ and $K^U_{\mathrm{k}1}$ modest, and thus $K_{M\mathrm{i}0}/M$, $K_{M\mathrm{i}1}/M^2$, $K_{M\mathrm{k}0}/M$ and $K_{M\mathrm{k}1}/M^2$ can also be made small with a modest M.

Regarding a guideline for taking the parameters N, M and L, we can summarize the above arguments as follows. It may be reasonable to take a relatively small L as long as $\|\mathcal{A}^L\| < 1$; this is to avoid undue increase of K_{NL}, or in particular $\|\mathcal{A}_{NL}\|$ (or the computation time for them). Once L is fixed, the next step would be to take an N such that K_{NL} is as small as we wish; this is always possible by taking N sufficiently large. For example, if

$$\mathcal{A} = P_{\mathcal{A}} \Lambda_{\mathcal{A}} P_{\mathcal{A}}^{-1} \tag{3.130}$$

with a diagonal $\Lambda_{\mathcal{A}}$, then it is easy to see that

$$K_{NL} \leq \rho^{N+1} K_{\mathcal{A}} \tag{3.131}$$

where

$$K_{\mathcal{A}} := L \frac{\|C_\Sigma P_{\mathcal{A}}\| \|P_{\mathcal{A}}^{-1}\|}{1 - \|\mathcal{A}^L\|} \|C_0\| e^{\|A_2\|h} h e^{\|A\|h} \|B_1\| \tag{3.132}$$

This implies that $K_{NL} \leq \epsilon$ whenever $N \geq N_\epsilon := (\log \epsilon - \log K_{\mathcal{A}})/\log \rho - 1$. Once N is also fixed, the uniform upper bounds $K_{\mathrm{i}0}^U$, $K_{\mathrm{i}1}^U$, $K_{\mathrm{k}0}$ and $K_{\mathrm{k}1}$ in (3.53), (3.75), (3.99) and (3.114), respectively, are determined, and thus the last step would be to take an M such that $K_{\mathrm{i}0}^U/M$, $K_{\mathrm{i}1}^U/M^2$, $K_{\mathrm{k}0}^U/M$ and $K_{\mathrm{k}1}^U/M^2$ are as small as we wish. It is obvious that following this kind of guideline leads to computation methods for the L_∞-induced norm of the sampled-data system Σ_{SD} (given by $\|\mathcal{P}\|$) to any degree of accuracy.

3.6 Comparison between the Input Approximation and Kernel Approximation Approaches

In this section, we compare effectiveness of the input approximation and kernel approximation approaches in the L_∞-induced norm analysis of sampled-data systems. We could see that $K_{M\mathrm{k}0}$ and $K_{M\mathrm{k}1}$ relevant to the approximation errors in the kernel approximation approach are smaller than $K_{M\mathrm{i}0}$ and $K_{M\mathrm{i}1}$, respectively, relevant to those for the input approximation approach. More precisely, we could see from Lemmas 3.1 and 3.7 that the error bound in the approximation of the operator $\mathbf{B}_1'$ through the kernel approximation approach is smaller than that through the input approximation approach, under the piecewise constant approximation scheme. Furthermore, by comparing Lemmas 3.4 and 3.9, we see that the error bound through the kernel approximation approach is smaller than that through the input approximation approach, under the piecewise linear approximation scheme. These observations are closely related to the L_∞-induced norm analysis of continuous-time FDLTI systems, in which the operator $\mathbf{B}'$, which is similar to $\mathbf{B}_1'$, is approximated by using the input approximation or kernel approximation approach. In other words, this chapter directly applied the approximation ideas developed in Chapter 2 to the approximation of the operator $\mathbf{B}_1'$ when we compute the L_∞-induced norm of sampled-data systems.

We also should note that the kernel approximation approach leads to smaller error bounds in the approximation of the operator $\mathcal{D}$ than those through the input approximation approach, while there is no difference between the input approximation and kernel approximation approaches in the approximation of the operator $\mathbf{M}_1'$. This could be confirmed

by comparing Lemmas 3.3 and 3.8 and Lemmas 3.6 and 3.10. From these observations, it is expected that the kernel approximation approach works more effectively than the input approximation approach in the L_∞-induced norm analysis of sampled-data systems. However, in contrast to the case of continuous-time FDLTI systems given in Chapter 2, for the piecewise constant approximation scheme, the error bound in (3.126) through the input approximation approach is smaller than that in (3.128) through the kernel approximation approach. This is because we could see from (3.39), (3.42), (3.52), (3.92), (3.95) and (3.98) that $\frac{1}{2}K_{M\mathrm{i}0} < K_{M\mathrm{k}0} < K_{M\mathrm{i}0}$. On the other hand, for the piecewise linear approximation scheme, the error bound in (3.129) through the kernel approximation approach is smaller than that in (3.127) through the input approximation approach since $K_{M\mathrm{k}1} < K_{M\mathrm{i}1}$. The discrepancy between the continuous-time systems and sampled-data systems in the use of the piecewise constant approximation scheme is interpreted as stemming from the following reason. The h-periodic nature of the input/output relation of the sampled-data system Σ_{SD} requires us to deal with not only the operator $\mathbf{B}_1$ but also the operators $\mathbf{M}_1$ and $\mathbf{D}_{11}$ (more precisely, not only the operators $\mathbf{B}_1'$ but also the operators $\mathbf{M}_1'$ and $\mathbf{D}_{11}'$) and the error bounds $K_{M\mathrm{i}0}$, $K_{M\mathrm{i}1}$, $K_{M\mathrm{k}0}$ and $K_{M\mathrm{k}1}$ are dependent on all the approximations of the operators $\mathbf{B}_1$, $\mathbf{M}_1$ and $\mathbf{D}_{11}$ while we only need to approximate the operator $\mathbf{B}$ (which is similar to $\mathbf{B}_1$ in sampled-data systems) in the continuous-time case.

To summarize, the gap between the upper and lower bounds in (3.129) for the kernel approximation approach is smaller than that in (3.127) for the input approximation approach when we use the piecewise linear approximation scheme. However, for the piecewise constant approximation scheme, the gap in (3.126) is smaller than that in (3.128). Meanwhile, it will be numerically demonstrated in the following section that the piecewise linear approximation scheme is superior to the piecewise constant approximation scheme in the computation of $\|\mathcal{P}\|$ under both the input approximation and kernel approximation approaches. Combining these observations clearly indicates an advantage of the method with a combined use of the piecewise linear approximation scheme and the kernel approximation approach over the other three methods.

3.7 Numerical Examples

In this section, we study two numerical examples and examine effectiveness of the computation methods developed in this chapter.

Let us first consider the stable SISO sampled-data system

$$A = \begin{bmatrix} 0 & -0.5 \\ 1 & -1.5 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 0 & 1.5 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}$$
$$D_{11} = 1, \quad D_{12} = 0 \tag{3.133}$$
$$A_\Psi = \begin{bmatrix} -0.4888 & 1.6687 \\ 0.0737 & -0.2547 \end{bmatrix}, \quad B_\Psi = \begin{bmatrix} -3.1180 \\ 0.4701 \end{bmatrix}$$
$$C_\Psi = \begin{bmatrix} -1.6601 & 5.7348 \end{bmatrix}, \quad D_\Psi = -7.5709 \tag{3.134}$$

with $h = 0.5$. We compute estimates of the L_∞-induced norm $\|\mathcal{P}\|$ by taking the fast-lifting parameter M ranging from 50 to 500 on the condition that $L = 10$ and then $N = 50$, which follow in this order by the guideline in Subsection 3.5.3, leading to $K_{NL} = 6.71 \times 10^{-8}$. The results for the upper and lower bounds of $\|\mathcal{P}\|$ obtained by Theorem 3.5 and the computation times under the piecewise constant approximation scheme are shown in Table 3.1, while those with the piecewise linear approximation scheme are shown in Table 3.2. We are mainly interested in the comparison between the input approximation approach and the kernel approximation approach discussed in this chapter. Hence, these (and the following) tables consist of Case (a) for the input approximation approach and Case (b) for the kernel approximation approach.

We next consider the stable MIMO sampled-data system

$$A = \begin{bmatrix} 1 & -2 \\ 2 & -2 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 & 0.5 \\ -1 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 1 & 0 \end{bmatrix}$$
$$D_{11} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad D_{12} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{3.135}$$
$$A_\Psi = \begin{bmatrix} 0.0010 & -0.1780 \\ 0.0043 & -0.7740 \end{bmatrix}, \quad B_\Psi = \begin{bmatrix} 0.2815 & 1.2240 \end{bmatrix}$$
$$C_\Psi = \begin{bmatrix} -0.0305 & 5.4482 \end{bmatrix}, \quad D_\Psi = -5.0894 \tag{3.136}$$

and $h = 0.5$. We compute the upper and lower bounds of its L_∞-induced norm by taking the fast-lifting parameter M ranging from 50 to 500 on the condition that $L = 10$ and then $N = 50$, which leads to $K_{NL} = 3.76 \times 10^{-7}$. The results are shown in Tables 3.3 and 3.4.

We can see from Tables 3.1–3.4 that the error bounds for the computation of $\|\mathcal{P}\|$ (i.e., the gaps between the upper and lower bounds) decrease by taking the fast-lifting parameter M larger for all estimates. Thus, we can confirm validity of all the four approximation methods provided in this chapter for computing the L_∞-induced norm $\|\mathcal{P}\|$ of sampled-data systems. A more important concern in this chapter, however, lies in the effectiveness comparison between (a) the input approximation approach and (b) the kernel approximation approach.

Table 3.1: Results with piecewise constant approximation scheme in SISO example.

Case (a): Input approximation approach

M	50	100	200	500
$\|P^-_{NM\mathrm{i}0}\| + \dfrac{K_{M\mathrm{i}0}}{M} + K_{NL}$	5.509782	4.504184	4.014698	3.725265
$\|P_{NM\mathrm{i}0}\| - K_{NL}$	3.533585	3.533967	3.533983	3.534023
time (sec)	0.033683	0.099336	0.436956	4.189501

Case (b): Kernel approximation approach

M	50	100	200	500
$\|P^-_{NM\mathrm{k}0}\| + \dfrac{K_{M\mathrm{k}0}}{M} + K_{NL}$	5.009958	4.256218	3.891201	3.675982
$\|P^-_{NM\mathrm{k}0}\| - \dfrac{K_{M\mathrm{k}0}}{M} - K_{NL}$	2.018439	2.792301	3.167054	3.388178
time (sec)	0.029470	0.097271	0.433973	3.942940

Table 3.2: Results with piecewise linear approximation scheme in SISO example.

Case (a): Input approximation approach

M	50	100	200	500
$\|P^-_{NM\mathrm{i}1}\| + \dfrac{K_{M\mathrm{i}1}}{M^2} + K_{NL}$	3.556322	3.539553	3.535367	3.534243
$\|P^-_{NM\mathrm{i}1}\| - \dfrac{K_{M\mathrm{i}1}}{M^2} - K_{NL}$	3.510848	3.528381	3.532598	3.533802
time (sec)	2.459881	9.838883	38.889801	259.028169

Case (b): Kernel approximation approach

M	50	100	200	500
$\|F^-_{hM\mathrm{k}1}\| + \dfrac{K_{M\mathrm{k}1}}{M^2} + K_{NL}$	3.548239	3.537549	3.534868	3.534164
$\|F^-_{hM\mathrm{k}1}\| - \dfrac{K_{M\mathrm{k}1}}{M^2} - K_{NL}$	3.518779	3.530347	3.533088	3.533881
time (sec)	2.361763	9.235672	37.372630	240.347713

In this regard, we had an earlier argument in Section 3.6, which implies that, under the piecewise constant approximation scheme, the kernel approximation approach can provide no advantage over the input approximation approach in reducing the gap between the computed upper and lower bounds. As seen from Tables 3.1 and 3.3, the convergence of this gap (common for the input and kernel approximation schemes) is not fast with respect to M. On the other hand, as seen from Tables 3.2 and 3.4, the piecewise linear approximation scheme exhibits much faster convergence. This demonstrates that the piecewise linear approximation scheme works much more effectively than the piecewise constant approximation scheme. In this respect, it should be observed that the piecewise linear approximation scheme requires much larger computation time than the piecewise constant approximation scheme under the

Table 3.3: Results with piecewise constant approximation scheme in MIMO example.

Case (a): Input approximation approach

M	50	100	200	500
$\|P^-_{NM\mathrm{i}0}\| + \dfrac{K_{M\mathrm{i}0}}{M} + K_{NL}$	9.334245	7.816323	7.097122	6.677878
$\|P_{NM\mathrm{i}0}\| - K_{NL}$	6.403292	6.403316	6.403397	6.403411
time (sec)	0.040395	0.135144	0.735836	7.968628

Case (b): Kernel approximation approach

M	50	100	200	500
$\|P^-_{NM\mathrm{k}0}\| + \dfrac{K_{M\mathrm{k}0}}{M} + K_{NL}$	8.757358	7.537009	6.959637	6.623409
$\|P^-_{NM\mathrm{k}0}\| - \dfrac{K_{M\mathrm{k}0}}{M} - K_{NL}$	4.176935	5.333536	5.878978	6.196126
time (sec)	0.037555	0.130486	0.689447	7.673816

Table 3.4: Results with piecewise linear approximation scheme in MIMO example.

Case (a): Input approximation approach

M	50	100	200	500
$\|P^-_{NM\mathrm{i}1}\| + \dfrac{K_{M\mathrm{i}1}}{M^2} + K_{NL}$	6.444728	6.413338	6.405846	6.403799
$\|P^-_{NM\mathrm{i}1}\| - \dfrac{K_{M\mathrm{i}1}}{M^2} - K_{NL}$	6.367203	6.394763	6.401300	6.403080
time (sec)	2.674662	10.671502	43.783685	287.790821

Case (b): Kernel approximation approach

M	50	100	200	500
$\|F^-_{hM\mathrm{k}1}\| + \dfrac{K_{M\mathrm{k}1}}{M^2} + K_{NL}$	6.427564	6.409233	6.404843	6.403640
$\|F^-_{hM\mathrm{k}1}\| - \dfrac{K_{M\mathrm{k}1}}{M^2} - K_{NL}$	6.385214	6.399080	6.402357	6.403247
time (sec)	2.646978	10.664925	43.601142	287.277921

same parameter M. However, we can also see from these tables that the gaps between the upper and lower bounds in the piecewise linear approximation scheme with $M = 50$ are much smaller than those in the piecewise constant approximation scheme with $M = 500$, while the computation time for the former is smaller than that for the latter. These observations suggest that the piecewise linear approximation scheme drastically outperforms the piecewise constant approximation scheme.

We can further see from these tables that once we switch to the piecewise linear approximation scheme, an advantage of the kernel approximation approach over the input approximation approach is prominent. This is because the range between the upper and lower bounds obtained by the kernel approximation approach is always contained in (and

thus less conservative than) that by the input approximation approach for the same parameter M. Furthermore, the computation times in the kernel approximation approach are slightly smaller than those in the input approximation approach under the same parameter M. As an overall evaluation, the kernel approximation approach with the piecewise linear approximation scheme exhibits the smallest range for the L_∞-induced norm estimates with relatively short computation times among the four methods developed in this chapter.

3.8 Concluding Remarks

In this chapter, we considered a difficult problem of accurately computing the L_∞-induced norm associated with a stable sampled-data system, which is very important in many control systems. To solve this problem, we first introduced lifted representation of sampled-data systems. In the lifted representation, the input and output operators of sampled-data systems are derived, and these operators should be approximated to compute the L_∞-induced norm. In this respect, we developed two approaches for computing the L_∞-induced norm of sampled-data systems by using ideas of input approximation and kernel approximation approaches. This is stimulated by the success in computing the L_∞-induced norm of a stable FDLTI system. In these two approximation approaches, piecewise constant approximation and piecewise linear approximation schemes are applied via the fast-lifting treatment of sampled-data systems, so that the input or kernel function of the input operator and the hold function of the output operator associated with sampled-data systems are approximated by piecewise constant or piecewise linear functions. We showed that an upper bound and lower bound of the L_∞-induced norm can be readily computed through such approximation methods and the gap between the upper and lower bounds in the piecewise constant approximation or piecewise linear approximation scheme is ensured to converge to 0 at the rate of $1/M$ or $1/M^2$, respectively, under both the input approximation and kernel approximation approaches, where M is the fast-lifting parameter. Even though these convergence rates are qualitatively the same in the two approximation approaches, our detailed analysis showed that the approximation errors through the kernel approximation approach are smaller than those through the input approximation approach, under the piecewise linear approximation scheme. We then examined the effectiveness of the two approximation approaches through numerical studies and confirmed that the kernel approximation approach with the piecewise linear approximation scheme derived the smallest range for the L_∞-induced norm estimates with relatively short computation times among the four methods developed in this chapter.

Finally, we give some remark on why this chapter confines itself to piecewise constant approximation and piecewise linear approximation schemes and does not deal with piecewise

higher-order-polynomial approximation. Simply constructing the pth-order approximant $\mathbf{B}'_{\mathrm{i}p}$ and $\mathbf{B}'_{\mathrm{k}p}$ to $\mathbf{B}'_1$ (with desired properties from the pth-order approximation viewpoint) could be carried out even for $p \geq 2$ by following the same line of arguments as in Subsection 2.9.3 and Section 3.4, respectively. The pth-order approximant $\mathbf{M}'_{\mathrm{a}p}$ to $\mathbf{M}'_1$ can also be introduced readily through the Taylor series expansion. Nevertheless, extension of the present studies to $p \geq 2$ is nontrivial because it seems very hard to find a way to uniquely fix the input of $\mathbf{M}'_{\mathrm{a}p}$ to such a value that is ensured to be 'the one we may assume in our induced-norm computation.' Hence, we cannot predetermine the timing $\theta' \in [0, h')$ such that the output of $\mathbf{M}'_{\mathrm{a}p}$ at θ' does correspond to our induced-norm computation. This is in sharp contrast with the present paper dealing only with $p = 0$ and $p = 1$ (i.e., constant and linear functions), in which it is obvious that considering only $\theta' = 0$ and $\theta' \to h'$ suffices *whatever the input of* $\mathbf{M}'_{\mathrm{a}p}$ *may be* (i.e., despite that even $p = 0$ or $p = 1$ does not allow us to uniquely fix its input, either). Note that this strong feature was the key in successfully circumventing the reference to θ' when (3.163) was reduced to (3.168) and (3.173) (and similarly for the θ' in (3.166)) and leading to finite-dimensional discretization. Another obstacle may be how to construct and deal with suitable approximants $\mathbf{D}'_{\mathrm{a}p}$ to $\mathbf{D}'_{11}$ for $p \geq 2$, which is also nontrivial. Resolving all these issues might lead to an extension of the results in this chapter to $p \geq 2$, and such a direction might be qualified as a possible future study.

3.9 Appendix

In this section, we give proofs of the lemmas given in this chapter. They are based on the Taylor expansion of the matrix exponential of $A\theta'$ (or Ah'), and the proofs of lemmas in the kernel approximation approach proceed in essentially the same way as those of lemmas in the input approximation approach. Hence, only the proofs of the lemmas in the input approximation approach are given. We further deal with the derivation of Theorem 3.2 which is concerned with computing $\|\mathcal{P}^{-}_{NMi1}\|$.

3.9.1 Proofs of Lemmas 3.2, 3.3, 3.5 and 3.6

As mentioned above, only the proofs of the lemmas relevant to the input approximation approach are given. Because the proofs of Lemmas 3.1 and 3.4 proceed in essentially the same way as those of Lemmas 2.1 and Lemma 2.3, respectively, we omit the proofs of these lemmas. Hence, only the proofs of Lemmas 3.2, 3.3, 3.5 and 3.6 are given here.

Proof of Lemma 3.2:

By the Taylor expansion of $\exp(A_2\theta')$, it readily follows that

$$\left((\mathbf{M}_1' - \mathbf{M}_{\mathrm{a}0}')\begin{bmatrix} x \\ u \end{bmatrix}\right)(\theta') = C_0 \sum_{i=1}^{\infty} \frac{(A_2\theta')^i}{i!}\begin{bmatrix} x \\ u \end{bmatrix} \tag{3.137}$$

Then, we can see that

$$\begin{aligned} \|\mathbf{M}_1' - \mathbf{M}_{\mathrm{a}0}'\| &\leq \|C_0 A_2\| h' \sum_{i=1}^{\infty} \frac{\|A_2\|^{i-1}(h')^{i-1}}{i!} \\ &\leq \|C_0 A_2\| h' \sum_{i=0}^{\infty} \frac{\|A_2\|^{i}(h')^{i}}{i!} \leq \|C_0 A_2\| h' e^{\|A_2\| h'} \end{aligned} \tag{3.138}$$

This completes the proof.

Proof of Lemma 3.3:

We first note that $(\overline{\mathbf{M}_1'}\Delta_M^0\overline{\mathbf{B}_1'} + \overline{\mathbf{D}_{11}'}) - (\overline{\mathbf{M}_{\mathrm{a}0}'}\Delta_M^0\overline{\mathbf{B}_{\mathrm{i}0}'} + \overline{\mathbf{D}_{\mathrm{a}0}'})$ has a Toeplitz structure because of the Toeplitz structure of Δ_{M0}. Thus, its $L_\infty[0,h')$-induced norm coincides with the $L_\infty[0,h')$-induced norm of its last block row, i.e.,

$$\mathbf{E}_{\mathrm{i}0}' := \begin{bmatrix} \mathbf{M}_1'(A_{2d}')^{M-2}\mathbf{B}_1' & \cdots & \mathbf{M}_1'\mathbf{B}_1' & \mathbf{D}_{11}' \end{bmatrix} - \begin{bmatrix} \mathbf{M}_{\mathrm{a}0}'(A_{2d}')^{M-2}\mathbf{B}_{\mathrm{i}0}' & \cdots & \mathbf{M}_{\mathrm{a}1}'\mathbf{B}_{\mathrm{i}0}' & \mathbf{D}_{\mathrm{a}0}' \end{bmatrix} \tag{3.139}$$

It readily follows that $\|\mathbf{E}_{\mathrm{i}0}\|$ is bounded from above by

$$\epsilon_{\mathrm{i}0}^U = \sum_{k=0}^{M-2} \left\{ \left\| \mathbf{M}_1'(A_{2d}')^k\mathbf{B}_1' - \mathbf{M}_{\mathrm{a}0}'(A_{2d}')^k\mathbf{B}_{\mathrm{i}0}' \right\| \right\} + \|\mathbf{D}_{11}' - \mathbf{D}_{\mathrm{a}0}'\| \tag{3.140}$$

We next consider an upper bound of $\epsilon_{\mathrm{i}0}^U$. By considering the Taylor expansion of $\exp(A\theta')$, it readily follows that

$$\begin{aligned} &\left(\left(\mathbf{M}_1'(A_{2d}')^k\mathbf{B}_1' - \mathbf{M}_{\mathrm{a}0}'(A_{2d}')^k\mathbf{B}_{\mathrm{i}0}'\right) w\right)(\theta') \\ =& C_1(A_d')^k \cdot \int_0^{h'} \left\{ \exp(A(h'-\tau')) - \frac{1}{h'}A_{0d}' \right\} B_1 w(\tau') d\tau' \\ &\qquad + C_1(A_d')^k \sum_{i=1}^{\infty} \frac{(At')^i}{i!} \int_0^{h'} \exp(A(h'-\tau'))B_1 w(\tau') d\tau' \end{aligned} \tag{3.141}$$

where

$$A_{0d}' = \int_0^{h'} \exp(A(h'-\theta'))d\theta' = A_d' \int_0^{h'} \exp(-A\theta')d\theta' \tag{3.142}$$

Since

$$\exp(A(h'-\tau')) - \frac{1}{h'}A'_{0d} = A'_d \sum_{i=1}^{\infty} \left\{ \frac{(-A)^i(\tau')^i}{i!} - \frac{(-A)^i(h')^i}{(i+1)!} \right\} \tag{3.143}$$

we can confirm that the inequalities

$$\begin{aligned}
&\left\|\mathbf{M}'_1(A'_{2d})^k\mathbf{B}'_1 - \mathbf{M}'_{\mathrm{a}0}(A'_{2d})^k\mathbf{B}'_{\mathrm{i}0}\right\| \\
\leq &\left\|C_1(A'_d)^{k+1}\right\| \|B_1\| \int_0^{h'} \sum_{i=1}^{\infty} \left\{ \frac{\|A\|^i(\tau')^i}{i!} + \frac{\|A\|^i(h')^i}{(i+1)!} \right\} d\tau' \\
+ &\left\|C_1(A'_d)^k\right\| \|B_1\| \sum_{i=1}^{\infty} \frac{\|A\|^i(h')^i}{i!} \cdot \int_0^{h'} e^{\|A\|h'} d\tau' \\
\leq &(h')^2\|A\| \cdot \|B_1\| e^{\|A\|h'} \left(\left\|C_1(A'_d)^{k+1}\right\| + \left\|C_1(A'_d)^k\right\| e^{\|A\|h'} \right)
\end{aligned} \tag{3.144}$$

are established. On the other hand, it is easily see that

$$\|\mathbf{D}'_{11} - \mathbf{D}'_{\mathrm{a}0}\| \leq \|C_1\| \cdot \|B_1\| \sup_{0\leq\theta'<h'} \int_0^{\theta'} \|\exp(A(\theta'-\tau'))\| d\tau' \leq h'\|C_1\| \cdot \|B_1\| e^{\|A\|h'} \tag{3.145}$$

Combining (3.144) and (3.145) leads to

$$\epsilon_{\mathrm{i}0}^U \leq \frac{K_{M\mathcal{D}\mathrm{i}0}}{M} \tag{3.146}$$

This completes the proof.

Proof of Lemma 3.5:

By the Taylor expansion of $\exp(A_2\theta')$, we can see that

$$\left((\mathbf{M}'_1 - \mathbf{M}'_{\mathrm{a}1}) \begin{bmatrix} x \\ u \end{bmatrix} \right)(\theta') = C_0 \sum_{i=2}^{\infty} \frac{(A_2\theta')^i}{i!} \begin{bmatrix} x \\ u \end{bmatrix} \tag{3.147}$$

It immediately follows that

$$\begin{aligned}
\|\mathbf{M}'_1 - \mathbf{M}'_{\mathrm{a}1}\| &\leq \frac{\|C_0A_2^2\|(h')^2}{2} \sum_{i=2}^{\infty} \frac{\|A_2\|^{i-2}(h')^{i-2}}{i!/2} \\
&\leq \frac{\|C_0A_2^2\|(h')^2}{2} \sum_{i=0}^{\infty} \frac{\|A_2\|^i(h')^i}{i!} \leq \frac{\|C_0A_2^2\|(h')^2}{2} e^{\|A_2\|h'}
\end{aligned} \tag{3.148}$$

This completes the proof.

Proof of Lemma 3.6:

Because $(\overline{\mathbf{M}'_1}\Delta_M^0\overline{\mathbf{B}'_1}+\overline{\mathbf{D}'_{11}})-(\overline{\mathbf{M}'_{\mathrm{a}1}}\Delta_M^0\overline{\mathbf{B}'_{\mathrm{i}1}}+\overline{\mathbf{D}'_{\mathrm{a}1}})$ has a Toeplitz structure, its $L_\infty[0,h')$-induced norm coincides with the $L_\infty[0,h')$-induced norm of its last block row, i.e.,

$$\mathbf{E}'_{\mathrm{i}1} := \begin{bmatrix}\mathbf{M}'_1(A'_{2d})^{M-2}\mathbf{B}'_1 & \cdots & \mathbf{M}'_1\mathbf{B}'_1 & \mathbf{D}'_{11}\end{bmatrix} - \begin{bmatrix}\mathbf{M}'_{\mathrm{a}1}(A'_{2d})^{M-2}\mathbf{B}'_{\mathrm{i}1} & \cdots & \mathbf{M}'_{\mathrm{a}1}\mathbf{B}'_{\mathrm{i}1} & \mathbf{D}'_{\mathrm{a}1}\end{bmatrix} \tag{3.149}$$

Obviously, $\|\mathbf{E}_{\mathrm{i}1}\|$ is bounded from above by

$$\epsilon_{\mathrm{i}1}^U = \sum_{k=0}^{M-2}\left\{\left\|\mathbf{M}'_1(A'_{2d})^k\mathbf{B}'_1-\mathbf{M}'_{\mathrm{a}1}(A'_{2d})^k\mathbf{B}'_{\mathrm{i}1}\right\|\right\}+\|\mathbf{D}'_{11}-\mathbf{D}'_{\mathrm{a}1}\| \tag{3.150}$$

We next consider an upper bound of $\epsilon_{\mathrm{i}1}^U$. Since f_0 and f_1 are scalar valued functions, it follows from the Taylor expansion of $\exp(A\theta')$ that

$$\begin{aligned}&\left(\left(\mathbf{M}'_1(A'_{2d})^k\mathbf{B}'-\mathbf{M}'_{\mathrm{a}1}(A'_{2d})^k\mathbf{B}'_{\mathrm{i}1}\right)w\right)(\theta')\\&=C_1(I+A\theta')(A'_d)^k\cdot\int_0^{h'}\left\{\exp(A(h'-\tau'))-A'_{0d}f_0(\tau')-A'_{1d}f_1(\tau')\right\}B_1w(\tau')d\tau'\\&+C_1\left(\sum_{i=2}^{\infty}\frac{(A\theta')^i}{i!}\right)(A'_d)^k\int_0^{h'}\exp(A(h'-\tau'))B_1w(\tau')d\tau'\end{aligned} \tag{3.151}$$

where

$$A'_{1d}=\int_0^{h'}\exp(A(h'-\theta'))(\theta')d\theta'=A'_d\int_0^{h'}\exp(-A\theta')(\theta')d\theta' \tag{3.152}$$

By the definition of f_0 and f_1, we can show that

$$\begin{aligned}&\exp(A(h'-\tau'))-A'_{0d}f_0(\tau')-A'_{1d}f_1(\tau')\\&=A'_d\sum_{i=2}^{\infty}\left\{\frac{(-A)^i(\tau')^i}{i!}-\left(\frac{6i}{(i+2)!}(-A)^i(h')^{i-1}\right)\tau'\right\}+A'_d\sum_{i=2}^{\infty}\frac{2(i-1)}{(i+2)!}(-A)^i(h')^i\\&=:A'_dL_A(\tau')\end{aligned} \tag{3.153}$$

Hence, we have

$$\begin{aligned}&\left\|\mathbf{M}'_1(A'_{2d})^k\mathbf{B}'_1-\mathbf{M}'_{\mathrm{a}1}(A'_{2d})^k\mathbf{B}'_{\mathrm{i}1}\right\|\\&\le\max_{\theta'\in\{0,h'\}}\left\|C_1(I+A\theta')(A'_d)^{k+1}\right\|\|B_1\|\int_0^{h'}\|L_A(\tau')\|d\tau'\\&\qquad+\left\|C_1(A'_d)^k\right\|\|B_1\|\sum_{i=2}^{\infty}\frac{\|A\|^i(h')^i}{i!}\cdot\int_0^{h'}e^{\|A\|h'}d\tau'\\&\le\frac{1}{2}(h')^3\|A\|^2\|B_1\|e^{\|A\|h'}\max_{\theta'\in\{0,h'\}}\|C_1(I+A\theta')(A'_d)^{k+1}\|\\&\qquad+\frac{1}{2}(h')^3\|A\|^2\|B_1\|e^{2\|A\|h'}\left\|C_1(A'_d)^k\right\|\end{aligned} \tag{3.154}$$

because we can establish the inequalities

$$\begin{aligned}
&\int_0^{h'} \|L_A(\tau')\| d\tau' \\
\leq &\int_0^{h'} \sum_{i=2}^{\infty} \left\{ \frac{\|A\|^i (\tau')^i}{i!} + \left(\frac{6i}{(i+2)!} \|A\|^i (h')^{i-1} \right) \tau' \right\} d\tau' + \int_0^{h'} \sum_{i=2}^{\infty} \frac{2(i-1)}{(i+2)!} \|A\|^i (h')^i d\tau' \\
\leq &\frac{1}{2} (h')^3 \|A\|^2 e^{\|A\| h'} \qquad (3.155)
\end{aligned}$$

from the definition of $L_A(\tau')$.

Remark 3.16 Because $L_A(\tau')$ in (3.153) is essentially the equivalent to $L_{Ac}(\theta')$ in (2.68), we have the same result (3.155) as (2.69).

On the other hand, it is obvious that

$$\begin{aligned}
&\|\mathbf{D}'_{11} - \mathbf{D}'_{\mathrm{a}1}\| \\
= &\sup_{\|w\| \leq 1} \sup_{0 \leq \theta' < h'} \left\| \int_0^{\theta'} C_1 \left(\exp(A(\theta' - \tau')) - I \right) B_1 w(\tau') d\tau' \right\| \\
= &\sup_{\|w\| \leq 1} \sup_{0 \leq \theta' < h'} \left\| \int_0^{\theta'} C_1 \left(\exp(As) - I \right) B_1 w(\theta' - s) ds \right\| \qquad (3.156)
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\|\mathbf{D}'_{11} - \mathbf{D}'_{\mathrm{a}1}\| &\leq \|C_1\| \cdot \|B_1\| \int_0^{h'} \|\exp(As) - I\| \, ds \\
&\leq \|C_1\| \cdot \|B_1\| \int_0^{h'} \sum_{i=1}^{\infty} \frac{\|A\|^i s^i}{i!} ds \leq \frac{1}{2} (h')^2 \|C_1\| \cdot \|A\| \cdot \|B_1\| e^{\|A\| h'} \qquad (3.157)
\end{aligned}$$

Combining (3.154) and (3.157) leads to

$$\epsilon_{\mathrm{i}1}^U \leq \frac{K_{M\mathcal{D}\mathrm{i}1}}{M^2} \qquad (3.158)$$

This completes the proof.

3.9.2 Computation Method for $\|\mathcal{P}^-_{NM\mathrm{i}1}\|$

This appendix is devoted to the derivation of Theorem 3.2. We begin by giving a concise way for representing $(\mathcal{P}^-_{NM\mathrm{i}1} w)(0)$ and $(\mathcal{P}^-_{NM\mathrm{i}1} w)(h')$. A direct computation shows that

$$\begin{aligned}
\mathbf{B}'_{\mathrm{i}1} w_j^{(i)} &= \int_0^{h'} (B'_{0d} f_0(\tau') + B'_{1d} f_1(\tau')) w_j^{(i)}(\tau') d\tau' \\
&= \int_0^{h'} (G_0 + G_1 \tau') w_j^{(i)}(\tau') d\tau' \qquad (3.159)
\end{aligned}$$

where

$$B'_{1d} := \int_0^{h'} \exp(A(h'-\theta'))\theta' B_1 d\theta' \tag{3.160}$$

$$G_0 := -\frac{6}{(h')^2}B'_{1d} + \frac{4}{h'}B'_{0d} \tag{3.161}$$

$$G_1 := \frac{12}{(h')^3}B'_{1d} - \frac{6}{(h')^2}B'_{0d} \tag{3.162}$$

Hence, noting (3.59), we readily see that the function $\left(\overline{\mathbf{M}'_{\mathrm{a1}}}\mathcal{A}_{Mj}\overline{\mathbf{B}'_{\mathrm{i1}}}w_{j+1}\right)(\theta')$ in (3.77) equals the linear function

$$\sum_{p=1}^{M}(H_{jp0} + H_{jp1}\theta')\int_0^{h'}(G_0 + G_1\tau')w_j^{(p)}(\tau')d\tau' \tag{3.163}$$

where the matrices H_{jp0} and H_{jp1} $(p = 1, \cdots, M; j = 0, \cdots, N)$ are defined as

$$H_{jp0} := \overline{C_0}A'_{2dM}C_\Sigma\mathcal{A}^j J_\Sigma(A'_d)^{M-p} \tag{3.164}$$

$$H_{jp1} := \overline{C_1[A\ B_2]}A'_{2dM}C_\Sigma\mathcal{A}^j J_\Sigma(A'_d)^{M-p} \tag{3.165}$$

Similarly, under the notation $w_{0k} = [(w_{0k}^{(1)})^T, \cdots, (w_{0k}^{(M)})^T]$ $(k = 1, 2)$, it follows that $(\overline{\mathbf{M}'_{\mathrm{a1}}}\Delta_M^0\overline{\mathbf{B}'_{\mathrm{i1}}}w_{02})(\theta')$ equals the linear function

$$\sum_{p=1}^{M}(S_{p0} + S_{p1}\theta')\int_0^{h'}(G_0 + G_1\tau')w_{02}^{(p)}(\tau')d\tau' \tag{3.166}$$

where

$$S_{p0} := \overline{C_0}\Delta_{Mp}^0, \quad S_{p1} := \overline{C_1[A\ B_2]}\Delta_{Mp}^0 \tag{3.167}$$

and Δ_{Mp}^0 is the pth block column of Δ_M^0.

It follows from a direct computation with (3.163) and (3.166) together with the definition of $\mathbf{D}'_{\mathrm{a10}}$ that $(\mathcal{P}_{NMi1}^{-}w)(0)$ with $w = [w_{00}^T, w_{01}^T, w_{02}^T, w_1^T, \cdots, w_{N+1}^T]^T$ is determined by the mappings

$$w_{j+1}^{(p)} \mapsto \int_0^{h'}(Y_{jp0}^{[0]} + Y_{jp1}^{[0]}\tau')w_{j+1}^{(p)}(\tau')d\tau' \quad (j = 0, \cdots, N) \tag{3.168}$$

$$w_{00}^{(p)} \mapsto 0, \quad w_{01}^{(p)} \mapsto 0 \tag{3.169}$$

$$w_{02}^{(p)} \mapsto \int_0^{h'}(Z_{p0}^{[0]} + Z_{p1}^{[0]}\tau')w_{02}^{(p)}(\tau')d\tau' \tag{3.170}$$

where

$$Y_{jp0}^{[0]} := H_{jp0}G_0, \quad Y_{jp1}^{[0]} := H_{jp0}G_1 \tag{3.171}$$

$$Z_{p0}^{[0]} := S_{p0}G_0, \quad Z_{p1}^{[0]} := S_{p0}G_1 \tag{3.172}$$

Similarly, since w_{01} could be assumed to be a constant function (whose value equals $w_{01}(0)$) by noting the signs of each elements in each rows of the matrix C_1B_1 when we deal with $\|\mathcal{P}_{NMi1}^{-}\|$, it follows that $(\mathcal{P}_{NMi1}^{-}w)(h')$ is determined by the mappings

$$w_j^{(p)} \mapsto \int_0^{h'} (Y_{jp0}^{[h']} + Y_{jp1}^{[h']}\tau')w_{j+1}^{(p)}(\tau')d\tau' \quad (j = 0, \cdots, N) \tag{3.173}$$

$$w_{00}^{(p)} \mapsto 0, \quad w_{01}^{(p)} \mapsto C_1B_1h'w_{01}^{(p)}(0) \tag{3.174}$$

$$w_{02}^{(p)} \mapsto \int_0^{h'} (Z_{p0}^{[h']} + Z_{p1}^{[h']}\tau')w_{02}^{(p)}(\tau')d\tau' \tag{3.175}$$

where

$$Y_{jp0}^{[h']} := (H_{jp0} + H_{jp1}h')G_0, \quad Y_{jp1}^{[h']} := (H_{jp0} + H_{jp0}h')G_1 \tag{3.176}$$

$$Z_{p0}^{[h']} := (S_{p0} + S_{p1}h')G_0, \quad Z_{p1}^{[h']} := (S_{p0} + S_{p1}h')G_1 \tag{3.177}$$

The above mappings immediately lead us to a procedure for the computation of $\|\mathcal{P}_{NMi1}^{-}\|$ given in (3.81). This can be summarized as follows if we note that computing the induced norm of the operator representing the action (3.168) would require us to compute the $L_1[0, h')$ norm of each entry of $Y_{jp0}^{[0]} + Y_{jp1}^{[0]}\tau'$; by the properties of the $L_\infty[0, h')$ norm, it suffices us to repeat essentially the same arguments:

Let $T_{jp\mathrm{i}}^{[0]}$ $(j = 0, \cdots, N;\ p = 1, \cdots, M)$ be the matrix consisting of the $L_1[0, h')$ norm of each entry of the matrix function $Y_{jp0}^{[0]} + Y_{jp1}^{[0]}\tau'$ involved in (3.168), while let $T_{jp\mathrm{i}}^{[h']}$ $(j = 0, \cdots, N;\ i = 1, \cdots, M)$ be the matrix constructed in the same way from $Y_{jp0}^{[h']} + Y_{jp1}^{[h']}\tau'$ involved in (3.173). Similarly, let $V_{p\mathrm{i}}^{[0]}$ be the matrix consisting of the $L_1[0, h')$ norm of each entry of the matrix function $Z_{p0}^{[0]} + Z_{p1}^{[0]}\tau'$ involved in (3.170), while let $V_{p\mathrm{i}}^{[h']}$ be the matrix made in the same way from $Z_{p0}^{[h']} + Z_{p1}^{[h']}\tau'$ involved in (3.175). Note that each $L_1[0, h')$ norm can easily be computed exactly, since we only deal with linear functions. Theorem 3.2 now follows immediately from Proposition 3.4 by applying the triangle inequality.

Chapter 4

Sampled-Data Controller Synthesis for L_∞-Induced Norm Minimization

4.1 Introduction

The disturbance rejection problem is one of the main issues in control, and system norms are used to evaluate the effect of disturbances. Among various system norms, the L_∞-induced (or l_∞-induced) norm is used to deal with the bounded persistent disturbances, such as steps and sinusoids, which are often encountered in control systems. Because this norm corresponds to the L_1 (or l_1) norm of the impulse response of the system in the continuous-time (or discrete-time) case, the study associated with the treatment of the L_∞-induced norm (or l_∞-induced norm) has been named the L_1 (or l_1) problem. Some special cases of the L_1 (or l_1) problem have been formulated but a general case was not dealt with in [70]. The general case of the continuous-time L_1 problem was discussed in [20], [62] while the discrete-time l_1 problem was dealt with in [19],[21]–[24],[47],[59]. Stimulated by the success in these studies, the L_1 problem of sampled-data systems (with intersample behavior taken into account) has been studied in [3],[28],[66]. However, in contrast to the case of sampled-data H_∞ problem [6],[13],[34],[36],[38],[60],[61],[67],[68] and H_2 problem [5],[15], [16],[33],[39],[60],[61], no exact solution has been obtained for the L_1 problem of sampled-data systems and only approximate methods have been provided. More precisely, a sampled-data system is approximately treated in [3],[28],[66] as a discrete-time system through the fast-sample/fast-hold (FSFH) approximation technique [46], and it is shown that the l_∞-induced norm of the resulting discrete-time system converges to the L_∞-induced norm of the original sampled-data system at the rate of $1/M$, as the FSFH approximation parameter M tends to infinity.

As a significant advance over the conventional methods through the FSFH approximation technique, we developed in Chapter 3 extended methods for the L_∞-induced norm analysis

problem of sampled-data systems by using the ideas of fast-lifting with the input approximation and kernel approximation approaches. Fast-lifting also has an integer parameter M as in the FSFH approximation technique. However, it is used only to subdivide the sampling interval $[0, h)$ into M smaller pieces, while the conventional FSFH approximation technique takes M equally spaced sampling points on the interval $[0, h)$; no information is hence lost as to signals on $[0, h)$ by fast-lifting. This feature is crucial in developing piecewise constant approximation and piecewise linear approximation schemes under both the input approximation and kernel approximation approaches, while the FSFH approximation technique corresponds to applying only piecewise constant approximation scheme under the input approximation approach. Methods for computing an upper bound and a lower bound of the L_∞-induced norm of sampled-data systems through the input approximation and kernel approximation approaches are provided and it is shown that the gap between the upper and lower bounds converges to 0 at the rate of $1/M$ or $1/M^2$, in the piecewise constant approximation or piecewise linear approximation scheme, respectively, for the fast-lifting parameter M.

Unfortunately, however, these methods are restricted to analysis and cannot be used directly for synthesis. This is because it requires to compute the $L_1[0, h/M)$ norms of kernel functions determined by the continuous-time finite-dimensional linear time-invariant (FDLTI) system and the discrete-time controller and the structure of the way the controller parameters are involved in the kernel functions is complicated. In contrast, this chapter aims at establishing two discretization procedures of the generalized plant P described by (3.1) for the L_1 optimal controller synthesis problem of sampled-data systems via the piecewise constant approximation and piecewise linear approximation schemes under the input approximation approach. These discretization procedures are achieved by introducing two types of 'constant approximation operators' or 'linear approximation operators' for signals on the interval $[0, h/M)$ obtained by applying fast-lifting, one for the input signals and the other for the output signals. By applying the arguments of preadjoint operators [10],[11], [58], which are reviewed in Chapter 1, we provide two important inequalities that form theoretical bases for the piecewise constant approximation and piecewise linear approximation schemes to the L_1 optimal controller synthesis problem. Here, we would like to note that not every operator has a preadjoint, but those operators we deal with in this thesis do; it suffices to note that for $X = (L_\infty[0, h))^\nu$ and $X = \mathbb{R}^n_\infty$, a unique X_* such that $(X_*)^* = X$ is $X_* = (L_1[0, h))^\nu$ and $X_* = \mathbb{R}^n_1$, respectively. On the other hand, these two inequalities show that the piecewise constant approximation and piecewise linear approximation schemes are in the convergence rates of $1/M$ and $1/M^2$, respectively, for the sampled-data L_1 optimal controller synthesis. In connection with these convergence rates, we further give two

discretization procedures of a generalized plant for the sampled-data L_1 optimal controller synthesis through the piecewise constant approximation and piecewise linear approximation treatment.

On the other hand, however, it is not clear whether the kernel approximation approach can be directly applied to the L_1 optimal controller synthesis problem of sampled-data systems while the input approximation approach can be. The reason is relevant to the question on the parallel convergence arguments for the operators $\mathbf{T}'_0$ and $\mathbf{T}'_1$ such that $\mathbf{B}'_{\mathrm{k}0} = \mathbf{B}'_1 \cdot \mathbf{T}'_0$ and $\mathbf{B}'_{\mathrm{k}1} = \mathbf{B}'_1 \cdot \mathbf{T}'_1$. The preadjoint arguments, which play a crucial role in tackling the L_1 optimal controller synthesis problem with the input approximation approach, would somehow apply to the operators $\mathbf{T}'_0$ and $\mathbf{T}'_1$ in the kernel approximation approach, but it is still unclear whether parallel convergence rates could be established. This, in turn, implies that developing a theoretical basis of the kernel approximation approach for this synthesis problem seems to be a nontrivial issue. This interesting topic is left for future topic.

The organization of this chapter is as follows. In Section 4.2, we first formulate the problem definition with the review of the lifted representation of sampled-data systems. In Section 4.3, fast-lifting treatment of sampled-data systems is reviewed. We next apply the idea of the piecewise constant approximation to the L_1 optimal sampled-data controller synthesis problem in Section 4.4. The L_1 optimal sampled-data controller synthesis problem in the piecewise constant approximation is reduced to the discrete-time l_1 optimal controller synthesis problem in Section 4.5. We further apply the idea of the piecewise linear approximation to the L_1 optimal sampled-data controller synthesis problem in Section 4.6. The L_1 optimal sampled-data controller synthesis problem in the piecewise linear approximation is reduced to the discrete-time l_1 optimal controller synthesis problem in Section 4.7. In Section 4.8, the effectiveness of the proposed methods is demonstrated through a numerical example. We give concluding remarks in Section 4.9. Finally, the proofs of the lemmas given in this chapter and the approximation of the vector set $\boldsymbol{\Phi}_M$, which is relevant to the discretization of the continuous-time generalized plant through the piecewise linear approximation, are provided in Section 4.10.

4.2 Problem Formulation

Let us consider the sampled-data system Σ_{SD} shown in Figure 3.1, where P denotes the continuous-time generalized plant, while Ψ, $\mathcal{H}$ and $\mathcal{S}$ denote the discrete-time LTI controller, the zero-order hold and the ideal sampler, respectively, operating with sampling period h in a synchronous fashion. Solid lines and dashed lines in Fig. 3.1 are used to represent continuous-time signals and discrete-time signals, respectively. Furthermore, we suppose that P and

Ψ are described respectively by (3.1) and (3.2) where $x(t) \in \mathbb{R}^n_\infty$, $w(t) \in \mathbb{R}^{n_w}_\infty$, $u(t) \in \mathbb{R}^{n_u}_\infty$, $z(t) \in \mathbb{R}^{n_z}_\infty$, $y(t) \in \mathbb{R}^{n_y}_\infty$, $\psi_k \in \mathbb{R}^{n_\Psi}_\infty$, $y_k = y(kh)$ and $u(t) = u_k$ $(kh \le t < (k+1)h)$.

This chapter studies the so-called L_1 optimal control problem of the sampled-data system Σ_{SD}, i.e., a synthesis method for the stabilizing controller Ψ such that the L_∞-induced norm of the mapping $\mathcal{F}(P, \mathcal{H}\Psi\mathcal{S})$ between $w \in (L_\infty)^{n_w}$ and $z \in (L_\infty)^{n_z}$ is minimized. Even computing the L_∞-induced norm when P and Ψ are given is a difficult problem, to which the ideas of the fast-sample/fast-hold (FSFH) approximation approach [46] and the input approximation and kernel approximation approaches in Chapter 3 have been applied. FSFH approximation approach leads to the associated approximate discretization of P [3], [28],[66], with which the computation problem is reduced to a discrete-time counterpart (i.e., an l_1 problem). An important role was played by the lifting technique [4],[6],[68], [71] in such a direction, through which a synthesis method of Ψ was also developed in [3], [28] with the FSFH approximation. Stimulated by the recent success of (approximate but asymptotically accurate) computation of the L_∞-induced norm via the piecewise constant approximation (which, under the input approximation approach, essentially coincides with the FSFH approximation approach) and piecewise linear approximation schemes introduced in Chapter 3, the present chapter aims at developing new theoretical frameworks for the L_1 optimal control problem of sampled-data systems. More precisely, we establish two inequalities by using the ideas of the piecewise constant approximation and piecewise linear approximation schemes under the input approximation approach together with the arguments of preadjoint operators, which form mathematical bases of these approximation schemes for the L_1 optimal sampled-data controller synthesis; the piecewise constant approximation and piecewise linear approximation schemes are in the convergence rates of $1/M$ and $1/M^2$, respectively, where M is the fast-lifting parameter. Even though the convergence rate in the piecewise constant approximation scheme is the same as that in [3] through the FSFH approximation, the arguments for the convergence proof in [3] are more involved, and our following arguments utilize a simpler inequality and circumvent such involved arguments. With these convergence rates, we develop two discretization procedures of the generalized plant P for the L_1 optimal sampled-data controller synthesis problem.

By applying lifting to w and z while discretizing u and y, the (partially) lifted representation of the continuous-time generalized plant P is described by

$$\widehat{P} : \begin{cases} x_{k+1} = A_d x_k + \mathbf{B}_1 \widehat{w}_k + B_{2d} u_k \\ \widehat{z}_k = \mathbf{C}_1 x_k + \mathbf{D}_{11} \widehat{w}_k + \mathbf{D}_{12} u_k \\ y_k = C_{2d} x_k \end{cases} \tag{4.1}$$

where

$$(\mathbf{C}_1 x)(\theta) = C_1 \exp(A\theta)x : \mathbb{R}^n_\infty \to (L_\infty[0,h))^{n_z} \tag{4.2}$$

$$(\mathbf{D}_{12} u_k)(\theta) = \int_0^\theta C_1 \exp(A(\theta-\tau))B_2 d\tau u_k + D_{12} u_k : \mathbb{R}^{n_u}_\infty \to (L_\infty[0,h))^{n_z} \tag{4.3}$$

and the other matrices and operators are the same as those in (3.10), (3.12) and (3.15). The above two operators $\mathbf{C}_1$ and $\mathbf{D}_{12}$ constitute the operator $\mathbf{M}_1$ in (3.13); $\mathbf{M}_1$ is divided by $\mathbf{M}_1 = [\mathbf{C}_1 \ \mathbf{D}_{12}]$, and these two portions are described by (4.2) and (4.3), respectively.

As shown in Figure 4.1, connecting Ψ to the above $\widehat{P}$ leads to the mapping between $\{\widehat{w}_k\}_{k=0}^\infty$ and $\{\widehat{z}_k\}_{k=0}^\infty$, which we denote by $\mathcal{F}(\widehat{P}, \Psi)$; it coincides with the lifted representation $\mathbf{W}_h \mathcal{F}(P, \mathcal{H}\Psi\mathcal{S})\mathbf{W}_h^{-1}$ for the mapping $\mathcal{F}(P, \mathcal{H}\Psi\mathcal{S})$. Since $\mathbf{W}_h$ is norm-preserving, we can see that the L_∞-induced norm $\|\mathcal{F}(P, \mathcal{H}\Psi\mathcal{S})\|$ of the sampled-data system Σ_{SD} coincides with the $l^\infty_{L_\infty[0,h)}$-induced norm $\|\mathcal{F}(\widehat{P}, \Psi)\|$, which is discussed in Chapter 1.

Remark 4.1 The mapping between $\{\widehat{w}_k\}_{k=0}^\infty$ and $\{\widehat{z}_k\}_{k=0}^\infty$ in Figure 4.1 is essentially the same as that in Figure 3.3; the closed-loop system obtained by connecting $\widehat{P}$ and Ψ in Figure 4.1 could be described by $\widehat{\Sigma}_{\mathrm{SD}}$ in Figure 3.3.

We further introduce the (standard lifting-free) discrete-time plant

$$P_d : \begin{cases} x_{k+1} = A_d x_k + \eta_k + B_{2d} u_k \\ \zeta_k = \begin{bmatrix} I \\ 0 \end{bmatrix} x_k + \begin{bmatrix} 0 \\ I \end{bmatrix} u_k \\ y_k = C_{2d} u_k \end{cases} \tag{4.4}$$

and denote by $\mathcal{F}(P_d, \Psi)$ the mapping between the discrete-time signals $\eta_k \in \mathbb{R}^n_\infty$ and $\zeta_k \in \mathbb{R}^{n+n_u}_\infty$ associated with the closed-loop system obtained by connecting Ψ to the above P_d. Then, by comparison between this P_d and the lifted generalized plant $\widehat{P}$, we see that $\mathcal{F}(\widehat{P}, \Psi)$ admits the representation

$$\mathcal{F}(\widehat{P}, \Psi) = \mathbf{M}_1 \mathcal{F}(P_d, \Psi)\mathbf{B}_1 + \mathbf{D}_{11} \tag{4.5}$$

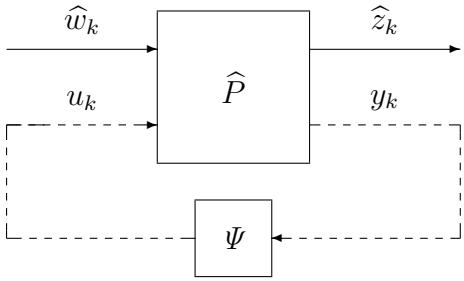


Figure 4.1: Closed-loop system by $\widehat{P}$ and Ψ.

and the mapping between $\widehat{w}_k$ and $\widehat{z}_k$ in (4.5) could be described through Figure 4.2.

Remark 4.2 In (4.5), since the left hand side denotes a dynamical system in discrete-time with the lifted input $\{\widehat{w}_k\}_{k=0}^{\infty}$ and output $\{\widehat{z}_k\}_{k=0}^{\infty}$, the operator $\mathbf{B}_1$ on the right hand side acts on every $\widehat{w}_k$, and $\widehat{z}_k$ is associated with the output of $\mathbf{M}_1$ for every k. Similarly for the interpretation of $\mathbf{D}_{11}$. Similar conventions apply to the following arguments.

Since $\mathcal{F}(\widehat{P}, \Psi) = \mathbf{W}_h \mathcal{F}(\widehat{P}, \mathcal{H}\Psi\mathcal{S})\mathbf{W}_h^{-1}$ and $\mathbf{W}_h$ is norm-preserving as mentioned above, it readily follows that the L_1 optimal controller synthesis problem of the sampled-data system Σ_{SD} reduces to the synthesis of Ψ minimizing $\|\mathcal{F}(\widehat{P}, \Psi)\|$. This, however, is still a difficult problem because the operators $\mathbf{B}_1$ and $\mathbf{M}_1$ (as well as $\mathbf{D}_{11}$) cannot be exactly discretized with finite-dimensional matrices in contrast to the H_2 and H_∞ problems of sampled-data systems. With this in mind, this chapter aims at approximating the operators $\mathbf{B}_1$, $\mathbf{M}_1$ and $\mathbf{D}_{11}$ by using the ideas of the piecewise constant approximation and piecewise linear approximation schemes under the input approximation approach, and establishing associated discretization procedures of the continuous-time generalized plant for the L_1 optimal control problem of sampled-data systems. More precisely, we provide, by introducing what we call L_1 discretization of the generalized plant, two (almost) equivalent discrete-time l_1-optimal control problems together with the associated approximation errors.

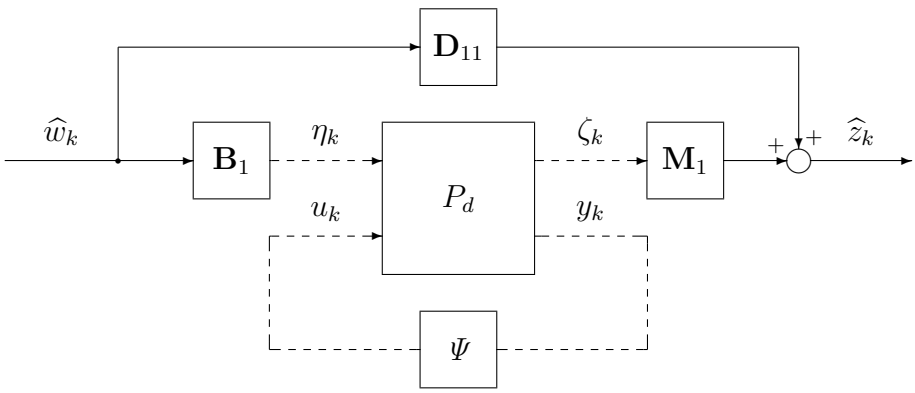


Figure 4.2: Operator representation of the sampled-data system Σ_{SD}.

4.3 Review of Fast-Lifting Treatment of the Sampled-Data System Σ_{SD}

As a preliminary step to apply the piecewise constant approximation and piecewise linear approximation schemes to the L_1 optimal sampled-data controller synthesis, we first review

the fast-lifting treatment of the sampled-data system $\Sigma_{\rm SD}$. Because the fast-lifting $\mathbf{L}_M$ is norm-preserving, it readily follows that

$$\|\mathcal{F}(\widehat{P},\, \Psi)\| = \|\mathbf{L}_M \mathcal{F}(\widehat{P},\, \Psi)\mathbf{L}_M^{-1}\| \tag{4.6}$$

where the right-hand side means the $l^{\infty}_{L_\infty[0,h')}$-induced norm with $h' := h/M$, which is discussed in Chapter 1. Let us consider applying fast-lifting on $\widehat{w}_k$ and $\widehat{z}_k$ in the (partially) lifted generalized plant $\widehat{P}$, and consider its fast-lifted counterpart

$$\widehat{P}_M = \mathrm{diag}[\mathbf{L}_M, I]\ \widehat{P}\ \mathrm{diag}[\mathbf{L}_M^{-1}, I] \tag{4.7}$$

Then, we see that $\mathbf{L}_M \mathcal{F}(\widehat{P},\, \Psi)\mathbf{L}_M^{-1} = \mathcal{F}(\widehat{P}_M,\, \Psi)$, and it admits the representation

$$\mathcal{F}(\widehat{P}_M,\, \Psi) = \mathbf{L}_M \mathbf{M}_1 \mathcal{F}(P_d,\, \Psi)\mathbf{B}_1 \mathbf{L}_M^{-1} + \mathbf{L}_M \mathbf{D}_{11} \mathbf{L}_M^{-1} \tag{4.8}$$

which we call the fast-lifted representation of the sampled-data system $\Sigma_{\rm SD}$.

We are in a position to review the piecewise constant approximation and piecewise linear approximation treatment of the operators $\mathbf{B}_1$, $\mathbf{M}_1$ and $\mathbf{D}_{11}$ in the above fast-lifted representation (4.8), which immediately leads us to piecewise constant and piecewise linear approximations of the sampled-data system $\Sigma_{\rm SD}$. Such approximations were developed in Chapter 3 for analysis problems, but it was far from straightforward to extend the methods in Chapter 3 in such a way that the L_1 optimal sampled-data controller synthesis problem can be dealt with. The present chapter employs essentially the same approximations of these operators as fundamental tools but aims at taking completely new approaches so that we can establish discretization procedures of the continuous-time generalized plant for the L_1 optimal sampled-data controller synthesis problem. As in existing studies, this is a problem that aims at minimizing the L_∞-induced norm of the sampled-data system $\Sigma_{\rm SD}$, for which our discretization procedures of the generalized plant, together with the associated approximation error analysis, provide two approaches that reduce the problem to an (almost) equivalent discrete-time l_1 optimal controller synthesis problem (which is the problem of minimizing the l_∞-induced norm).

4.4 Piecewise Constant Approximation to the L_1 Optimal Sampled-Data Controller Synthesis Problem

In this section, we apply the piecewise constant approximation (via the input approximation approach) introduced in Chapter 3 to operators $\mathbf{B}_1$, $\mathbf{M}_1$ and $\mathbf{D}_{11}$. We first introduce the operator $\mathbf{J}_0' : (L_\infty[0,h'))^{n_w} \to (L_\infty[0,h'))^{n_w}$ defined as (2.19), by which $\mathbf{J}_0' w$ is always

a constant function. $\mathbf{J}_0'$ is used to approximate the input of $\mathbf{B}_1$ with a piecewise constant function and is tailored to possess important properties in terms of some Taylor expansion arguments (see Section 3.9 for details), in addition to the property that $\mathbf{J}_0' w = w$ for any constant function w. We next introduce the operator $\mathbf{H}_0' : (L_\infty[0,h'))^{n_z} \to (L_\infty[0,h'))^{n_z}$ described by

$$(\mathbf{H}_0' z)(\theta') = z(0) \quad (0 \le \theta' < h') \tag{4.9}$$

as well as the operator $\mathbf{D}_{\mathrm{a}0}' : (L_\infty[0,h'))^{n_w} \to (L_\infty[0,h'))^{n_z}$ given in (3.34). Obviously, $\mathbf{H}_0'$ could be interpreted as an operator producing a constant function that preserves the value of the input at $\theta' = 0$, and thus $\mathbf{H}_0' z = z$ for any constant function z on $[0,h')$. Strictly speaking, $\mathbf{H}_0'$ is not an operator on $(L_\infty[0,h'))^{n_z}$ but on its subspace of functions continuous at time 0. However, this issue causes no problems since $\mathbf{H}_0'$ is used for approximating $\mathbf{M}_1$ (or its output with a piecewise constant function) and operates only on its output. In contrast, approximation of $\mathbf{B}_1$ should take into account that its input may be discontinuous, and this is why the other more involved operator $\mathbf{J}_0'$ in (2.19) is used. $\mathbf{D}_{\mathrm{a}0}'$ is used for approximating $\mathbf{D}_{11}$ as in Chapter 3, and it corresponds to ignoring the compact portion of the compression operator defined on $[0,h')$. The details of our approximation treatment employing these operators are as follows.

Following the basic ideas mentioned above, we consider replacing $\mathbf{L}_M\mathbf{M}_1$ and $\mathbf{B}_1\mathbf{L}_M^{-1}$ in (4.8) with $\overline{\mathbf{H}_0'}\mathbf{L}_M\mathbf{M}_1$ and $\mathbf{B}_1\mathbf{L}_M^{-1}\overline{\mathbf{J}_0'}$, respectively, (i.e., $\mathbf{M}_1$ and $\mathbf{B}_1$ are approximated by $\mathbf{L}_M^{-1}\overline{\mathbf{H}_0'}\mathbf{L}_M\mathbf{M}_1$ and $\mathbf{B}_1\mathbf{L}_M^{-1}\overline{\mathbf{J}_0'}\mathbf{L}_M$, respectively) where $\overline{(\cdot)}$ denotes $\mathrm{diag}[(\cdot),\cdots,(\cdot)]$ consisting of M copies of $(\cdot)$. To facilitate such treatment, let us introduce the operators $\mathbf{H}_{M0}$ and $\mathbf{J}_{M0}$ given respectively by

$$\mathbf{H}_{M0} = \overline{\mathbf{H}_0'}\mathbf{L}_M : (L_\infty[0,h))^{n_z} \to (L_\infty[0,h'))^{Mn_z} \tag{4.10}$$

$$\mathbf{J}_{M0} = \mathbf{L}_M^{-1}\overline{\mathbf{J}_0'} : (L_\infty[0,h'))^{Mn_w} \to (L_\infty[0,h))^{n_w} \tag{4.11}$$

Remark 4.3 Because $\mathbf{B}_1\mathbf{L}_M^{-1}$ can be described as $A_{dM}'\overline{\mathbf{B}_1'}$, approximating $\mathbf{B}_1\mathbf{L}_M^{-1}$ with $\mathbf{B}_1\mathbf{L}_M^{-1}\overline{\mathbf{J}_0'}$ is equivalent to approximating $\mathbf{B}_1'$ with $\mathbf{B}_1'\mathbf{J}_0' = \mathbf{B}_{\mathrm{i}0}'$, which is defined as (3.32) (i.e., approximating the input of $\mathbf{B}_1'$ with a constant function). Similarly, approximating $\mathbf{L}_M\mathbf{M}_1$ with $\overline{\mathbf{H}_0'}\mathbf{L}_M\mathbf{M}_1$ is equivalent to approximating $\mathbf{M}_1'$ with $\mathbf{H}_0'\mathbf{M}_1' = \mathbf{M}_{\mathrm{a}0}'$, which is defined as (3.33) (i.e., approximating the output of $\mathbf{M}_1'$ with a constant function).

Next, to facilitate the treatment of $\mathbf{L}_M\mathbf{D}_{11}\mathbf{L}_M^{-1}$ in (4.8), we again note from (3.26) that $\mathbf{L}_M\mathbf{D}_{11}\mathbf{L}_M^{-1} (= \mathbf{L}_M\mathcal{D}\mathbf{L}_M^{-1})$ is described by

$$\mathbf{L}_M\mathbf{D}_{11}\mathbf{L}_M^{-1} = \overline{\mathbf{M}_1'}\Delta_M^0\overline{\mathbf{B}_1'} + \overline{\mathbf{D}_{11}'} \tag{4.12}$$

Applying once again the aforementioned ideas to (4.12), we further define the operator

$$\mathbf{D}_{M0} = \overline{\mathbf{H}_0'}\,\overline{\mathbf{M}_1'}\,\Delta_M^0\overline{\mathbf{B}_1'}\,\overline{\mathbf{J}_0'} + \overline{\mathbf{D}_{\mathrm{a}0}'} : (L_\infty[0,h'))^{Mn_w} \to (L_\infty[0,h'))^{Mn_z} \tag{4.13}$$

What has been done up to now is that the input and output of $\mathbf{M}_1\mathcal{F}(P_d,\,\Psi)\mathbf{B}_1$ in (4.8) are approximated by piecewise constant functions, similar treatment has been done on the first term on the right hand side of (4.12), and the second term of (4.12) was approximated by $\overline{\mathbf{D}_{\mathrm{a}0}'}$. This treatment has followed the same arguments in Chapter 3. To summarize, we have introduced the following approximation of $\mathcal{F}(\widehat{P}_M,\,\Psi)$:

$$\mathcal{F}(\widehat{P}_{M0},\,\Psi) := \mathbf{H}_{M0}\mathbf{M}_1\mathcal{F}(P_d,\,\Psi)\mathbf{B}_1\mathbf{J}_{M0} + \mathbf{D}_{M0} \tag{4.14}$$

We call it piecewise constant approximation of the sampled-data system Σ_{SD}, which alleviates the difficulty in designing the discrete-time controller Ψ minimizing $\|\mathcal{F}(\widehat{P},\,\Psi)\| = \|\mathcal{F}(\widehat{P}_M,\,\Psi)\|$, and the mapping between $\widehat{w}_k$ and $\widehat{z}_k$ in (4.14) could be described through Figure 4.3.

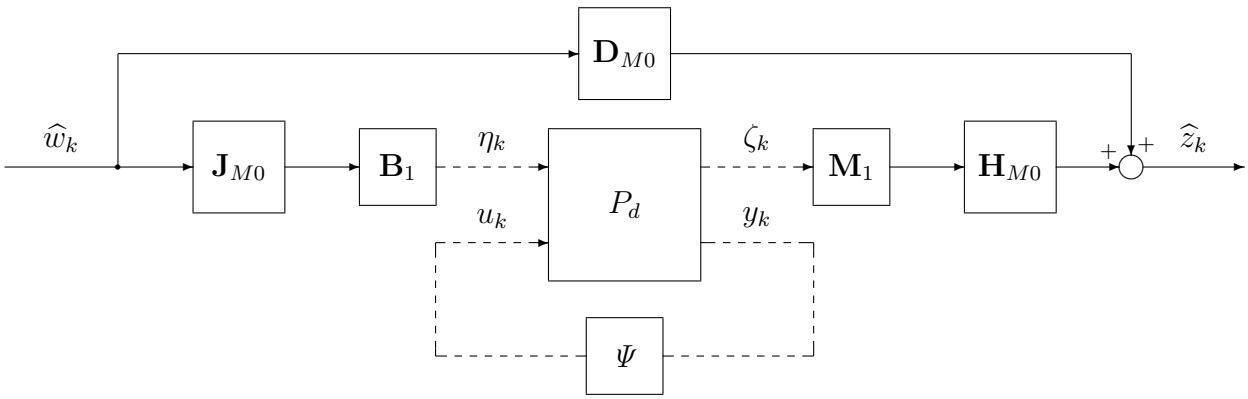


Figure 4.3: Piecewise constant approximation of the sampled-data system Σ_{SD}.

4.4.1 Error Analysis of Piecewise Constant Approximation

This subsection is devoted to showing that the error in the piecewise constant approximation converges to 0 at the rate of $1/M$ as $M \to \infty$ when we design the discrete-time controller Ψ through $\mathcal{F}(\widehat{P}_{M0},\,\Psi)$. To evaluate the error in the approximation of $\|\mathcal{F}(\widehat{P},\,\Psi)\| = \|\mathcal{F}(\widehat{P}_M,\,\Psi)\|$ by $\|\mathcal{F}(\widehat{P}_{M0},\,\Psi)\|$, we first introduce 'finite-rank portions' of $\mathcal{F}(\widehat{P}_M,\,\Psi)$ in (4.8) and $\mathcal{F}(\widehat{P}_{M0},\,\Psi)$ in (4.14) given respectively by

$$\mathcal{F}^0(\widehat{P}_M,\,\Psi) := \mathbf{L}_M\mathbf{M}_1\mathcal{F}(P_d,\,\Psi)\mathbf{B}_1\mathbf{L}_M^{-1} = \mathcal{F}(\widehat{P}_M,\,\Psi) - \mathbf{L}_M\mathbf{D}_{11}\mathbf{L}_M^{-1} \tag{4.15}$$

$$\mathcal{F}^0(\widehat{P}_{M0},\,\Psi) := \mathbf{H}_{M0}\mathbf{M}_1\mathcal{F}(P_d,\,\Psi)\mathbf{B}_1\mathbf{J}_{M0} = \mathcal{F}(\widehat{P}_{M0},\,\Psi) - \mathbf{D}_{M0} \tag{4.16}$$

By comparing the above equations, we could see that evaluating $\mathbf{J}_{M0} - \mathbf{L}_M^{-1}$ and $\mathbf{H}_{M0} - \mathbf{L}_M$ is important in the error analysis. The following lemma is relevant to such evaluation and plays a key role in our discussions.

Lemma 4.1 Suppose that (A, B_1) is controllable and (C_0, A_2) is observable, where these matrices are relevant to the continuous-time generalized plant P in (3.1). Then, we have the following properties regarding the preadjoints $\mathbf{B}_{1*}$ and $\mathbf{J}_{M0*}$ and the operators $\mathbf{M}_1$ and $\mathbf{H}_{M0}$.

a) There exists a constant K_{B0} such that

$$\|(\mathbf{L}_M - \mathbf{J}_{M0*})|_{R(\mathbf{B}_{1*})}\|_1 \leq \frac{K_{B0}}{M} \tag{4.17}$$

where $R(\mathbf{B}_{1*})$ denotes the range of $\mathbf{B}_{1*}$ as mentioned in Chapter 1 and is viewed as a subset of $(L_1[0,h))^{n_w}$.

b) There exists a constant K_{C0} such that

$$\|(\mathbf{L}_M - \mathbf{H}_{M0})|_{R(\mathbf{M}_1)}\| \leq \frac{K_{C0}}{M} \tag{4.18}$$

where $R(\mathbf{M}_1)$ denotes the range of $\mathbf{M}_1$ as mentioned in Chapter 1 and is viewed as a subset of $(L_\infty[0,h))^{n_z}$.

Remark 4.4 The two norms $\|\cdot\|_1$ and $\|\cdot\|$ in Lemma 4.1 mean the $L_1[0,h')$-induced norm and the $L_\infty[0,h')$-induced norm, respectively. From the definition of the preadjoint in Chapter 1, $\mathbf{B}_{1*} : \mathbb{R}_1^n \to (L_1[0,h'))^{n_w}$ and $\mathbf{J}_{M0*} : (L_1[0,h))^{n_w} \to (L_1[0,h'))^{Mn_w}$ are given respectively by

$$(\mathbf{B}_{1*}x)(\theta') := B_1^T \exp(A^T(h-\theta'))x \quad (0 \leq \theta' < h') \tag{4.19}$$

$$\mathbf{J}_{M0*} := \overline{\mathbf{J}'_{0*}}\mathbf{L}_M \tag{4.20}$$

where the preadjoint $\mathbf{J}'_{0*} : (L_1[0,h'))^{n_w} \to (L_1[0,h'))^{n_w}$ is given by

$$(\mathbf{J}'_{0*}w)(\theta') = \frac{1}{h'}\int_0^{h'} w(\tau')d\tau' \quad (0 \leq \theta' < h') \tag{4.21}$$

Remark 4.5 If we note (4.20), it is not hard to see that the claim (4.17) can be roughly restated as the assertion that $\|(I-\overline{\mathbf{J}'_{0*}})\mathbf{L}_M\mathbf{B}_{1*}\|_1 = \|\mathbf{B}_1\mathbf{L}_M^{-1}(I-\overline{\mathbf{J}'_0})\|$ can be made arbitrarily small with the order $1/M$ as M tends to ∞. Roughly speaking, the latter assertion could be interpreted as a result of the following observation: if M is large enough, the input of $\mathbf{B}_1$ could be approximated by a piecewise constant function with M segments, causing only slight effects on its output, because of the 'low pass' nature of the integral operator $\mathbf{B}_1$. The claim (4.18) also has a similar interpretation. Regarding rigorous proof of Lemma 4.1, we mostly follow similar arguments to the proof of Lemma 4 in [3], which is a result concerned with the FSFH approximation technique [46]. In [3], however, integral inequalities are used to establish the associated convergence rate (see (18), (19) and (20) in [3] for details). We do not need to use these integral inequalities for establishing (4.17) and (4.18), and the proof of Lemma 4.1 could be established instead by a Taylor expansion technique. Similarly for Lemma 4.3 relevant to piecewise linear approximation given in a subsequent subsection.

Remark 4.6 The controllability and observability assumptions in Lemma 4.1 are only for the ease in the proof, and can in fact be removed. This is because we can always replace these pairs with controllable and observable ones, without changing the ranges $R(\mathbf{B}_{1*})$ and $R(\mathbf{M}_1)$. However, the proof of Lemma 4.1 is omitted because it is essentially the same as that of Lemma 4.3, which is associated with the piecewise linear approximation.

We have the following important result from Lemma 4.1.

Proposition 4.1 There exists a constant K_0 independent of Ψ, such that

$$\|\mathcal{F}^0(\widehat{P}_{M0},\, \Psi) - \mathcal{F}^0(\widehat{P}_M,\, \Psi)\| \leq \frac{K_0}{M}\|\mathcal{F}^0(\widehat{P}_M,\, \Psi)\| \tag{4.22}$$

Remark 4.7 A similar inequality with $\mathcal{F}^0(\widehat{P}_M,\, \Psi)$ replaced by $\mathcal{F}^0(\widehat{P}_{M0},\, \Psi)$ on the right hand side is established in [3] by using the arguments of the integral inequalities mentioned in Remark 4.5 together with slightly modified versions of inequalities (4.17) and (4.18). However, the proof of the modified inequalities becomes more involved. We would like to stress that our following arguments successfully utilize the simpler inequality (4.22) and circumvent such involved arguments.

Proof. We first deal with the approximation on the output side. Let us introduce the following 'finite-rank portion' of $\mathcal{F}(\widehat{P},\, \Psi)$:

$$\mathcal{F}^0(\widehat{P},\, \Psi) := \mathbf{M}_1 \mathcal{F}(P_d,\, \Psi)\mathbf{B}_1 \tag{4.23}$$

From the second assertion of Lemma 4.1, we have

$$\begin{aligned}
&\|\mathcal{F}^0(\widehat{P}_M,\, \Psi) - \mathbf{H}_{M0}\mathcal{F}^0(\widehat{P},\, \Psi)\mathbf{L}_M^{-1}\| \\
=&\|\mathbf{L}_M\mathbf{M}_1\mathcal{F}(P_d,\, \Psi)\mathbf{B}_1\mathbf{L}_M^{-1} - \mathbf{H}_{M0}\mathbf{M}_1\mathcal{F}(P_d,\, \Psi)\mathbf{B}_1\mathbf{L}_M^{-1}\| \\
=&\|(\mathbf{L}_M - \mathbf{H}_{M0})|_{R(\mathbf{M}_1)}\, \mathbf{M}_1\mathcal{F}(P_d,\, \Psi)\mathbf{B}_1\mathbf{L}_M^{-1}\| \\
\leq&\frac{K_{C0}}{M}\|\mathbf{M}_1\mathcal{F}(P_d,\, \Psi)\mathbf{B}_1\mathbf{L}_M^{-1}\| \\
=&\frac{K_{C0}}{M}\|\mathcal{F}^0(\widehat{P}_M,\, \Psi)\|
\end{aligned} \tag{4.24}$$

In particular, this implies

$$\|\mathbf{H}_{M0}\mathcal{F}^0(\widehat{P},\, \Psi)\mathbf{L}_M^{-1}\| \leq \left(1 + \frac{K_{C0}}{M}\right)\|\mathcal{F}^0(\widehat{P}_M,\, \Psi)\| \tag{4.25}$$

We next deal with the approximation on the input side. It follows from the first assertion of Lemma 4.1 that

$$\begin{aligned}
&\|\mathbf{H}_{M0}\mathcal{F}^0(\widehat{P},\, \Psi)\mathbf{L}_M^{-1} - \mathcal{F}^0(\widehat{P}_{M0},\, \Psi)\| \\
=&\|\mathbf{H}_{M0}\mathbf{M}_1\mathcal{F}(P_d,\, \Psi)\mathbf{B}_1\mathbf{L}_M^{-1} - \mathbf{H}_{M0}\mathbf{M}_1\mathcal{F}(P_d,\, \Psi)\mathbf{B}_1\mathbf{J}_{M0}\| \\
=&\|(\mathbf{H}_{M0}\mathbf{M}_1\mathcal{F}(P_d,\, \Psi))\mathbf{B}_1(\mathbf{L}_M^{-1} - \mathbf{J}_{M0})\| \\
=&\|(\mathbf{L}_M - \mathbf{J}_{M0*})\mathbf{B}_{1*}(\mathbf{H}_{M0}\mathbf{M}_1\mathcal{F}(P_d,\, \Psi))_*\|_1 \\
=&\|(\mathbf{L}_M - \mathbf{J}_{M0*})|_{R(\mathbf{B}_{1*})}(\mathbf{H}_{M0}\mathbf{M}_1\mathcal{F}(P_d,\, \Psi)\mathbf{B}_1)_*\|_1 \\
\leq&\frac{K_{B0}}{M}\|\mathbf{H}_{M0}\mathbf{M}_1\mathcal{F}(P_d,\, \Psi)\mathbf{B}_1\| \\
=&\frac{K_{B0}}{M}\|\mathbf{H}_{M0}\mathcal{F}^0(\widehat{P},\, \Psi)\mathbf{L}_M^{-1}\|
\end{aligned} \tag{4.26}$$

Combining (4.24), (4.26) and (4.25) leads to

$$\begin{aligned}
&\|\mathcal{F}^0(\widehat{P}_{M0},\, \Psi) - \mathcal{F}^0(\widehat{P}_M,\, \Psi)\| \\
\leq&\|\mathcal{F}^0(\widehat{P}_M,\, \Psi) - \mathbf{H}_{M0}\mathcal{F}^0(\widehat{P},\, \Psi)\mathbf{L}_M^{-1}\| + \|\mathbf{H}_{M0}\mathcal{F}^0(\widehat{P},\, \Psi)\mathbf{L}_M^{-1} - \mathcal{F}^0(\widehat{P}_{M0},\, \Psi)\| \\
\leq&\frac{K_{C0}}{M}\|\mathcal{F}^0(\widehat{P}_M,\, \Psi)\| + \frac{K_{B0}}{M}\|\mathbf{H}_{M0}\mathcal{F}^0(\widehat{P},\, \Psi)\mathbf{L}_M^{-1}\| \\
\leq&\left\{\frac{K_{C0}}{M} + \frac{K_{B0}}{M}\left(1 + \frac{K_{C0}}{M}\right)\right\}\|\mathcal{F}^0(\widehat{P}_M,\, \Psi)\| \\
\leq&\frac{K_{B0}(1 + K_{C0}) + K_{C0}}{M}\|\mathcal{F}^0(\widehat{P}_M,\, \Psi)\| =: \frac{K_0}{M}\|\mathcal{F}^0(\widehat{P}_M,\, \Psi)\|
\end{aligned} \tag{4.27}$$

This completes the proof. Q.E.D.

In view of (4.15) and (4.16), it is also important to evaluate $\mathbf{D}_{M0} - \mathbf{L}_M \mathbf{D}_{11} \mathbf{L}_M^{-1}$, for which we quote the result of Lemma 3.3 as follows.

Lemma 4.2 The inequality

$$\|\mathbf{D}_{M0} - \mathbf{L}_M \mathbf{D}_{11} \mathbf{L}_M^{-1}\| \leq \frac{K_{\mathcal{D}\mathrm{i}0}^U}{M} \tag{4.28}$$

holds, where $K_{\mathcal{D}\mathrm{i}0}^U$ is defined as (3.40).

We are in a position to give the following main result on the error analysis of piecewise constant approximation.

Theorem 4.1 The following inequality holds:

$$\left(1 - \frac{K_0}{M}\right) \|\mathcal{F}(\widehat{P}, \Psi)\| - \frac{K_{\mathcal{D}\mathrm{i}0}^U}{M} \leq \|\mathcal{F}(\widehat{P}_{M0}, \Psi)\| \leq \|\mathcal{F}(\widehat{P}, \Psi)\| \tag{4.29}$$

Proof. By noting that the piecewise constant approximation is norm-contractive, it readily follows that $\|\mathcal{F}(\widehat{P}_{M0}, \Psi)\| \leq \|\mathcal{F}(\widehat{P}, \Psi)\|$. A key in the proof is to show that $\|\mathcal{F}^0(\widehat{P}, \Psi)\| \leq \|\mathcal{F}(\widehat{P}, \Psi)\|$. This inequality follows from the properties of $L_\infty[0, h)$ if we note that the infinite (Toeplitz) matrix representation of the input/output relation of $\mathcal{F}^0(\widehat{P}, \Psi) = \mathbf{M}_1 \mathcal{F}(P_d, \Psi) \mathbf{B}_1$ is strictly block lower triangular (with respect to the partitioning associated with $\widehat{w}_k$ and $\widehat{z}_k$) because of the structure of P_d (note that (4.4) has no direct feedthrough matrix between η_k and ζ_k); this infinite matrix obviously has no overlap of nonzero entries with the infinite matrix representation of $\mathcal{F}(\widehat{P}, \Psi) - \mathcal{F}^0(\widehat{P}, \Psi) = \mathbf{D}_{11}$ (which is nothing but the infinite block diagonal matrix with all diagonal entries given by $\mathbf{D}_{11}$).

It follows form Proposition 4.1 and Lemma 4.2 that

$$\begin{aligned}
&\|\mathcal{F}(\widehat{P}_{M0}, \Psi) - \mathcal{F}(\widehat{P}_M, \Psi)\| \\
\leq &\|\mathcal{F}^0(\widehat{P}_{M0}, \Psi) - \mathcal{F}^0(\widehat{P}_M, \Psi)\| + \|\mathbf{D}_{M0} - \mathbf{L}_M \mathbf{D}_{11} \mathbf{L}_M^{-1}\| \\
\leq &\frac{K_0}{M} \|\mathcal{F}^0(\widehat{P}_M, \Psi)\| + \frac{K_{\mathcal{D}\mathrm{i}0}^U}{M} \\
\leq &\frac{K_0}{M} \|\mathcal{F}(\widehat{P}_M, \Psi)\| + \frac{K_{\mathcal{D}\mathrm{i}0}^U}{M}
\end{aligned} \tag{4.30}$$

Since $\|\mathcal{F}(\widehat{P}_M, \Psi)\| = \|\mathcal{F}(\widehat{P}, \Psi)\|$, the assertion follows immediately. Q.E.D.

4.4.2 Features of Piecewise Constant Approximation

Here, we provide the validity of the piecewise constant approximation in the L_1 optimal control problem of sampled-data systems by using the assertion in Theorem 4.1.

Theorem 4.1 clearly implies that the approximation error in the piecewise constant approximation converges to 0 at the rate of $1/M$ as $M \to \infty$, when the L_∞-induced norm of the sampled-data system Σ_{SD} is computed for a fixed controller Ψ. Theorem 4.1 also gives a theoretical basis for such an indirect and approximate approach to L_1 optimal controller synthesis for the sampled-data system Σ_{SD} that seeks for Ψ minimizing $\|\mathcal{F}(\widehat{P}_{M0}, \Psi)\|$ for a sufficiently large M. To see this, let

$$\gamma_{\mathrm{opt}} := \inf_{\Psi} \|\mathcal{F}(\widehat{P}, \Psi)\| \tag{4.31}$$

and take an M. Suppose that Ψ_{M0} is an ε-suboptimal controller with respect to $\|\mathcal{F}(\widehat{P}_{M0}, \Psi)\|$, i.e., $\|\mathcal{F}(\widehat{P}_{M0}, \Psi_{M0})\| \leq \gamma_{M0} + \epsilon$ ($\epsilon > 0$), where $\gamma_{M0} := \inf_{\Psi} \|\mathcal{F}(\widehat{P}_{M0}, \Psi)\|$. Let $M_0 \in \mathbb{N}$ be the minimum such that $M_0 > K_0$. Then, for $M \geq M_0$, the first inequality of (4.29) implies that

$$\begin{aligned}
\gamma_{\mathrm{opt}} &\leq \|\mathcal{F}(\widehat{P}, \Psi_{M0})\| \\
&\leq \left(1 + \frac{K_0}{M - K_0}\right) \|\mathcal{F}(\widehat{P}_{M0}, \Psi_{M0})\| + \frac{K^U_{\mathcal{D}\mathrm{i}0}}{M - K_0} \\
&\leq \left(1 + \frac{K_0}{M - K_0}\right) (\gamma_{M0} + \epsilon) + \frac{K^U_{\mathcal{D}\mathrm{i}0}}{M - K_0}
\end{aligned} \tag{4.32}$$

On the other hand, it follows from the second inequality of (4.29) that

$$\gamma_{M0} \leq \gamma_{\mathrm{opt}} \tag{4.33}$$

Substituting this into (4.32) and taking a sufficiently large M such that $M \geq M_0$, we see that

$$\gamma_{\mathrm{opt}} \leq \|\mathcal{F}(\widehat{P}, \Psi_{M0})\| \leq \gamma_{\mathrm{opt}} + \epsilon + \frac{X_0}{M} \tag{4.34}$$

where

$$X_0 := \frac{K_0 \gamma_{\mathrm{opt}}}{1 - K_0/M_0} + \frac{K^U_{\mathcal{D}\mathrm{i}0}}{1 - K_0/M_0} + \frac{K_0 \varepsilon}{1 - K_0/M_0} \tag{4.35}$$

Since $\epsilon > 0$ is arbitrary, letting M sufficiently large and taking a suboptimal Ψ_{M0} with respect to $\|\mathcal{F}(\widehat{P}_{M0}, \Psi)\|$ (sufficiently close to the infimal performance) is ensured to lead to a method for L_1 optimal controller synthesis for the sampled-data system Σ_{SD}. By (4.34), we could say that the convergence of Ψ_{M0} is in the order of $1/M$. We will further show in Section 4.5 that the synthesis problem of a suboptimal Ψ_{M0} can be equivalently reduced to a discrete-time l_1 optimal control problem.

4.5 Main Results in Piecewise Constant Approximation: Reduction to the Discrete-Time l_1 Optimal Control Problem

The preceding subsection showed a mathematical basis of the piecewise constant approximation in the L_1 optimal sampled-data controller synthesis problem. To exploit this approximation scheme, we need to have an explicit method for computing $\|\mathcal{F}(\widehat{P}_{M0}, \Psi)\|$, and this section provides such a method. More precisely, we derive a discretized generalized plant (for the continuous-time generalized plant P) that is useful for computing $\|\mathcal{F}(\widehat{P}_{M0}, \Psi)\|$. We further show that the discretized generalized plant together with the associated error analysis converts the synthesis problem of an L_1 optimal controller Ψ for the sampled-data system Σ_{SD} into the discrete-time synthesis problem of an l_1 optimal controller.

To derive such a discretized generalized plant, we need to consider replacing the operators $\mathbf{B}_1\mathbf{J}_{M0}$, $\mathbf{H}_{M0}\mathbf{M}_1$ and $\mathbf{D}_{M0}$ with finite-dimensional matrices. Here, we note that $\|\mathcal{F}(\widehat{P}_{M0}, \Psi)\|$ coincides with $\|P^{-}_{NMi0}\|$ in (3.57) when we take $N \to \infty$. In connection with this, we could derive from (3.31) and (3.57) the matrices B_{M10}, H_{M0} and D_{M10} defined respectively by

$$B_{M10} = A'_{dM}\overline{B'_{0d}} \tag{4.36}$$

$$H_{M0} = \overline{C_0}A'_{2dM} \tag{4.37}$$

$$D_{M10} = \begin{bmatrix}\overline{D_{11}} & \overline{C_0}\Delta^0_M\overline{B'_{0d}}\end{bmatrix} \tag{4.38}$$

and we easily see that $\|\mathcal{F}(\widehat{P}_{M0}, \Psi)\|$ coincides with the ∞-norm of the infinite-dimensional matrix

$$P_{M0} := \begin{bmatrix}D_{M10} & H_{M0}C_\Sigma J_\Sigma B_{M10} & H_{M0}C_\Sigma \mathcal{A} J_\Sigma B_{M10} & H_{M0}C_\Sigma \mathcal{A}^2 J_\Sigma B_{M10} & \cdots\end{bmatrix} \tag{4.39}$$

Here, we further note again that Δ^0_M is strictly block lower triangular. Based on the property of l_∞, the matrix D_{M10} may be redefined as

$$D_{M10} := \overline{C_0}\Delta^0_M\overline{B'_{0d}} + \overline{D_{11}} \tag{4.40}$$

without changing $\|P_{M0}\|$. The matrix P_{M0} with the modified form in (4.40) corresponds to the "last block row" of the input/output relation of a discrete-time system. Thus, the L_1 problem of $\mathcal{F}(\widehat{P}_{M0}, \Psi)$ is reducible to the following discrete-time l_1 problem.

Let us introduce the discrete-time plant given by

$$P_{M0d} : \begin{cases} x_{k+1} = A_d x_k + B_{M10} w_k + B_{2d} u_k \\ z_k = C_{M10} x_k + D_{M10} w_k + D_{M20} u_k \\ y_k = C_{2d} x_k \end{cases} \tag{4.41}$$

where the matrices $C_{M10} \in \mathbb{R}_{\infty}^{Mn_z \times n}$ and $D_{M20} \in \mathbb{R}_{\infty}^{Mn_z \times n_u}$ are given respectively by

$$\begin{bmatrix} C_{M10} & D_{M20} \end{bmatrix} := H_{M0} \tag{4.42}$$

The other matrices are given in (3.10), (4.36) and (4.40). Then, we readily see that the closed-loop system obtained by connecting Ψ to the above discrete-time plant P_{M0d} has the state-space representation $(\mathcal{A}, J_{\Sigma}B_{M10}, H_{M0}C_{\Sigma}, D_{M10})$. Since the "last block row" of the infinite (Toeplitz) matrix representation of the input/output relation of this closed-loop system is nothing but P_{M0} given in (4.39) with the modified form in (4.40), it follows readily that computing $\|P_{M0}\|$ is equivalent to computing the l_{∞}-induced norm of the above closed-loop system as shown in Figure 4.4. This implies that the L_1 problem of $\mathcal{F}(\widehat{P}_{M0}, \Psi)$ is exactly reducible to the discrete-time l_1 problem for the discretized generalized plant P_{M0d} in (4.41).

Let us denote by $\|\mathcal{F}(P_{M0d}, \Psi)\|$ the discrete-time l_{∞}-induced norm. Then, by the preceding arguments, $\|\mathcal{F}(P_{M0d}, \Psi)\|$ coincides with $\|\mathcal{F}(\widehat{P}_{M0}, \Psi)\|$. More precisely, we have the following result regarding approximately solving the L_1 optimal sampled-data controller synthesis problem such as $\inf_{\Psi} \|\mathcal{F}(\widehat{P}, \Psi)\|$ through the discrete-time l_1 optimal controller synthesis problem such as $\inf_{\Psi} \|\mathcal{F}(P_{M0d}, \Psi)\|$.

Theorem 4.2 The following inequality holds:

$$\left(1 - \frac{K_0}{M}\right) \|\mathcal{F}(\widehat{P}, \Psi)\| - \frac{K_{\mathcal{D}\mathrm{i}0}^{U}}{M} \leq \|\mathcal{F}(P_{M0d}, \Psi)\| \leq \|\mathcal{F}(\widehat{P}, \Psi)\| \tag{4.43}$$

Even though the central part of (4.43) (i.e., $\|\mathcal{F}(P_{M0d}, \Psi)\|$) essentially coincides with a conventional method in [3] via FSFH approximation [46] after all, the above inequality (4.43) is different from that in [3]. As mentioned in Remarks 4.5 and 4.7, we do not need to use the involved arguments of the integral inequalities in [3] for establishing (4.17), (4.18) and (4.22) as well as (4.43), and we successfully circumvent such involved arguments in this chapter.

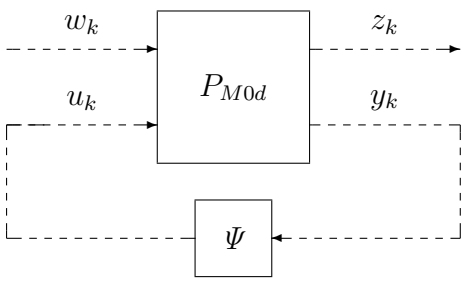


Figure 4.4: Discrete-time system obtained by piecewise constant approximation.

4.6 Piecewise Linear Approximation to the L_1 Optimal Control Sampled-Data Controller Synthesis Problem

This section is devoted to providing a method for the L_1 optimal sampled-data controller synthesis by using the idea of piecewise linear approximation (via the input approximation approach) introduced in Chapter 3. We first introduce the operator $\mathbf{J}_1' : (L_\infty[0,h'))^{n_w} \to (L_\infty[0,h'))^{n_w}$ defined as (2.29). $\mathbf{J}_1'$ is used to approximate the input of $\mathbf{B}_1$ with a piecewise linear function and is tailored to possess important properties in terms of some Taylor expansion arguments (see Section 3.9 for details), in addition to the property that $\mathbf{J}_1' w = w$ for any linear function w. We further introduce the operator $\mathbf{H}_1' : (L_\infty[0,h'))^{n_z} \to (L_\infty[0,h'))^{n_z}$ described by

$$(\mathbf{H}_1' z)(\theta') = z(0) + \theta' \frac{dz(\theta')}{d\tau'}|_{\theta'=0} \quad (0 \le \theta' < h') \tag{4.44}$$

as well as the operator $\mathbf{D}_{\mathrm{a1}}' : (L_\infty[0,h'))^{n_w} \to (L_\infty[0,h'))^{n_z}$ given in (3.60). Obviously, $\mathbf{H}_1'$ could be interpreted as an operator producing a linear function that preserves the value and derivative of the input at $\theta' = 0$, and thus $\mathbf{H}_1' z = z$ for any linear function z on $[0,h')$. Similarly for $\mathbf{H}_0'$ in Section 4.4, strictly speaking, $\mathbf{H}_1'$ is not an operator on $(L_\infty[0,h'))^{n_z}$ but on its subspace of functions continuous and (right) differentiable at time 0. However, this issue causes no problems since $\mathbf{H}_1'$ is used for approximating $\mathbf{M}_1$ (or its output with a piecewise linear function) and operates only on its output. However, similarly for $\mathbf{J}_0'$, approximating $\mathbf{B}_1$ should take into account that its input may be discontinuous, and this leads to the more involved definition of $\mathbf{J}_1'$ in (2.29). $\mathbf{D}_{\mathrm{a1}}'$ is used for approximating $\mathbf{D}_{11}$ as in Chapter 3, and it corresponds to applying constant approximation to the kernel function associated with the compact portion of the compression operator defined on $[0,h')$. The details of our approximation treatment employing these operators are as follows.

We consider replacing $\mathbf{L}_M\mathbf{M}_1$ and $\mathbf{B}_1\mathbf{L}_M^{-1}$ in (4.8) with $\overline{\mathbf{H}_1'}\mathbf{L}_M\mathbf{M}_1$ and $\mathbf{B}_1\mathbf{L}_M^{-1}\overline{\mathbf{J}_1'}$, respectively, (i.e., $\mathbf{M}_1$ and $\mathbf{B}_1$ are approximated by $\mathbf{L}_M^{-1}\overline{\mathbf{H}_1'}\mathbf{L}_M\mathbf{M}_1$ and $\mathbf{B}_1\mathbf{L}_M^{-1}\overline{\mathbf{J}_1'}\mathbf{L}_M$, respectively). To facilitate such treatment, we introduce the operators $\mathbf{H}_{M1}$ and $\mathbf{J}_{M1}$ described respectively by

$$\mathbf{H}_{M1} = \overline{\mathbf{H}_1'}\mathbf{L}_M : (L_\infty[0,h))^{n_z} \to (L_\infty[0,h'))^{Mn_z} \tag{4.45}$$

$$\mathbf{J}_{M1} = \mathbf{L}_M^{-1}\overline{\mathbf{J}_1'} : (L_\infty[0,h'))^{Mn_w} \to (L_\infty[0,h))^{n_w} \tag{4.46}$$

Remark 4.8 Approximating $\mathbf{B}_1\mathbf{L}_M^{-1}$ with $\mathbf{B}_1\mathbf{L}_M^{-1}\overline{\mathbf{J}_1'}$ is equivalent to approximating $\mathbf{B}_1'$ with $\mathbf{B}_1'\mathbf{J}_1' = \mathbf{B}_{\mathrm{i1}}'$, which is defined as (3.58) (i.e., approximating the input of $\mathbf{B}_1'$ with a linear

function). Similarly, approximating $\mathbf{L}_M\mathbf{M}_1$ with $\overline{\mathbf{H}_1'}\mathbf{L}_M\mathbf{M}_1$ is a equivalent to approximating $\mathbf{M}_1'$ with $\mathbf{H}_1'\mathbf{M}_1' = \mathbf{M}_{\mathrm{a}1}'$, which is defined as (3.59) (i.e., approximating the output of $\mathbf{M}_1'$ with a linear function).

Next, to facilitate the treatment of $\mathbf{L}_M\mathbf{D}_{11}\mathbf{L}_M^{-1}$ in (4.8), we define the operator

$$\mathbf{D}_{M1} = \overline{\mathbf{H}_1'}\,\overline{\mathbf{M}_1'}\Delta_M^0\overline{\mathbf{B}_1'}\,\overline{\mathbf{J}_1'} + \overline{\mathbf{D}_{\mathrm{a}1}'} : (L_\infty[0.h'))^{Mn_w} \to (L_\infty[0,h'))^{Mn_z} \tag{4.47}$$

What has been done in the above treatment is that the input and output of $\mathbf{M}_1\mathcal{F}(P_d,\,\Psi)\mathbf{B}_1$ in (4.8) are approximated by piecewise linear functions, similar treatment has been done on the first term of (4.12), and the second term of (4.12) was approximated by $\overline{\mathbf{D}_{\mathrm{a}1}'}$. This treatment has followed the same arguments in Chapter 3. To summarize, we have introduced the following approximation of $\mathcal{F}(\widehat{P}_M,\,\Psi)$:

$$\mathcal{F}(\widehat{P}_{M1},\,\Psi) := \mathbf{H}_{M1}\mathbf{M}_1\mathcal{F}(P_d,\,\Psi)\mathbf{M}_1\mathbf{J}_{M1} + \mathbf{D}_{M1} \tag{4.48}$$

We call it piecewise linear approximation of the sampled-data system Σ_{SD}, which alleviates the difficulty in designing the discrete-time controller Ψ minimizing $\|\mathcal{F}(\widehat{P},\,\Psi)\| = \|\mathcal{F}(\widehat{P}_M,\,\Psi)\|$, and the mapping between $\widehat{w}_k$ and $\widehat{z}_k$ in (4.48) could be described through Figure 4.5.

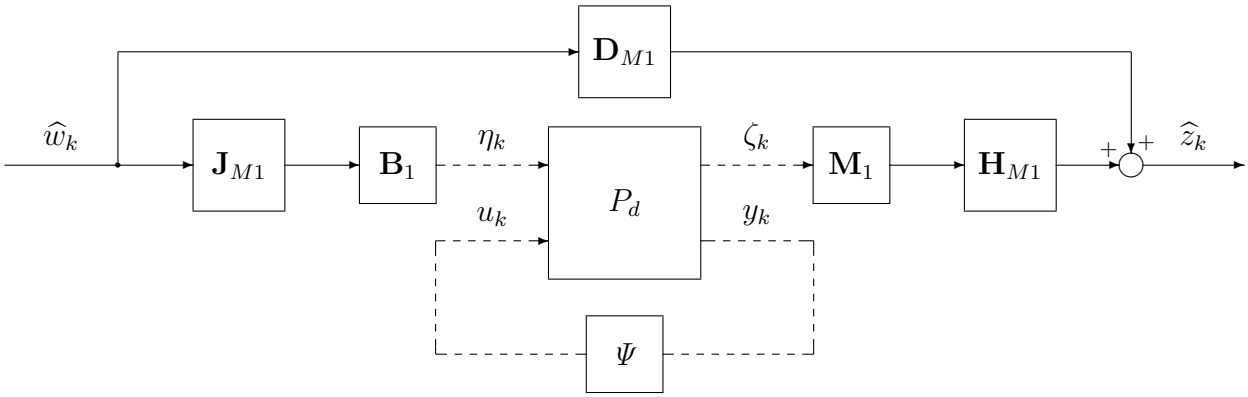


Figure 4.5: Piecewise linear approximation of the sampled-data system Σ_{SD}.

4.6.1 Error Analysis of Piecewise Linear Approximation

This subsection is devoted to showing that the error in piecewise linear approximation converges to 0 at the rate of $1/M^2$ as $M \to \infty$. To evaluate the error in the approximation

of $\|\mathcal{F}(\widehat{P}, \Psi)\| = \|\mathcal{F}(\widehat{P}_M, \Psi)\|$ by $\|\mathcal{F}(\widehat{P}_{M1}, \Psi)\|$, we first introduce 'finite-rank portion' of $\mathcal{F}(\widehat{P}_{M1}, \Psi)$ in (4.48) given by

$$\mathcal{F}^0(\widehat{P}_{M1}, \Psi) := \mathbf{H}_{M1}\mathbf{M}_1\mathcal{F}(P_d, \Psi)\mathbf{B}_1\mathbf{J}_{M1} = \mathcal{F}(\widehat{P}_{M1}, \Psi) - \mathbf{D}_{M1} \tag{4.49}$$

Comparing the above equation with $\mathcal{F}^0(\widehat{P}_M, \Psi)$ in (4.15), we see that evaluating $\mathbf{J}_{M1} - \mathbf{L}_M^{-1}$ and $\mathbf{H}_{M1} - \mathbf{L}_M$ is important in the error analysis. The following lemma is relevant to such evaluation and plays a key role in our discussions.

Lemma 4.3 Suppose that (A, B_1) is controllable and (C_0, A_2) is observable, where these matrices are relevant to the continuous-time generalized plant P in (3.1). Then, we have the following properties regarding the preadjoints $\mathbf{J}_{M1*}$ and $\mathbf{B}_{1*}$ and the operators $\mathbf{H}_{M1}$ and $\mathbf{M}_1$.

a) There exists a constant K_{B1} such that

$$\|(\mathbf{L}_M - \mathbf{J}_{M1*})|_{R(\mathbf{B}_{1*})}\|_1 \leq \frac{K_{B1}}{M^2} \tag{4.50}$$

where $R(\mathbf{B}_{1*})$ denotes the range of $\mathbf{B}_{1*}$ as mentioned in Chapter 1 and is viewed as a subset of $(L_1[0, h))^{n_w}$.

b) There exists a constant K_{C1} such that

$$\|(\mathbf{L}_M - \mathbf{H}_{M1})|_{R(\mathbf{M}_1)}\| \leq \frac{K_{C1}}{M^2} \tag{4.51}$$

where $R(\mathbf{M}_1)$ denotes the range of $\mathbf{M}_1$ as mentioned in Chapter 1 and is viewed as a subset of $(L_\infty[0, h))^{n_z}$.

Remark 4.9 The two norms $\|\cdot\|_1$ and $\|\cdot\|$ in Lemma 4.3 mean the $L_1[0, h')$-induced norm and the $L_\infty[0, h')$-induced norm, respectively. From the definition of the preadjoint in Chapter 1, $\mathbf{J}_{M1*} : (L_1[0, h))^{n_w} \to (L_1[0, h'))^{Mn_w}$ is given by

$$\mathbf{J}_{M1*} := \overline{\mathbf{J}'_{1*}}\mathbf{L}_M \tag{4.52}$$

where the preadjoint $\mathbf{J}'_{1*} : (L_1[0, h'))^{n_w} \to (L_1[0, h'))^{n_w}$ is given by

$$(\mathbf{J}'_{1*}w)(\theta') = f_0(\theta')\int_0^{h'} w(\tau')d\tau' + f_1(\theta')\int_0^{h'} \tau' w(\tau')d\tau' \tag{4.53}$$

Remark 4.10 If we note (4.52), it is not hard to see that the claim (4.50) can be roughly restated as the assertion that $\|(I - \overline{\mathbf{J}'_{1*}})\mathbf{L}_M\mathbf{B}_{1*}\|_1 = \|\mathbf{B}_1\mathbf{L}_M^{-1}(I - \overline{\mathbf{J}'_1})\|$ can be made arbitrarily small with the order $1/M^2$ as M tends to ∞. The claim (4.51) also has a similar interpretation.

The proof of Lemma 4.3 is given in Subsection 4.10.1 since it is very technical. We have the following important result from Lemma 4.3.

Proposition 4.2 There exists a constant K_1 independent of Ψ, such that

$$\|\mathcal{F}^0(\widehat{P}_{M1}, \Psi) - \mathcal{F}^0(\widehat{P}_M, \Psi)\| \leq \frac{K_1}{M^2}\|\mathcal{F}^0(\widehat{P}_M, \Psi)\| \tag{4.54}$$

Remark 4.11 Similarly for Lemma 4.1, we could also establish a similar inequality with $\mathcal{F}^0(\widehat{P}_M, \Psi)$ replaced by $\mathcal{F}^0(\widehat{P}_{M1}, \Psi)$ on the right hand side, if we use slightly modified versions of inequalities (4.50) and (4.51) (see (28), (29) and (32) in [52] for details). However, the proof of the modified inequalities becomes more involved. We stress that our following arguments successfully utilize the simpler inequality (4.54) and circumvent such involved arguments.

Proof. We first deal with the approximation on the output side. From the second assertion of Lemma 4.3, we have

$$\begin{aligned}
&\|\mathcal{F}^0(\widehat{P}_M, \Psi) - \mathbf{H}_{M1}\mathcal{F}^0(\widehat{P}, \Psi)\mathbf{L}_M^{-1}\| \\
=&\|\mathbf{L}_M\mathbf{M}_1\mathcal{F}(P_d, \Psi)\mathbf{B}_1\mathbf{L}_M^{-1} - \mathbf{H}_{M1}\mathbf{M}_1\mathcal{F}(P_d, \Psi)\mathbf{B}_1\mathbf{L}_M^{-1}\| \\
=&\|(\mathbf{L}_M - \mathbf{H}_{M1})|_{R(\mathbf{M}_1)}\,\mathbf{M}_1\mathcal{F}(P_d, \Psi)\mathbf{B}_1\mathbf{L}_M^{-1}\| \\
\leq&\frac{K_{C1}}{M^2}\|\mathbf{M}_1\mathcal{F}(P_d, \Psi)\mathbf{B}_1\mathbf{L}_M^{-1}\| \\
=&\frac{K_{C1}}{M^2}\|\mathcal{F}^0(\widehat{P}_M, \Psi)\|
\end{aligned} \tag{4.55}$$

In particular, this implies

$$\|\mathbf{H}_{M1}\mathcal{F}^0(\widehat{P}, \Psi)\mathbf{L}_M^{-1}\| \leq \left(1 + \frac{K_{C1}}{M^2}\right)\|\mathcal{F}^0(\widehat{P}_M, \Psi)\| \tag{4.56}$$

We next deal with the approximation on the input side. It follows from the first assertion of Lemma 4.3 that

$$
\begin{aligned}
&\|\mathbf{H}_{M1}\mathcal{F}^0(\widehat{P},\,\Psi)\mathbf{L}_M^{-1}-\mathcal{F}^0(\widehat{P}_{M1},\,\Psi)\| \\
=&\|\mathbf{H}_{M1}\mathbf{M}_1\mathcal{F}(P_d,\,\Psi)\mathbf{B}_1\mathbf{L}_M^{-1}-\mathbf{H}_{M1}\mathbf{M}_1\mathcal{F}(P_d,\,\Psi)\mathbf{B}_1\mathbf{J}_{M1}\| \\
=&\|(\mathbf{H}_{M1}\mathbf{M}_1\mathcal{F}(P_d,\,\Psi))\mathbf{B}_1(\mathbf{L}_M^{-1}-\mathbf{J}_{M1})\| \\
=&\|(\mathbf{L}_M-\mathbf{J}_{M1*})\mathbf{B}_{1*}(\mathbf{H}_{M1}\mathbf{M}_1\mathcal{F}(P_d,\,\Psi))_*\|_1 \\
=&\|(\mathbf{L}_M-\mathbf{J}_{M1*})|_{R(\mathbf{B}_{1*})}(\mathbf{H}_{M1}\mathbf{M}_1\mathcal{F}(P_d,\,\Psi)\mathbf{B}_1)_*\|_1 \\
\leq&\frac{K_{B1}}{M^2}\|\mathbf{H}_{M1}\mathbf{M}_1\mathcal{F}(P_d,\,\Psi)\mathbf{B}_1\| \\
=&\frac{K_{B1}}{M^2}\|\mathbf{H}_{M1}\mathcal{F}^0(\widehat{P},\,\Psi)\mathbf{L}_M^{-1}\|
\end{aligned}
\tag{4.57}
$$

Combining (4.55), (4.57) and (4.56) leads to

$$
\begin{aligned}
&\|\mathcal{F}^0(\widehat{P}_{M1},\,\Psi)-\mathcal{F}^0(\widehat{P}_M,\,\Psi)\| \\
\leq&\|\mathcal{F}^0(\widehat{P}_M,\,\Psi)-\mathbf{H}_{M1}\mathcal{F}^0(\widehat{P},\,\Psi)\mathbf{L}_M^{-1}\|+\|\mathbf{H}_{M1}\mathcal{F}^0(\widehat{P},\,\Psi)\mathbf{L}_M^{-1}-\mathcal{F}^0(\widehat{P}_{M1},\,\Psi)\| \\
\leq&\frac{K_{C1}}{M^2}\|\mathcal{F}^0(\widehat{P}_M,\,\Psi)\|+\frac{K_{B1}}{M^2}\|\mathbf{H}_{M1}\mathcal{F}^0(\widehat{P},\,\Psi)\mathbf{L}_M^{-1}\| \\
\leq&\left\{\frac{K_{C1}}{M^2}+\frac{K_{B1}}{M^2}\left(1+\frac{K_{C1}}{M^2}\right)\right\}\|\mathcal{F}^0(\widehat{P}_M,\,\Psi)\| \\
\leq&\frac{K_{B1}(1+K_{C1})+K_{C1}}{M^2}\|\mathcal{F}^0(\widehat{P}_M,\,\Psi)\|=:\frac{K_1}{M^2}\|\mathcal{F}^0(\widehat{P}_M,\,\Psi)\|
\end{aligned}
\tag{4.58}
$$

This completes the proof. Q.E.D.

In view of (4.15) and (4.49), it is also important to evaluate $\mathbf{D}_{M1}-\mathbf{L}_M\mathbf{D}_{11}\mathbf{L}_M^{-1}$, for which we quote the result of Lemma 3.6 as follows.

Lemma 4.4 The inequality

$$
\|\mathbf{D}_{M1}-\mathbf{L}_M\mathbf{D}_{11}\mathbf{L}_M^{-1}\|\leq\frac{K_{\mathcal{D}\mathrm{i}1}^U}{M^2}
\tag{4.59}
$$

holds, where $K_{\mathcal{D}\mathrm{i}1}^U$ is defined as (3.66).

We are in a position to give the following main result on the error analysis of piecewise linear approximation.

Theorem 4.3 The following inequality holds:

$$\left(1-\frac{K_1}{M^2}\right)\|\mathcal{F}(\widehat{P},\,\Psi)\|-\frac{K^U_{\mathcal{D}\mathrm{i}1}}{M^2}\leq\|\mathcal{F}(\widehat{P}_{M1},\,\Psi)\|\leq\left(1+\frac{K_1}{M^2}\right)\|\mathcal{F}(\widehat{P},\,\Psi)\|+\frac{K^U_{\mathcal{D}\mathrm{i}1}}{M^2} \tag{4.60}$$

Proof. We note from the proof of Theorem 4.1 that $\|\mathcal{F}^0(\widehat{P},\,\Psi)\|\leq\|\mathcal{F}(\widehat{P},\,\Psi)\|$. Then, it readily follows form Proposition 4.2 and Lemma 4.4 that

$$\begin{aligned}
&\|\mathcal{F}(\widehat{P}_{M1},\,\Psi)-\mathcal{F}(\widehat{P}_{M},\,\Psi)\|\\
\leq&\|\mathcal{F}^0(\widehat{P}_{M1},\,\Psi)-\mathcal{F}^0(\widehat{P}_{M1},\,\Psi)\|+\|\mathbf{D}_{M1}-\mathbf{L}_M\mathbf{D}_{11}\mathbf{L}_M^{-1}\|\\
\leq&\frac{K_1}{M^2}\|\mathcal{F}^0(\widehat{P}_{M},\,\Psi)\|+\frac{K^U_{\mathcal{D}\mathrm{i}1}}{M^2}\\
\leq&\frac{K_1}{M^2}\|\mathcal{F}(\widehat{P}_{M},\,\Psi)\|+\frac{K^U_{\mathcal{D}\mathrm{i}1}}{M^2}
\end{aligned} \tag{4.61}$$

Since $\|\mathcal{F}(\widehat{P}_M,\,\Psi)\|=\|\mathcal{F}(\widehat{P},\,\Psi)\|$, the assertion follows immediately. Q.E.D.

4.6.2 Features of Piecewise Linear Approximation

This subsection provides the validity of piecewise linear approximation in the L_1 optimal control problem of sampled-data system by using the argument in Theorem 4.3.

Theorem 4.3 clearly implies that the approximation error in the piecewise linear approximation converges to 0 at the rate of $1/M^2$ as $M\to\infty$, when the L_∞-induced norm of the sampled-data system Σ_{SD} is computed for a fixed controller Ψ. Theorem 4.3 also provides a theoretical basis for such an indirect and approximate approach to L_1 optimal controller synthesis for the sampled-data system Σ_{SD} that seeks for Ψ minimizing $\|\mathcal{F}(\widehat{P}_{M1},\,\Psi)\|$ for a sufficiently large M. We consider γ_{opt} given in (4.31) and take an M. Suppose that Ψ_{M1} is an ε-suboptimal controller with respect to $\|\mathcal{F}(\widehat{P}_{M1},\,\Psi)\|$, i.e., $\|\mathcal{F}(\widehat{P}_{M1},\,\Psi_{M1})\|\leq\gamma_{M1}+\epsilon$ ($\epsilon>0$), where $\gamma_{M1}:=\inf_{\Psi}\|\mathcal{F}(\widehat{P}_{M1},\,\Psi)\|$. Let $M_1\in\mathbb{N}$ be the minimum such that $M_1^2>K_1$. Then, for $M\geq M_1$, the first inequality of (4.60) implies that

$$\begin{aligned}
\gamma_{\mathrm{opt}}&\leq\|\mathcal{F}(\widehat{P},\,\Psi_{M1})\|\\
&\leq\left(1+\frac{K_1}{M^2-K_1}\right)\|\mathcal{F}(\widehat{P}_{M1},\,\Psi_M)\|+\frac{K^U_{\mathcal{D}\mathrm{i}1}}{M^2-K_1}\\
&\leq\left(1+\frac{K_1}{M^2-K_1}\right)(\gamma_{M1}+\epsilon)+\frac{K^U_{\mathcal{D}\mathrm{i}1}}{M^2-K_1}
\end{aligned} \tag{4.62}$$

On the other hand, it follows from the second inequality of (4.60) that

$$\gamma_{M1} \leq \left(1 + \frac{K_1}{M^2}\right)\gamma_{\rm opt} + \frac{K^U_{\mathcal{D}\rm i1}}{M^2} \tag{4.63}$$

Substituting this into (4.62) and taking a sufficiently large M such that $M \geq M_1$, we see that

$$\gamma_{\rm opt} \leq \|\mathcal{F}(\widehat{P}, \Psi_{M1})\| \leq \gamma_{\rm opt} + \epsilon + \frac{X_1}{M^2} \tag{4.64}$$

where

$$X_1 := \frac{2K_1\gamma_{\rm opt}}{1 - K_1/M_1^2} + \frac{2K^U_{\mathcal{D}\rm i1}}{1 - K_1/M_1^2} + \frac{K_1\epsilon}{1 - K_1/M_1^2} \tag{4.65}$$

Since $\epsilon > 0$ is arbitrary, letting M sufficiently large and taking a suboptimal Ψ_{M1} with respect to $\|\mathcal{F}(\widehat{P}_{M1}, \Psi)\|$ (sufficiently close to the infimal performance) is ensured to lead to a method for L_1 optimal controller synthesis for the sampled-data system $\Sigma_{\rm SD}$. By (4.64), we could say that the convergence of Ψ_{M1} is in the order of $1/M^2$. We will further show in Section 4.7 that the synthesis problem of a suboptimal Ψ_{M1} can be (almost equivalently) reduced to a discrete-time l_1 optimal control problem.

4.7 Main Results in Piecewise Linear Approximation: Reduction to the Discrete-Time l_1 Optimal Control Problem

The preceding section is concerned with a promising aspect of dealing with the piecewise linear approximation $\mathcal{F}(\widehat{P}_{M1}, \Psi)$. To exploit this approximation, however, we obviously need to have an explicit method for computing $\|\mathcal{F}(\widehat{P}_{M1}, \Psi)\|$, and this section is devoted to giving such a method. More precisely, we show that a discretized generalized plant (for the continuous-time generalized plant P) can be introduced that is useful for computing $\|\mathcal{F}(\widehat{P}_{M1}, \Psi)\|$. We further show that the discretized generalized plant together with the associated error analysis converts the synthesis problem of an L_1 optimal controller Ψ for the sampled-data system $\Sigma_{\rm SD}$ into the discrete-time synthesis problem of an l_1 optimal controller.

Remark 4.12 Even though $\|\mathcal{F}(\widehat{P}_{M1}, \Psi)\|$ coincides with $\|P^-_{NM\rm i1}\|$ in (3.83) when we take $N \to \infty$, the arguments in this section are completely different from those in Chapter 3. More precisely, the arguments in Chapter 3 could not lead to discretization of the continuous-time generalized plant P, but the arguments in this section lead to such a discretization.

To derive such a discretized generalized plant, we recall that $\mathcal{F}(\widehat{P}_{M1},\, \Psi)$ is given by (4.48) and consider replacing $\mathbf{B}_1\mathbf{J}_{M1}, \mathbf{H}_{M1}\mathbf{M}_1$ and $\mathbf{D}_{M1}$ with finite-dimensional matrices.

4.7.1 Approximation of the Unit Ball Image of $\mathbf{B}_1\mathbf{J}_{M1}$

This subsection provides a method for approximating the closed unit ball image of $\mathbf{B}_1\mathbf{J}_{M1}$ with that of a matrix in the l_∞ sense. To this end, we first review the operation of $\mathbf{B}_1\mathbf{J}_{M1} = \mathbf{B}_1\mathbf{L}_M^{-1}\overline{\mathbf{J}_1'}$. For $\mathbf{L}_M w = [(w^{(1)})^T, \cdots, (w^{(M)})^T]^T \in (L_\infty[0,h'))^{Mn_w}$, we have

$$\begin{aligned}\mathbf{B}_1\mathbf{J}_{M1}w &= \begin{bmatrix}(A_d')^{M-1} & \cdots & I\end{bmatrix}\overline{\mathbf{B}_1'}\,\overline{\mathbf{J}_1'}\begin{bmatrix}w^{(1)}\\ \vdots \\ w^{(M)}\end{bmatrix}\\ &= \begin{bmatrix}(A_d')^{M-1} & \cdots & I\end{bmatrix}\overline{B_{hd}'}\phi_M \end{aligned} \tag{4.66}$$

where

$$B_{hd}' := \begin{bmatrix}4B_{0d}' - \dfrac{6}{h'}B_{1d}' & -3B_{0d}' + \dfrac{6}{h'}B_{1d}'\end{bmatrix} \tag{4.67}$$

$$\phi_M := \begin{bmatrix}(\phi_0^{(1)})^T & (\phi_1^{(1)})^T & \cdots & (\phi_0^{(M)})^T & (\phi_1^{(M)})^T\end{bmatrix}^T \tag{4.68}$$

$$\phi_0^{(i)} := \frac{1}{h'}\int_0^{h'} w^{(i)}(\tau')d\tau',\ \phi_1^{(i)} := \frac{2}{(h')^2}\int_0^{h'} w^{(i)}(\tau')\tau' d\tau' \tag{4.69}$$

with $A_d',\ B_{0d}'$ and B_{1d}' defined as (3.25), (3.55) and (3.160), respectively. Hence, it is expected that considering the set of the vectors ϕ_M for all w in the unit ball of $(L_\infty[0,h'))^{Mn_w}$ and replacing $\mathbf{B}_1\mathbf{J}_{M1}$ with the matrix $[(A_d')^{M-1}\ \cdots\ I]\overline{B_{hd}'}$ in (4.66) may be helpful in computing $\|\mathcal{F}(\widehat{P}_{M1},\, \Psi)\|$. The following result is associated with $\mathbf{B}_1'\mathbf{J}_1'$ in (4.66) (or, more precisely, with each $[(\phi_0^{(i)})^T,\ (\phi_1^{(i)})^T]^T$ in (4.68)) and plays an important role in representing the above vector set, denoted by $\boldsymbol{\Phi}_M$.

Theorem 4.4 Let

$$\phi_0 := \frac{1}{h'}\int_0^{h'} w(t)dt, \quad \phi_1 := \frac{2}{(h')^2}\int_0^{h'} w(t)tdt \tag{4.70}$$

where $w \in L_\infty[0,h')$ is a scalar function. Then, the set of (ϕ_0, ϕ_1) corresponding to all w such that $\|w\| \le 1$ is characterized by

$$-1 \le \phi_0 \le 1 \tag{4.71}$$

$$\frac{\phi_0^2 + 2\phi_0 - 1}{2} \le \phi_1 \le \frac{-\phi_0^2 + 2\phi_0 + 1}{2} \tag{4.72}$$

Proof. It is obvious that (4.71) holds for every w such that $\|w\| \leq 1$. Let us take an arbitrary ϕ_0 satisfying (4.71), and take the following $w(t)$, which satisfies $\|w\| \leq 1$ and the first equation of (4.70).

$$w(t) = \begin{cases} -1 & \left(0 \leq t < \dfrac{(1-\phi_0)h'}{2}\right) \\ 1 & \left(\dfrac{(1-\phi_0)h'}{2} \leq t < h'\right) \end{cases} \tag{4.73}$$

Because of a particular waveform of this w, together with the fact that t is strictly increasing, it is obvious that this w attains the maximum of ϕ_1 among those satisfying the first equation of (4.70). Similarly, the above w with ϕ_0 replaced by $-\phi_0$ and with the sign inverted attains the minimum of ϕ_1 among those satisfying the first equation of (4.70). For these particular w, we see that ϕ_1 takes the extreme values in (4.72). Taking convex combinations of the above two extreme inputs completes the proof. Q.E.D.

The area associated with (4.71) and (4.72) is shown in Fig. 4.6, which is obviously convex. Hence, it is expected that this area can be approximated in the l_∞ sense, with an arbitrary degree of accuracy, by using the tangents for the boundary curves. This approximation procedure can be described as follows. We introduce the approximation parameter $Q \in \mathbb{N}$, and draw the tangents for the boundary curves at the points $\phi_0 = 2(i-1)/(Q-1) - 1$ for $i = 1, \cdots, Q$ (≥ 2) in Fig. 4.6. The area in Fig. 4.6 can then be approximated with these tangents by a convex polygon as shown in Fig. 4.7, and this technique can be applied[1] to the l_∞ approximation of the set $\boldsymbol{\Phi}_M$. For example, when $Q = 3$ in Fig. 4.7, the convex polygon is determined by

$$-1 \leq \phi_0 \leq 1 \tag{4.74}$$

$$-1 \leq -2\phi_0 + 2\phi_1 \leq 1 \tag{4.75}$$

$$-1 \leq -2\phi_0 + \phi_1 \leq 1 \tag{4.76}$$

and the set $\boldsymbol{\Phi}_M$ is outer-approximated with such M $2Q$-polygons, or more precisely, by the set (denoted by $\boldsymbol{\Phi}_M^{[Q]}$) of all vectors whose image by $\overline{\Delta^{[Q]}}$ lies in the closed unit ball of $\mathbb{R}_\infty^{MQn_w}$, where

$$\Delta^{[Q]} = \Delta^{[3]} := \begin{bmatrix} 0_{n_w} & I_{n_w} \\ -2I_{n_w} & 2I_{n_w} \\ -2I_{n_w} & I_{n_w} \end{bmatrix} \in \mathbb{R}_\infty^{Qn_w \times 2n_w} \tag{4.77}$$

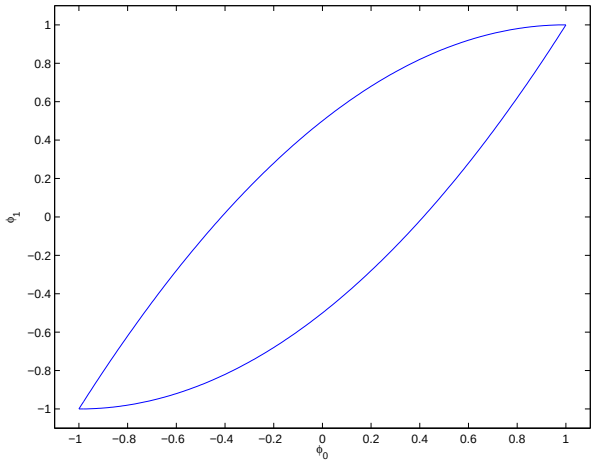


Figure 4.6: The set of (ϕ_0, ϕ_1) given by (4.70) with $\|w\| \leq 1$.

In the following discussions, it is convenient to describe the outer-approximated set $\boldsymbol{\Phi}_M^{[Q]}$ as the unit ball image of $\overline{\Omega^{[Q]}}$ with a suitable matrix $\Omega^{[Q]}$. To give the representation of such $\Omega^{[Q]}$, we take $v_1, \cdots, v_Q$ to be the non-negative vector representations of the Q left-upper edges of the polygon, aligned from the rightmost one to the leftmost one[2]. For example, when $Q = 3$ in Fig. 4.7,

$$v_1 = \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}, \ v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \ v_3 = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} \tag{4.78}$$

By using these vectors $v_i = [v_{i1}, \ v_{i2}]^T$, the matrix $\Omega^{[Q]}$ is defined as

$$\Omega^{[Q]} = \frac{1}{2} \begin{bmatrix} v_{11} I_{n_w} & \cdots & v_{Q1} I_{n_w} \\ v_{12} I_{n_w} & \cdots & v_{Q2} I_{n_w} \end{bmatrix} \in \mathbb{R}_\infty^{2n_w \times Q n_w} \tag{4.79}$$

Then, the outer-approximated set $\boldsymbol{\Phi}_M^{[Q]}$ coincides with the set of all vectors $\overline{\Omega^{[Q]}} w_d$ with $\|w_d\| \leq 1$ (see Section 4.10 for the proof). Summarizing the above arguments, we see from (4.66) that the operators $\mathbf{B}_1' \mathbf{J}_1'$ and $\mathbf{B}_1 \mathbf{J}_{M1}$ may respectively be replaced by the matrices

$$B_1^{[Q]'} := B_{hd}' \Omega^{[Q]}, \quad B_{M11}^{[Q]} := \begin{bmatrix} (A_d')^{M-1} & \cdots & I \end{bmatrix} \overline{B_1^{[Q]'}} \tag{4.80}$$

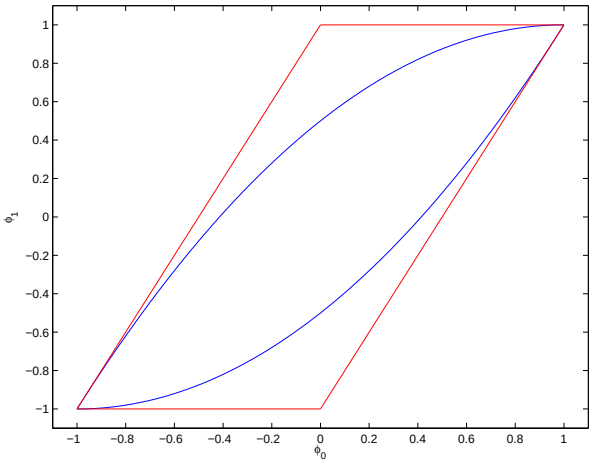
Furthermore, it is obvious from the above arguments that $\boldsymbol{\Phi}_M^{[Q]}$ converges to $\boldsymbol{\Phi}_M$ as $Q \to \infty$.

4.7.2 Replacing $\mathbf{H}_{M1}\mathbf{M}_1$ and $\mathbf{D}_{M1}$ with Appropriate Matrices

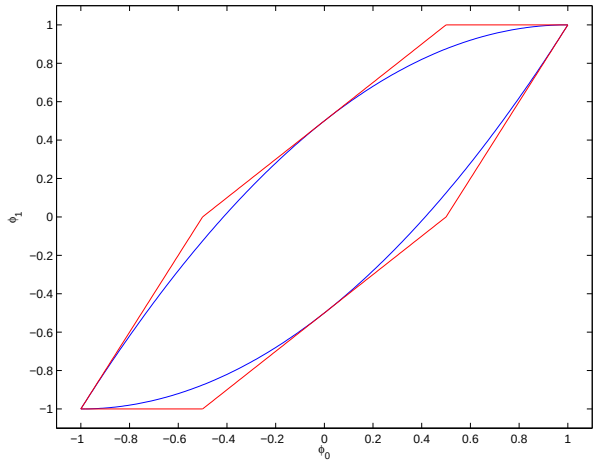
In this subsection, we give a method for replacing the operators $\mathbf{H}_{M1}\mathbf{M}_1$ and $\mathbf{D}_{M1}$ with appropriate matrices so that $\|\mathcal{F}(\widehat{P}_{M1}, \Psi)\|$ can be computed as the l_∞-induced norm of a

[1]We can apply Theorem 4.4 to each entry of the vector function w because (4.69) can be computed entrywise. This leads to the formulation with the identity matrices I_{n_w} in (4.77) and (4.79).

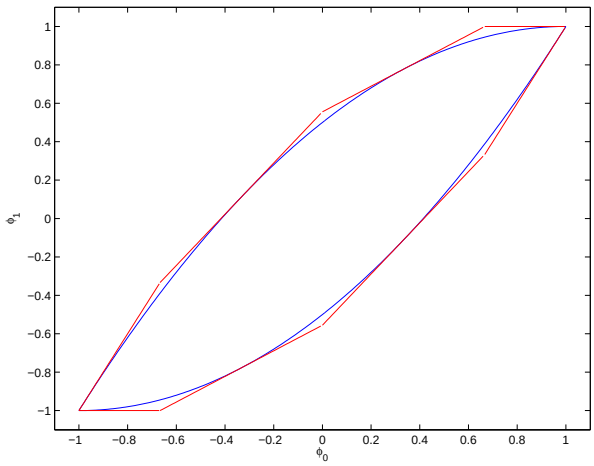
[2]We remark that the ith block row of $\Delta^{[Q]}$ corresponds, by definition, to the strip determined by two lines, each of which contains the edge corresponding to the vector v_i and the other parallel edge.



(a) $Q = 2$



(b) $Q = 3$



(c) $Q = 4$

Figure 4.7: Approximation of ϕ_0 and ϕ_1.

discrete-time system (if the approximation error associated with the parameter Q in the preceding subsection were ignored; the error will be considered later). Here, note that the infinite (Toeplitz) matrix representation of the input/output relation of $\mathcal{F}(\widehat{P}_{M1}, \Psi) = \mathbf{H}_{M1}\mathbf{M}_1\mathcal{F}(P_d, \Psi)\mathbf{B}_1\mathbf{J}_{M1} + \mathbf{D}_{M1}$ is block lower triangular (with respect to the partitioning associated with $\widehat{w}_k$ and $\widehat{z}_k$) because of the structure of P_d, as mentioned in the preceding section. We then see that $\|\mathcal{F}(\widehat{P}_{M1}, \Psi)\|$ coincides with the $L_\infty[0, h')$-induced norm of the "last block row" of the above infinite matrix. Note that $\mathcal{F}(\widehat{P}_{M1}, \Psi)$ admits the representation

$$\begin{cases} \xi_{k+1} &= \mathcal{A}\xi_k + J_\Sigma \mathbf{B}_1 \mathbf{J}_{M1}\widehat{w}_k \\ \widehat{z}_k &= \mathbf{H}_{M1}\mathbf{M}_1 C_\Sigma \xi_k + \mathbf{D}_{M1}\widehat{z}_k \end{cases} \tag{4.81}$$

Hence, by the "last block row" mentioned above, we mean

$$\mathcal{P}_{M1} := [\mathbf{D}_{M1} \;\; \mathbf{H}_{M1}\mathbf{M}_1 C_\Sigma J_\Sigma \mathbf{B}_1 \mathbf{J}_{M1} \;\; \mathbf{H}_{M1}\mathbf{M}_1 C_\Sigma \mathcal{A} J_\Sigma \mathbf{B}_1 \mathbf{J}_{M1} \\ \mathbf{H}_{M1}\mathbf{M}_1 C_\Sigma \mathcal{A}^2 J_\Sigma \mathbf{B}_1 \mathbf{J}_{M1} \;\; \mathbf{H}_{M1}\mathbf{M}_1 C_\Sigma \mathcal{A}^3 J_\Sigma \mathbf{B}_1 \mathbf{J}_{M1} \;\; \cdots] \tag{4.82}$$

(where we have reversed the order of the operator entries for notational simplicity). Here, recall that $\mathbf{D}_{M1} = \overline{\mathbf{H}'_1}\,\overline{\mathbf{M}'_1}\Delta^0_M\overline{\mathbf{B}'_1}\,\overline{\mathbf{J}'_1} + \overline{\mathbf{D}'_{\mathrm{a}1}}$, where Δ^0_M is strictly block lower triangular. Hence, it follows that $\overline{\mathbf{D}'_{\mathrm{a}1}}$ can be handled separately from $\overline{\mathbf{H}'_1}\,\overline{\mathbf{M}'_1}\Delta^0_M\overline{\mathbf{B}'_1}\,\overline{\mathbf{J}'_1}$. Furthermore, to facilitate the treatment of the operator $\overline{\mathbf{D}'_{\mathrm{a}1}}$, we use again the notation $\mathbf{D}'_{\mathrm{a}10} := \mathbf{D}'_{\mathrm{a}1} - D_{11}$, which is an integral operator by the definition of $\mathbf{D}'_{\mathrm{a}1}$ in (3.60). Then, by using the properties of $L_\infty[0, h')$, $\mathcal{P}_{M1}$ may be redefined as

$$\mathcal{P}_{M1} := [\overline{D_{11}} \;\; \overline{\mathbf{D}'_{\mathrm{a}10}} \;\; \overline{\mathbf{H}'_1}\,\overline{\mathbf{M}'_1}\Delta^0_M\overline{\mathbf{B}'_1}\,\overline{\mathbf{J}'_1} \;\; \mathbf{H}_{M1}\mathbf{M}_1 C_\Sigma J_\Sigma \mathbf{B}_1 \mathbf{J}_{M1} \\ \mathbf{H}_{M1}\mathbf{M}_1 C_\Sigma \mathcal{A} J_\Sigma \mathbf{B}_1 \mathbf{J}_{M1} \;\; \mathbf{H}_{M1}\mathbf{M}_1 C_\Sigma \mathcal{A}^2 J_\Sigma \mathbf{B}_1 \mathbf{J}_{M1} \;\; \cdots] \tag{4.83}$$

without changing its norm.

For simplicity, let us suppose $D_{11} = 0$ for a while; we will return to the general case with $D_{11} \neq 0$ around the end of this subsection. Then, based on the property of $L_\infty[0, h')$, we can apply essentially the same arguments as those in Chapter 3 to show that the input of the operator $\mathbf{D}'_{\mathrm{a}10}$ may be restricted to a constant (vector) function in its treatment for the computation of $\|\mathcal{P}_{M1}\|$ in (4.83); this is because $\mathbf{D}'_{\mathrm{a}10}$ is simply an integral operator and the outputs of the operators in the other entries in $\mathcal{P}_{M1}$ are linear functions. An immediate consequence of this restriction is that the output of $\mathcal{P}_{M1}$ itself also becomes a linear function, so that the $L_\infty[0, h')$ norm of the output of $\mathcal{P}_{M1}$ can be evaluated by considering only its values at $\theta' = 0$ and $\theta' = h' - 0$. More precisely, since the operations of $\mathbf{H}_{M1}\mathbf{M}_1$ and $\overline{\mathbf{D}'_{\mathrm{a}10}}$ are respectively given by

$$\left(\mathbf{H}_{M1}\mathbf{M}_1 \begin{bmatrix} x \\ u \end{bmatrix}\right)(\theta') = \overline{\mathbf{H}'_1}\,\overline{\mathbf{M}'_1} \begin{bmatrix} I \\ \vdots \\ (A'_{2d})^{M-1} \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}(\theta') = \overline{[C_0(I + A_2\theta')]} \begin{bmatrix} I \\ \vdots \\ (A'_{2d})^{M-1} \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \tag{4.84}$$

$$\overline{(\mathbf{D}'_{\mathrm{a}10}w)}(\theta') = \overline{C_1 B_1 w(0)\theta'} \tag{4.85}$$

(where w is assumed to be a constant function in (4.85) by the preceding arguments), the operators $\mathbf{H}'_1\mathbf{M}'_1$ and $\mathbf{D}'_{\mathrm{a}10}$ in $\mathcal{P}_{M1}$ may be replaced by the matrices

$$H'_1 := \begin{bmatrix} C_0 \\ C_0(I + A_2 h') \end{bmatrix} \in \mathbb{R}^{2n_z \times n}_\infty, \tag{4.86}$$

$$D'_{\mathrm{a}10} := \begin{bmatrix} 0 \\ C_1 B_1 h' \end{bmatrix} \in \mathbb{R}^{2n_z \times n_w}_\infty \tag{4.87}$$

respectively, without changing $\|\mathcal{P}_{M1}\|$. After such replacement, the only operators remaining in $\mathcal{P}_{M1}$ are $\mathbf{B}_1\mathbf{J}_{M1}$, but the preceding arguments suggest that it can be further replaced by the matrix $B_{M11}^{[Q]}$ in (4.80).

By using the matrices $B_1^{[Q]'}$, $B_{M11}^{[Q]}$, H_1' and $D_{\mathrm{a}10}'$, the $L_\infty[0,h')$-induced norm $\|\mathcal{P}_{M1}\|$ and thus $\|\mathcal{F}(\widehat{P}_{M1},\,\Psi)\|$ is obtained by computing the l_∞-induced norm of the matrix $P_{M1}^{[Q]}$ (with an error associated with the approximation of $\boldsymbol{\Phi}_M$ by $\boldsymbol{\Phi}_M^{[Q]}$)[3] given by

$$\begin{aligned} P_{M1}^{[Q]} := [\ & \overline{\widetilde{D}_{11}}\ \ \overline{D_{\mathrm{a}10}'}\ \ \overline{H_1'}\Delta_M^0\overline{B_1^{[Q]'}}\ \ H_{M1}C_\Sigma J_\Sigma B_{M11}^{[Q]} \\ & H_{M1}C_\Sigma\mathcal{A}J_\Sigma B_{M11}^{[Q]}\ \ H_{M1}C_\Sigma\mathcal{A}^2J_\Sigma B_{M11}^{[Q]}\ \ \cdots] \end{aligned} \tag{4.88}$$

where

$$H_{M1} := \overline{H_1'}\begin{bmatrix} I \\ \vdots \\ (A_{2d}')^{M-1} \end{bmatrix}, \quad \widetilde{D}_{11} := \begin{bmatrix} D_{11} \\ D_{11} \end{bmatrix} \tag{4.89}$$

Here, we note again that Δ_M^0 is strictly block lower triangular and $Q \geq 2$. Then, based on the property of $L_\infty[0,h')$, the matrix $P_{M1}^{[Q]}$ may be redefined as

$$P_{M1}^{[Q]} := \begin{bmatrix} D_{M11}^{[Q]} & H_{M1}C_\Sigma J_\Sigma B_{M11}^{[Q]} & H_{M1}C_\Sigma\mathcal{A}J_\Sigma B_{M11}^{[Q]} & \cdots \end{bmatrix} \tag{4.90}$$

without changing its norm, with the matrices

$$D_{M11}^{[Q]} := \overline{H_1'}\Delta_M^0\overline{B_1^{[Q]'}} + \overline{D_{\mathrm{a}1}^{[Q]'}} \in \mathbb{R}_\infty^{2Mn_z\times MQn_w} \tag{4.91}$$

$$D_{\mathrm{a}1}^{[Q]'} := \begin{bmatrix} D_{\mathrm{a}10}' & \widetilde{D}_{11} & 0 \end{bmatrix} \in \mathbb{R}_\infty^{2n_z\times Qn_w} \tag{4.92}$$

Because the matrix $P_{M1}^{[Q]}$ (in the modified form in (4.90)) corresponds to the "last block row" of the input/output relation of a discrete-time system, the L_1 problem of $\mathcal{F}(\widehat{P}_{M1},\,\Psi)$ is reducible to the discrete-time l_1 problem described in the following subsection.

Remark 4.13 We have recovered the general case with $D_{11} \neq 0$ in (4.88) and thereafter. Indeed, the preceding arguments immediately lead to (4.88) with the first entry $\overline{\widetilde{D}_{11}}$ removed, and it is not hard to see that dealing with $D_{11} \neq 0$ leads to (4.88) as it is; the treatment of D_{11} is essentially the same as that in [66].

4.7.3 Discretization of the Continuous-Time Generalized Plant

This subsection is devoted to showing that the arguments in the preceding subsection immediately lead us to a discretization procedure for the continuous-time generalized plant

[3]Because $\boldsymbol{\Phi}_M^{[Q]}$ converges to $\boldsymbol{\Phi}_M$ as $Q \to \infty$, we can have an arbitrary degree of accuracy. See Theorem 4.5.

P that can be used in dealing with the L_1 problem of $\mathcal{F}(\widehat{P}_{M1}, \Psi)$ through a discrete-time l_1 problem.

Indeed, let us consider the discrete-time plant given by

$$P_{M1d}^{[Q]} : \begin{cases} x_{k+1} = A_d x_k + B_{M11}^{[Q]} w_k + B_{2d} u_k \\ z_k = C_{M11} x_k + D_{M11}^{[Q]} w_k + D_{M21} u_k \\ y_k = C_{2d} x_k \end{cases} \tag{4.93}$$

where the matrices $C_{M11} \in \mathbb{R}_\infty^{2Mn_z \times n}$ and $D_{M21} \in \mathbb{R}_\infty^{2Mn_z \times n_u}$ are given by

$$\begin{bmatrix} C_{M11} & D_{M21} \end{bmatrix} := H_{M1} \tag{4.94}$$

The other matrices are given in (3.10), (4.80) and (4.91). Then, we readily see that the closed-loop system obtained by connecting Ψ to the above discrete-time plant $P_{M1d}^{[Q]}$ as shown in Figure 4.8 has the state-space representation $(\mathcal{A}, J_\Sigma B_{M11}^{[Q]}, H_{M1} C_\Sigma, D_{M11}^{[Q]})$. Since the "last block row" of the infinite (Toeplitz) matrix representation of the input/output relation of this closed-loop system (with the entries aligned in the reverse order) is nothing but $P_{M1}^{[Q]}$ given in (4.90), it follows readily that computing $\|P_{M1}^{[Q]}\|$ is equivalent to computing the l_∞-induced norm of the above closed-loop system. This implies that the L_1 problem of $\mathcal{F}(\widehat{P}_{M1}, \Psi)$ with the relevant outer-approximation of $\boldsymbol{\Phi}_M$ by $\boldsymbol{\Phi}_M^{[Q]}$ is exactly reducible to the discrete-time l_1 problem for the discretized generalized plant $P_{M1d}^{[Q]}$.

Let us denote by $\|\mathcal{F}(P_{M1d}^{[Q]}, \Psi)\|$ the l_∞-induced norm computed through this outer-approximation. Then, by the preceding arguments, $\|\mathcal{F}(P_{M1d}^{[Q]}, \Psi)\|$ can be made arbitrarily close to $\|\mathcal{F}(\widehat{P}_{M1}, \Psi)\|$ by taking a sufficiently large Q. More precisely, we have the following result regarding approximately solving the L_1 optimal sampled-data controller synthesis problem such as $\inf_{\Psi} \|\mathcal{F}(\widehat{P}, \Psi)\|$ through the discrete-time l_1 optimal controller synthesis problem such as $\inf_{\Psi} \|\mathcal{F}(P_{M1d}^{[Q]}, \Psi)\|$, where $\kappa^{[Q]}$ is defined as the largest $\kappa > 0$ such that $\kappa \boldsymbol{\Phi}_M^{[Q]} \subset \boldsymbol{\Phi}_M$ (which can be computed easily for each Q).

Theorem 4.5 The inequality

$$\begin{aligned} \left(1 - \frac{K_1}{M^2}\right) \|\mathcal{F}(\widehat{P}, \Psi)\| - \frac{K_{\mathcal{D}i1}^U}{M^2} &\le \|\mathcal{F}(P_{M1d}^{[Q]}, \Psi)\| \\ &\le \frac{1}{\kappa^{[Q]}} \left[\left(1 + \frac{K_1}{M^2}\right) \|\mathcal{F}(\widehat{P}, \Psi)\| + \frac{K_{\mathcal{D}i1}^U}{M^2} \right] \end{aligned} \tag{4.95}$$

holds and $\kappa^{[Q]}$ converges to 1 as $Q \to \infty$.

Note that the above lower bound of $\|\mathcal{F}(P_{M1d}^{[Q]}, \Psi)\|$ is the same as that of $\|\mathcal{F}(\widehat{P}_{M1}, \Psi)\|$ in (4.60), while the upper bound is larger than that of $\|\mathcal{F}(\widehat{P}_{M1}, \Psi)\|$ in (4.60) by the factor

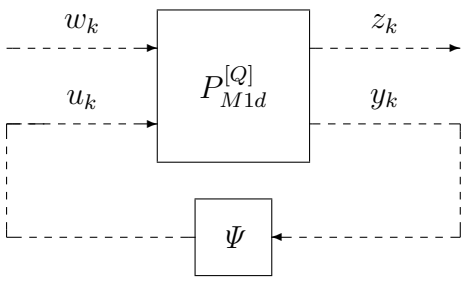


Figure 4.8: Discrete-time system obtained by piecewise linear approximation.

of $1/\kappa^{[Q]}$; this follows immediately from $\boldsymbol{\Phi}_M \subset \boldsymbol{\Phi}_M^{[Q]}$ and $\kappa^{[Q]}\boldsymbol{\Phi}_M^{[Q]} \subset \boldsymbol{\Phi}_M$. The assertion on $\kappa^{[Q]}$ is also obvious (see Fig. 4.7). This theorem obviously ensures that the L_1 optimal sampled-data controller synthesis can be carried out through $\mathcal{F}(P_{M1d}^{[Q]}, \Psi)$ with sufficiently large M and Q.

4.8 Numerical Example

This section examines the effectiveness of the discretization methods developed in this chapter through a numerical example.

We consider the continuous-time generalized plant

$$A = \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}, \; C_2 = \begin{bmatrix} 1 & 0 \end{bmatrix}$$
$$D_{11} = D_{12} = 0 \tag{4.96}$$

and take the sampling period $h = 0.3$. Let Ψ_{M0} and $\Psi_{M1}^{[Q]}$ be the discrete-time l_1 optimal controllers minimizing the l_∞-induced norms $\|\mathcal{F}(P_{M0d}, \Psi)\|$ through the L_1-discretization P_{M0d} of P via the piecewise constant approximation and $\|\mathcal{F}(P_{M1d}^{[Q]}, \Psi)\|$ through the L_1-discretization $P_{M1d}^{[Q]}$ of P via the piecewise linear approximation, respectively. They are designed under the fast-lifting parameter $M = 2, 3, 4$ together with $Q = 3$. The results of the l_∞-induced norm of the discrete-time system $\mathcal{F}(P_{M0d}, \Psi_{M0})$ and the L_∞-induced norm of the sampled-data system Σ_{SD} with $\Psi = \Psi_{M0}$ (i.e., $\|\mathcal{F}(\widehat{P}, \Psi_{M0})\|$) are shown in Table 4.1. Furthermore, the results of the l_∞-induced norm of the discrete-time system $\mathcal{F}(P_{M1d}^{[Q]}, \Psi_{M1}^{[Q]})$ and the L_∞-induced norm of the sampled-data system Σ_{SD} with $\Psi = \Psi_{M1}^{[Q]}$ (i.e., $\|\mathcal{F}(\widehat{P}, \Psi_{M1}^{[Q]})\|$) are shown in Table 4.2. These values are computed so that they are accurate up to the digits shown therein[4]. We can see from these tables that $\|\mathcal{F}(\widehat{P}, \Psi_{M1}^{[Q]})\|$ successfully becomes much

[4]Even though the arguments of Theorem 4.5 can be used also for analysis, the L_∞-induced norm computation for the designed controllers is carried out through the arguments in Chapter 3. This is because the computation of the upper and lower bounds therein is much simpler, even though its extension to controller synthesis is very hard.

smaller than $\|\mathcal{F}(\widehat{P}, \Psi_{M0})\|$ under the same approximation parameter M. Furthermore, we can also see from these tables that $\|\mathcal{F}(\widehat{P}, \Psi_{M1}^{[Q]})\|/\|\mathcal{F}(P_{M1d}^{[Q]}, \Psi_{M1}^{[Q]})\|$ is much closer to 1 than $\|\mathcal{F}(\widehat{P}, \Psi_{M0})\|/\|\mathcal{F}(P_{M0d}, \Psi_{M0})\|$ under the same parameter M. This implies that taking the intersample behavior of the sampled-data system Σ_{SD} into account can be more accurately reflected on the controller design even with a relatively small M through the piecewise linear approximation than the piecewise constant approximation. These observations clearly suggest that the piecewise linear approximation method is very effective and drastically outperforms the piecewise constant approximation method in the L_1 optimal controller synthesis of sampled-data systems.

Table 4.1: Results of the L_1 optimal sampled-data controller synthesis through the piecewise constant approximation.

M	2	3	4
$\|\mathcal{F}(P_{M0d}, \Psi_{M0})\|$	0.4835	0.5362	0.5552
$\|\mathcal{F}(\widehat{P}, \Psi_{M0})\|$	3.7012	1.7814	1.6169

Table 4.2: Results of the L_1 optimal sampled-data controller synthesis through the piecewise linear approximation ($Q = 3$).

M	2	3	4
$\|\mathcal{F}(P_{M1d}^{[N]}, \Psi_{M1}^{[N]})\|$	0.6617	0.6226	0.5931
$\|\mathcal{F}(\widehat{P}, \Psi_{M1}^{[N]})\|$	1.5048	0.6453	0.6351

4.9 Concluding Remarks

In this chapter, we developed two discretization methods for the L_1 optimal controller synthesis problem of sampled-data systems by using the ideas of piecewise constant and piecewise linear approximations discussed in chapter 3. The key techniques in these developments were the application of fast-lifting and the introduction of two types of 'constant approximation operators' or 'linear approximation operators,' one for input signals and the other for output signals. With these approximation operators, the continuous-time signals (i.e., the input signals and the output signals) are approximated by piecewise constant or piecewise linear functions, and this leads to the operator approximations of the input and output operators associated with the lifted sampled-data systems. To demonstrate the benefit of these operator approximations in the L_1 optimal controller synthesis problem of sampled-data systems, we first established Theorem 4.1 or the inequality (4.29) through the arguments of preadjoint operators. This inequality was shown to play an important role

in the piecewise constant approximation treatment for the L_1 optimal controller synthesis problem of sampled-data systems. In particular, it was shown that the convergence rate associated with the piecewise constant approximation is $1/M$ with respect to the fast-lifting parameter M. We then provided a discretization procedure for the continuous-time generalized plant through the piecewise constant approximation. This procedure enables us to give an 'equivalent' discrete-time generalized plant to the continuous-time generalized plant with any degree of accuracy, and thus the L_1 optimal controller synthesis problem of sampled-data systems can be reduced to the discrete-time l_1 optimal controller synthesis problem. Even though the arguments of such an equivalent discrete-time generalized plant together with the associated convergence rate essentially coincide with a method in [3] through the FSFH approximation technique [46], the arguments in this chapter lead to the simpler inequalities (4.17), (4.18), (4.22) and (4.29) and circumvent involved arguments in [3]. We further established Theorem 4.3 or the inequality (4.60) through the arguments of preadjoint operators. This inequality was shown to play an important role in the piecewise linear approximation treatment for the L_1 optimal controller synthesis problem of sampled-data systems, and it was shown that the convergence rate associated with the piecewise linear approximation is $1/M^2$ with respect to the fast-lifting parameter M. We also provided a discretization procedure for the continuous-time generalized plant through the piecewise linear approximation, and this procedure enables us to give an '(almost) equivalent' discrete-time generalized plant to the continuous-time generalized plant. Thus, the L_1 optimal controller synthesis problem of sampled-data systems can be reduced to the discrete-time l_1 optimal controller synthesis problem through the piecewise linear approximation. Finally, we examined the effectiveness of the developed methods through a numerical study, and it was confirmed that the piecewise linear approximation method works far more effectively than the piecewise constant approximation method.

4.10 Appendix

In this section, we give proof of Lemma 4.3 and deal with the approximation of $\boldsymbol{\Phi}_M$.

4.10.1 Proof of Lemma 4.3

Because the proof of Lemma 4.1 proceed in essentially the same way as that of Lemma 4.3, we omit the proof of Lemma 4.1 and only the proof of Lemma 4.3 is given here.

a) As a preliminary step to obtain the bound (4.50), we first show the existence of a constant

c_1 independent of M such that

$$h\|x\|_1 \leq c_1\|\mathbf{B}_{1*}x\|_1, \quad \forall x \in \mathbb{R}_1^n \tag{4.97}$$

To show this, let us introduce the operator $\mathbf{T}_B : (L_1[0,h))^{n_w} \to \mathbb{R}_1^n$ given by

$$\mathbf{T}_B w = \int_0^h \exp(A(h-\theta))B_1 w(\theta)d\theta \tag{4.98}$$

(Note that $\mathbf{T}_B$ is formally the same as $\mathbf{B}_1$ except that the domain is not $(L_\infty[0,h))^{n_w}$ but $(L_1[0,h))^{n_w}$ and the codomain is not $\mathbb{R}_\infty^n$ but $\mathbb{R}_1^n$.) Then, since $\mathbf{B}_{1*}$ is given by $(\mathbf{B}_{1*}x)(\theta) = B_1^T \exp(A^T(h-\theta))x$, we have

$$\mathbf{T}_B\mathbf{B}_{1*} = \int_0^h \exp(A(h-\theta))B_1B_1^T \exp(A^T(h-\theta))d\theta \tag{4.99}$$

Note that (4.99) implies that $\mathbf{T}_B\mathbf{B}_{1*}$ coincides with the controllability Grammian, which we denote by W_B. We then have

$$\begin{aligned} h\|x\|_1 = h\|W_B^{-1}\mathbf{T}_B\mathbf{B}_{1*}x\|_1 &\leq h\|W_B^{-1}\|_1\|\mathbf{T}_B\|_1\|\mathbf{B}_{1*}x\|_1 \\ &\leq h\|W_B^{-1}\|_1 e^{\|A\|_1 h}\|B_1\|_1\|\mathbf{B}_{1*}x\|_1 =: c_1\|\mathbf{B}_{1*}x\|_1 \end{aligned} \tag{4.100}$$

because W_B^{-1} exists by the controllability assumption.

Next, let $g \in R(\mathbf{B}_{1*})$, i.e., $g(\theta) = B_1^T \exp(A^T(h-\theta))x$ for some $x \in \mathbb{R}_1^n$. Then, a direct computation with (4.52) and (4.53) leads to

$$(\mathbf{J}_{M1*}g)(\theta') = \begin{bmatrix} B_1^T(A'_{dT})^{M-1}f(A,\theta')^T \\ \vdots \\ B_1^T f(A,\theta')^T \end{bmatrix} x \tag{4.101}$$

where

$$A'_{dT} := (A'_d)^T \tag{4.102}$$

$$f(A,\theta') := f_0(\theta')A'_{0d} + f_1(\theta')A'_{1d} \tag{4.103}$$

with A'_{0d} and A'_{1d} defined as (3.142) and (3.152), respectively. Because $h\|x\|_1 \leq c_1\|g\|_1$ by (4.97), a direct computation together with the Taylor expansion arguments leads to

$$\begin{aligned} &\|(\mathbf{L}_M - \mathbf{J}_{M1*})g\|_1 = \|\mathbf{L}_M\mathbf{B}_{1*}x - \mathbf{J}_{M1*}\mathbf{B}_{1*}x\|_1 \\ \leq &\|(I - \overline{\mathbf{J}'_{1*}})\mathbf{L}_M\mathbf{B}_{1*}x\|_1 \\ \leq &\|B_1^T\|_1 e^{\|A^T\|_1 h}M \cdot \int_0^{h'} \|f(A,\theta')^T - \exp(A^T(h'-\theta'))\|_1 d\theta'\|x\|_1 \\ \leq &\|B_1^T\|_1 e^{\|A^T\|_1 h}M\frac{1}{2}\frac{h^3}{M^3}\|A'_{dT}\|_1 \cdot \|A^T\|_1^2 e^{\|A^T\|_1 h'}\|x\|_1 \\ \leq &\frac{h^2}{2M^2}e^{3\|A\|h}\|A\|^2\|B_1\|c_1\|g\|_1 =: \frac{K_{B1}}{M^2}\|g\|_1 \end{aligned} \tag{4.104}$$

This implies (4.50) and the proof of part a) is completed.

b) As a preliminary step to obtain the bound (4.51), we show the existence of a constant c_∞ independent of M such that

$$\|p\| \leq c_\infty \|\mathbf{M}_1 p\|, \quad \forall p \in \mathbb{R}_\infty^{n+n_u} \tag{4.105}$$

To show this, we introduce the operator $\mathbf{T}_C : (L_\infty[0,h))^{n_z} \to \mathbb{R}_\infty^{n+n_u}$ given by

$$\mathbf{T}_C z = \int_0^h \exp(A_2^T \theta) C_0^T z(\theta) d\theta \tag{4.106}$$

Then, we have

$$\mathbf{T}_C \mathbf{M}_1 = \int_0^h \exp(A_2^T \theta) C_0^T C_0 \exp(A_2 \theta) d\theta =: W_C \tag{4.107}$$

Since W_C is the observability Grammian, the remaining part of the proof proceeds in a similar way to part a) with $\|\cdot\|_1$ replaced by $\|\cdot\|$; by using essentially the same arguments as part a), we have

$$\begin{aligned} \|p\| &= \|W_C^{-1} \mathbf{T}_C \mathbf{M}_1 p\| \leq \|W_C^{-1}\| \cdot \|\mathbf{T}_C\| \cdot \|\mathbf{M}_1 p\| \\ &\leq \|W_C^{-1}\| h e^{\|A_2^T\| h} \|C_0^T\| \cdot \|\mathbf{M}_1 p\| =: c_\infty \|\mathbf{M}_1 p\| \end{aligned} \tag{4.108}$$

From (4.105) and a direct computation together with the Taylor expansion arguments, we can obtain

$$\|(\mathbf{L}_M - \mathbf{H}_{M1})|_{R(\mathbf{M}_1)}\| \leq \frac{h^2}{2M^2} \|C_0\| \cdot \|A_2\|^2 e^{2\|A_2\| h} c_\infty =: \frac{K_{C1}}{M^2} \tag{4.109}$$

4.10.2 Approximation of the Vector Set $\boldsymbol{\Phi}_M$

In Section 4.7, we showed that the vector set $\boldsymbol{\Phi}_M$ can be outer-approximated by the set $\boldsymbol{\Phi}_M^{[Q]}$ of all vectors whose image by $\Delta^{[Q]}$ lies in the closed unit ball of $\mathbb{R}_\infty^{Qn_w}$. In this subsection, we show that $\boldsymbol{\Phi}_M^{[Q]}$ can be alternatively represented as the image of the closed unit ball of $\mathbb{R}_\infty^{Qn_w}$ by the matrix $\Omega^{[Q]}$, i.e.,

$$\{w_\Delta \mid \|\Delta^{[Q]} w_\Delta\| \leq 1\} = \{\Omega^{[Q]} w_d \mid \|w_d\| \leq 1\} \tag{4.110}$$

In the proof, we assume $n_w = 1$ without loss of generality.

It is obvious from the definition of $v_1, \cdots, v_Q$ that, for $w_d = [1, \cdots, 1]^T \in \mathbb{R}_\infty^Q$, we have $\Omega^{[Q]} w_d = [1, 1]^T$, which coincides with the right-upper vertex of the $2Q$-polygon associated with the outer-approximation of the set $\boldsymbol{\Phi}_M$ (recall Fig. 4.7). Reversing the sign of the

first entry of w_d, we readily see from the definition of v_1 that $\Omega^{[Q]}w_d$ then coincides with the vertex of the $2Q$-polygon to the left of the aforementioned vertex. Reversing the other entries of w_d one by one from the upper ones (until we take Q-different values of w_d) and further considering the (completely) sign-reversed versions of these Q values of w_d, we readily see that all vertices of the $2Q$-polygon lie in the unit ball image of $\Omega^{[Q]}$. Since the unit ball image is convex, we readily see that the $2Q$-polygon is contained in the unit ball image, i.e.,

$$\{w_\Delta \mid \|\Delta^{[Q]}w_\Delta\| \leq 1\} \subset \{\Omega^{[Q]}w_d \mid \|w_d\| \leq 1\} \tag{4.111}$$

To show the opposite inclusion relation, we first represent the matrix $\Delta^{[Q]}$ by

$$\Delta^{[Q]} = \begin{bmatrix} u_1 \\ \vdots \\ u_Q \end{bmatrix} \tag{4.112}$$

where $u_i^T \in \mathbb{R}^2_\infty$ $(i = 1, \cdots, Q)$. Then, we consider the matrix

$$\Lambda^{[Q]} := \frac{1}{2} \begin{bmatrix} u_1 \\ \vdots \\ u_Q \end{bmatrix} \begin{bmatrix} v_1 & \cdots & v_Q \end{bmatrix} \tag{4.113}$$

By the definition of u_i, the (i, i) entry of $\Lambda^{[Q]}$ is 0 for $i = 1, \cdots, Q$ by orthogonality. Furthermore, since all entries of the vectors $v_1, \cdots, v_Q$ are non-negative and the slope v_{i2}/v_{i1} of the ith edge increases as i increases, the (i, j) entry of the matrix $\Lambda^{[Q]}$ is positive for $i > j$ while it is negative for $i < j$. Thus, the absolute sum of the ith row of $\Lambda^{[Q]}$ is given by

$$\begin{aligned} &\frac{1}{2}\{|u_i v_1| + \cdots + |u_i v_{i-1}| + |u_i v_i| + |u_i v_{i+1}| + \cdots + |u_i v_N|\} \\ =&\frac{1}{2}\{-u_i(v_1 + \cdots + v_i) + u_i(v_{i+1} + \cdots + v_N)\} \\ =&u_i \left\{[1\ 1]^T - (v_1 + \cdots + v_i)\right\} \end{aligned} \tag{4.114}$$

since $v_1 + \cdots + v_Q = [2\ 2]^T$. If we note that $[1\ 1]^T - (v_1 + \cdots + v_i)$ is nothing but the ith vertex to the left from the right-upper vertex $[1, 1]^T$ along the left-upper edges of the polygon and thus lies on the ith edge of the $2Q$-polygon, we see from the definition of u_i that the above absolute sum equals 1 for each $i = 1, \cdots, Q$. This implies that the unit ball image of $\Omega^{[N]}$ is contained in the $2Q$-polygon, i.e.,

$$\{w_\Delta \mid \|\Delta^{[Q]}w_\Delta\| \leq 1\} \supset \{\Omega^{[Q]}w_d \mid \|w_d\| \leq 1\} \tag{4.115}$$

This completes the proof.

Chapter 5

Conclusion

This thesis studied performance analysis and sampled-data controller synthesis for bounded persistent disturbances. The L_∞-induced norm was used to deal with bounded persistent disturbances, and methods for computing the L_∞-induced norm of continuous-time and sampled-data systems were provided by using ideas of input approximation and kernel approximation approaches. Furthermore, the sampled-data controller synthesis problem for L_∞-induced norm minimization was dealt with. We summarize the main contributions of this thesis in the following.

In Chapter 2, we tackled a difficult problem of accurately computing the L_∞-induced norm associated with a stable continuous-time finite-dimensional linear time-invariant (FDLTI) system. To solve this problem, we applied a truncation idea with a sufficiently large h, which mostly reduces the problem to the induced-norm computation of the compression operator defined on the time interval $[0, h)$. We first developed an input approximation approach to L_∞-induced norm computation based on the fast-lifting treatment, by which the input of a continuous-time FDLTI system was approximated by piecewise constant or piecewise linear functions. We next developed a kernel approximation approach to the L_∞-induced norm computation problem, by which the kernel function associated with the convolution formula of continuous-time FDLTI systems was approximated by piecewise constant or piecewise linear functions, and this approach was also based on fast-lifting. Even though they are two different approximation approaches in terms of the viewpoint behind approximations, they share a common technical feature that they employ a piecewise constant approximation or piecewise linear approximation scheme of functions. We then showed that the approximation errors in these two approaches converge to 0 at the rates of $1/M$ and $1/M^2$ in the piecewise constant approximation and piecewise linear approximation schemes, respectively, as the fast-lifting parameter M tends to infinity. Through these evaluations, we gave methods for computing an upper bound and lower bound of the L_∞-induced norm of continuous-time

FDLTI systems to any degree of accuracy. We examined effectiveness of the two approximation approaches through numerical studies and confirmed that the kernel approximation approach works far more effectively than the input approximation approach, not only in accuracy but also in computation times, especially when the piecewise linear approximation scheme is taken.

In Chapter 3, we considered a difficult problem of accurately computing the L_∞-induced norm of sampled-data systems. We first applied the lifting technique to sampled-data systems and derived the input and output operators in the lifted representation of sampled-data systems. To compute the L_∞-induced norm, we approximated these operators by using the ideas of input approximation and kernel approximation approaches discussed in Chapter 2. Both approximation approaches use ideas of piecewise constant approximation and piecewise linear approximation schemes via the fast-lifting treatment of sampled-data systems. In particular, the input or the kernel function of the input operator as well as the hold function of the output operator associated with sampled-data systems are approximated by piecewise constant or piecewise linear functions. Through these ideas, we gave methods that can readily compute an upper bound and lower bound of the L_∞-induced norm. Furthermore, we showed that the gap between the upper and lower bounds in the piecewise constant approximation or piecewise linear approximation scheme is ensured to converge to 0 at the rate of $1/M$ or $1/M^2$, respectively, under both the input approximation and kernel approximation approaches, as the fast-lifting parameter M tends to infinity. We further clarified that even though these convergence rates are qualitatively the same in the two approximation approaches, the approximation errors through the kernel approximation approach are smaller than those through the input approximation approach. We finally examined the effectiveness of the two approximation approaches through numerical studies and confirmed that the kernel approximation approach with the piecewise linear approximation scheme derived the smallest range for the L_∞-induced norm estimates with relatively short computation times among the four methods developed in this chapter.

In Chapter 4, we provided discretization methods of the continuous-time generalized plant for the L_1 optimal sampled-data control problem, which allows us to carry out sampled-data controller synthesis for L_∞-induced norm minimization through existing synthesis methods for the discrete-time controller minimizing the l_∞-induced norm. We applied the ideas of the piecewise constant approximation or piecewise linear approximation scheme together with the input approximation approach discussed in Chapter 3 to the L_1 optimal sampled-data control problem. Through these applications, the continuous-time signals (i.e., the input signals and the output signals) were approximated by piecewise constant or piecewise linear functions, and this led to the approximations of the input and output operators associated

with the lifted sampled-data systems. We further established two important inequalities through the arguments of preadjoint operators, which formed mathematical bases for the piecewise constant approximation and piecewise linear approximation schemes to the L_1 optimal sampled-data controller synthesis problem. More precisely, it was shown from these inequalities that the convergence rate for the L_1 optimal sampled-data controller synthesis in piecewise constant or piecewise linear approximation is $1/M$ or $1/M^2$, respectively. We further gave discretization procedures for the continuous-time generalized plant through piecewise constant and piecewise linear approximations. These procedures enabled us to give an 'equivalent' discrete-time generalized plant to the continuous-time generalized plant with any degree of accuracy, and thus the L_1 optimal sampled-data controller synthesis problem was reduced to the discrete-time l_1 optimal controller synthesis problem through piecewise constant and piecewise linear approximations. We finally examined the effectiveness of the developed methods through a numerical study and confirmed that the piecewise linear approximation method works far more effectively than the piecewise constant approximation method.

To summarize, we studied performance analysis and sampled-data controller synthesis for bounded persistent disturbances by taking the L_∞-induced norm as a measure for quantifying the effect of the disturbances. The input approximation and kernel approximation approaches played key roles in computing the L_∞-induced norm of continuous-time and sampled-data systems. We further provided methods for the L_1 optimal sampled-data controller synthesis, by which the L_∞-induced norm of sampled-data systems is minimized. This synthesis problem was tackled only through the input approximation approach, unlike the analysis problems for continuous-time and sampled-data systems, for which the kernel approximation approach was also studied and, more importantly, outperformed the input approximation approach. However, it is not clear whether the kernel approximation approach can be directly applied to the L_1 optimal controller synthesis problem of sampled-data systems while the input approximation approach can be. The reason is relevant to the question on the parallel convergence arguments for the operators $\mathbf{T}_0'$ and $\mathbf{T}_1'$ such that $\mathbf{B}_{\mathrm{k}0}' = \mathbf{B}_1' \cdot \mathbf{T}_0'$ and $\mathbf{B}_{\mathrm{k}1}' = \mathbf{B}_1' \cdot \mathbf{T}_1'$. The preadjoint arguments, which play a crucial role in solving the L_1 optimal controller synthesis problem with the input approximation approach, would somehow apply to the operators $\mathbf{T}_0'$ and $\mathbf{T}_1'$ in the kernel approximation approach, but it is still unclear whether parallel convergence rates could be established. This, in turn, implies that developing a theoretical basis of the kernel approximation approach for this synthesis problem seems to be a nontrivial issue. This interesting topic is left for future studies.

Another issue worth noting in the arguments of this thesis is that only piecewise constant and piecewise linear functions were used in the approximation treatment of systems. Roughly

speaking, these functions correspond to jth order polynomials with $j = 0$ and $j = 1$, and one might ask whether higher order polynomials with $j \geq 2$ could be used to develop parallel frameworks and derive more effective methods. In fact, constructing the jth order approximations for the L_∞-induced norm computation of continuous-time and sampled-data systems could, in principle, be carried out even for $j \geq 2$ by following the same line of arguments as in Chapters 2 and 3. Nevertheless, the overall performance improvement by taking $j \geq 2$ may not be definite since it would take a longer time to accurately compute the $L_1[0, h')$ norms of jth-order polynomials when $j \geq 2$. Furthermore, extension to $j \geq 2$ in sampled-data systems is nontrivial because we cannot predetermine the timing $\theta' \in [0, h')$ such that the output of $\mathbf{M}'_{\mathrm{a}j}$ at θ' does correspond to our induced-norm computation. This is in sharp contrast with the arguments in this thesis dealing only with $j = 0$ and $j = 1$ (i.e., piecewise constant and piecewise linear approximations), in which it is obvious that considering only $\theta' = 0$ and $\theta' \to h'$ is sufficient. Analyzing such an aspect and developing an effective computation method exploiting a jth-order approximation idea for $j \geq 2$ may be an interesting future topic.

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