Rough path theory via fractional calculus

Yu Ito
Department of Applied Analysis and Complex Dynamical Systems
Graduate School of Informatics
Kyoto University

Abstract

In this paper, we develop an alternative approach to the fundamental theory of rough paths on the basis of fractional calculus. First, using fractional derivatives, we introduce integration along β-Hölder rough paths for any roughness $\beta \in (0, 1]$ and prove that this integral coincides with the first level path of the rough integral along geometric β-Hölder rough paths that were introduced by Lyons [19]. Next, we generalize the formulation to adapt for the concept of controlled paths introduced by Gubinelli [9]. As an application, we provide an alternative proof of Lyons' extension theorem for geometric β-Hölder rough paths together with an explicit expression of the extension map. Finally, using the integration of controlled paths based on fractional derivatives, we formulate rough differential equations and establish existence and uniqueness results of solutions to rough differential equations driven by geometric β-Hölder rough paths with $\beta \in (1/3, 1/2]$.

Contents

List of symbols

1 Introduction

The theory of rough paths introduced by Lyons [19] has produced a framework of multidimensional controlled differential equations driven by non-smooth functions, known as rough differential equations. These differential equations have led to useful methods for studying stochastic calculus; in particular, it has enabled us to take a pathwise approach to classical stochastic calculus and provided a convenient tool for the study of a large class of stochastic processes that are not semimartingales, such as fractional Brownian motions. After this revolution, several different approaches have been proposed for the study of the theory of rough paths (e.g. [4, 5, 7, 9–11, 13]). Among other things, controlling rough paths introduced by Gubinelli [9] gave a natural extension of rough integration, the integration of 1-forms along rough paths by Lyons; this extension yielded another formulation of rough differential equations. Recent researches have shown that the rough differential equations in the sense of Gubinelli [9] have produced new methods for studying stochastic partial differential equations as well as stochastic differential equations; indeed, it has provided appropriate frameworks for a number of classically ill-posed stochastic partial differential equations, including Burgers type equations and the KPZ equations.

On the other hand, Hu and Nualart [13] introduced another different approach, which relies on fractional calculus. They defined integration along Hölder continuous functions of order $\beta \in (1/3, 1/2)$ generalizing Riemann–Stieltjes type integrals for more regular functions in terms of fractional derivatives by Zähle [25]. This integral can be regarded as an alternative definition of rough integrals and provides an additional tool to study multidimensional controlled differential equations driven by Hölder continuous functions; for example, Besalú and Nualart [2] made use of this concept for a study of stochastic differential equations driven by fractional Brownian motions with Hurst parameter $H \in (1/3, 1/2)$. Furthermore, Besalú, Márquez-Carreras, and Rovira [1] applied this integral to a study of stochastic delay equations driven by fractional Brownian motions. The results of [2, 13] and [1] with $\beta, H \in (1/3, 1/2)$ can be considered as extensions of their previous works [12, 22] and [3] in the case $\beta, H > 1/2$. Here it should be noted that, in the case $\beta, H > 1/2$, there is no need to use the theory of rough paths because of sufficient regularities of the functions under consideration. This approach is beneficial in that the integration is not based on any approximation arguments, in contrast to the rough integration of Lyons [19] as the limit of a type of Riemann sums, and this explicit formula straightforwardly leads to quantitative estimates of the integration. Therefore, we expect further developments in this direction to provide sophisticated access to the fundamental theory of rough paths.

Motivated by these preceding studies, this paper develops the approach by Hu and Nualart [13] to more general rough paths. Treating β-Hölder continuous functions with β less than $1/3$ is much more involved since we have to consider rough paths up to the Nth level path, where N is the unique integer such that $N \leq 1/\beta < N + 1$. We first define the integral along β-Hölder rough paths for any roughness $\beta \in (0, 1]$ using fractional derivatives (Definition 3.1), which is explicitly given by ordinary Lebesgue integrals. The definition is entirely new and generalizes preceding studies [13, 25]. To ensure the definition is reasonable, we prove that the integral is consistent with the Riemann–Stieltjes integral along smooth curves (Theorem 3.3) and is a continuous functional with respect to the β-Hölder rough path metric (Theorem 3.4). As a result, this integral coincides with the first level path of the rough integral along geometric β-Hölder rough paths in the sense of Lyons [19] (Theorem 3.5).

One of the key ingredients for the definition of our integral is the integration by parts of fractional orders as described by Hu and Nualart [13, Theorem 3.3]. Due to less regularity of functions, the integrand has to be decomposed into the regular part and the remainder part for the integration to make sense. The latter is then replaced by the higher level path of the rough path. In this procedure, we have to take care of additional terms that successively arise from integration by parts formulas and the multiplicative property of the rough path. For this reason, the resulting formula involves some complicated terms. Once we have explicit expressions for the integral, however, it is not difficult to provide quantitative estimates for proving the continuity of the integration operator.

We next generalize the integral of 1-forms along rough paths introduced in Definition 3.1 to that of controlled paths. The concept of controlled paths was introduced by Gubinelli [9] to produce a more general framework of rough integrals and differential equations. This generalization has an application to Lyons' extension theorem (also called the first fundamental result in the theory of rough paths) as follows. Let $X = (1, X^1, \dots, X^N)$ be a β-Hölder rough path, that is, a multiplicative functional of degree N with finite β-Hölder estimates (see Eqs. (2.3) and (2.4)). Lyons' extension theorem states that for any integer $k \geq N + 1$, the rough path $X = (1, X^1, \dots, X^N)$ extends to the unique multiplicative functional of degree k that possesses β-Hölder estimates (see [19, Theorem 2.2.1] for the exact statement of the claim). This extension map has been constructed by a discrete approximation similar to the Riemann sums [19]. By using our integration, the extension map induced by geometric Hölder rough paths is expressed explicitly by ordinary Lebesgue integrals using fractional derivatives (Definition 3.17). This result can also be regarded as an alternative proof of Lyons' extension theorem for geometric Hölder rough paths. Gubinelli also proved Lyons' extension theorem in his framework (cf. [9, Proposition 10]), but our approach is different from his and the results are not comparable.

Lastly, we formulate rough differential equations driven by β-Hölder rough paths with $\beta \in (1/3, 1/2]$ in our framework (Definition 4.1). Our definition of the solutions is consistent with that of Gubinelli [9]. We first solve rough differential equations driven by geometric β-Hölder rough paths on a small interval by a classical fixed point argument in a suitable complete metric space of controlled paths (Proposition 4.16). Next, concatenating the local solutions, we construct a solution on the whole interval and then show uniqueness of the global solutions. As a result, we establish global existence and uniqueness of solutions to rough differential equations driven by geometric β-Hölder rough paths (Theorem 4.2). For the fixed point argument, it is essential to provide quantitative estimates of the integration of controlled paths. This follows easily from the explicit formula of our integration. Accordingly, our arguments are more straightforward than the original ones of Lyons [19] on the basis of discrete approximations. The main difference with the formulation of Hu and Nualart [13] is described as follows. Their definition of the solutions consists of a closed system of three integral equations and these are all defined by ordinary Lebesgue integrals using fractional derivatives (see Eqs. (4.3), (4.4), and (4.5) of [13]), while ours is given by a closed system of only two equations. One of them is an integral equation and is defined by the integral of controlled paths based on fractional derivatives, but another equation is simply defined without any integrations and can be regarded as the derivative of the former integral equation with respect to (the first level path of) the rough paths. In this sense, our formulation is more concise than that of Hu and Nualart.

The remainder of this paper is organized as follows. The basic framework is arranged in Section 2. In Subsection 2.1, we provide a brief review of the concepts of rough paths and fractional calculus. Our version of Lyons' extension theorem is also described here. In Subsection 2.2, we introduce some fractional operators and prove their continuity properties for later use.

The results concerning the rough integration are arranged in Section 3. In Subsection 3.1, using the fractional operators, we define integrals of 1-forms along rough paths and state the main theorems. In Subsection 3.2, we first provide a definition of controlled paths and their examples. We next define integrals of controlled paths along rough paths and state some of their properties. The application to Lyons' extension theorem is also described. The proofs are given in Subsection 3.3.

We discuss rough differential equations in Section 4. In Subsection 4.1, we formulate the concept of solutions to rough differential equations and state the theorem on existence and uniqueness results of the solutions. For the proof, we prepare some basic estimates and lemmas in the subsequent subsection. The last subsection is devoted to the proof of the main theorem.

Acknowledgements. First and foremost, I would like to thank my supervisor, Professor Masanori Hino for his guidance from my master's course at the graduate school. I wish to express my gratitude to Professor Yuzuru Inahama for his valuable advice, in particular, suggesting the generalization of [14] to integration of controlled paths. I am indebted to Professor Hiroshi Kawabi for many valuable and instructive discussions of the theory of rough paths. His intensive courses on rough paths and controlling rough paths at Ritsumeikan University and Tsukuba University also helped me understand the theory more deeply. Finally, I would like to thank all the members of Sub-department of Applied Analysis for their continuous support.

Sections 2 and 3 of this paper are based on [14, 15]. The final publication of [14] is available at Springer via http://dx.doi.org/10.1007/s11118-014-9428-3.

2 Framework

2.1 Preliminaries

In this subsection, we briefly review some concepts of rough paths [6, 8, 17–21] and fractional operators [23, 25]. Our version of Lyons' extension theorem is also described.

2.1.1 Notation

Throughout this paper, C denotes a positive constant, which may change line by line. Let V and W denote finite-dimensional normed spaces with norms $\|\cdot\|_V$ and $\|\cdot\|_W$, respectively. Although the fundamental theory of rough paths is valid for suitable infinite-dimensional Banach spaces, we consider only finite-dimensional cases in this paper to avoid technical difficulties that are not relevant to our theme. We use $L(V,W)$ to denote the set of all linear maps from V to W. Let U be a subset of V. We use $C(U,W)$ to denote the space of all W-valued continuous functions on U. Let λ be a real number with $0<\lambda\le 1$. We use $C^{\lambda\text{-Höl}}(U,W)$ to denote the space of all W-valued λ-Hölder continuous functions on U and define the corresponding semi-norm by $\|\cdot\|_{\lambda\text{-Höl};U}$, namely

$$\|f\|_{\lambda\text{-Höl};U} := \sup_{x,y\in U,\ x\neq y} \frac{\|f(x)-f(y)\|_W}{\|x-y\|_V^\lambda}.$$

We also use $\|\cdot\|_{\infty;U}$ to denote the supremum norm of a W-valued function on U, namely

$$\|f\|_{\infty;U} := \sup_{x\in U} \|f(x)\|_W.$$

We will omit U from the notation if there is no ambiguity; that is, we write $\|f\|_{\lambda\text{-Höl}}$ and $\|f\|_\infty$ instead of $\|f\|_{\lambda\text{-Höl};U}$ and $\|f\|_{\infty;U}$, respectively. For a subset U_0 of U, we denote the restriction of f on U_0 by $f|_{U_0}$. We can then write $\|f\|_{\lambda\text{-Höl};U_0}$ and $\|f\|_{\infty;U_0}$ for $\|f|_{U_0}\|_{\lambda\text{-Höl};U_0}$ and $\|f|_{U_0}\|_{\infty;U_0}$, respectively. Let l be a non-negative integer. We denote by $C^{l,\lambda}(V,W)$ the space of all W-valued l-times continuously Fréchet differentiable functions on V whose lth derivative is λ-Hölder continuous on V. For $f \in C^{l,\lambda}(V,W)$ such that f, $\nabla f, \dots, \nabla^l f$ are all bounded on V in addition, we set

$$\|f\|_{C^{l,\lambda}} := \bigl(\max_{0\le k\le l} \|\nabla^k f\|_{\infty;V}\bigr) \vee \|\nabla^l f\|_{\lambda\text{-Höl};V}. \tag{2.1}$$

Here, $p \vee q$ denotes the maximum of real numbers p and q. We also use $p\wedge q$ to denote the minimum of p and q. Furthermore, let $\lfloor p\rfloor$ and $\lceil p\rceil$ denote the largest integer less than or equal to p and the smallest integer more than or equal to p, respectively.

Let T denote a positive constant. This constant will be fixed throughout Sections 2 and 3. The simplex $\{(s,t)\in\mathbb{R}^2 : 0\le s\le t\le T\}$ is denoted by Δ_T, which is a closed subset of $\mathbb{R}^2$. Let $\mathcal{C}_1(V)$ and $\mathcal{C}_2(V)$ denote $C([0,T],V)$ and $C(\Delta_T,V)$, respectively. For $f\in\mathcal{C}_1(\mathbb{C})$ and $g\in\mathcal{C}_2(\mathbb{C})$, we define $fg\in\mathcal{C}_2(\mathbb{C})$ and $gf\in\mathcal{C}_2(\mathbb{C})$ by

$$(fg)_{s,t} := f_s g_{s,t} \quad \text{and} \quad (gf)_{s,t} := g_{s,t} f_t \quad \text{for } (s,t)\in\Delta_T. \tag{2.2}$$

For $g\in\mathcal{C}_2(V)$, $\mu>0$, and $(a,b)\in\Delta_T$ with $a<b$, we set

$$\|\!|g|\!\|_{\mu;[a,b]} := \sup_{a\le s<t\le b} \frac{\|g_{s,t}\|_V}{(t-s)^\mu}$$

and write $\|\!|g|\!\|_\mu$ instead of $\|\!|g|\!\|_{\mu;[0,T]}$. Furthermore, we set $\mathcal{C}_2^\mu(V) := \{g\in\mathcal{C}_2(V) : \|\!|g|\!\|_\mu<\infty\}$ and $\mathcal{C}_1^\lambda(V) := C^{\lambda\text{-Höl}}([0,T],V)$.

Hereafter, E and F denote the Euclidean spaces $\mathbb{R}^d$ and $\mathbb{R}^e$ respectively and $|\cdot|$ denotes the Euclidean norms of E, F, and their tensor spaces. For a positive integer k, $T^{(k)}(E)$ denotes $\bigoplus_{j=0}^k E^{\otimes j}$ and we define the norm on $T^{(k)}(E)$ as

$$\|\boldsymbol{a}\|_{T^{(k)}(E)} := \sum_{j=0}^k |a^j| \quad \text{for } \boldsymbol{a} = (a^0,a^1,\dots,a^k)\in T^{(k)}(E).$$

The set of all $X=(X^0,X^1,\dots,X^k)\in C(\Delta_T,T^{(k)}(E))$ such that $X^0_{s,t}=1$ for all $(s,t)\in\Delta_T$ is denoted by $C_0(\Delta_T,T^{(k)}(E))$.

2.1.2 Rough paths and Lyons' extension theorem

Let k be a positive integer. We say that $X=(1,X^1,\dots,X^k)\in C_0(\Delta_T,T^{(k)}(E))$ is a multiplicative functional of degree k in E if

$$\sum_{i=0}^j X^i_{s,u}\otimes X^{j-i}_{u,t} = X^j_{s,t} \tag{2.3}$$

for each $j = 1, \dots, k$ and $s, t, u \in [0, T]$ with $s \le u \le t$. Let β be a real number with $0 < \beta \le 1$. We say that $X = (1, X^1, \dots, X^k) \in C_0(\Delta_T, T^{(k)}(E))$ has finite β-Hölder estimates if

$$\sup_{0 \le s < t \le T} \frac{|X^j_{s,t}|}{(t-s)^{j\beta}} < \infty \tag{2.4}$$

for each $j = 1, \dots, k$. We denote by $C_{0,\beta}(\Delta_T, T^{(k)}(E))$ the space of all $X = (1, X^1, \dots, X^k) \in C_0(\Delta_T, T^{(k)}(E))$ with finite β-Hölder estimates and define the distance on $C_{0,\beta}(\Delta_T, T^{(k)}(E))$ as

$$d_{\beta,k}(X, \tilde{X}) := \max_{1 \le j \le k} \| X^j - \tilde{X}^j \|_{j\beta;[0,T]} \quad \text{for } X, \tilde{X} \in C_{0,\beta}(\Delta_T, T^{(k)}(E)).$$

Let $x \in \mathcal{C}^1_1(E)$. We set

$$X^j_{s,t} = \int_{s < u_1 < \cdots < u_j < t} dx_{u_1} \otimes \cdots \otimes dx_{u_j} \tag{2.5}$$

for each $j = 1, \dots, k$ and $(s,t) \in \Delta_T$. Then we see that $X = (1, X^1, \dots, X^k)$ is a multiplicative functional of degree k in E with finite 1-Hölder estimates and we call this the step-k signature of x. Let N denote $\lfloor 1/\beta \rfloor$. A multiplicative functional of degree N in E with finite β-Hölder estimates is called a β-Hölder rough path in E. A step-N signature is called a smooth rough path and the elements in the closure of the set of all smooth rough paths with respect to the distance $d_{\beta,N}$ are called geometric β-Hölder rough paths. The spaces of all β-Hölder rough paths, smooth rough paths, and geometric β-Hölder rough paths in E are denoted by $\Omega_{\beta,T}(E)$, $S\Omega_{\beta,T}(E)$, and $G\Omega_{\beta,T}(E)$, respectively. We will omit T from the notations $\Omega_{\beta,T}(E)$, $S\Omega_{\beta,T}(E)$, and $G\Omega_{\beta,T}(E)$ if there is no ambiguity. The following property of geometric β-Hölder rough paths $X = (1, X^1, \dots, X^N) \in G\Omega_\beta(E)$ is used in Section 3: for each $j = 1, \dots, k$ and $(s,t) \in \Delta_T$,

$$\text{the symmetric part of } X^j_{s,t} \text{ is equal to } (X^1_{s,t})^{\otimes j}/j!. \tag{2.6}$$

Let us now introduce our version of Lyons' extension theorem.

Theorem 2.1 (cf. [19, Theorem 2.2.1])**.** *Let $X = (1, X^1, \dots, X^N) \in \Omega_\beta(E)$. For any integer $k \ge N+1$, there exists a unique extension of the rough path X to a multiplicative functional of degree k in E with finite β-Hölder estimates.*

In [19, Theorem 2.2.1], rough paths X of finite p-variation with $p := 1/\beta$ are treated and the exact claim includes quantitative estimates for the extension of X by using control functions ω. For Theorem 2.1 and the alternative proof of the theorem for geometric β-Hölder rough paths $X \in G\Omega_\beta(E)$ given in Section 3, we consider only a particular case where ω is given by $\omega(s, t) = C(t-s)$ for some constant C for simplicity and are not concerned with uniform estimates for the continuity of the extension map.

2.1.3 Fractional integrals and derivatives

Let a and b be real numbers with $a < b$. For $p \in [1, \infty)$, $L^p(a,b)$ denotes the real L^p-space on the interval $[a,b]$ with respect to the Lebesgue measure. Let $f \in L^1(a,b)$ and $\alpha \in (0,\infty)$. The left-and right-sided Riemann–Liouville fractional integrals of f of order α are defined for almost all $t \in (a,b)$ by

$$I^{\alpha}_{a+} f(t) := \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)\, ds$$

and

$$I^{\alpha}_{b-} f(t) := \frac{(-1)^{-\alpha}}{\Gamma(\alpha)} \int_t^b (s-t)^{\alpha-1} f(s)\, ds,$$

respectively, where $(-1)^{-\alpha} := e^{-i\pi\alpha}$ and $\Gamma(\alpha)$ denotes the gamma function, namely $\Gamma(\alpha) := \int_0^\infty r^{\alpha-1} e^{-r}\, dr$. We use $I^{\alpha}_{\substack{a+\\(b-)}}(L^p)$ to denote the image of $L^p(a,b)$ by the operator $I^{\alpha}_{\substack{a+\\(b-)}}$. Here, we note a simple criterion for functions to belong to $I^{\alpha}_{\substack{a+\\(b-)}}(L^p)$. This criterion is used frequently in Section 3 without being explicitly noted: if $f \in C^{\lambda\text{-Höl}}([a,b],\mathbb{R})$ with $\alpha < \lambda \le 1$, then $f \in I^{\alpha}_{a+}(L^p) \cap I^{\alpha}_{b-}(L^p)$ for any $1 \le p < \infty$. Let $f \in I^{\alpha}_{\substack{a+\\(b-)}}(L^1)$ with $0 < \alpha < 1$. The left- and right-sided Weyl–Marchaud fractional derivatives of f of order α are defined for almost all $t \in (a,b)$ by

$$D^{\alpha}_{a+} f(t) := \frac{1}{\Gamma(1-\alpha)} \left(\frac{f(t)}{(t-a)^{\alpha}} + \alpha \int_a^t \frac{f(t)-f(s)}{(t-s)^{\alpha+1}}\, ds \right) \tag{2.7}$$

and

$$D^{\alpha}_{b-} f(t) := \frac{(-1)^{\alpha}}{\Gamma(1-\alpha)} \left(\frac{f(t)}{(b-t)^{\alpha}} + \alpha \int_t^b \frac{f(t)-f(s)}{(s-t)^{\alpha+1}}\, ds \right), \tag{2.8}$$

respectively. The integrals above are well-defined for almost all $t \in (a,b)$. The following three formulas are important in the subsequent sections. The first is the composition formula:

$$D^{\alpha}_{\substack{a+\\(b-)}} (D^{\beta}_{\substack{a+\\(b-)}} f) = D^{\alpha+\beta}_{\substack{a+\\(b-)}} f \tag{2.9}$$

for $f \in I^{\alpha+\beta}_{\substack{a+\\(b-)}}(L^1)$, $0 < \alpha < 1$, and $0 < \beta < 1$ with $\alpha + \beta < 1$. The second is the basic integration by parts formula of order α:

$$(-1)^{\alpha} \int_a^b D^{\alpha}_{a+} f(t) g(t)\, dt = \int_a^b f(t) D^{\alpha}_{b-} g(t)\, dt \tag{2.10}$$

for $f \in I^{\alpha}_{a+}(L^p)$, $g \in I^{\alpha}_{b-}(L^q)$, $0 < \alpha < 1$, $1 \le p < \infty$, and $1 \le q < \infty$ with $1/p + 1/q \le 1 + \alpha$. The third is also regarded as an integration by parts formula of order α. Let $f \in C^{\lambda\text{-Höl}}([a,b],\mathbb{R})$ and $g \in C^{\mu\text{-Höl}}([a,b],\mathbb{R})$ with $\lambda + \mu > 1$. Then, the Riemann–Stieltjes integral $\int_a^b f(t)\, dg(t)$ exists [24] and is expressed as follows: for $\alpha \in (1-\mu, \lambda)$,

$$\int_a^b f(t)\, dg(t) = (-1)^{\alpha} \int_a^b D^{\alpha}_{a+} f_{a+}(t) D^{1-\alpha}_{b-} g_{b-}(t)\, dt + f(a)(g(b) - g(a)) \tag{2.11}$$

$$= (-1)^{\alpha} \int_a^b D^{\alpha}_{a+} f(t) D^{1-\alpha}_{b-} g_{b-}(t)\, dt, \tag{2.12}$$

where $f_{a+}(t) := f(t) - f(a)$ and $g_{b-}(t) := g(t) - g(b)$. For proofs of Eqs. (2.11) and (2.12), see [25, Theorem 4.2.1 and Proposition 2.2].

2.2 Some fractional operators

In this subsection, we introduce some variants of the fractional derivatives and integral operators for later use. Throughout this subsection, we will assume the following: (a,b) is an element of Δ_T with $a<b$, β is a real number with $0<\beta\le 1$, k is a positive integer, and γ is a real number with $0<\gamma<\min\{1/k,\beta\}$. We also recall V is a finite-dimensional normed space with norm $\|\cdot\|_V$.

2.2.1 Definition of the operators and their properties

Let $\mu>0$ and $\Psi\in\mathcal{C}_2^{\mu}(V)$. For $\alpha\in(0,\mu\wedge 1)$, we define $\mathcal{D}_{a+}^{\alpha}\Psi$ and $\mathcal{D}_{b-}^{\alpha}\Psi$ as $\mathcal{D}_{a+}^{\alpha}\Psi(a):=0$,

$$\mathcal{D}_{a+}^{\alpha}\Psi(u):=\frac{1}{\Gamma(1-\alpha)}\left(\frac{\Psi_{a,u}}{(u-a)^{\alpha}}+\alpha\int_a^u\frac{\Psi_{v,u}}{(u-v)^{\alpha+1}}\,dv\right)\quad\text{for } u\in(a,T] \tag{2.13}$$

and $\mathcal{D}_{b-}^{\alpha}\Psi(b):=0$,

$$\mathcal{D}_{b-}^{\alpha}\Psi(r):=\frac{(-1)^{1+\alpha}}{\Gamma(1-\alpha)}\left(\frac{\Psi_{r,b}}{(b-r)^{\alpha}}+\alpha\int_r^b\frac{\Psi_{r,v}}{(v-r)^{\alpha+1}}\,dv\right)\quad\text{for } r\in[0,b). \tag{2.14}$$

It is straightforward to show that, for each $u\in[a,T]$ and $r\in[0,b]$,

$$\|\mathcal{D}_{a+}^{\alpha}\Psi(u)\|_V\le\frac{1}{\Gamma(1-\alpha)}\frac{\mu}{\mu-\alpha}\|\Psi\|_{\mu;[a,u]}(u-a)^{\mu-\alpha} \tag{2.15}$$

and

$$\|\mathcal{D}_{b-}^{\alpha}\Psi(r)\|_V\le\frac{1}{\Gamma(1-\alpha)}\frac{\mu}{\mu-\alpha}\|\Psi\|_{\mu;[r,b]}(b-r)^{\mu-\alpha}. \tag{2.16}$$

If $\Psi\in\mathcal{C}_2^{\lambda}(V)$ is of the form $\Psi_{s,t}=\psi(t)-\psi(s)$ for some $\psi\in\mathcal{C}_1^{\lambda}(V)$ with $0<\lambda\le 1$, then the identity $\mathcal{D}^{\alpha}_{\substack{a+\\(b-)}}\Psi=D^{\alpha}_{\substack{a+\\(b-)}}\psi_{\substack{a+\\(b-)}}$ holds for $\alpha\in(0,\lambda)$ from the definition. Using these functions, we further introduce the following.

Definition 2.2. Let $X=(1,X^1,\dots,X^k)\in C_{0,\beta}(\Delta_T,T^{(k)}(E))$ and $\{\gamma_l\}_{l=1}^k$ be a set of positive numbers that satisfy $\gamma_l<\beta$ for each $l=1,\dots,k$ and $\sum_{l=1}^k\gamma_l\le 1$. Then, for each $j=1,\dots,k$, we define a function $\mathcal{R}_{b-}^{\gamma_1,\dots,\gamma_j}X$ on $[0,b]$ as follows: for each $r\in[0,b]$,

$$\mathcal{R}_{b-}^{\gamma_1}X(r):=\mathcal{D}_{b-}^{\gamma_1}X^1(r)$$

and

$$\mathcal{R}_{b-}^{\gamma_1,\dots,\gamma_j}X(r):=\mathcal{D}_{b-}^{\sum_{l=1}^j\gamma_l}X^j(r)-\sum_{i=1}^{j-1}\mathcal{D}_{b-}^{\sum_{l=i+1}^j\gamma_l}(X^{j-i}\otimes\mathcal{R}_{b-}^{\gamma_1,\dots,\gamma_i}X)(r)$$

for $j=2,\dots,k$, inductively. If $\gamma_1,\dots,\gamma_k$ are all the same value γ, we write $\mathcal{R}_{b-}^{(j,\gamma)}X$ for $\mathcal{R}_{b-}^{\gamma_1,\dots,\gamma_j}X$. Specifically, $\mathcal{R}_{b-}^{(1,\gamma)}X(r):=\mathcal{D}_{b-}^{\gamma}X^1(r)$ and

$$\mathcal{R}_{b-}^{(j,\gamma)}X(r):=\mathcal{D}_{b-}^{j\gamma}X^j(r)-\sum_{i=1}^{j-1}\mathcal{D}_{b-}^{(j-i)\gamma}(X^{j-i}\otimes\mathcal{R}_{b-}^{(i,\gamma)}X)(r)$$

for $j=2,\dots,k$.

We note that $\mathcal{R}^{(j,\gamma)}_{b-}X$ is well-defined by the assumption that $0 < \gamma < \min\{1/k, \beta\}$. Furthermore, with regard to the second terms of $\mathcal{R}^{\gamma_1,\dots,\gamma_j}_{b-}X(r)$ and $\mathcal{R}^{(j,\gamma)}_{b-}X(r)$,

$$\mathcal{D}^{\sum_{l=i+1}^{j}\gamma_l}_{b-}(X^{j-i}\otimes\mathcal{R}^{\gamma_1,\dots,\gamma_i}_{b-}X)(r) = \frac{(-1)^{1+\sum_{l=i+1}^{j}\gamma_l}\sum_{l=i+1}^{j}\gamma_l}{\Gamma(1-\sum_{l=i+1}^{j}\gamma_l)}\int_r^b \frac{X^{j-i}_{r,v}\otimes\mathcal{R}^{\gamma_1,\dots,\gamma_i}_{b-}X(v)}{(v-r)^{\sum_{l=i+1}^{j}\gamma_l+1}}\,dv \quad (2.17)$$

and

$$\mathcal{D}^{(j-i)\gamma}_{b-}(X^{j-i}\otimes\mathcal{R}^{(i,\gamma)}_{b-}X)(r) = \frac{(-1)^{1+(j-i)\gamma}(j-i)\gamma}{\Gamma(1-(j-i)\gamma)}\int_r^b \frac{X^{j-i}_{r,v}\otimes\mathcal{R}^{(i,\gamma)}_{b-}X(v)}{(v-r)^{(j-i)\gamma+1}}\,dv \quad (2.18)$$

hold for each $i = 1,\dots,j-1$ from Eqs. (2.2), (2.14), and $\mathcal{R}^{\gamma_1,\dots,\gamma_i}_{b-}X(b) = \mathcal{R}^{(i,\gamma)}_{b-}X(b) = 0$.

Definition 2.3. Let $X = (1, X^1,\dots,X^k) \in C_{0,\beta}(\Delta_T, T^{(k)}(E))$, $j = 1,\dots,k$, μ a real number with $\mu > 1 - j\gamma$ and Ψ a function in $\mathcal{C}^{\mu}_2(L(E^{\otimes(j-1)}, L(E,F)))$. An F-valued function $\mathcal{I}^{j,\gamma}_X(\Psi)$ on Δ_T is defined as

$$\mathcal{I}^{j,\gamma}_X(\Psi)_{s,t} := (-1)^{1-j\gamma}\int_s^t \mathcal{D}^{1-j\gamma}_{s+}\Psi(u)\mathcal{R}^{(j,\gamma)}_{t-}X(u)\,du \quad \text{for } (s,t)\in\Delta_T. \quad (2.19)$$

These functions possess the following continuity properties.

Proposition 2.4. *In the setting of Definition 2.3, the map $\Psi \mapsto \mathcal{I}^{j,\gamma}_X(\Psi)$ is bounded linear from $\mathcal{C}^{\mu}_2(L(E^{\otimes(j-1)}, L(E,F)))$ to $\mathcal{C}^{\mu+j\beta}_2(F)$; in particular, it is Lipschitz continuous.*

Proposition 2.5. *In the setting of Definition 2.3, the map $(1, X^1,\dots,X^j) \mapsto \mathcal{I}^{j,\gamma}_X(\Psi)$ is locally Lipschitz continuous from $C_{0,\beta}(\Delta_T, T^{(j)}(E))$ to $\mathcal{C}^{\mu+j\beta}_2(F)$.*

We will prove them in the remainder of this subsection.

2.2.2 Proof of Propositions 2.4 and 2.5

Let $X, \tilde{X} \in C_{0,\beta}(\Delta_T, T^{(k)}(E))$. For each $j = 1,\dots,k$, we set $K^{(j)}_{a,b} := \max_{1\le i\le j}\|X^i\|_{i\beta;[a,b]}$ and $\tilde{K}^{(j)}_{a,b} := \max_{1\le i\le j}\|\tilde{X}^i\|_{i\beta;[a,b]}$.

Lemma 2.6. *Under the above notation and assumptions, for each $j = 2,\dots,k$,*

$$\|\mathcal{R}^{(j,\gamma)}_{b-}X - \mathcal{R}^{(j,\gamma)}_{b-}\tilde{X}\|_{\infty;[a,b]} \le C(1 + C(K^{(j-1)}_{a,b} + \tilde{K}^{(j-1)}_{a,b}))^{j-1}\max_{1\le i\le j}\|X^i - \tilde{X}^i\|_{i\beta;[a,b]}(b-a)^{j(\beta-\gamma)}, \quad (2.20)$$

where $C = (\beta/(\beta-\gamma))\Gamma(1-\gamma)^{-1}$. If $\tilde{X} = (1,0,\dots,0)$, then, for each $j = 2,\dots,k$,

$$\|\mathcal{R}^{(j,\gamma)}_{b-}X\|_{\infty;[a,b]} \le C(1 + CK^{(j-1)}_{a,b})^{j-1}K^{(j)}_{a,b}(b-a)^{j(\beta-\gamma)}. \quad (2.21)$$

Proof. We prove Eq. (2.20) by induction on j. We set $r \in [a,b]$ with $a \le r < b$ since $\mathcal{R}_{b-}^{(j,\gamma)}X(b) = \mathcal{R}_{b-}^{(j,\gamma)}\tilde{X}(b) = 0$ holds from the definition. From Eq. (2.18),

$$\begin{aligned}
|\mathcal{R}_{b-}^{(2,\gamma)}X(r) - \mathcal{R}_{b-}^{(2,\gamma)}\tilde{X}(r)| \le |\mathcal{D}_{b-}^{2\gamma}X^2(r) - \mathcal{D}_{b-}^{2\gamma}\tilde{X}^2(r)| \\
+ \frac{\gamma}{\Gamma(1-\gamma)}\int_r^b \frac{|X_{r,v}^1 - \tilde{X}_{r,v}^1||\mathcal{R}_{b-}^{(1,\gamma)}X(v)|}{(v-r)^{\gamma+1}}\,dv \\
+ \frac{\gamma}{\Gamma(1-\gamma)}\int_r^b \frac{|\tilde{X}_{r,v}^1||\mathcal{R}_{b-}^{(1,\gamma)}X(v) - \mathcal{R}_{b-}^{(1,\gamma)}\tilde{X}(v)|}{(v-r)^{\gamma+1}}\,dv \\
=: A_1 + A_2 + A_3.
\end{aligned}$$

From Eq. (2.16), we have

$$A_1 \le \frac{1}{\Gamma(1-2\gamma)}\frac{2\beta}{2\beta-2\gamma}\|X^2 - \tilde{X}^2\|_{2\beta;[r,b]}(b-r)^{2(\beta-\gamma)} \le C\|X^2 - \tilde{X}^2\|_{2\beta;[r,b]}(b-r)^{2(\beta-\gamma)}$$

and

$$\begin{aligned}
A_2 &\le \frac{\gamma}{\Gamma(1-\gamma)}\int_r^b (v-r)^{\beta-\gamma-1}\,dv\|X^1 - \tilde{X}^1\|_{\beta;[r,b]}\frac{1}{\Gamma(1-\gamma)}\frac{\beta}{\beta-\gamma}\|X^1\|_{\beta;[r,b]}(b-r)^{\beta-\gamma} \\
&= \frac{\gamma}{\Gamma(1-\gamma)}\frac{(b-r)^{\beta-\gamma}}{\beta-\gamma}\|X^1 - \tilde{X}^1\|_{\beta;[r,b]}C\|X^1\|_{\beta;[r,b]}(b-r)^{\beta-\gamma} \\
&\le C^2\|X^1\|_{\beta;[r,b]}\|X^1 - \tilde{X}^1\|_{\beta;[r,b]}(b-r)^{2(\beta-\gamma)}.
\end{aligned}$$

In a similar way, we get

$$A_3 \le C^2\|\tilde{X}^1\|_{\beta;[r,b]}\|X^1 - \tilde{X}^1\|_{\beta;[r,b]}(b-r)^{2(\beta-\gamma)}.$$

By combining these estimates, we obtain

$$A_1 + A_2 + A_3 \le C(1 + (C(\|X^1\|_{\beta;[r,b]} + \|\tilde{X}^1\|_{\beta;[r,b]}))) \max_{1\le l\le 2}\|X^l - \tilde{X}^l\|_{l\beta;[r,b]}(b-r)^{2(\beta-\gamma)}.$$

Hence, Eq. (2.20) holds for $j = 2$. Suppose that Eq. (2.20) holds for each $j = 2,\dots,J$ with $J \le k-1$. By using the induction hypothesis and calculations similar to those shown above, we have

$$\begin{aligned}
&|\mathcal{R}_{b-}^{(J+1,\gamma)}X(r) - \mathcal{R}_{b-}^{(J+1,\gamma)}\tilde{X}(r)| \\
&= |\mathcal{D}_{b-}^{(J+1)\gamma}X^{J+1}(r) - \mathcal{D}_{b-}^{(J+1)\gamma}\tilde{X}^{J+1}(r)| \\
&\quad + \sum_{i=1}^{J}\left\{\frac{(J+1-i)\gamma}{\Gamma(1-(J+1-i)\gamma)}\int_r^b \frac{|X_{r,v}^{J+1-i} - \tilde{X}_{r,v}^{J+1-i}||\mathcal{R}_{b-}^{(i,\gamma)}X(v)|}{(v-r)^{(J+1-i)\gamma+1}}\,dv\right. \\
&\quad \left. + \frac{(J+1-i)\gamma}{\Gamma(1-(J+1-i)\gamma)}\int_r^b \frac{|\tilde{X}_{r,v}^{J+1-i}||\mathcal{R}_{b-}^{(i,\gamma)}X(v) - \mathcal{R}_{b-}^{(i,\gamma)}\tilde{X}(v)|}{(v-r)^{(J+1-i)\gamma+1}}\,dv\right\} \\
&\le C\|X^{J+1} - \tilde{X}^{J+1}\|_{(J+1)\beta;[r,b]}(b-r)^{(J+1)(\beta-\gamma)}
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=1}^{J}\Bigg\{C\|X^{J+1-i}-\tilde{X}^{J+1-i}\|_{(J+1-i)\beta;[r,b]}(b-r)^{(J+1-i)(\beta-\gamma)}\\
&\quad\times C(1+CK_{r,b}^{(i-1)})^{i-1}K_{r,b}^{(i)}(b-r)^{i(b-r)}\\
&+C\|\tilde{X}^{J+1-i}\|_{(J+1-i)\beta;[r,b]}(b-r)^{(J+1-i)(\beta-\gamma)}\\
&\quad\times C(1+C(K_{r,b}^{(i-1)}+\tilde{K}_{r,b}^{(i-1)}))^{i-1}\max_{1\le l\le i}\|X^l-\tilde{X}^l\|_{l\beta;[r,b]}(b-r)^{i(b-r)}\Bigg\}\\
&\quad\text{(from the induction hypothesis)}\\
&\le C\max_{1\le l\le J+1}\|X^l-\tilde{X}^l\|_{l\beta;[r,b]}(b-r)^{(J+1)(\beta-\gamma)}\\
&\quad\times\left(1+C\sum_{i=1}^{J}\left(K_{r,b}^{(i)}+\|\tilde{X}^{J+1-i}\|_{(J+1-i)\beta;[r,b]}\right)(1+C(K_{r,b}^{(i-1)}+\tilde{K}_{r,b}^{(i-1)}))^{i-1}\right)\\
&\le C\max_{1\le l\le J+1}\|X^l-\tilde{X}^l\|_{l\beta;[r,b]}(b-r)^{(J+1)(\beta-\gamma)}\\
&\quad\times\left(1+C(K_{r,b}^{(J)}+\tilde{K}_{r,b}^{(J)})\sum_{i=1}^{J}(1+C(K_{r,b}^{(J)}+\tilde{K}_{r,b}^{(J)}))^{i-1}\right)\\
&=C(1+C(K_{r,b}^{(J)}+\tilde{K}_{r,b}^{(J)}))^{J}\max_{1\le l\le J+1}\|X^l-\tilde{X}^l\|_{l\beta;[r,b]}(b-r)^{(J+1)(b-r)},
\end{aligned}
$$

as desired. Therefore, Eq. (2.20) holds for $j=J+1$. □

Proof of Propositions 2.4 and 2.5. We first prove Proposition 2.4. The linearity of $\mathcal{I}_X^{j,\gamma}$ follows immediately from the definition. From Eq. (2.21) and the relation $C\le\beta/(\beta-\gamma)$, we have

$$
\|\mathcal{R}_{b-}^{(j,\gamma)}X\|_{\infty;[a,b]}\le C_{j,\beta,\gamma}(1+\max_{1\le i\le j-1}\|X^i\|_{i\beta;[a,b]})^{j-1}\max_{1\le i\le j}\|X^i\|_{i\beta;[a,b]}(b-a)^{j(\beta-\gamma)}, \tag{2.22}
$$

where $C_{j,\beta,\gamma}:=(\beta/(\beta-\gamma))^j$. Then, from Eq. (2.15), for each $(s,t)\in\Delta_T$ with $a\le s<t\le b$,

$$
\begin{aligned}
|\mathcal{I}_X^{j,\gamma}(\Psi)_{s,t}|&\le\|\mathcal{D}_{s+}^{1-j\gamma}\Psi\|_{\infty;[s,t]}\|\mathcal{R}_{b-}^{(j,\gamma)}X\|_{\infty;[s,t]}(t-s)\\
&\le C_{j,\beta,\gamma,\mu}\|\Psi\|_{\mu;[s,t]}(1+\max_{1\le i\le j-1}\|X^i\|_{i\beta;[s,t]})^{j-1}\max_{1\le i\le j}\|X^i\|_{i\beta;[s,t]}(t-s)^{\mu+j\beta},
\end{aligned}
$$

where $C_{j,\beta,\gamma,\mu}:=(\mu/(\mu-(1-j\gamma)))\Gamma(j\gamma)^{-1}C_{j,\beta,\gamma}$. Therefore,

$$
\|\mathcal{I}_X^{j,\gamma}(\Psi)\|_{\mu+j\beta;[a,b]}\le C_{j,\beta,\gamma,\mu}\|\Psi\|_{\mu;[a,b]}(1+\max_{1\le i\le j-1}\|X^i\|_{i\beta;[a,b]})^{j-1}\max_{1\le i\le j}\|X^i\|_{i\beta;[a,b]}. \tag{2.23}
$$

It is also straightforward to show that $\mathcal{I}_X^{j,\gamma}(\Psi)$ belongs to $\mathcal{C}_2(F)$. Hence, $\mathcal{I}_X^{j,\gamma}(\Psi)\in\mathcal{C}_2^{\mu+j\beta}(F)$ and $\mathcal{I}_X^{j,\gamma}$ is bounded. Thus we obtain the claim of Proposition 2.4. We next prove Proposition 2.5 in a similar way. From Eq. (2.20) and the relation $C\le\beta/(\beta-\gamma)$,

$$
\begin{aligned}
&\|\mathcal{R}_{b-}^{(j,\gamma)}X-\mathcal{R}_{b-}^{(j,\gamma)}\tilde{X}\|_{\infty;[a,b]}\\
&\le C_{j,\beta,\gamma}(1+\max_{1\le i\le j-1}\|X^i\|_{i\beta;[a,b]}+\max_{1\le i\le j-1}\|\tilde{X}^i\|_{i\beta;[a,b]})^{j-1}\max_{1\le i\le j}\|X^i-\tilde{X}^i\|_{i\beta;[a,b]}(b-a)^{j(\beta-\gamma)}
\end{aligned}
\tag{2.24}
$$

and from Eq. (2.15),

$$
\begin{aligned}
&\|\mathcal{I}_X^{j,\gamma}(\Psi) - \mathcal{I}_{\tilde{X}}^{j,\gamma}(\Psi)\|_{\mu+j\beta;[a,b]} \\
&\leq C_{j,\beta,\gamma,\mu}\|\Psi\|_{\mu;[a,b]}(1 + \max_{1\leq i\leq j-1}\|X^i\|_{i\beta;[a,b]} + \max_{1\leq i\leq j-1}\|\tilde{X}^i\|_{i\beta;[a,b]})^{j-1}\max_{1\leq i\leq j}\|X^i - \tilde{X}^i\|_{i\beta;[a,b]}.
\end{aligned}
\tag{2.25}
$$

This yields Proposition 2.5. □

3 Integrals along rough paths via fractional calculus

3.1 Integration of 1-forms

We introduce our definition of integrals of 1-forms along rough paths as well as the main theorems of this section. Throughout this subsection, we will assume the following: (a, b) is an element of Δ_T with $a < b$ and β is a real number with $0 < \beta \leq 1$. We also recall $N = \lfloor 1/\beta \rfloor$ and V is a finite-dimensional normed space with norm $\|\cdot\|_V$.

3.1.1 Definition of the integral

We introduce two symbols for the definition of our integral. For $X = (1, X^1, \ldots, X^N) \in \Omega_\beta(E)$ and $\xi \in E$, we define $X^{1,\xi} \in \mathcal{C}_1^\beta(E)$ as

$$X_t^{1,\xi} := \xi + X_{0,t}^1 \quad \text{for } t \in [0, T]. \tag{3.1}$$

Let l be an integer with $0 \leq l \leq N - 1$ and λ a real number with $0 < \lambda \leq 1$. For $f \in C^{l,\lambda}(E, V)$ and $x \in \mathcal{C}_1^\beta(E)$, we define a V-valued function $R^l(f, x)$ on Δ_T as

$$R^l(f, x)_{s,t} := f(x_t) - \sum_{i=0}^{l} \frac{1}{i!}\nabla^i f(x_s)(x_t - x_s)^{\otimes i} \quad \text{for } (s, t) \in \Delta_T.$$

Then, $R^l(f, x)$ belongs to $\mathcal{C}_2^{(l+\lambda)\beta}(V)$; indeed, it is easy to prove that there exists a positive constant $C_{l,\lambda}$ such that

$$\|R^l(f, x)\|_{(l+\lambda)\beta;[a,b]} \leq C_{l,\lambda}\|\nabla^l f\|_{\lambda\text{-Höl}}\|x\|_{\beta\text{-Höl};[a,b]}^{l+\lambda}. \tag{3.2}$$

We are now ready to define the integral of 1-forms along X.

Definition 3.1. Let $X \in \Omega_\beta(E)$ and $\xi \in E$. Let λ be a real number with $1/\beta - N < \lambda \leq 1$ and $\varphi \in C^{N-1,\lambda}(E, L(E, F))$. Take γ such that $(1 - \lambda\beta)/N < \gamma < \beta$. Then, for each $(s, t) \in \Delta_T$, we define $I_\varphi^\gamma(X, \xi)_{s,t} \in F$ as

$$I_\varphi^\gamma(X, \xi)_{s,t} := \sum_{n=1}^{N} \nabla^{n-1}\varphi(X_s^{1,\xi})X_{s,t}^n + \sum_{n=1}^{N} \mathcal{I}_X^{n,\gamma}(R^{N-n}(\nabla^{n-1}\varphi, X^{1,\xi}))_{s,t}. \tag{3.3}$$

Remark 3.2. Let us make a few comments about the definition above.

(1) The inequality $1 - n\gamma < (N - n + \lambda)\beta$ follows from the assumption $(1 - \lambda\beta)/N < \gamma < \beta$. Then, $\mathcal{I}_X^{n,\gamma}(R^{N-n}(\nabla^{n-1}\varphi, X^{1,\xi}))$ is well-defined from Eq. (3.2) and so is $I_\varphi^\gamma(X, \xi)$. Moreover, we see from Proposition 2.4 that $I_\varphi^\gamma(X, \xi)$ belongs to $\mathcal{C}_2^\beta(F)$.

(2) If $N = 1$, then $I_\varphi^\gamma(X, \xi)_{a,b}$ coincides with the right-hand side of Eq. (2.11) with $f(t) = \varphi(X_t^{1,\xi})$, $g(t) = X_{0,t}^1$ and $\alpha = 1 - \gamma$. Hence, the equality $I_\varphi^\gamma(X, \xi)_{a,b} = \int_a^b \varphi(X_t^{1,\xi})\, dX_{0,t}^1$ holds. In particular, this value is independent of the choice of γ. If $N \geq 2$, from Theorems 3.3 and 3.4 stated below, $I_\varphi^\gamma(X, \xi)_{a,b}$ is independent of the choice of γ for $X \in G\Omega_\beta(E)$. However, it is uncertain whether such a property holds for non-geometric Hölder rough paths $X \in \Omega_\beta(E)$.

(3) Although the definition of $\mathcal{R}_{t-}^{(n,\gamma)}X$ looks complicated, as seen in Subsection 3.3, this term naturally comes out from the integration by parts of fractional orders and the multiplicative property of the rough path as described by Hu and Nualart [13, Theorem 3.3]. Indeed, if $N = 2$, the equality $\mathcal{R}_{b-}^{(2,\gamma)}X = D_{b-}^\gamma(\mathcal{D}_{b-}^\gamma X^2)$ holds for $X \in S\Omega_\beta(E)$ as seen from the proof of Proposition 3.30. The right-hand side of this equality is also well-defined for every $X \in \Omega_\beta(E)$ and this appears in the integral introduced by Hu and Nualart [13, Definition 3.2]. On the other hand, the left-hand side of this equality and its generalizations $\mathcal{R}_{t-}^{(n,\gamma)}X$ in our integration appear for the first time.

3.1.2 Statement of main theorems

The following are the main theorems of this subsection.

Theorem 3.3. *Let $X \in S\Omega_\beta(E)$, $\xi \in E$, and $\varphi \in C^{N-1,1}(E, L(E, F))$. Take $\gamma \in ((1 - \beta)/N, \beta)$. Then, for each $(s, t) \in \Delta_T$, $I_\varphi^\gamma(X, \xi)_{s,t}$ coincides with the Riemann–Stieltjes integral $\int_s^t \varphi(X_u^{1,\xi})\, dX_{0,u}^1$.*

Theorem 3.4. *Let $0 < \lambda \leq 1$ and $\varphi \in C^{N,\lambda}(E, L(E, F))$ such that $\nabla\varphi, \ldots, \nabla^N\varphi$ are bounded on E. Take $\gamma \in ((1 - \beta)/N, \beta)$. Then, the map $(X, \xi) \mapsto I_\varphi^\gamma(X, \xi)$ is locally Lipschitz continuous from $\Omega_\beta(E) \times E$ to $\mathcal{C}_2^\beta(F)$.*

We prove Theorems 3.3 and 3.4 in Subsection 3.3. From these theorems, $I_\varphi^\gamma(X, \xi)$ is closely related to the rough integral introduced in Lyons [19]. We refer to Section 4.3 of [20] for the definition of the rough integral and the details of the construction. Let $X \in G\Omega_\beta(E)$, $\xi \in E$, and $1/\beta - N < \lambda \leq 1$. Let $\varphi \in C^{N-1,\lambda}(E, L(E, F))$ such that φ, $\nabla\varphi, \ldots, \nabla^{N-1}\varphi$ are bounded on E. Then the rough integral $\int \varphi(X_u^{1,\xi})\, dX$ is defined and the map $X \mapsto \int \varphi(X_u^{1,\xi})\, dX$ is continuous from $G\Omega_\beta(E)$ to $G\Omega_\beta(F)$ [20, Definition 4.9 and Theorem 4.12]. Moreover, if $X \in S\Omega_\beta(E)$, then, for each $(s, t) \in \Delta_T$, the first level path of the rough integral $\int_s^t \varphi(X_u^{1,\xi})\, dX^1$ coincides with the Riemann–Stieltjes integral $\int_s^t \varphi(X_u^{1,\xi})\, dX_{0,u}^1$. Using these properties of $\int \varphi(X_u^{1,\xi})\, dX$ and Theorems 3.3 and 3.4, we obtain the following:

Theorem 3.5. *Let $X \in G\Omega_\beta(E)$, $\xi \in E$, and $0 < \lambda \leq 1$. Let $\varphi \in C^{N,\lambda}(E, L(E, F))$ such that φ, $\nabla\varphi, \ldots, \nabla^N\varphi$ are bounded on E. Take $\gamma \in ((1 - \beta)/N, \beta)$. Then, for each $(s, t) \in \Delta_T$, $I_\varphi^\gamma(X, \xi)_{s,t}$ coincides with the first level path of the rough integral $\int_s^t \varphi(X_u^{1,\xi})\, dX^1$.*

Remark 3.6. Let us make a few conceptual comments about our integration.

(1) Because the definition of our integration is based on fractional derivatives, it is essential for the rough paths to possess the finite Hölder estimates (2.4). In the usual integration theory of rough paths, the finite p-variation condition for $p \geq 1$ is imposed on rough paths instead of (2.4), which are called p-rough paths. The space of p-rough paths is a complete metric space with respect to the p-variation norm and this space contains the space of β-Hölder rough paths with $p = 1/\beta$. On the other hand, p-rough paths are identified with β-Hölder rough paths via re-parameterizations. In this sense, our framework is not very restricted as compared with the usual theory of rough paths.

(2) The identification mentioned above is also valid for geometric p-rough paths. More precisely, given a geometric p-rough path X in E, there exists a continuous increasing function τ from $[0, T]$ to itself such that $\tilde{X} \in G\Omega_\beta(E)$, where $\tilde{X}_{s,t} := X_{\tau(s),\tau(t)}$ for $(s,t) \in \Delta_T$. Then, from Theorems 3.3 and 3.4, for each $(s,t) \in \Delta_T$, $I^\gamma_\varphi(\tilde{X}, \xi)_{s,t}$ coincides with the first level path of the rough integral $\int_s^t \varphi(X_u^{1,\xi})\, dX^1$ along p-rough path X and the value of $I^\gamma_\varphi(\tilde{X}, \xi)_{s,t}$ is independent of the choice of re-parameterization τ. Similarly, $I^\gamma_\varphi(\tilde{X}, \xi)_{s,t}$ is well-defined for non-geometric p-rough paths X and the corresponding re-parameterizations. However, it is unknown whether $I^\gamma_\varphi(\tilde{X}, \xi)_{s,t}$ possesses such two properties.

(3) The relation to the integral introduced by Hu and Nualart [13] is stated as follows. For $f \in C(F, L(E, F))$, we define $\hat{f} \in C(E \oplus F, L(E \oplus F, E \oplus F))$ as

$$\hat{f}(x,y)(u,v) := (u, f(y)u) \quad \text{for } (x,y), (u,v) \in E \oplus F.$$

Let $\beta \in (1/3, 1/2)$, $\hat{\xi} \in E \oplus F$, and $Z \in G\Omega_\beta(E \oplus F)$. Take $\gamma \in ((1-\beta)/2, \beta)$. Then, for sufficiently smooth f, the projection of $I^\gamma_{\hat{f}}(Z, \hat{\xi})_{a,b} \in E \oplus F$ onto F is identical to the integral in Definition 3.2 given by Hu and Nualart [13]. This follows from Theorems 3.3, 3.4, and the corresponding results [13, Theorem 3.3, Propositions 3.4, and 6.4]. However, it is not known whether such identification is true for non-geometric Hölder rough paths $Z \in \Omega_\beta(E \oplus F)$.

3.2 Integration of controlled paths and its application

In this subsection, we introduce integrals of controlled paths along rough paths. This is a generalization of the integral introduced in Definition 3.1. As an application, we provide an alternative proof of Lyons' extension theorem for geometric Hölder rough paths together with an explicit expression of the extension map. Throughout this subsection, we will assume the following: (a,b) is an element of Δ_T with $a < b$, β is a real number with $0 < \beta \leq 1$, k is a positive number, and γ is a real number with $0 < \gamma < \min\{1/k, \beta\}$.

3.2.1 Controlled paths

Let $X = (1, X^1, \dots, X^k) \in C_{0,\beta}(\Delta_T, T^{(k)}(E))$. We say that a k-tuple $Y = (Y^{(0)}, Y^{(1)}, \dots, Y^{(k-1)})$ is a path controlled by X with values in F if Y satisfies the following two properties:

(1) for each $l = 0, \dots, k-1$, $Y^{(l)} \in \mathcal{C}^\beta_1(L(E^{\otimes l}, F))$;

(2) for each $l = 0, \dots, k-1$, $R_l^{k-1-l}(X,Y) \in \mathcal{C}_2^{(k-l)\beta}(L(E^{\otimes l}, F))$, where

$$R_l^{k-1-l}(X,Y)_{s,t} := Y_t^{(l)} - \sum_{i=0}^{k-1-l} Y_s^{(l+i)} X_{s,t}^i \quad \text{for } (s,t) \in \Delta_T. \tag{3.4}$$

It is sometimes referred to as a controlled path for $X \in C_{0,\beta}(\Delta_T, T^{(k)}(E))$. The space of all paths controlled by $X \in C_{0,\beta}(\Delta_T, T^{(k)}(E))$ with values in F is denoted by $\mathcal{Q}_{X,T}^{\beta,k}(F)$, which is a normed space under the norm $Y \mapsto \sum_{l=0}^{k-1} |Y_0^{(l)}| + \|Y\|_{X,\beta,k;[0,T]}$. Here, in general, $\|Y\|_{X,\beta;[a,b]}$ is defined by

$$\|Y\|_{X,\beta,k;[a,b]} := \sum_{l=0}^{k-1} \|R_l^{k-1-l}(X,Y)\|_{(k-l)\beta;[a,b]} \quad \text{for } Y \in \mathcal{Q}_{X,T}^{\beta,k}(F). \tag{3.5}$$

If there is no ambiguity, we omit T and $[0,T]$ from the notations $\mathcal{Q}_{X,T}^{\beta,k}(F)$ and $\|Y\|_{X,\beta,k;[0,T]}$, respectively. We also write $\|Y\|_{X,\beta;[a,b]}$ instead of $\|Y\|_{X,\beta,N;[a,b]}$. Although the highest level path X^k is not necessary for our definition of paths controlled by $X \in C_{0,\beta}(\Delta_T, T^{(k)}(E))$, we need it in applications. The multiplicative property (2.3) is not assumed for $X \in C_{0,\beta}(\Delta_T, T^{(k)}(E))$ in the definition of paths controlled by X. In the following examples, however, such properties play an essential role in confirming property (2) in the definition above.

Example 3.7. Let $\varphi \in C^{N-1,1}(E, L(E,F))$ and $X = (1, X^1, \dots, X^N) \in G\Omega_\beta(E)$. For $\xi \in E$, we define $X^{1,\xi} \in \mathcal{C}_1^\beta(E)$ as in Eq. (3.1). For each $l = 0, \dots, N-1$, we set $Y^{(l)} \in \mathcal{C}_1^\beta(L(E^{\otimes l}, L(E,F)))$ as

$$Y_t^{(l)} := \nabla^l \varphi(X_t^{1,\xi}) \quad \text{for } t \in [0,T]. \tag{3.6}$$

From the property (2.6) and the symmetry of the derivatives of φ,

$$R_l^{N-1-l}(X,Y)_{s,t} = R^{N-1-l}(\nabla^l \varphi, X^{1,\xi})_{s,t} \tag{3.7}$$

holds for each $l = 0, \dots, N-1$ and $(s,t) \in \Delta_T$. Then, from Eq. (3.2), we have

$$|R_l^{N-1-l}(X,Y)_{s,t}| \leq C_{l,\lambda} \|\nabla^{N-1}\varphi\|_{1\text{-Höl}} \|X^1\|_\beta^{N-l} (t-s)^{(N-l)\beta}.$$

Thus, $Y = (Y^{(0)}, Y^{(1)}, \dots, Y^{(N-1)})$ belongs to $\mathcal{Q}_X^{\beta,N}(L(E,F))$. In addition, if $1/3 < \beta \leq 1$, then Y belongs to $\mathcal{Q}_X^{\beta,N}(L(E,F))$ for every $X \in \Omega_\beta(E)$.

Example 3.8 (cf. [9, Proposition 4]). Let $1/3 < \beta \leq 1/2$, $X = (1, X^1, X^2) \in C_{0,\beta}(\Delta_T, T^{(2)}(E))$, $Y = (Y^{(0)}, Y^{(1)}) \in \mathcal{Q}_X^{\beta,2}(F)$, and $\varphi \in C^{1,1}(F, L(E,F))$ such that $\nabla\varphi$ is bounded on F. We set $\varphi(Y)^{(0)} \in \mathcal{C}_1^\beta(L(E,F))$ and $\varphi(Y)^{(1)} \in \mathcal{C}_1^\beta(L(E, L(E,F)))$ as

$$\varphi(Y)_t^{(0)} := \varphi(Y_t^{(0)}) \quad \text{and} \quad \varphi(Y)_t^{(1)} := \nabla\varphi(Y_t^{(0)}) Y_t^{(1)} \quad \text{for } t \in [0,T].$$

Then, $\varphi(Y) := (\varphi(Y)^{(0)}, \varphi(Y)^{(1)})$ belongs to $\mathcal{Q}_X^{\beta,2}(L(E,F))$. For the proof, see Lemma 4.3.

Example 3.9. Let X be a multiplicative functional of degree k in E with finite β-Hölder estimates. For each $l = 0, \dots, k-1$, we set $Y^{(l)} \in \mathcal{C}_1^\beta(L(E^{\otimes l}, L(E, E^{\otimes(k+1)})))$ as

$$(Y_t^{(l)}(\eta))(\xi) := (X_{0,t}^{k-l} \otimes \eta) \otimes \xi \quad \text{for } t \in [0,T], \tag{3.8}$$

where $\eta \in E^{\otimes l}$ and $\xi \in E$. From Eq. (2.3), for each $l = 0, \dots, k-1$ and $(s,t) \in \Delta_T$,

$$R_l^{k-1-l}(X,Y)_{s,t} = X_{0,t}^{k-l} - \sum_{i=0}^{k-1-l} X_{0,s}^{k-l-i} \otimes X_{s,t}^i = X_{s,t}^{k-l}. \tag{3.9}$$

Then, from Eq. (2.4), $Y = (Y^{(0)}, Y^{(1)}, \dots, Y^{(k-1)})$ belongs to $\mathcal{Q}_X^{\beta,k}(L(E, E^{\otimes(k+1)}))$.

The construction of multiplicative functionals with finite β-Hölder estimates in Example 3.9 is used in the proof of Lyons' extension theorem (Theorem 3.20). The controlled path in Example 3.8 is used in the definition of rough differential equations (Definition 4.1).

3.2.2 Definition of the integral and its properties

We assume that $(1-\beta)/N < \gamma < \beta$ in this subsection. We set

$$M_{\beta,T}(E,F) := \{(X,Y) : X \in C_{0,\beta}(\Delta_T, T^{(N)}(E)), Y \in \mathcal{Q}_{X,T}^{\beta,N}(F)\}$$

equipped with a distance

$$m_\beta((X,Y),(\tilde{X},\tilde{Y})) := d_{\beta,N}(X,\tilde{X}) + \sum_{j=1}^N |Y_0^{(j-1)} - \tilde{Y}_0^{(j-1)}| + d_{X,\tilde{X},\beta}(Y,\tilde{Y}) \tag{3.10}$$

for $(X,Y),(\tilde{X},\tilde{Y}) \in M_{\beta,T}(E,F)$. Here,

$$d_{X,\tilde{X},\beta}(Y,\tilde{Y}) := \sum_{j=1}^N \| R_{j-1}^{N-j}(X,Y) - R_{j-1}^{N-j}(\tilde{X},\tilde{Y}) \|_{(N-j+1)\beta;[0,T]}. \tag{3.11}$$

We define the subset $S_{\beta,T}(E,F)$ of $M_{\beta,T}(E,F)$ by $S_{\beta,T}(E,F) := \{(X,Y) : X \in S\Omega_{\beta,T}(E), Y \in \mathcal{Q}_{X,T}^{1,N}(F)\}$ and let $\overline{S_{\beta,T}}(E,F)$ denote the closure of $S_{\beta,T}(E,F)$ with respect to the distance m_β. We will omit T from the notations $M_{\beta,T}(E,F)$, $S_{\beta,T}(E,F)$, and $\overline{S_{\beta,T}}(E,F)$ if there is no ambiguity. In Example 3.7, if φ is sufficiently smooth and all derivatives are bounded on E, then the pair (X,Y) belongs to $\overline{S_\beta}(E, L(E,F))$. This is proved by straightforward calculation. In Example 3.8, if the pair (X,Y) is in $\overline{S_\beta}(E,F)$ and $\varphi \in C^{2,1}(E, L(E,F))$ such that $\nabla\varphi$ and $\nabla^2\varphi$ are bounded on F, then the pair $(X, \varphi(Y))$ belongs to $\overline{S_\beta}(E, L(E,F))$. See Proposition 4.7 for the proof. The following is our definition of the integral of controlled paths along rough paths.

Definition 3.10. For $(X,Y) \in M_\beta(E, L(E,F))$, an F-valued function $I^\gamma(X,Y)$ on Δ_T is defined by

$$I^\gamma(X,Y)_{s,t} := \sum_{n=1}^N Y_s^{(n-1)} X_{s,t}^n + \sum_{n=1}^N \mathcal{I}_X^{n,\gamma}(R_{n-1}^{N-n}(X,Y))_{s,t} \quad \text{for } (s,t) \in \Delta_T.$$

We note that the inequality $1-n\gamma < (N-n+1)\beta$ follows from the assumption that $(1-\beta)/N < \gamma$. Therefore, $\mathcal{I}_X^{n,\gamma}(R_{n-1}^{N-n}(X,Y))_{s,t}$ is well-defined and so is $I^\gamma(X,Y)_{s,t}$. The following theorem justifies treating $I^\gamma(X,Y)$ as the integral of Y along X.

Theorem 3.11. *Let $(X,Y) \in S_\beta(E, L(E,F))$. Then, for each $(s,t) \in \Delta_T$, $I^\gamma(X,Y)_{s,t}$ coincides with the Riemann–Stieltjes integral $\int_s^t Y_u^{(0)} dX_{0,u}^1$.*

We prove Theorem 3.11 in Subsection 3.3.

Theorem 3.12. *The map $(X,Y) \mapsto I^\gamma(X,Y)$ is locally Lipschitz continuous from $M_\beta(E, L(E,F))$ to $\mathcal{C}_2^\beta(F)$.*

Proof. From Proposition 2.4, $I^\gamma(X,Y)$ belongs to $\mathcal{C}_2^\beta(F)$. Set $(s,t) \in \Delta_T$ with $s < t$. For (X,Y), $(\tilde{X},\tilde{Y}) \in M_\beta(E, L(E,F))$,

$$\begin{aligned}|I^\gamma(X,Y)_{s,t} - I^\gamma(\tilde{X},\tilde{Y})_{s,t}| \le \sum_{n=1}^N \Big\{ &|Y_s^{(n-1)} - \tilde{Y}_s^{(n-1)}||X_{s,t}^n| + |\tilde{Y}_s^{(n-1)}||X_{s,t}^n - \tilde{X}_{s,t}^n| \\ &+ |\mathcal{I}_X^{n,\gamma}(R_{n-1}^{N-n}(X,Y) - R_{n-1}^{N-n}(\tilde{X},\tilde{Y}))_{s,t}| \\ &+ |\mathcal{I}_X^{n,\gamma}(R_{n-1}^{N-n}(\tilde{X},\tilde{Y}))_{s,t} - \mathcal{I}_{\tilde{X}}^{n,\gamma}(R_{n-1}^{N-n}(\tilde{X},\tilde{Y}))_{s,t}| \Big\}.\end{aligned} \tag{3.12}$$

By the definition of controlled paths, we have

$$\begin{aligned}|Y_s^{(n-1)} - \tilde{Y}_s^{(n-1)}| &\le |Y_0^{(n-1)} - \tilde{Y}_0^{(n-1)}| + |R_{n-1}^{N-n}(X,Y)_{0,s} - R_{n-1}^{N-n}(\tilde{X},\tilde{Y})_{0,s}| \\ &\quad + \sum_{i=1}^{N-n} |Y_0^{(n-1+i)} X_{0,s}^i - \tilde{Y}_0^{(n-1+i)} \tilde{X}_{0,s}^i| \\ &\le |Y_0^{(n-1)} - \tilde{Y}_0^{(n-1)}| + \|R_{n-1}^{N-n}(X,Y) - R_{n-1}^{N-n}(\tilde{X},\tilde{Y})\|_{(N-n+1)\beta} T^{(N-n+1)\beta} \\ &\quad + \sum_{i=1}^{N-n} \Big\{ |Y_0^{(n-1+i)} - \tilde{Y}_0^{(n-1+i)}||X_{0,s}^i| + |\tilde{Y}_0^{(n-1+i)}||X_{0,s}^i - \tilde{X}_{0,s}^i| \Big\} \\ &\le \left(1 + T^{(N-n+1)\beta} + \sum_{i=1}^{N-n} \left(\|X^i\|_{i\beta} + |\tilde{Y}_0^{(n-1+i)}| \right) T^{i\beta} \right) m_\beta((X,Y),(\tilde{X},\tilde{Y}))\end{aligned} \tag{3.13}$$

for each $n = 1, \dots, N$. Then, from Eqs. (2.23), (2.25), (3.10), (3.11), (3.12), and (3.13), we obtain the statement of the theorem immediately. $\square$

Corollary 3.13. *Let $X \in C_{0,\beta}(\Delta_T, T^{(N)}(E))$. Then, the map $Y \mapsto I^\gamma(X,Y)$ is locally Lipschitz continuous from $\mathcal{Q}_X^{\beta,N}(L(E,F))$ to $\mathcal{C}_2^\beta(F)$.*

Proof. Apply Eqs. (2.23) and (3.5) to Eq. (3.12) with $X = \tilde{X}$. $\square$

From Theorems 3.11 and 3.12, we see that, for each $s,t,u \in [0,T]$ with $s \le u \le t$, the identity

$$I^\gamma(X,Y)_{s,u} + I^\gamma(X,Y)_{u,t} = I^\gamma(X,Y)_{s,t} \tag{3.14}$$

holds for $(X,Y) \in \overline{S_\beta}(E, L(E,F))$. Using this relation, we obtain the following propositions.

Proposition 3.14. *Let $(X,Y) \in \overline{S_\beta}(E, L(E,F))$. Then, for each $(s,t) \in \Delta_T$,*

$$I^\gamma(X,Y)_{s,t} = \lim_{|\mathcal{P}_{s,t}|\to 0} \sum_{i=0}^{m-1} \sum_{n=1}^{N} Y_{u_i}^{(n-1)} X_{u_i,u_{i+1}}^n,$$

where the limit is taken over all finite partitions $\mathcal{P}_{s,t} = \{u_0, u_1, \dots, u_m\}$ of the interval $[s,t]$ such that $s = u_0 \le u_1 \le \cdots \le u_m = t$ and $|\mathcal{P}_{s,t}| := \max_{0\le i\le m-1}|u_{i+1} - u_i|$.

Proof. From Eq. (3.14) and Definition 3.10, for any partition $\mathcal{P}_{s,t} = \{u_0, u_1, \dots, u_m\}$,

$$\begin{aligned} I^\gamma(X,Y)_{s,t} &= \sum_{i=0}^{m-1} I^\gamma(X,Y)_{u_i,u_{i+1}} \\ &= \sum_{i=0}^{m-1} \left\{ \sum_{n=1}^{N} Y_{u_i}^{(n-1)} X_{u_i,u_{i+1}}^n + \sum_{n=1}^{N} \mathcal{I}_X^{n,\gamma}(R_{n-1}^{N-n}(X,Y))_{u_i,u_{i+1}} \right\}. \end{aligned}$$

It then suffices to show that, for each $n = 1, \dots, N$,

$$\lim_{|\mathcal{P}_{s,t}|\to 0} \sum_{i=0}^{m-1} |\mathcal{I}_X^{n,\gamma}(R_{n-1}^{N-n}(X,Y))_{u_i,u_{i+1}}| = 0. \tag{3.15}$$

From Eq. (2.23), we have

$$\begin{aligned} &|\mathcal{I}_X^{n,\gamma}(R_{n-1}^{N-n}(X,Y))_{u_i,u_{i+1}}| \\ &\le C_{n,\beta,\gamma} \|R_{n-1}^{N-n}(X,Y)\|_{(N-n+1)\beta} (1 + \max_{1\le j\le n-1} \|X^j\|_{j\beta})^{n-1} \max_{1\le j\le n} \|X^j\|_{j\beta} (u_{i+1} - u_i)^{(N+1)\beta}. \end{aligned}$$

Thus, from the relation $(N+1)\beta > 1$,

$$\sum_{i=0}^{m-1} |\mathcal{I}_X^{n,\gamma}(R_{n-1}^{N-n}(X,Y))_{u_i,u_{i+1}}| \le C \sum_{i=0}^{m-1} (u_{i+1} - u_i)^{(N+1)\beta} \le C|\mathcal{P}_{s,t}|^{(N+1)\beta-1}(t-s) \to 0$$

as $|\mathcal{P}_{s,t}| \to 0$. Here, C is a positive constant that does not depend on $\mathcal{P}_{s,t}$. Therefore, Eq. (3.15) holds. Thus we obtain the claim of the proposition. $\square$

Proposition 3.15. *Let $(X,Y) \in \overline{S_\beta}(E, L(E,F))$. We set $Z = (Z^{(0)}, Z^{(1)}, \dots, Z^{(N-1)})$ as*

$$Z_t^{(0)} := I^\gamma(X,Y)_{0,t} \quad \textit{and} \quad Z_t^{(l)} := Y_t^{(l-1)} \quad \textit{for } t \in [0,T]$$

and each $l = 1, \dots, N-1$. Then, (X,Z) belongs to $\overline{S_\beta}(E, L(E,F))$.

Proposition 3.15 will be used in Section 4 and the proof is given in Subsection 3.3.

Remark 3.16. Let us make a few comments about our integration.

(1) Take $X \in G\Omega_\beta(E)$ and $Y \in \mathcal{Q}_X^{\beta,N}(L(E,F))$ as in Example 3.7. Then, $I^\gamma(X,Y)$ is the same as the integral introduced in Definition 3.1. Thus we see from Theorem 3.5 that $I^\gamma(X,Y)$ coincides with the first level path of the rough integral along $X \in G\Omega_\beta(E)$.

(2) The relation to the integration of controlled paths introduced by Gubinelli [9] is stated as follows. Let $1/3 < \beta \le 1/2$. Based on Proposition 3.14 and [9, Corollaries 2 and 3], we see that $I^\gamma(X,Y)$ coincides with the integral introduced in [9, Corollary 3] on Δ_T if $I^\gamma(X,Y)$ satisfies Eq. (3.14). Therefore, for $(X,Y) \in \overline{S_\beta}(E, L(E,F))$, $I^\gamma(X,Y)$ is consistent with the integral introduced in [9, Corollary 3] on Δ_T. However, it is unknown whether Eq. (3.14) is true for every $X \in \Omega_\beta(E)$ and $Y \in \mathcal{Q}_X^{\beta;2}(L(E,F))$.

(3) If $N = 1$, then $I^\gamma(X,Y)_{a,b}$ coincides with the Riemann–Stieltjes integral $\int_a^b Y_t^{(0)} dX_{0,t}^1$ for $(X,Y) \in M_\beta(X,Y)$, which follows from Eq. (2.11) with $f = Y^{(0)}$, $g = X_{0,\cdot}^1$ and $\alpha = 1-\gamma$. In particular, $I^\gamma(X,Y)_{a,b}$ is independent of the choice of γ. If $N \ge 2$, then this value is independent of the choice of γ for $(X,Y) \in \overline{S_\beta}(E, L(E,F))$ from Proposition 3.14. However, it is unknown whether such a property holds for every $(X,Y) \in M_\beta(E, L(E,F))$.

3.2.3 Application: Lyons' extension theorem via fractional calculus

We assume the following: j is an integer with $j \ge N$ and γ_j is a real number with $(1-\beta)/j < \gamma_j < \min\{1/j, \beta\}$. To construct the extension map we first define the following functional.

Definition 3.17. For $X = (1, X^1, \dots, X^j) \in C_{0,\beta}(\Delta_T, T^{(j)}(E))$, an $E^{\otimes(j+1)}$-valued function $\hat{X}^{j+1}$ on Δ_T is defined by

$$\hat{X}_{s,t}^{j+1} := \sum_{n=1}^{j} (-1)^{1-n\gamma_j} \int_s^t \mathcal{D}_{s+}^{1-n\gamma_j} X^{j+1-n}(u) \otimes \mathcal{R}_{t-}^{(n,\gamma_j)} X(u)\, du \quad \text{for } (s,t) \in \Delta_T.$$

We note that the inequality $1 - n\gamma_j < (j+1-n)\beta$ follows from the assumption that $(1-\beta)/j < \gamma_j$. Thus, $\hat{X}^{j+1}$ is well-defined and for each $(s,t) \in \Delta_T$,

$$|\hat{X}_{s,t}^{j+1}| \le C_{j,\beta,\gamma_j} (\max_{1 \le i \le j} \|X^i\|_{i\beta})^2 ((1 + \max_{1 \le i \le j-1} \|X^i\|_{i\beta})^j - 1)(\max_{1 \le i \le j-1} \|X^i\|_{i\beta})^{-1} (t-s)^{(j+1)\beta}$$

from Eqs. (2.4), (2.15), (2.19), and (2.23). Furthermore, from Propositions 2.4 and 2.5, we obtain the following proposition.

Proposition 3.18. *For $X = (1, X^1, \dots, X^j) \in C_{0,\beta}(\Delta_T, T^{(j)}(E))$, the map $X \mapsto \hat{X}^{j+1}$ is locally Lipschitz continuous from $C_{0,\beta}(\Delta_T, T^{(j)}(E))$ to $\mathcal{C}_2^{(j+1)\beta}(E^{\otimes(j+1)})$.*

The following is a key proposition for the proof of Theorem 3.20.

Proposition 3.19. *Let $X = (1, X^1, \dots, X^j)$ be a step-j signature in E. Then, $(1, X^1, \dots, X^j, \hat{X}^{j+1})$ is the step-$(j+1)$ signature, that is, for each $(s,t) \in \Delta_T$, $\hat{X}_{s,t}^{j+1}$ coincides with the Riemann–Stieltjes integral $\int_s^t X_{s,u}^j \otimes dX_{0,u}^1$.*

We prove Proposition 3.19 in Subsection 3.3. From Propositions 3.18 and 3.19, for geometric β-Hölder rough paths $X \in G\Omega_\beta(E)$, we can see that the definition of $\hat{X}^{j+1}$ is independent of the choice of γ_j. The following is our version of Lyons' extension theorem for $X \in G\Omega_\beta(E)$.

Theorem 3.20. *Let $X = (1, X^1, \dots, X^N) \in G\Omega_\beta(E)$. For any integer $k \ge N+1$, there exists an extension of the rough path X to a multiplicative functional of degree k in E with finite β-Hölder estimates.*

Proof. We define $\hat{X}^{N+1}$ as in Definition 3.17 and set $\hat{X}^{(N+1)} := (1, X^1, \dots, X^N, \hat{X}^{N+1})$. Here, we take an arbitrary γ_N such that $(1-\beta)/N < \gamma_N < \min\{1/N, \beta\} = \beta$ to define $\hat{X}^{N+1}$. From Proposition 3.18, $\hat{X}^{(N+1)}$ belongs to $C_{0,\beta}(\Delta_T, T^{(N+1)}(E))$. By the definition of $X \in G\Omega_\beta(E)$, there exists a sequence of smooth rough paths $X(m)$ which converges to X with respect to the distance $d_{\beta,N}$. Hence, from Propositions 3.18 and 3.19, $\lim_{m\to\infty} d_{\beta,N+1}(X(m)^{(N+1)}, \hat{X}^{(N+1)}) = 0$, where $X(m)^{(N+1)}$ is the step-$(N+1)$ signature of $X(m)^1_{0,\cdot} \in \mathcal{C}^1_1(E)$. Thus, $\hat{X}^{(N+1)}$ is a multiplicative functional of degree $(N+1)$ in E. This implies the statement of the theorem for $k = N+1$. By repeating this argument with the parameters $\gamma_{N+1}, \dots, \gamma_{k-1}$, the desired statement is proven for any $k \geq N+1$. □

We remark that [19, Theorem 2.2.1] implies the uniqueness of extensions even for $X \in \Omega_\beta(E)$. In particular, for $X \in G\Omega_\beta(E)$, the extension by Theorem 3.20 coincides with those introduced by Lyons [19, Theorem 2.2.1] and by Gubinelli [9, Proposition 10]. However, it is unknown for non-geometric Hölder rough paths $X \in \Omega_\beta(E)$ whether $\hat{X}^{(k)}$ defined as in Theorem 3.20 is a multiplicative functional of degree k in E.

3.3 Proofs

In this subsection, we prove Theorems 3.3, 3.4, 3.11, and Proposition 3.19. Throughout this subsection, we assume the following: (a, b) is an element of Δ_T with $a < b$, β is a real number with $0 < \beta \leq 1$, and k is a positive integer. We also recall $N = \lfloor 1/\beta \rfloor$ and V is a finite-dimensional normed space with norm $\|\cdot\|_V$.

3.3.1 Proof of Theorem 3.4

We prove Theorem 3.4 along with some estimates of $I^\gamma_\varphi(X, \xi)$. Let $X, \tilde{X} \in \Omega_\beta(E)$. For each $n = 1, \dots, N$, we set $K^{(n)}_{a,b} := \max_{1\leq i\leq n} \|\!|X^i|\!\|_{i\beta;[a,b]}$ and $\tilde{K}^{(n)}_{a,b} := \max_{1\leq i\leq n} \|\!|\tilde{X}^i|\!\|_{i\beta;[a,b]}$.

Proposition 3.21. *In the setting of Definition 3.1, $I^\gamma_\varphi(X, \xi)$ belongs to $\mathcal{C}^\beta_2(F)$ and there exists a positive constant C depending only on β and γ such that*

$$\|\!|I^\gamma_\varphi(X, \xi)|\!\|_{\beta;[a,b]} \leq C C^1_{a,b} \big(\|\nabla^{N-1}\varphi\|_{\lambda\text{-Höl}} \vee \max_{1\leq n\leq N} |\nabla^{n-1}\varphi(\xi)|\big) K^{(N)}_{a,b}, \tag{3.16}$$

where

$$C^1_{a,b} := \big(1 \vee b^{(N-1+\lambda)\beta}\big) \sum_{n=1}^{N} \left\{ \|\!|X^1|\!\|^{N-n+\lambda}_{\beta;[0,b]} + \sum_{k=0}^{N-n} \|\!|X^k|\!\|_{k\beta;[0,b]} + \|\!|X^1|\!\|^{N-n+\lambda}_{\beta;[a,b]} \big(1 + K^{(n-1)}_{a,b}\big)^{n-1} \right\}.$$

Proof. From Proposition 2.4, $I^\gamma_\varphi(X, \xi)$ belongs to $\mathcal{C}^\beta_2(F)$. We prove Eq. (3.16). Set $(s, t) \in \Delta_T$ with $a \leq s < t \leq b$. Then,

$$\begin{aligned}
&|I^\gamma_\varphi(X, \xi)_{s,t}| \\
&\leq \sum_{n=1}^{N} \left\{ |\nabla^{n-1}\varphi(X^{1,\xi}_s)||X^n_{s,t}| + \|\mathcal{D}^{1-n\gamma}_{s+} R^{N-n}(\nabla^{n-1}\varphi, X^{1,\xi})\|_{\infty;[s,t]} \|\mathcal{R}^{(n,\gamma)}_{t-} X\|_{\infty;[s,t]} (t-s) \right\}.
\end{aligned}$$

Using Eqs. (2.15), (2.22) and (3.2), we get

$$\begin{aligned}
&\|\mathcal{D}_{s+}^{1-n\gamma}R^{N-n}(\nabla^{n-1}\varphi, X^{1,\xi})\|_{\infty;[s,t]}\|\mathcal{R}_{t-}^{(n,\gamma)}X\|_{\infty;[s,t]}(t-s)\\
&\le C\|R^{N-n}(\nabla^{n-1}\varphi, X^{1,\xi})\|_{(N-n+\lambda)\beta;[s,t]}\big(1+K_{s,t}^{(n-1)}\big)^{n-1}K_{s,t}^{(n)}(t-s)^{(N+\lambda)\beta}\\
&\le C\|\nabla^{N-1}\varphi\|_{\lambda\text{-Höl}}\|X^1\|_{\beta;[s,t]}^{N-n+\lambda}\big(1+K_{s,t}^{(n-1)}\big)^{n-1}K_{s,t}^{(n)}(t-s)^{(N+\lambda)\beta}.
\end{aligned}$$

From Eq. (3.2), we have

$$\begin{aligned}
&|\nabla^{n-1}\varphi(X_s^{1,\xi})||X_{s,t}^n|\\
&\le \left(|R^{N-n}(\nabla^{n-1}\varphi, X^{1,\xi})_{0,s}| + \sum_{k=0}^{N-n}|\nabla^{n-1+k}\varphi(X_0^{1,\xi})||X_{0,s}^k|\right)\|X^n\|_{n\beta;[s,t]}(t-s)^{n\beta}\\
&\le \left(C\|\nabla^{N-1}\varphi\|_{\lambda\text{-Höl}}\|X^1\|_{\beta;[0,s]}^{N-n+\lambda}s^{(N-n+\lambda)\beta} + \sum_{k=0}^{N-n}|\nabla^{n-1+k}\varphi(\xi)|\|X^k\|_{k\beta;[0,s]}s^{k\beta}\right)K_{s,t}^{(n)}(t-s)^{n\beta}.
\end{aligned}$$

Combining these estimates implies Eq. (3.16). □

Let l be an integer with $0 \le l \le N-1$, λ a real number with $0 < \lambda \le 1$, and $f \in C^{l+1,\lambda}(E, V)$ such that $\nabla^{l+1}f$ is bounded on E. From simple estimates, there exists a positive constant $C_{l,\lambda}$ such that, for each $x, \tilde{x} \in C^{\beta\text{-Höl}}([0,T], E)$,

$$\begin{aligned}
&\|R^l(f,x) - R^l(f,\tilde{x})\|_{(l+1)\beta;[a,b]}\\
&\le C_{l,\lambda}\|\nabla^{l+1}f\|_{\lambda\text{-Höl}}\|x\|_{\beta\text{-Höl};[a,b]}^{l+1}\big(\|x-\tilde{x}\|_{\infty;[a,b]} + \|x-\tilde{x}\|_{\beta\text{-Höl};[a,b]}(b-a)^{\beta}\big)^{\lambda}\\
&\quad + C_{l,\lambda}\|\nabla^{l+1}f\|_{\infty}\left(\sum_{k=0}^{l}\|x\|_{\beta\text{-Höl};[a,b]}^{k}\|\tilde{x}\|_{\beta\text{-Höl};[a,b]}^{l-k}\right)\|x-\tilde{x}\|_{\beta\text{-Höl};[a,b]}. \qquad (3.17)
\end{aligned}$$

We recall Eq. (2.1) for the meaning of the symbol $\|\nabla\varphi\|_{C^{N-1,\lambda}}$ below.

Proposition 3.22. *Let $X, \tilde{X} \in \Omega_\beta(E)$ and $\xi \in E$. Under the assumptions of Theorem 3.4, there exists a constant C depending only on β and γ such that*

$$\|I_\varphi^\gamma(X,\xi) - I_\varphi^\gamma(\tilde{X},\xi)\|_{\beta;[a,b]} \le CC_{a,b}^2\big(|\varphi(\xi)| \vee \|\nabla\varphi\|_{C^{N-1,\lambda}}\big)\big(\|X^1-\tilde{X}^1\|_{\beta;[a,b]}^{\lambda} \vee d_{\beta,N}(X,\tilde{X})\big), \qquad (3.18)$$

where

$$\begin{aligned}
C_{a,b}^2 := \big(1 \vee (b-a)^{(N+\lambda)\beta} \vee b^{N\beta}\big)\sum_{n=1}^{N}\Bigg\{&\|X^n\|_{n\beta;[a,b]} + \|\tilde{X}^1\|_{\beta;[0,b]}^{N-n+1} + \sum_{k=0}^{N-n}\|\tilde{X}^k\|_{k\beta;[0,b]}\\
&+ \left(\|X^1\|_{\beta;[a,b]}^{N-n+1} + \left(\sum_{k=0}^{N-n}\|X^1\|_{\beta;[a,b]}^{k}\|\tilde{X}^1\|_{\beta;[a,b]}^{N-n-k}\right)\right)\big(1+K_{a,b}^{(n-1)}\big)^{n-1}K_{a,b}^{(n)}\\
&+ \|\tilde{X}^1\|_{\beta;[a,b]}^{N-n+1}\big(1+K_{a,b}^{(n-1)}+\tilde{K}_{a,b}^{(n-1)}\big)^{n-1}\Bigg\}.
\end{aligned}$$

Proof. Set $(s,t) \in \Delta_T$ with $a \le s < t \le b$. Then,

$$
\begin{aligned}
&|I_\varphi^\gamma(X,\xi)_{s,t} - I_\varphi^\gamma(\tilde{X},\xi)_{s,t}| \\
&\le \sum_{n=1}^N \Big\{ |\nabla^{n-1}\varphi(X_s^{1,\xi}) - \nabla^{n-1}\varphi(\tilde{X}_s^{1,\xi})||X_{s,t}^n| + |\nabla^{n-1}\varphi(\tilde{X}_s^{1,\xi})||X_{s,t}^n - \tilde{X}_{s,t}^n| \\
&\quad + \|\mathcal{D}_{s+}^{1-n\gamma}(R^{N-n}(\nabla^{n-1}\varphi, X^{1,\xi}) - R^{N-n}(\nabla^{n-1}\varphi, \tilde{X}^{1,\xi}))\|_{\infty;[s,t]}\|\mathcal{R}_{t-}^{(n,\gamma)}X\|_{\infty;[s,t]}(t-s) \\
&\quad + \|\mathcal{D}_{s+}^{1-n\gamma}R^{N-n}(\nabla^{n-1}\varphi, \tilde{X}^{1,\xi})\|_{\infty;[s,t]}\|\mathcal{R}_{t-}^{(n,\gamma)}X - \mathcal{R}_{t-}^{(n,\gamma)}\tilde{X}\|_{\infty;[s,t]}(t-s) \Big\}.
\end{aligned}
$$

Using Eqs. (2.15), (2.22) and (3.17), we obtain

$$
\begin{aligned}
&\|\mathcal{D}_{s+}^{1-n\gamma}(R^{N-n}(\nabla^{n-1}\varphi, X^{1,\xi}) - R^{N-n}(\nabla^{n-1}\varphi, \tilde{X}^{1,\xi}))\|_{\infty;[s,t]}\|\mathcal{R}_{t-}^{(n,\gamma)}X\|_{\infty;[s,t]}(t-s) \\
&\le C\| R^{N-n}(\nabla^{n-1}\varphi, X^{1,\xi}) - R^{N-n}(\nabla^{n-1}\varphi, \tilde{X}^{1,\xi})\|_{(N-n+1)\beta;[s,t]}\big(1+K_{s,t}^{(n-1)}\big)^{n-1}K_{s,t}^{(n)}(t-s)^{(N+1)\beta} \\
&\le C(\|\nabla^N\varphi\|_{\lambda\text{-Höl}} \vee \|\nabla^N\varphi\|_\infty)\Bigg(\| X^1\|_{\beta;[s,t]}^{N-n+1}\| X^1 - \tilde{X}^1\|_{\beta;[s,t]}^\lambda (t-s)^{\lambda\beta} \\
&\quad + \left(\sum_{k=0}^{N-n} \| X^1\|_{\beta;[s,t]}^k \|\tilde{X}^1\|_{\beta;[s,t]}^{N-n-k}\right)\|X^1 - \tilde{X}^1\|_{\beta\text{-Höl};[s,t]}\Bigg)\big(1+K_{s,t}^{(n-1)}\big)^{n-1}K_{s,t}^{(n)}(t-s)^{(N+1)\beta}.
\end{aligned}
$$

Using Eqs. (2.15), (2.24) and (3.2), we have

$$
\begin{aligned}
&\|\mathcal{D}_{s+}^{1-n\gamma}R^{N-n}(\nabla^{n-1}\varphi, \tilde{X}^{1,\xi})\|_{\infty;[s,t]}\|\mathcal{R}_{t-}^{(n,\gamma)}X - \mathcal{R}_{t-}^{(n,\gamma)}\tilde{X}\|_{\infty;[s,t]}(t-s) \\
&\le C\| R^{N-n}(\nabla^{n-1}\varphi, \tilde{X}^{1,\xi})\|_{(N-n+1)\beta;[s,t]}\big(1+K_{s,t}^{(n-1)} + \tilde{K}_{s,t}^{(n-1)}\big)^{n-1}d_{\beta,N}(X,\tilde{X})(t-s)^{(N+1)\beta} \\
&\le C\|\nabla^N\varphi\|_\infty \|\tilde{X}^1\|_{\beta;[s,t]}^{N-n+1}\big(1+K_{s,t}^{(n-1)} + \tilde{K}_{s,t}^{(n-1)}\big)^{n-1}d_{\beta,N}(X,\tilde{X})(t-s)^{(N+1)\beta}.
\end{aligned}
$$

We also have

$$
|\nabla^{n-1}\varphi(X_s^{1,\xi}) - \nabla^{n-1}\varphi(\tilde{X}_s^{1,\xi})||X_{s,t}^n| \le \|\nabla^n\varphi\|_\infty \|X^1 - \tilde{X}^1\|_{\beta;[0,s]}s^\beta \|X^n\|_{n\beta;[s,t]}(t-s)^{n\beta}
$$

and

$$
\begin{aligned}
&|\nabla^{n-1}\varphi(\tilde{X}_s^{1,\xi})||X_{s,t}^n - \tilde{X}_{s,t}^n| \\
&\le \left(|R^{N-n}(\nabla^{n-1}\varphi, \tilde{X}^{1,\xi})_{0,s}| + \sum_{k=0}^{N-n} |\nabla^{n-1+k}\varphi(\tilde{X}_0^{1,\xi})||\tilde{X}_{0,s}^k| \right)\|X^n - \tilde{X}^n\|_{n\beta;[s,t]}(t-s)^{n\beta} \\
&\le \left(C\|\nabla^N\varphi\|_\infty \|\tilde{X}^1\|_{\beta;[0,s]}^{N-n+1}s^{(N-n+1)\beta} + \sum_{k=0}^{N-n} |\nabla^{n-1+k}\varphi(\xi)|\|\tilde{X}^k\|_{k\beta;[0,s]}s^{k\beta}\right)d_{\beta,N}(X,\tilde{X})(t-s)^{n\beta}.
\end{aligned}
$$

Combining these estimates implies Eq. (3.18). □

Proposition 3.23. *Let $X \in \Omega_\beta(E)$ and $\xi, \tilde{\xi} \in E$. Under the assumptions of Theorem 3.4, there exists a constant C depending only on β and γ such that*

$$
\|I_\varphi^\gamma(X,\xi) - I_\varphi^\gamma(X,\tilde{\xi})\|_{\beta;[a,b]} \le CC_{a,b}^3\|\nabla\varphi\|_{C^{N-1,\lambda}}K_{a,b}^{(N)}\big(|\xi-\tilde{\xi}|^\lambda \vee |\xi - \tilde{\xi}|\big), \tag{3.19}
$$

where

$$C^3_{a,b} := \big(1 \vee (b-a)^{N\beta}\big) \sum_{n=1}^{N} \Big\{ 1 + \|X^1\|^{N-n+1}_{\beta;[a,b]} \big(1 + K^{(n-1)}_{a,b}\big)^{n-1} \Big\}.$$

Proof. Set $(s,t) \in \Delta_T$ with $a \le s < t \le b$. Then,

$$\begin{aligned}
&|I^\gamma_\varphi(X,\xi)_{s,t} - I^\gamma_\varphi(X,\tilde{\xi})_{s,t}| \\
&\le \sum_{n=1}^{N} \Big\{ |\nabla^{n-1}\varphi(X^{1,\xi}_s) - \nabla^{n-1}\varphi(X^{1,\tilde{\xi}}_s)||X^n_{s,t}| \\
&\quad + \|\mathcal{D}^{1-n\gamma}_{s+}(R^{N-n}(\nabla^{n-1}\varphi, X^{1,\xi}) - R^{N-n}(\nabla^{n-1}\varphi, X^{1,\tilde{\xi}}))\|_{\infty;[s,t]} \|\mathcal{R}^{(n,\gamma)}_{t-}X\|_{\infty;[s,t]}(t-s) \Big\}.
\end{aligned}$$

Using (2.15), (2.22) and (3.17), we get

$$\begin{aligned}
&\|\mathcal{D}^{1-n\gamma}_{s+}(R^{N-n}(\nabla^{n-1}\varphi, X^{1,\xi}) - R^{N-n}(\nabla^{n-1}\varphi, X^{1,\tilde{\xi}}))\|_{\infty;[s,t]} \|\mathcal{R}^{(n,\gamma)}_{t-}X\|_{\infty;[s,t]}(t-s) \\
&\le C \| R^{N-n}(\nabla^{n-1}\varphi, X^{1,\xi}) - R^{N-n}(\nabla^{n-1}\varphi, X^{1,\tilde{\xi}}) \|_{(N-n+1)\beta;[s,t]} \big(1 + K^{(n-1)}_{s,t}\big)^{n-1} K^{(n)}_{s,t} (t-s)^{(N+1)\beta} \\
&\le C \|\nabla^N \varphi\|_{\lambda\text{-Höl}} \|X^1\|^{N-n+1}_{\beta;[s,t]} |\xi - \tilde{\xi}|^\lambda \big(1 + K^{(n-1)}_{s,t}\big)^{n-1} K^{(n)}_{s,t} (t-s)^{(N+1)\beta}.
\end{aligned}$$

We also have

$$|\nabla^{n-1}\varphi(X^{1,\xi}_s) - \nabla^{n-1}\varphi(X^{1,\tilde{\xi}}_s)||X^n_{s,t}| \le \|\nabla^n \varphi\|_\infty |\xi - \tilde{\xi}| \|X^n\|_{n\beta;[s,t]} (t-s)^{n\beta}.$$

Combining these estimates implies (3.19). □

Proof of Theorem 3.4. Using Eqs. (3.16), (3.18), and (3.19) proved above, we obtain the statement of the theorem immediately. □

3.3.2 Proof of Proposition 3.15

We now prove Proposition 3.15. Let $(X,Y) \in M_\beta(E, L(E,F))$. We say that $I^\gamma(X,Y)$ is additive on Δ_T if $I^\gamma(X,Y)$ satisfies Eq. (3.14), that is, the identity

$$I^\gamma(X,Y)_{s,u} + I^\gamma(X,Y)_{u,t} = I^\gamma(X,Y)_{s,t}$$

holds for each $s,t,u \in [0,T]$ with $s \le u \le t$. Also, we set $Z = (Z^{(0)}, Z^{(1)}, \dots, Z^{(N-1)})$ as

$$Z^{(0)}_t := I^\gamma(X,Y)_{0,t} \quad \text{and} \quad Z^{(l)}_t := Y^{(l-1)}_t \quad \text{for } t \in [0,T] \tag{3.20}$$

and each $l = 1, \dots, N-1$.

Lemma 3.24. *Let $(X,Y) \in M_\beta(E, L(E,F))$. Suppose that $I^\gamma(X,Y)$ is additive on Δ_T. Then, Z belongs to $\mathcal{Q}^{\beta,N}_X(F)$.*

Proof. From the additivity of $I^\gamma(X,Y)$ and Theorem 3.12, $Z^{(0)}$ belongs to $\mathcal{C}_1^\beta(F)$. Also, from the definition of controlled paths, $Z^{(l)}$ is in $\mathcal{C}_1^\beta(L(E^{\otimes l},F))$ for each $l=1,\ldots,N-1$. It then suffices to show that $R_l^{N-1-l}(X,Z)\in\mathcal{C}_2^{(N-l)\beta}(L(E^{\otimes l},F))$ for each $l=0,\ldots,N-1$. Set $(s,t)\in\Delta_T$ with $s<t$. From the additivity of $I^\gamma(X,Y)$, we have

$$R_0^{N-1}(X,Z)_{s,t}=Y_s^{(N-1)}X_{s,t}^N+\sum_{n=1}^N\mathcal{I}_X^{n,\gamma}(R_{n-1}^{N-n}(X,Y))_{s,t}. \tag{3.21}$$

Then, from Eq. (2.23), $\mathcal{I}_X^{n,\gamma}(R_{n-1}^{N-n}(X,Y))$ belongs to $\mathcal{C}_2^{(N+1)\beta}(F)$ and so $R_0^{N-1}(X,Z)\in\mathcal{C}_2^{N\beta}(F)$. Furthermore, for each $l=1,\ldots,N-1$, from the definition of controlled paths,

$$R_l^{N-1-l}(X,Z)_{s,t}=R_{l-1}^{N-l}(X,Y)_{s,t}+Y_s^{(N-1)}X_{s,t}^{N-l}. \tag{3.22}$$

Then, $R_l^{N-1-l}(X,Z)\in\mathcal{C}_2^{(N-l)\beta}(L(E^{\otimes l},F))$ and thus Z belongs to $\mathcal{Q}_X^{\beta,N}(F)$. □

Remark 3.25. By an argument similar to the proof above, it is easy to verify that if $(X,Y)\in S_\beta(E,L(E,F))$, then Z belongs to $\mathcal{Q}_X^{1,N}(F)$. This fact is used in the proof of Proposition 3.15.

Lemma 3.26. *Let $(X,Y),(\tilde X,\tilde Y)\in M_\beta(E,L(E,F))$ and M be a positive constant such that*

$$\sum_{n=1}^N\left\{\|X^n\|_{n\beta}+\|\tilde X^n\|_{n\beta}+|Y_0^{(n-1)}|+|\tilde Y_0^{(n-1)}|\right\}+\|Y\|_{X,\beta}+\|\tilde Y\|_{\tilde X,\beta}\le M.$$

Suppose that $I^\gamma(X,Y)$ and $I^\gamma(\tilde X,\tilde Y)$ are additive on Δ_T. For $(\tilde X,\tilde Y)$, we set $\tilde Z=(\tilde Z^{(0)},\tilde Z^{(1)},\ldots,\tilde Z^{(N-1)})$ as in Eq. (3.20). Then, we have a local Lipschitz estimate

$$d_{X,\tilde X,\beta}(Z,\tilde Z)\le Lm_\beta((X,Y),(\tilde X,\tilde Y))$$

for a suitable constant L which depends only on β, γ, T, and M.

Proof. From Eq. (3.21),

$$\begin{aligned}&R_0^{N-1}(X,Z)_{s,t}-R_0^{N-1}(\tilde X,\tilde Z)_{s,t}\\&=Y_s^{(N-1)}X_{s,t}^N-\tilde Y_s^{(N-1)}\tilde X_{s,t}^N+\sum_{n=1}^N\left\{\mathcal{I}_X^{n,\gamma}(R_{n-1}^{N-n}(X,Y))_{s,t}-\mathcal{I}_{\tilde X}^{n,\gamma}(R_{n-1}^{N-n}(\tilde X,\tilde Y))_{s,t}\right\}.\end{aligned}$$

By inequalities of the form $|ab-\tilde a\tilde b|\le|a-\tilde a||b|+|\tilde a||b-\tilde b|$, we get

$$\begin{aligned}&|Y_s^{(N-1)}X_{s,t}^N-\tilde Y_s^{(N-1)}\tilde X_{s,t}^N|/(t-s)^{N\beta}\\&\le(|Y_0^{(N-1)}-\tilde Y_0^{(N-1)}|+\|R_{N-1}^0(X,Y)-R_{N-1}^0(\tilde X,\tilde Y)\|_\beta T^\beta)\|X^N\|_{N\beta}\\&\quad+(|\tilde Y_0^{(N-1)}|+\|R_{N-1}^0(\tilde X,\tilde Y)\|_\beta T^\beta)\|X^N-\tilde X^N\|_{N\beta}\\&\le((1+T^\beta)\|X^N\|_{N\beta}+(|\tilde Y_0^{(N-1)}|+\|R_{N-1}^0(\tilde X,\tilde Y)\|_\beta T^\beta))m_\beta((X,Y),(\tilde X,\tilde Y))\\&\le(1+T^\beta)Mm_\beta((X,Y),(\tilde X,\tilde Y)).\end{aligned}$$

Also, from Eqs. (2.23) and (2.25),

$$
\begin{aligned}
&|\mathcal{I}_X^{n,\gamma}(R_{n-1}^{N-n}(X,Y))_{s,t} - \mathcal{I}_{\tilde{X}}^{n,\gamma}(R_{n-1}^{N-n}(\tilde{X},\tilde{Y}))_{s,t}|/(t-s)^{(N+1)\beta} \\
&\le |\mathcal{I}_X^{n,\gamma}(R_{n-1}^{N-n}(X,Y) - R_{n-1}^{N-n}(\tilde{X},\tilde{Y}))_{s,t}|/(t-s)^{(N+1)\beta} \\
&\quad + |\mathcal{I}_X^{n,\gamma}(R_{n-1}^{N-n}(\tilde{X},\tilde{Y}))_{s,t} - \mathcal{I}_{\tilde{X}}^{n,\gamma}(R_{n-1}^{N-n}(\tilde{X},\tilde{Y}))_{s,t}|/(t-s)^{(N+1)\beta} \\
&\le C_{n,\beta,\gamma}\|R_{n-1}^{N-n}(X,Y) - R_{n-1}^{N-n}(\tilde{X},\tilde{Y})\|_{(N-n+1)\beta}(1+\max_{1\le i\le n-1}\|X^i\|_{i\beta})^{n-1}\max_{1\le i\le n}\|X^i\|_{i\beta} \\
&\quad + C_{n,\beta,\gamma}\|R_{n-1}^{N-n}(\tilde{X},\tilde{Y})\|_{\mu}(1+\max_{1\le i\le n-1}\|X^i\|_{i\beta}+\max_{1\le i\le n-1}\|\tilde{X}^i\|_{i\beta})^{n-1}\max_{1\le i\le n}\|X^i-\tilde{X}^i\|_{i\beta} \\
&\le C_{N,\beta,\gamma}M(1+M)^{N-1}m_\beta((X,Y),(\tilde{X},\tilde{Y}))
\end{aligned}
$$

for each $n = 1, \ldots, N$. Furthermore, from Eq. (3.22),

$$
\begin{aligned}
&|R_l^{N-1-l}(X,Z)_{s,t} - R_l^{N-1-l}(\tilde{X},\tilde{Z})_{s,t}|/(t-s)^{(N-l)\beta} \\
&\le (|R_{l-1}^{N-l}(X,Y)_{s,t} - R_{l-1}^{N-l}(\tilde{X},\tilde{Y})_{s,t}| + |Y_s^{(N-1)}X_{s,t}^{N-l} - \tilde{Y}_s^{(N-1)}\tilde{X}_{s,t}^{N-l}|)/(t-s)^{(N-l)\beta} \\
&\le \|R_{l-1}^{N-l}(X,Y)_{s,t} - R_{l-1}^{N-l}(\tilde{X},\tilde{Y})\|_{(N-l+1)\beta}(t-s)^\beta \\
&\quad + (|Y_0^{(N-1)} - \tilde{Y}_0^{(N-1)}| + \|R_{N-1}^0(X,Y) - R_{N-1}^0(\tilde{X},\tilde{Y})\|_\beta T^\beta)\|X^{N-l}\|_{(N-l)\beta} \\
&\quad + (|\tilde{Y}_0^{(N-1)}| + \|R_{N-1}^0(\tilde{X},\tilde{Y})\|_\beta T^\beta)\|X^{N-l} - \tilde{X}^{N-l}\|_{(N-l)\beta} \\
&\le (T^\beta + (1+T^\beta)M + (1+T^\beta)M)m_\beta((X,Y),(\tilde{X},\tilde{Y}))
\end{aligned}
$$

for each $l = 1, \ldots, N-1$. Then, combining these estimates, we obtain the statement of the proposition immediately. □

Proof of Proposition 3.15. Under the assumption of Proposition 3.15, i.e., $(X,Y) \in \overline{S_\beta}(E, L(E, F))$, $I^\gamma(X,Y)$ is additive on Δ_T. This follows from Theorems 3.11 and 3.12. Then, from Lemma 3.24, (X,Z) belongs to $M_\beta(E,F)$. By the definition of $\overline{S_\beta}(E, L(E,F))$, there exists a sequence $\{(X(n),Y(n))\}_{n=1}^\infty \subset S_\beta(E, L(E,F))$ which converges to (X,Y) with respect to the distance m_β. For each $n = 1, 2, \ldots$, we set $Z(n) = (Z(n)^{(0)}, Z(n)^{(1)}, \ldots, Z(n)^{(N-1)})$ as in Eq. (3.20). From Remark 3.25, $(X(n), Z(n))$ belongs to $S_\beta(E,F)$. Furthermore, we see from Lemma 3.26 that $(X(n), Z(n))$ converges to (X,Z) with respect to m_β. Thus, (X,Z) belongs to $\overline{S_\beta}(E,F)$. □

3.3.3 Proof of Theorems 3.3, 3.11, and Proposition 3.19

Using Proposition 3.27 stated below, we prove Theorems 3.3, 3.11, and Proposition 3.19. Let $X \in C_{0,\beta}(\Delta_T, T^{(k)}(E))$ and $Y \in \mathcal{Q}_X^{\beta,k}(L(E,F))$. For each $l = 0, \ldots, k-1$, $m = 0, \ldots, k-1-l$, and $(s,t) \in \Delta_T$, we set

$$
R_l^m(X,Y)_{s,t} := Y_t^{(l)} - \sum_{i=0}^m Y_s^{(l+i)} X_{s,t}^i. \tag{3.23}
$$

Proposition 3.27. *Let X be a step-k signature in E and $Y \in \mathcal{Q}_X^{1,k}(L(E,F))$. Take $\gamma \in (0, \min\{1/k, \beta\})$. Then, for each $l = 0, \ldots, k-1$, $m = 0, \ldots, k-1-l$, and $(s,t) \in \Delta_T$,*

$$
\int_s^t R_l^m(X,Y)_{s,u}\, dX_{0,u}^1 = \sum_{n=1}^{m+1} \mathcal{I}_X^{n,\gamma}(R_{l+n-1}^{m-n+1}(X,Y))_{s,t}, \tag{3.24}
$$

where the left-hand side is the Riemann–Stieltjes integral of $R_l^m(X,Y)_{s,\cdot}$ along $X_{0,\cdot}^1$.

Proof of Theorem 3.3. Under the assumptions of Theorem 3.3, we can take $Y = (Y^{(0)}, Y^{(1)}, \dots, Y^{(N-1)}) \in \mathcal{Q}_X^{1,N}(L(E,F))$ as in Example 3.7. Then, from Eq. (3.4) for $l = 0$ and $k = N$ and Proposition 3.27 for $l = 0$ and $m = N-1$,

$$\begin{aligned}\int_s^t Y_u^{(0)} dX_{0,u}^1 &= \sum_{i=0}^{N-1} \left\{ Y_s^{(i)} \int_s^t X_{s,u}^i \otimes dX_{0,u}^1 \right\} + \int_s^t R_0^{N-1}(X,Y)_{s,u}\, dX_{0,u}^1 \\ &= \sum_{n=1}^{N} Y_s^{(n-1)} X_{s,t}^n + \sum_{n=1}^{N} \mathcal{I}_X^{n,\gamma}(R_{n-1}^{N-n}(X,Y))_{s,t}. \end{aligned} \tag{3.25}$$

From Eq. (3.7), we obtain the claim of the theorem. □

Proof of Theorem 3.11. In the same way as in the proof of Theorem 3.3, that is, from Eq. (3.4) for $l = 0$ and $k = N$ and Proposition 3.27 for $l = 0$ and $m = N-1$, we have the same identity as Eq. (3.25). This is the claim of the theorem. □

Proof of Proposition 3.19. Under the assumptions of Proposition 3.19, we can take $Y = (Y^{(0)}, Y^{(1)}, \dots, Y^{(k-1)}) \in \mathcal{Q}_X^{1,k}(L(E, E^{\otimes(k+1)}))$ as in Example 3.9. Then, from Eq. (3.9) for $l = n-1$ and $k = j$ and Proposition 3.27 for $l = 0$ and $m = j-1$,

$$\hat{X}_{s,t}^{j+1} = \sum_{n=1}^{j} \mathcal{I}_X^{n,\gamma_j}(R_{n-1}^{j-n}(X,Y))_{s,t} = \int_s^t R_0^{j-1}(X,Y)_{s,u}\, dX_{0,u}^1.$$

From Eq. (3.9) for $l = 0$ and $k = j$, we obtain the claim of the proposition. □

3.3.4 Proof of Proposition 3.27

We first show the Hölder continuity of $\mathcal{R}_{b-}^{\gamma_1,\dots,\gamma_n} X$ for the proof of Proposition 3.30 stated below, which is essential for the proof of Proposition 3.27. Although only the case $\gamma_1 = \dots = \gamma_n$ appears in Proposition 3.27, we need general cases for Propositon 3.30.

The following lemma is a slight reformulation of Lemmas 6.1 and 6.2 in [13].

Lemma 3.28. *For $0 < \delta < \varepsilon \le 1$, there exists a positive constant $C_{\delta,\epsilon}$ such that*

$$y^\delta - x^\delta \le C_{\delta,\varepsilon} x^{\delta-\varepsilon}(y-x)^\varepsilon \quad \textit{for } 0 < x < y. \tag{3.26}$$

For $\delta, \varepsilon > 0$ with $0 < \varepsilon - \delta < 1$, there exists a positive constant $C_{\delta,\epsilon}$ such that

$$\int_0^1 u^\varepsilon \left(u^{-\delta-1} - (u+z)^{-\delta-1}\right) du \le C_{\delta,\varepsilon} z^{\varepsilon-\delta} \quad \textit{for } 0 \le z < \infty. \tag{3.27}$$

Proof. First, we prove Eq. (3.26). It suffices to prove this inequality with $x = 1$ by homogeneity. We define

$$h(y) := \frac{y^\delta - 1}{(y-1)^\varepsilon} \quad \text{for } y \in (1,\infty)$$

and note that h is positive on $(1,\infty)$. If $\varepsilon = 1$, then $\lim_{y\downarrow 1} h(y) = (y^\delta)'|_{y=1} = \delta$. If $\varepsilon < 1$, then, from l'Hôpital's rule, we get $\lim_{y\downarrow 1} h(y) = 0$. Also, from the relation $0 < \delta < \varepsilon$, we get $\lim_{y\to\infty} h(y) = 0$.

Hence, h is bounded on $(1,\infty)$ and Eq. (3.26) holds with $x=1$. Thus we obtain Eq. (3.26). We next prove Eq. (3.27). We may assume $z>0$. Using the change of variables $u=zv$, we get

$$\begin{aligned}
\text{(left-hand side of (3.27))} &= z^{\varepsilon-\delta}\int_0^{1/z} v^{\varepsilon}\bigl(v^{-\delta-1}-(v+1)^{-\delta-1}\bigr)\,dv\\
&\leq z^{\varepsilon-\delta}\int_0^{\infty} v^{\varepsilon}\bigl(v^{-\delta-1}-(v+1)^{-\delta-1}\bigr)\,dv.
\end{aligned}$$

Then, from the relation $\varepsilon-\delta>0$,

$$\int_0^1 v^{\varepsilon}\bigl(v^{-\delta-1}-(v+1)^{-\delta-1}\bigr)\,dv \leq \int_0^1 v^{\varepsilon-\delta-1}\,dv<\infty.$$

Also, from the relation $\varepsilon-\delta<1$,

$$\begin{aligned}
\int_1^{\infty} v^{\varepsilon}\bigl(v^{-\delta-1}-(v+1)^{-\delta-1}\bigr)\,dv &= \int_1^{\infty} v^{\varepsilon}\biggl(\int_v^{v+1}(\delta+1)w^{-\delta-2}\,dw\biggr)\,dv\\
&\leq (\delta+1)\int_1^{\infty} v^{\varepsilon-\delta-2}\,dv<\infty.
\end{aligned}$$

Hence, Eq. (3.27) holds with $z>0$. Therefore, we obtain Eq. (3.27). $\square$

Using this lemma, we show the Hölder continuity of $\mathcal{R}_{b-}^{\gamma_1,\dots,\gamma_n}X$.

Lemma 3.29. *In the setting of Definition 2.2, for each $n=1,\dots,k$, the function $\mathcal{R}_{b-}^{\gamma_1,\dots,\gamma_n}X$ is $\min_{0\leq i<j\leq n}\{\min\{(j-i)\beta,1\}-\sum_{l=i+1}^{j}\gamma_l\}$-Hölder continuous on the interval $[0,b]$. In particular, for $\gamma\in(0,\min\{1/k,\beta\})$, $\mathcal{R}_{b-}^{(n,\gamma)}X$ is $\min\{\beta-\gamma,1-n\gamma\}$-Hölder continuous on the interval $[0,b]$.*

Proof. Let $\beta_p=\min\{p\beta,1\}$ and $\sigma_p^q=\sum_{l=p}^{q}\gamma_l$. From Definition 2.2 and Eq. (2.17), it suffices to prove the following:

(1) for each $n=1,\dots,k$, $\mathcal{D}_{b-}^{\sigma_1^n}X^n$ is $\min_{1\leq j\leq n}\{\beta_j-\sigma_1^j\}$-Hölder continuous on $[0,b]$;

(2) for each $n=2,\dots,k$ and $m=1,\dots,n-1$,

$$\int_r^b \frac{X_{r,v}^{n-m}\otimes\mathcal{R}_{b-}^{\gamma_1,\dots,\gamma_m}X(v)}{(v-r)^{\sigma_{m+1}^n+1}}\,dv$$

is $\min_{1\leq i<j\leq n}\{\beta_{j-i}-\sigma_{i+1}^{j}\}$-Hölder continuous in r on $[0,b]$.

We first prove claim (1). Take the real numbers s and t such that $0\leq s<t\leq b$. Concerning the first term in the definition of $\mathcal{D}_{b-}^{\sigma_1^n}X^n$ (see Eq. (2.14)), we have

$$\left|\frac{X_{t,b}^n}{(b-t)^{\sigma_1^n}}-\frac{X_{s,b}^n}{(b-s)^{\sigma_1^n}}\right| \leq \left|\frac{X_{t,b}^n}{(b-t)^{\sigma_1^n}}-\frac{X_{t,b}^n}{(b-s)^{\sigma_1^n}}\right| + \left|\frac{X_{t,b}^n}{(b-s)^{\sigma_1^n}}-\frac{X_{s,b}^n}{(b-s)^{\sigma_1^n}}\right| =: A_1+A_2.$$

Using Eq. (3.26), we have

$$\begin{aligned}
A_1 &\leq \|X^n\|_{\beta_n;[t,b]}(b-t)^{\beta_n}(b-t)^{-\sigma_1^n}(b-s)^{-\sigma_1^n}\bigl((b-s)^{\sigma_1^n}-(b-t)^{\sigma_1^n}\bigr)\\
&\leq \|X^n\|_{\beta_n}(b-t)^{\beta_n-\sigma_1^n}(b-s)^{-\sigma_1^n}C_{\sigma_1^n,\beta_n}(b-t)^{\sigma_1^n-\beta_n}(t-s)^{\beta_n}\\
&\leq C_{\sigma_1^n,\beta_n}\|X^n\|_{\beta_n}(t-s)^{\beta_n-\sigma_1^n}.
\end{aligned}$$

Using Eq. (2.3), we also obtain

$$
\begin{aligned}
A_2 &\le \left(\sum_{j=1}^{n} \|X^j\|_{\beta_j;[s,t]} \|X^{n-j}\|_{\beta_{n-j};[t,b]} (t-s)^{\beta_j} (b-t)^{\beta_{n-j}}\right)(b-s)^{-\sigma_1^n} \\
&\le \|X^n\|_{\beta_n} (t-s)^{\beta_n-\sigma_1^n} + \sum_{j=1}^{n-1} \|X^j\|_{\beta_j} \|X^{n-j}\|_{\beta_{n-j}} (b-s)^{\beta_{n-j}-\sigma_{j+1}^n} (t-s)^{\beta_j-\sigma_1^j}.
\end{aligned}
$$

With regard to the second term in the definition of $\mathcal{D}_{b-}^{\sigma_1^n} X^n$ (see Eq. (2.14)), we have

$$
\begin{aligned}
&\left|\int_t^b \frac{X_{t,v}^n}{(v-t)^{\sigma_1^n+1}}\, dv - \int_s^b \frac{X_{s,v}^n}{(v-s)^{\sigma_1^n+1}}\, dv\right| \\
&\le \int_s^t \frac{|X_{s,v}^n|}{(v-s)^{\sigma_1^n+1}}\, dv + \int_t^b \left|\frac{X_{t,v}^n}{(v-t)^{\sigma_1^n+1}} - \frac{X_{s,v}^n}{(v-s)^{\sigma_1^n+1}}\right| dv =: B_1 + B_2.
\end{aligned}
$$

Then,

$$
B_1 \le \int_s^t \|X^n\|_{\beta_n;[s,v]} (v-s)^{\beta_n-\sigma_1^n-1}\, dv \le \|X^n\|_{\beta_n} \frac{1}{\beta_n - \sigma_1^n} (t-s)^{\beta_n-\sigma_1^n}
$$

and

$$
B_2 \le \int_t^b \left|\frac{X_{t,v}^n}{(v-t)^{\sigma_1^n+1}} - \frac{X_{t,v}^n}{(v-s)^{\sigma_1^n+1}}\right| dv + \int_t^b \left|\frac{X_{t,v}^n}{(v-s)^{\sigma_1^n+1}} - \frac{X_{s,v}^n}{(v-s)^{\sigma_1^n+1}}\right| dv =: B_{21} + B_{22}.
$$

By using the change of variables $u = (v-t)/(b-t)$ and Eq. (3.27) with $z = (t-s)/(b-t)$, we get

$$
\begin{aligned}
B_{21} &\le \int_t^b \|X^n\|_{\beta_n;[t,v]} (v-t)^{\beta_n} \left((v-t)^{-\sigma_1^n-1} - (v-s)^{-\sigma_1^n-1}\right) dv \\
&\le \|X^n\|_{\beta_n} (b-t)^{\beta_n-\sigma_1^n} \int_0^1 u^{\beta_n} \left(u^{-\sigma_1^n-1} - (u+z)^{-\sigma_1^n-1}\right) du \\
&\le \|X^n\|_{\beta_n} (b-t)^{\beta_n-\sigma_1^n} C_{\sigma_1^n,\beta_n} z^{\beta_n-\sigma_1^n} \\
&= \|X^n\|_{\beta_n} C_{\sigma_1^n,\beta_n} (t-s)^{\beta_n-\sigma_1^n}
\end{aligned}
$$

and, from Eq. (2.3),

$$
\begin{aligned}
B_{22} &\le \int_t^b \left(\sum_{j=1}^{n} \|X^j\|_{\beta_j;[s,t]} \|X^{n-j}\|_{\beta_{n-j};[t,v]} (t-s)^{\beta_j} (v-t)^{\beta_{n-j}}\right)(v-s)^{-\sigma_1^n-1}\, dv \\
&\le \|X^n\|_{\beta_n} \frac{1}{\sigma_1^n} (t-s)^{\beta_n-\sigma_1^n} + \sum_{j=1}^{n-1} \|X^j\|_{\beta_j} \|X^{n-j}\|_{\beta_{n-j}} \int_t^b (v-s)^{\beta_{n-j}-\sigma_{j+1}^n-1}\, dv\, (t-s)^{\beta_j-\sigma_1^j}.
\end{aligned}
$$

Combining these estimates, we obtain claim (1).

Turning to claim (2), let ψ denote $\mathcal{R}_{b-}^{\gamma_1,\dots,\gamma_m} X$. We note that ψ is bounded on $[0,b]$, which is proved as in the proof of Eq. (2.22). Select real numbers s and t such that $0 \le s < t \le b$. We then have

$$\left|\int_t^b \frac{X_{t,v}^{n-m}\otimes\psi(v)}{(v-t)^{\sigma_{m+1}^n+1}}\,dv - \int_s^b \frac{X_{s,v}^{n-m}\otimes\psi(v)}{(v-s)^{\sigma_{m+1}^n+1}}\,dv\right|$$
$$\leq \int_s^t \frac{|X_{s,v}^{n-m}\otimes\psi(v)|}{(v-s)^{\sigma_{m+1}^n+1}}\,dv + \int_t^b \left|\frac{X_{t,v}^{n-m}\otimes\psi(v)}{(v-t)^{\sigma_{m+1}^n+1}} - \frac{X_{s,v}^{n-m}\otimes\psi(v)}{(v-s)^{\sigma_{m+1}^n+1}}\right|dv =: C_1 + C_2.$$

Then,

$$\begin{aligned}C_1 &\leq \|\psi\|_{\infty;[0,b]}\int_s^t |\!|\!|X^{n-m}|\!|\!|_{\beta_{n-m};[s,v]}(v-s)^{\beta_{n-m}-\sigma_{m+1}^n-1}\,dv\\ &\leq \|\psi\|_\infty |\!|\!|X^{n-m}|\!|\!|_{\beta_{n-m}}\frac{1}{\beta_{n-m}-\sigma_{m+1}^n}(t-s)^{\beta_{n-m}-\sigma_{m+1}^n}\end{aligned}$$

and

$$\begin{aligned}C_2 &\leq \int_t^b \left|\frac{X_{t,v}^{n-m}\otimes\psi(v)}{(v-t)^{\sigma_{m+1}^n+1}} - \frac{X_{t,v}^{n-m}\otimes\psi(v)}{(v-s)^{\sigma_{m+1}^n+1}}\right|dv + \int_t^b \left|\frac{X_{t,v}^{n-m}\otimes\psi(v)}{(v-s)^{\sigma_{m+1}^n+1}} - \frac{X_{s,v}^{n-m}\otimes\psi(v)}{(v-s)^{\sigma_{m+1}^n+1}}\right|dv\\ &=: C_{21}+C_{22}.\end{aligned}$$

By using the change of variables $u=(v-t)/(b-t)$ and Eq. (3.27) with $z=(t-s)(b-t)$, we get

$$\begin{aligned}C_{21} &\leq \|\psi\|_{\infty;[0,b]}\int_t^b |\!|\!|X^{n-m}|\!|\!|_{\beta_{n-m};[t,v]}(v-t)^{\beta_{n-m}}\left((v-t)^{-\sigma_{m+1}^n-1}-(v-s)^{-\sigma_{m+1}^n-1}\right)dv\\ &\leq \|\psi\|_\infty |\!|\!|X^{n-m}|\!|\!|_{\beta_{n-m}}(b-t)^{\beta_{n-m}-\sigma_{m+1}^n}\int_0^1 u^{\beta_{n-m}}\left(u^{-\sigma_{m+1}^n-1}-(u+z)^{-\sigma_{m+1}^n-1}\right)du\\ &\leq \|\psi\|_\infty |\!|\!|X^{n-m}|\!|\!|_{\beta_{n-m}}(b-t)^{\beta_{n-m}-\sigma_{m+1}^n}C_{\sigma_{m+1}^n,\beta_{n-m}}z^{\beta_{n-m}-\sigma_{m+1}^n}\\ &= \|\psi\|_\infty |\!|\!|X^{n-m}|\!|\!|_{\beta_{n-m}}C_{\sigma_{m+1}^n,\beta_{n-m}}(t-s)^{\beta_{n-m}-\sigma_{m+1}^n}\end{aligned}$$

and, from Eq. (2.3),

$$\begin{aligned}C_{22} &\leq \|\psi\|_{\infty;[t,b]}\int_t^b \Bigg(|\!|\!|X^{n-m}|\!|\!|_{\beta_{n-m};[s,t]}(t-s)^{\beta_{n-m}}\\ &\qquad + \sum_{j=m+1}^{n-1}|\!|\!|X^{j-m}|\!|\!|_{\beta_{j-m};[s,t]}|\!|\!|X^{n-j}|\!|\!|_{\beta_{n-j};[t,v]}(t-s)^{\beta_{j-m}}(v-t)^{\beta_{n-j}}\Bigg)(v-s)^{-\sigma_{m+1}^n-1}\,dv\\ &\leq \|\psi\|_\infty\Bigg(|\!|\!|X^{n-m}|\!|\!|_{\beta_{n-m}}\frac{1}{\sigma_{m+1}^n}(t-s)^{\beta_{n-m}-\sigma_{m+1}^n}\\ &\qquad + \sum_{j=m+1}^{n-1}|\!|\!|X^{j-m}|\!|\!|_{\beta_{j-m}}|\!|\!|X^{n-j}|\!|\!|_{\beta_{n-j}}\int_t^b (v-s)^{\beta_{n-j}-\sigma_{j+1}^n-1}\,dv\,(t-s)^{\beta_{j-m}-\sigma_{m+1}^j}\Bigg).\end{aligned}$$

Combining these estimates, we obtain claim (2). $\square$

We remark the following identities for later use. Let $f, g \in C^{\lambda\text{-Höl}}([0,T],\mathbb{R})$ and $0 < \alpha < \lambda$. From Eqs. (2.13) and (2.14), for each $t \in (a,b)$,

$$D^{\alpha}_{a+}(fg)(t) = D^{\alpha}_{a+}f(t)g(t) + \frac{\alpha}{\Gamma(1-\alpha)}\int_a^t \frac{f(s)(g(t)-g(s))}{(t-s)^{\alpha+1}}\,ds \tag{3.28}$$

and

$$D^{\alpha}_{b-}(fg)(t) = f(t)D^{\alpha}_{b-}g(t) - \frac{(-1)^{\alpha}\alpha}{\Gamma(1-\alpha)}\int_t^b \frac{(f(s)-f(t))g(s)}{(s-t)^{\alpha+1}}\,ds. \tag{3.29}$$

Proposition 3.30. *Let $X = (1, X^1, \ldots, X^k)$ be a multiplicative functional of degree k in E with finite 1-Hölder estimates. Take positive numbers $\{\gamma_l\}_{l=1}^k$ such that $\sum_{l=1}^k \gamma_l < 1$. Then, for each $n = 2, \ldots, k$ and $r \in (a,b)$,*

$$\mathcal{R}^{\gamma_1,\ldots,\gamma_n}_{b-}X(r) = D^{\sum_{l=1}^n \gamma_l}_{b-}X^n_{a,\cdot,b-}(r) - \sum_{j=1}^{n-1} D^{\sum_{l=j+1}^n \gamma_l}_{b-}(X^{n-j}_{a,\cdot} \otimes \mathcal{R}^{\gamma_1,\ldots,\gamma_j}_{b-}X)(r). \tag{3.30}$$

In particular, for $\gamma \in (0, 1/k)$,

$$\mathcal{R}^{(n,\gamma)}_{b-}X(r) = D^{n\gamma}_{b-}X^n_{a,\cdot,b-}(r) - \sum_{j=1}^{n-1} D^{(n-j)\gamma}_{b-}(X^{n-j}_{a,\cdot} \otimes \mathcal{R}^{(j,\gamma)}_{b-}X)(r). \tag{3.31}$$

Proof. From Lemma 3.29, the right-hand side of Eq. (3.30) is well-defined under our assumptions. Let σ_p^q denote $\sum_{l=p}^q \gamma_l$. We note that Eq. (3.30) is equivalent to the following identity: for each $r \in (a,b)$,

$$\begin{aligned} &D^{\sigma_1^n}_{b-}X^n_{a,\cdot,b-}(r) - \mathcal{D}^{\sigma_1^n}_{b-}X^n(r) \\ &= \sum_{j=1}^{n-1}\left\{D^{\sigma_{j+1}^n}_{b-}(X^{n-j}_{a,\cdot} \otimes \mathcal{R}^{\gamma_1,\ldots,\gamma_j}_{b-}X)(r) + \frac{(-1)^{\sigma_{j+1}^n}\sigma_{j+1}^n}{\Gamma(1-\sigma_{j+1}^n)}\int_r^b \frac{X^{n-j}_{r,v} \otimes \mathcal{R}^{\gamma_1,\ldots,\gamma_j}_{b-}X(v)}{(v-r)^{\sigma_{j+1}^n+1}}\,dv\right\}. \end{aligned} \tag{3.32}$$

We prove this equation by induction with respect to n. From Eq. (2.3), we have

$$(\text{left-hand side of (3.32) with } n=2) = X^1_{a,r} \otimes \mathcal{D}^{\sigma_1^2}_{b-}X^1(r).$$

By using Eqs. (2.3), (3.29) and (2.9), we obtain

$$(\text{right-hand side of (3.32) with } n=2) = X^1_{a,r} \otimes D^{\gamma_2}_{b-}(\mathcal{R}^{\gamma_1}_{b-}X)(r) = X^1_{a,r} \otimes \mathcal{D}^{\gamma_1+\gamma_2}_{b-}X^1(r).$$

Hence, Eq. (3.32) holds for $n = 2$.

Suppose that Eq. (3.32) holds for each $n = 2, \ldots, m$ with $m \le k-1$. By using the induction hypothesis and Eq. (2.9), we have the following identity: for each $n = 1, \ldots, m$, $\alpha \in (0, 1-\sigma_1^n)$ and $r \in (a,b)$,

$$D^{\alpha}_{b-}\mathcal{R}^{\gamma_1,\ldots,\gamma_{n-1},\gamma_n}_{b-}X(r) = \mathcal{R}^{\gamma_1,\ldots,\gamma_{n-1},\gamma_n+\alpha}_{b-}X(r). \tag{3.33}$$

Using Eq. (2.3), we first have

$$\text{(left-hand side of (3.32) with } n=m+1) = \sum_{j=1}^{m} X_{a,r}^{m+1-j} \otimes \mathcal{D}_{b-}^{\sigma_1^{m+1}} X^j(r).$$

We then calculate the right-hand side of Eq. (3.32) with $n=m+1$. For each $j=1,\dots,m$,

$$\begin{aligned}
&D_{b-}^{\sigma_{j+1}^{m+1}}(X_{a,\cdot}^{m+1-j} \otimes \mathcal{R}_{b-}^{\gamma_1,\dots,\gamma_j} X)(r) + \frac{(-1)^{\sigma_{j+1}^{m+1}} \sigma_{j+1}^{m+1}}{\Gamma(1-\sigma_{j+1}^{m+1})} \int_r^b \frac{X_{r,v}^{m+1-j} \otimes \mathcal{R}_{b-}^{\gamma_1,\dots,\gamma_j} X(v)}{(v-r)^{\sigma_{j+1}^{m+1}+1}} \, dv \\
&= D_{b-}^{\sigma_{j+1}^{m+1}}(X_{a,\cdot}^{m+1-j} \otimes \mathcal{R}_{b-}^{\gamma_1,\dots,\gamma_j} X)(r) \\
&\quad + \frac{(-1)^{\sigma_{j+1}^{m+1}} \sigma_{j+1}^{m+1}}{\Gamma(1-\sigma_{j+1}^{m+1})} \int_r^b \frac{(X_{a,v}^{m+1-j} - X_{a,r}^{m+1-j}) \otimes \mathcal{R}_{b-}^{\gamma_1,\dots,\gamma_j} X(v)}{(v-r)^{\sigma_{j+1}^{m+1}+1}} \, dv \\
&\quad - \sum_{i=1}^{m-j} X_{a,r}^i \otimes \frac{(-1)^{\sigma_{j+1}^{m+1}} \sigma_{j+1}^{m+1}}{\Gamma(1-\sigma_{j+1}^{m+1})} \int_r^b \frac{X_{r,v}^{m+1-j-i} \otimes \mathcal{R}_{b-}^{\gamma_1,\dots,\gamma_j} X(v)}{(v-r)^{\sigma_{j+1}^{m+1}+1}} \, dv \quad \text{(from (2.3))} \\
&= X_{a,r}^{m+1-j} \otimes D_{b-}^{\sigma_{j+1}^{m+1}}(\mathcal{R}_{b-}^{\gamma_1,\dots,\gamma_j} X)(r) \\
&\quad - \sum_{i=1}^{m-j} X_{a,r}^i \otimes \frac{(-1)^{\sigma_{j+1}^{m+1}} \sigma_{j+1}^{m+1}}{\Gamma(1-\sigma_{j+1}^{m+1})} \int_r^b \frac{X_{r,v}^{m+1-j-i} \otimes \mathcal{R}_{b-}^{\gamma_1,\dots,\gamma_j} X(v)}{(v-r)^{\sigma_{j+1}^{m+1}+1}} \, dv. \quad \text{(from (3.29))}
\end{aligned} \tag{3.34}$$

Therefore, we obtain

$$\begin{aligned}
&\text{(right-hand side of (3.32) with } n=m+1) \\
&= \sum_{j=1}^{m} X_{a,r}^{m+1-j} \otimes \mathcal{R}_{b-}^{\gamma_1,\dots,\gamma_{j-1},\sigma_j^{m+1}} X(r) \\
&\quad - \sum_{j=1}^{m} \sum_{i=1}^{m-j} X_{a,r}^i \otimes \frac{(-1)^{\sigma_{j+1}^{m+1}} \sigma_{j+1}^{m+1}}{\Gamma(1-\sigma_{j+1}^{m+1})} \int_r^b \frac{X_{r,v}^{m+1-j-i} \otimes \mathcal{R}_{b-}^{\gamma_1,\dots,\gamma_j} X(v)}{(v-r)^{\sigma_{j+1}^{m+1}+1}} \, dv \quad \text{(from (3.33) and (3.34))} \\
&= X_{a,r}^m \otimes \mathcal{R}_{b-}^{\sigma_1^{m+1}} X(r) + \sum_{j=2}^{m} X_{a,r}^{m+1-j} \otimes \mathcal{R}_{b-}^{\gamma_1,\dots,\gamma_{j-1},\sigma_j^{m+1}} X(r) \\
&\quad - \sum_{j=2}^{m} X_{a,r}^{m+1-j} \otimes \sum_{j=1}^{j-1} \frac{(-1)^{\sigma_{j+1}^{m+1}} \sigma_{j+1}^{m+1}}{\Gamma(1-\sigma_{j+1}^{m+1})} \int_r^b \frac{X_{r,v}^{j-j} \otimes \mathcal{R}_{b-}^{\gamma_1,\dots,\gamma_j} X(v)}{(v-r)^{\sigma_{j+1}^{m+1}+1}} \, dv \\
&= X_{a,r}^m \otimes \mathcal{D}_{b-}^{\sigma_1^{m+1}} X^1(r) + \sum_{j=2}^{m} X_{a,r}^{m+1-j} \otimes \mathcal{D}_{b-}^{\sigma_1^{m+1}} X^j(r). \quad \text{(from Definition 2.2 and (2.17))}
\end{aligned}$$

Hence, Eq. (3.32) holds for $n=m+1$. Consequently, we obtain the claim of the proposition. □

Let us introduce one more notation for the proof of Proposition 3.27. Let X be a multiplicative functional of degree k. For $j=1,\dots,k$, we set $\mathcal{T}(X)^j \in \mathcal{C}_2(E^{\otimes j})$ as follows: for each $(s,t) \in \Delta_T$,

$\mathcal{T}(X)^1_{s,t} := X^1_{s,t}$ and

$$\mathcal{T}(X)^j_{s,t} := X^j_{s,t} - \sum_{i=1}^{j-1} \mathcal{T}(X)^i_{s,t} \otimes X^{j-i}_{s,t} \tag{3.35}$$

for $j = 2, \dots, k$, inductively. Then, for each $j = 2, \dots, k$ and $(s,t) \in \Delta_T$, the identity

$$\sum_{i=1}^{j-1} \mathcal{T}(X)^i_{s,t} \otimes X^{j-i}_{s,t} = \sum_{i=1}^{j-i} X^i_{s,t} \otimes \mathcal{T}(X)^{j-i}_{s,t} \tag{3.36}$$

holds. This is proved by simple calculation and induction on j. By using Eq. (3.36) and induction on j, we can show that, for each $s, u, t \in [0,T]$ with $s \le u \le t$, the identity

$$X^j_{u,t} = X^j_{s,t} - X^j_{s,u} - \sum_{i=1}^{j-1} \mathcal{T}(X)^i_{s,u} \otimes (X^{j-i}_{s,t} - X^{j-i}_{s,u}) \tag{3.37}$$

holds for $j = 2, \dots, k$. We now have all the tools to prove Proposition 3.27.

Proof of Proposition 3.27. Fix l with $0 \le l \le k-1$. We prove Eq. (3.24) by induction on m. Using Eqs. (3.23) and (2.11), we have

$$\int_s^t R^0_l(X,Y)_{s,u}\, dX^1_{0,u} = \mathcal{I}^{1,\gamma}_X (R^0_l(X,Y))_{s,t}.$$

Hence, Eq. (3.24) holds for $m = 0$. Suppose that Eq. (3.24) holds for $m = M$ with $0 \le M \le k-2-l$. Using Eqs. (3.23) and (2.5) and the induction hypothesis, we have

$$\begin{aligned}\int_s^t R^{M+1}_l(X,Y)_{s,u}\, dX^1_{0,u} &= \int_s^t R^M_l(X,Y)_{s,u}\, dX^1_{0,u} - Y^{(l+M+1)}_s \int_s^t X^{M+1}_{s,u} \otimes dX^1_{0,u} \\ &= \sum_{n=1}^{M+1} \mathcal{I}^{n,\gamma}_X (R^{M-n+1}_{l+n-1}(X,Y))_{s,t} - Y^{(l+M+1)}_s X^{M+2}_{s,t}.\end{aligned}$$

For the proof of Eq. (3.24) for $m = M+1$, it then suffices to show the following identity:

$$\sum_{n=1}^{M+1} \mathcal{I}^{n,\gamma}_X (R^{M-n+1}_{l+n-1}(X,Y) - R^{M+1-n+1}_{l+n-1}(X,Y))_{s,t} = \mathcal{I}^{M+2,\gamma}_X (R^0_{l+M+1}(X,Y))_{s,t} + Y^{(l+M+1)}_s X^{M+2}_{s,t}.$$

By the definition of $\mathcal{I}^{n,\gamma}_X$ (see Eq. (2.19)), for each $n = 1, \dots, M+1$, we have

$$\begin{aligned}&\mathcal{I}^{n,\gamma}_X (R^{M-n+1}_{l+n-1}(X,Y) - R^{M+1-n+1}_{l+n-1}(X,Y))_{s,t} \\ &= (-1)^{1-n\gamma} \int_s^t \mathcal{D}^{1-n\gamma}_{s+} (R^{M-n+1}_{l+n-1}(X,Y) - R^{M+1-n+1}_{l+n-1}(X,Y))(u) \mathcal{R}^{(n,\gamma)}_{t-} X(u)\, du.\end{aligned}$$

We calculate the integrand as follows: for each $u \in (s,t)$,

$$
\begin{aligned}
&\mathcal{D}_{s+}^{1-n\gamma}(R_{l+n-1}^{M-n+1}(X,Y) - R_{l+n-1}^{M+1-n+1}(X,Y))(u)\\
&= \frac{1}{\Gamma(n\gamma)}\left(\frac{Y_s^{(l+M+1)}X_{s,u}^{M+1-n+1}}{(u-s)^{1-n\gamma}} + (1-n\gamma)\int_s^u \frac{Y_v^{(l+M+1)}X_{v,u}^{M+1-n+1}}{(u-v)^{(1-n\gamma)+1}}\,dv\right) \quad \text{(from (3.23))}\\
&= \frac{1}{\Gamma(n\gamma)}\frac{Y_s^{(l+M+1)}X_{s,u}^{M+1-n+1}}{(u-s)^{1-n\gamma}} + \frac{1-n\gamma}{\Gamma(n\gamma)}\int_s^u \frac{Y_v^{(l+M+1)}(X_{s,u}^{M+1-n+1} - X_{s,v}^{M+1-n+1})}{(u-v)^{(1-n\gamma)+1}}\,dv\\
&\quad - \sum_{i=1}^{M+1-n}\frac{1-n\gamma}{\Gamma(n\gamma)}\int_s^u \frac{Y_v^{(l+M+1)}(\mathcal{T}(X)_{s,v}^i \otimes (X_{s,u}^{M+1-n+1-i} - X_{s,v}^{M+1-n+1-i}))}{(u-v)^{(1-n\gamma)+1}}\,dv \quad \text{(from (3.37))}\\
&= \frac{1}{\Gamma(n\gamma)}\frac{Y_s^{(l+M+1)}X_{s,u}^{M+1-n+1}}{(u-s)^{1-n\gamma}}\\
&\quad + D_{s+}^{1-n\gamma}(Y_\cdot^{(l+M+1)}X_{s,\cdot}^{M+1-n+1})(u) - D_{s+}^{1-n\gamma}Y_\cdot^{(l+M+1)}(u)X_{s,u}^{M+1-n+1}\\
&\quad - \sum_{i=1}^{M+1-n}\Big\{D_{s+}^{1-n\gamma}(Y_\cdot^{(l+M+1)}\mathcal{T}(X)_{s,\cdot}^i X_{s,\cdot}^{M+1-n+1-i})(u)\\
&\qquad - D_{s+}^{1-n\gamma}(Y_\cdot^{(l+M+1)}\mathcal{T}(X)_{s,\cdot}^i)(u)X_{s,u}^{M+1-n+1-i}\Big\} \quad \text{(from (3.28))}\\
&= -D_{s+}^{1-n\gamma}(Y_\cdot^{(l+M+1)} - Y_s^{(l+M+1)})(u)X_{s,u}^{M+1-n+1}\\
&\quad + D_{s+}^{1-n\gamma}(Y_\cdot^{(l+M+1)}(X_{s,\cdot}^{M+1-n+1} - \sum_{i=1}^{M+1-n}\mathcal{T}(X)_{s,\cdot}^i \otimes X_{s,\cdot}^{M+1-n+1-i}))(u)\\
&\quad + \sum_{i=1}^{M+1-n} D_{s+}^{1-n\gamma}(Y_\cdot^{(l+M+1)}\mathcal{T}(X)_{s,\cdot}^i)(u)X_{s,u}^{M+1-n+1-i}\\
&= -D_{s+}^{1-n\gamma}(Y_\cdot^{(l+M+1)} - Y_s^{(l+M+1)})(u)X_{s,u}^{M+1-n+1}\\
&\quad + D_{s+}^{1-n\gamma}(Y_\cdot^{(l+M+1)}\mathcal{T}(X)_{s,\cdot}^{M+1-n+1})(u) \quad \text{(from (3.35))}\\
&\quad + \sum_{i=1}^{M+1-n} D_{s+}^{1-n\gamma}(Y_\cdot^{(l+M+1)}\mathcal{T}(X)_{s,\cdot}^i)(u)X_{s,u}^{M+1-n+1-i}.
\end{aligned}
$$

Therefore, for each $n = 1, \dots, M+1$, we obtain

$$
\begin{aligned}
&\mathcal{I}_X^{n,\gamma}(R_{l+n-1}^{M-n+1}(X,Y) - R_{l+n-1}^{M+1-n+1}(X,Y))_{s,t}\\
&= -(-1)^{1-(M+2)\gamma}\int_s^t D_{s+}^{1-(M+2)\gamma}(Y_\cdot^{(l+M+1)} - Y_s^{(l+M+1)})(u)\\
&\quad \times D_{t-}^{(M+1-n+1)\gamma}(X_{s,\cdot}^{M+1-n+1} \otimes \mathcal{R}_{t-}^{(n,\gamma)}X)(u)\,du\\
&\qquad \text{(from (2.9), Lemma 3.29, and (2.10) with } \alpha = (M+1-n+1)\gamma)\\
&\quad + A_n^{M+1-n+1}\\
&\quad + \sum_{i=1}^{M+1-n}(-1)^{1-(M+2-i)\gamma}\int_s^t D_{s+}^{1-(M+2-i)\gamma}(Y_\cdot^{(l+M+1)}\mathcal{T}(X)_{s,\cdot}^i)(u)
\end{aligned}
$$

$$\times D_{t-}^{(M+1-n+1-i)\gamma}(X_{s,\cdot}^{M+1-n+1-i} \otimes \mathcal{R}_{t-}^{(n,\gamma)}X)(u)\,du.$$
$$\text{(from (2.9), Lemma 3.29, and (2.10) with } \alpha = (M+1-n+1-i)\gamma) \tag{3.38}$$

Here, $A_n^{M+1-n+1}$ is defined by

$$A_n^j := (-1)^{1-n\gamma}\int_s^t D_{s+}^{1-n\gamma}(Y_\cdot^{(l+M+1)}\mathcal{T}(X)_{s,\cdot}^j)(u)\mathcal{R}_{t-}^{(n,\gamma)}X(u)\,du$$

for each $n = 1, \ldots, M+1$ and $j = 1, \ldots, M+1$. Also, we have

$$\begin{aligned}
A_1^{M+1} &= \int_s^t Y_u^{(l+M+1)}\mathcal{T}(X)_{s,u}^{M+1}\,dX_{0,u}^1 \quad \text{(from (2.12))} \\
&= \int_s^t Y_u^{(l+M+1)}\,dX_{s,u}^{M+2} - \sum_{i=1}^{M}\int_s^t Y_u^{(l+M+1)}\mathcal{T}(X)_{s,u}^i\,dX_{s,u}^{M+2-i} \quad \text{(from (3.35) and (2.3))} \\
&= (-1)^{1-(M+2)\gamma}\int_s^t D_{s+}^{1-(M+2)\gamma}(Y_\cdot^{(l+M+1)} - Y_s^{(l+M+1)})(u)D_{t-}^{(M+2)\gamma}X_{s,\cdot,t-}^{M+2}(u)\,du \\
&\quad + Y_s^{(l+M+1)}(X_{s,t}^{M+2} - X_{s,s}^{M+2}) \\
&\quad - \sum_{i=1}^{M}(-1)^{1-(M+2-i)\gamma}\int_s^t D_{s+}^{1-(M+2-i)\gamma}(Y_\cdot^{(l+M+1)}\mathcal{T}(X)_{s,\cdot}^i)(u)D_{t-}^{(M+2-i)\gamma}X_{s,\cdot,t-}^{M+2-i}(u)\,du. \\
&\quad \text{(from (2.11) and (2.12))}
\end{aligned} \tag{3.39}$$

Hence, by combining Eqs. (3.38) and (3.39), we have

$$\begin{aligned}
&\sum_{n=1}^{M+1}\mathcal{I}_X^{n,\gamma}(R_{l+n-1}^{M-n+1}(X,Y) - R_{l+n-1}^{M+1-n+1}(X,Y))_{s,t} \\
&= (-1)^{1-(M+2)\gamma}\int_s^t D_{s+}^{1-(M+2)\gamma}(Y_\cdot^{(l+M+1)} - Y_s^{(l+M+1)})(u) \\
&\quad \times \left(D_{t-}^{(M+2)\gamma}X_{s,\cdot,t-}^{M+2}(u) - \sum_{n=1}^{M+1}D_{t-}^{(M+1-n+1)\gamma}(X_{s,\cdot}^{M+1-n+1}\otimes\mathcal{R}_{t-}^{(n,\gamma)}X)(u)\right)du \\
&\quad + Y_s^{(l+M+1)}X_{s,t}^{M+2} \\
&\quad + \sum_{n=2}^{M+1}A_n^{M+1-n+1} \\
&\quad - \sum_{i=1}^{M}(-1)^{1-(M+2-i)\gamma}\int_s^t D_{s+}^{1-(M+2-i)\gamma}(Y_\cdot^{(l+M+1)}\mathcal{T}(X)_{s,\cdot}^i)(u) \\
&\qquad \times \left(D_{t-}^{(M+2-i)\gamma}X_{s,\cdot,t-}^{M+2-i}(u) - \sum_{n=1}^{M+1-i}D_{t-}^{(M+1-n+1-i)\gamma}(X_{s,\cdot}^{M+1-n+1-i}\otimes\mathcal{R}_{t-}^{(n,\gamma)}X)(u)\right)du \\
&= \mathcal{I}_X^{M+2,\gamma}(R_{l+M+1}^0(X,Y))_{s,t} + Y_s^{(l+M+1)}X_{s,t}^{M+2} + \sum_{n=2}^{M+1}A_n^{M+1-n+1} - \sum_{i=1}^{M}A_{M+2-i}^i \quad \text{(from (3.31))} \\
&= \mathcal{I}_X^{M+2,\gamma}(R_{l+M+1}^0(X,Y))_{s,t} + Y_s^{(l+M+1)}X_{s,t}^{M+2},
\end{aligned}$$

as desired. Therefore, Eq. (3.24) holds for $m = M + 1$ and thus the claim of the proposition holds by induction. $\square$

4 Differential equation driven by rough paths via fractional calculus

In this section, using the integral introduced in Definition 3.10, we study differential equations driven by geometric β-Hölder rough paths $X = (1, X^1, X^2) \in G\Omega_{\beta,T}(\mathbb{R}^d)$ with $\beta \in (1/3, 1/2]$. Throughout this section, we assume that β is a real number with $1/3 < \beta \leq 1/2$ and γ is a real number with $(1-\beta)/2 < \gamma < \beta$.

4.1 Rough differential equations

We define the concept of solutions to rough differential equations on the basis of [6, 9] and present the main theorem of this section. We recall Example 3.8, where $\varphi(Y) = (\varphi(Y)^{(0)}, \varphi(Y)^{(1)})$ was introduced for $Y = (Y^{(0)}, Y^{(1)}) \in \mathcal{Q}^{\beta,2}_{X,T}(F)$ with $X = (1, X^1, X^2) \in C_{0,\beta}(\Delta_T, T^{(2)}(E))$ and $\varphi \in C^{1,1}(F, L(E, F))$ such that $\nabla\varphi$ is bounded on F. In Lemma 4.3 below, we prove $\varphi(Y) \in \mathcal{Q}^{\beta,2}_{X,T}(L(E, F))$.

4.1.1 Definition of the solution

The following is our definition of solutions to rough differential equations.

Definition 4.1. Let $\xi \in F$, $\varphi \in C^{1,1}(F, L(E, F))$ such that $\nabla\varphi$ is bounded on F, and $X = (1, X^1, X^2) \in C_{0,\beta}(\Delta_T, T^{(2)}(E))$. We say that $Y = (Y^{(0)}, Y^{(1)})$ is a solution to the rough differential equation

$$dY_t = \varphi(Y_t)\, dX_t, \quad Y_0 = \xi \tag{4.1}$$

driven by $X = (1, X^1, X^2)$ along φ and starting at ξ, if $Y = (Y^{(0)}, Y^{(1)})$ is an element of $\mathcal{Q}^{\beta,2}_{X,T}(F)$ such that

$$Y^{(0)}_t = \xi + I^\gamma(X, \varphi(Y))_{0,t} \quad \text{and} \quad Y^{(1)}_t = \varphi(Y^{(0)}_t) \quad \text{for all } t \in [0, T]. \tag{4.2}$$

Here, $I^\gamma(X, \varphi(Y))$ is the integral of $\varphi(Y) = (\varphi(Y)^{(0)}, \varphi(Y)^{(1)}) \in \mathcal{Q}^{\beta,2}_{X,T}(L(E, F))$ along $X = (1, X^1, X^2) \in C_{0,\beta}(\Delta_T, T^{(2)}(E))$ in the sense of Definition 3.10, namely, for each $(s, t) \in \Delta_T$,

$$I^\gamma(X, \varphi(Y))_{s,t} = \varphi(Y^{(0)}_s)X^1_{s,t} + \nabla\varphi(Y^{(0)}_s)Y^{(1)}_s X^2_{s,t} + \mathcal{I}^{1,\gamma}_X(R^1_0(X, \varphi(Y)))_{s,t} + \mathcal{I}^{2,\gamma}_X(R^0_1(X, \varphi(Y)))_{s,t}.$$

4.1.2 Existence and uniqueness of solutions

Let $X = (1, X^1, X^2) \in G\Omega_{\beta,T}(E)$. We define the subset $\Pi_X(\overline{S_{\beta,T}}(E, F))$ of $\mathcal{Q}^{\beta,2}_{X,T}(F)$ by

$$\Pi_X(\overline{S_{\beta,T}}(E, F)) := \{Y : (X, Y) \in \overline{S_{\beta,T}}(E, F)\}.$$

It is straightforward to show that $\Pi_X(\overline{S_{\beta,T}}(E, F))$ is a complete metric space under the distance

$$m_{X,\beta}(Y, \tilde{Y}) := m_\beta((X, Y), (X, \tilde{Y})) \quad \text{for } Y, \tilde{Y} \in \Pi_X(\overline{S_{\beta,T}}(E, F)).$$

The following is a result on existence and uniqueness of the solutions to Eq. (4.1).

Theorem 4.2. *Given $\xi \in F$, $\varphi \in C^{2,1}(F, L(E,F))$ such that φ, $\nabla\varphi$, $\nabla^2\varphi$ are all bounded on F, and $X = (1, X^1, X^2) \in G\Omega_{\beta,T}(\mathbb{R}^d)$, the rough differential equation (4.1) admits a unique solution $Y = (Y^{(0)}, Y^{(1)})$ in $\Pi_X(\overline{S_{\beta,T}}(E,F))$.*

4.2 Basic estimates

For the proof of Theorem 4.2, we will provide several estimates of $\varphi(Y) = (\varphi(Y)^{(0)}, \varphi(Y)^{(1)})$ and $I^\gamma(X, \varphi(Y))$, and derive a priori estimates for solutions to rough differential equations. Also, several lemmas for shift operators will be provided for later use. Throughout this subsection, we use the following symbols: for $X, \tilde{X} \in C_{0,\beta}(\Delta_T, T^{(2)}(E))$,

$$C_{X^1} := 1 + \|X^1\|_\beta + \|\tilde{X}^1\|_\beta \quad \text{and} \quad C_X := C_{X^1} + \|X^2\|_{2\beta} + \|\tilde{X}^2\|_{2\beta};$$

for $Y \in \mathcal{Q}_X^{\beta,2}(F)$ and $\tilde{Y} \in \mathcal{Q}_{\tilde{X}}^{\beta,2}(F)$,

$$C_Y := 1 + |Y_0^{(1)}| + |\tilde{Y}_0^{(1)}| + \|Y\|_{X,\beta} + \|\tilde{Y}\|_{\tilde{X},\beta}.$$

Recall Eq. (3.5) for the definition of $\|Y\|_{X,\beta}$ and $\|\tilde{Y}\|_{\tilde{X},\beta}$ above.

4.2.1 Estimates of $\varphi(Y)$

Based on [8, Lemmas 7.3, 8.2, Theorem 7.5] and [9, Proposition 4], we provide several estimates of $\varphi(Y) = (\varphi(Y)^{(0)}, \varphi(Y)^{(1)})$. We remark that, in the assumptions of Lemmas 4.3, 4.5, 4.6, Remark 4.4, and Proposition 4.7 stated below, X and $\tilde{X}$ possess the second level paths X^2 and $\tilde{X}^2$ but these are not needed for the proofs. We also recall Eq. (2.1) for the meaning of $\|\nabla\varphi\|_{C^{0,1}}$ below.

Lemma 4.3. *Let $Y \in \mathcal{Q}_X^{\beta,2}(F)$ for some $X \in C_{0,\beta}(\Delta_T, T^{(2)}(E))$ and $\varphi \in C^{1,1}(F, L(E,F))$ such that $\nabla\varphi$ is bounded on E. Then, $\varphi(Y) = (\varphi(Y)^{(0)}, \varphi(Y)^{(1)})$ belongs to $\mathcal{Q}_X^{\beta,2}(L(E,F))$ and there exists a positive constant C which depends only on β and T such that*

$$\|\varphi(Y)\|_{X,\beta} \leq C\|\nabla\varphi\|_{C^{0,1}}(1 + \|X^1\|_\beta)^2(1 + |Y_0^{(1)}| + \|Y^{(1)}\|_{\beta\text{-}H\ddot{o}l})(|Y_0^{(1)}| + \|Y\|_{X,\beta}). \tag{4.3}$$

Moreover, C can be taken independently with respect to T in each finite interval.

Proof. By definition, $\varphi(Y)^{(0)} = \varphi(Y^{(0)})$ and $\varphi(Y)^{(1)} = \nabla\varphi(Y^{(0)})Y^{(1)}$ are β-Hölder continuous on $[0,T]$. We now prove that $R_0^1(X, \varphi(Y))$ belongs to $\mathcal{C}_2^\beta(L(E,F))$. Set $(s,t) \in \Delta_T$ with $s < t$ and decompose $R_0^1(X, \varphi(Y))_{s,t}$ as follows:

$$\begin{aligned}
R_0^1(X, \varphi(Y))_{s,t} &= \int_0^1 \nabla\varphi(Y_s^{(0)} + \tau(Y_t^{(0)} - Y_s^{(0)}))\, d\tau (Y_t^{(0)} - Y_s^{(0)}) - \nabla\varphi(Y_s^{(0)})Y_s^{(1)}X_{s,t}^1 \\
&= \int_0^1 \nabla\varphi(Y_s^{(0)} + \tau(Y_t^{(0)} - Y_s^{(0)}))\, d\tau (Y_s^{(1)}X_{s,t}^1 + R_0^1(X,Y)_{s,t}) - \nabla\varphi(Y_s^{(0)})Y_s^{(1)}X_{s,t}^1 \\
&= \int_0^1 (\nabla\varphi(Y_s^{(0)} + \tau(Y_t^{(0)} - Y_s^{(0)})) - \nabla\varphi(Y_s^{(0)}))\, d\tau Y_s^{(1)}X_{s,t}^1 \\
&\quad + \int_0^1 \nabla\varphi(Y_s^{(0)} + \tau(Y_t^{(0)} - Y_s^{(0)}))\, d\tau R_0^1(X,Y)_{s,t}.
\end{aligned}$$

Then,

$$|R_0^1(X,\varphi(Y))_{s,t}|/(t-s)^{2\beta}$$
$$\le \|\nabla\varphi\|_{\text{1-Höl}}\|Y^{(0)}\|_{\beta\text{-Höl}}(|Y_0^{(1)}|+\|Y^{(1)}\|_{\beta\text{-Höl}}T^\beta)\|X^1\|_\beta+\|\nabla\varphi\|_\infty\|R_0^1(X,Y)\|_{2\beta}.$$

Thus, $R_0^1(X,\varphi(Y))\in\mathcal{C}_2^\beta(L(E,F))$ and so $\varphi(Y)$ belongs to $\mathcal{Q}_X^{\beta,2}(L(E,F))$. We next prove Eq. (4.3). By inequalities of the form $|ab-\tilde a\tilde b|\le|a-\tilde a||b|+|\tilde a||b-\tilde b|$,

$$\begin{aligned}|R_1^0(X,\varphi(Y))_{s,t}|/(t-s)^\beta &\le (|\nabla\varphi(Y_t^{(0)})-\nabla\varphi(Y_s^{(0)})||Y_t^{(1)}|+|\nabla\varphi(Y_s^{(0)})||Y_t^{(1)}-Y_s^{(1)}|)/(t-s)^\beta\\ &\le \|\nabla\varphi\|_{\text{1-Höl}}\|Y^{(0)}\|_{\beta\text{-Höl}}(|Y_0^{(1)}|+\|Y^{(1)}\|_{\beta\text{-Höl}}T^\beta)+\|\nabla\varphi\|_\infty\|Y^{(1)}\|_{\beta\text{-Höl}}.\end{aligned}$$

Also, from the definition of controlled paths,

$$\begin{aligned}\|Y^{(0)}\|_{\beta\text{-Höl}} &\le (|Y_0^{(1)}|+\|Y^{(1)}\|_{\beta\text{-Höl}}T^\beta)\|X^1\|_\beta+\|R_0^1(X,Y)\|_{2\beta}T^\beta\\ &\le (1+\|X^1\|_\beta)(|Y_0^{(1)}|+\|Y\|_{X,\beta}T^\beta).\end{aligned}$$

Then, combining these estimates, we get

$$\begin{aligned}\|\varphi(Y)\|_{X,\beta} &= \|R_0^1(X,\varphi(Y))\|_{2\beta}+\|R_1^0(X,\varphi(Y))\|_\beta\\ &\le \|\nabla\varphi\|_{\text{1-Höl}}\|Y^{(0)}\|_{\beta\text{-Höl}}(|Y_0^{(1)}|+\|Y^{(1)}\|_{\beta\text{-Höl}}T^\beta)(1+\|X^1\|_\beta)\\ &\quad+\|\nabla\varphi\|_\infty(\|Y^{(1)}\|_{\beta\text{-Höl}}+\|R_0^1(X,Y)\|_{2\beta})\\ &\le \|\nabla\varphi\|_{C^{0,1}}((1+\|X^1\|_\beta)^2(|Y_0^{(1)}|+\|Y\|_{X,\beta}T^\beta)(|Y_0^{(1)}|+\|Y^{(1)}\|_{\beta\text{-Höl}}T^\beta)+\|Y\|_{X,\beta}).\end{aligned}$$

This yields Eq. (4.3) immediately. □

Remark 4.4. By an argument similar to the proof above, it is easy to verify that if $Y\in\mathcal{Q}_X^{1,2}(F)$ for some $X\in S\Omega_\beta(E)$, then $\varphi(Y)$ belongs $\mathcal{Q}_X^{1,2}(L(E,F))$. This fact is used later in this subsection.

Lemma 4.5. *Let $Y\in\mathcal{Q}_X^{\beta,2}(F)$, $\tilde Y\in\mathcal{Q}_{\tilde X}^{\beta,2}(F)$ for some $X,\tilde X\in C_{0,\beta}(\Delta_T,T^{(2)}(E))$, and $\varphi\in C^{1,1}(F, L(E,F))$ such that $\nabla\varphi$ is bounded on F. Then, there exists a positive constant C which depends only on β and T such that*

$$\|\varphi(Y^{(0)})-\varphi(\tilde Y^{(0)})\|_{\beta\text{-}H\ddot{o}l}\le C\|\nabla\varphi\|_{C^{0,1}}C_{X^1}C_Y(|Y_0^{(0)}-\tilde Y_0^{(0)}|+\|Y^{(0)}-\tilde Y^{(0)}\|_{\beta\text{-}H\ddot{o}l}). \tag{4.4}$$

Moreover, C can be taken independently with respect to T in each finite interval.

Proof. Set $(s,t)\in\Delta_T$ with $s<t$. Then,

$$\begin{aligned}&(\varphi(Y_t^{(0)})-\varphi(Y_s^{(0)}))-(\varphi(\tilde Y_t^{(0)})-\varphi(\tilde Y_s^{(0)}))\\ &=\int_0^1\nabla\varphi(Y_s^{(0)}+\tau(Y_t^{(0)}-Y_s^{(0)}))\,d\tau(Y_t^{(0)}-Y_s^{(0)})\\ &\quad-\int_0^1\nabla\varphi(\tilde Y_s^{(0)}+\tau(\tilde Y_t^{(0)}-\tilde Y_s^{(0)}))\,d\tau(\tilde Y_t^{(0)}-\tilde Y_s^{(0)})\\ &=\int_0^1(\nabla\varphi(Y_s^{(0)}+\tau(Y_t^{(0)}-Y_s^{(0)}))-\nabla\varphi(\tilde Y_s^{(0)}+\tau(\tilde Y_t^{(0)}-\tilde Y_s^{(0)})))\,d\tau(Y_t^{(0)}-Y_s^{(0)})\\ &\quad+\int_0^1\nabla\varphi(\tilde Y_s^{(0)}+\tau(\tilde Y_t^{(0)}-\tilde Y_s))\,d\tau((Y_t^{(0)}-Y_s^{(0)})-(\tilde Y_t^{(0)}-\tilde Y_s^{(0)})).\end{aligned}$$

So, we get

$$\begin{aligned}
&|(\varphi(Y_t^{(0)}) - \varphi(Y_s^{(0)})) - (\varphi(\tilde{Y}_t^{(0)}) - \varphi(\tilde{Y}_s^{(0)}))|/(t-s)^\beta \\
&\le \|\nabla\varphi\|_{1\text{-Höl}}(|Y_0^{(0)} - \tilde{Y}_0^{(0)}| + 2\|Y^{(0)} - \tilde{Y}^{(0)}\|_{\beta\text{-Höl}}T^\beta)\|Y^{(0)}\|_{\beta\text{-Höl}} + \|\nabla\varphi\|_\infty \|Y^{(0)} - \tilde{Y}^{(0)}\|_{\beta\text{-Höl}} \\
&\le C\|\nabla\varphi\|_{C^{0,1}}(1 + \|Y^{(0)}\|_{\beta\text{-Höl}})(|Y_0^{(0)} - \tilde{Y}_0^{(0)}| + \|Y^{(0)} - \tilde{Y}^{(0)}\|_{\beta\text{-Höl}}).
\end{aligned}$$

Furthermore, from the definition of controlled paths, we have

$$1 + \|Y^{(0)}\|_{\beta\text{-Höl}} \le (1 + \|X^1\|_\beta)(1 + |Y_0^{(1)}| + \|Y\|_{X,\beta}T^\beta) \le C_{X^1}C_Y.$$

Combining these estimates, we obtain Eq. (4.4) immediately. □

We recall Eq. (3.11) for the meaning of $d_{X,\tilde{X},\beta}(\varphi(Y), \varphi(\tilde{Y}))$ below.

Lemma 4.6. *Let $Y \in \mathcal{Q}_X^{\beta,2}(F)$, $\tilde{Y} \in \mathcal{Q}_{\tilde{X}}^{\beta,2}(F)$ for some $X, \tilde{X} \in C_{0,\beta}(\Delta_T, T^{(2)}(E))$, and $\varphi \in C^{2,1}(F, L(E,F))$ such that $\nabla\varphi$ and $\nabla^2\varphi$ are bounded on F. Then, there exists a positive constant C which depends only on β and T such that*

$$\begin{aligned}
d_{X,\tilde{X},\beta}(\varphi(Y), \varphi(\tilde{Y})) \le C\|\nabla\varphi\|_{C^{1,1}}\big(&C_{X^1}^2 C_Y^3(|Y_0^{(0)} - \tilde{Y}_0^{(0)}| + \|X^1 - \tilde{X}^1\|_\beta) \\
&+ C_{X^1}^3 C_Y^2(|Y_0^{(1)} - \tilde{Y}_0^{(1)}| + d_{X,\tilde{X},\beta}(Y,\tilde{Y}))\big). \qquad (4.5)
\end{aligned}$$

Moreover, C can be taken independently with respect to T in each finite interval.

Proof. Set

$$\delta := |Y_0^{(0)} - \tilde{Y}_0^{(0)}| + \|Y^{(0)} - \tilde{Y}^{(0)}\|_{\beta\text{-Höl}} + \|X^1 - \tilde{X}^1\|_\beta + |Y_0^{(1)} - \tilde{Y}_0^{(1)}| + d_{X,\tilde{X},\beta}(Y,\tilde{Y}).$$

We first provide an estimate of $\|R_0^1(X,\varphi(Y)) - R_0^1(\tilde{X},\varphi(\tilde{Y}))\|_{2\beta}$. Set $(s,t) \in \Delta_T$ with $s < t$. Then, in the same way as in the proof of Lemma 4.3, we have

$$\begin{aligned}
R_0^1(X,\varphi(Y))_{s,t} - R_0^1(\tilde{X},\varphi(\tilde{Y}))_{s,t} &= \bigg(\int_0^1 (\nabla\varphi(Y_s^{(0)} + \tau(Y_t^{(0)} - Y_s^{(0)})) - \nabla\varphi(Y_s^{(0)}))\,d\tau Y_s^{(1)} X_{s,t}^1 \\
&\qquad - \int_0^1 (\nabla\varphi(\tilde{Y}_s^{(0)} + \tau(\tilde{Y}_t^{(0)} - \tilde{Y}_s^{(0)})) - \nabla\varphi(\tilde{Y}_s^{(0)}))\,d\tau \tilde{Y}_s^{(1)} \tilde{X}_{s,t}^1\bigg) \\
&\quad + \bigg(\int_0^1 \nabla\varphi(Y_s^{(0)} + \tau(Y_t^{(0)} - Y_s^{(0)}))\,d\tau R_0^1(X,Y)_{s,t} \\
&\qquad - \int_0^1 \nabla\varphi(\tilde{Y}_s^{(0)} + \tau(\tilde{Y}_t^{(0)} - \tilde{Y}_s^{(0)}))\,d\tau R_0^1(\tilde{X},\tilde{Y})_{s,t}\bigg) \\
&=: A_{s,t}^1 + A_{s,t}^2.
\end{aligned}$$

We decompose $A^1_{s,t}$ as follows:

$$
\begin{aligned}
A^1_{s,t} &= \int_0^1 \int_0^{\tau_1} \nabla^2\varphi(Y^{(0)}_s + \tau_2(Y^{(0)}_t - Y^{(0)}_s))\, d\tau_2\, d\tau_1 (Y^{(0)}_t - Y^{(0)}_s) Y^{(1)}_s X^1_{s,t} \\
&\quad - \int_0^1 \int_0^{\tau_1} \nabla^2\varphi(\tilde{Y}^{(0)}_s + \tau_2(\tilde{Y}^{(0)}_t - \tilde{Y}^{(0)}_s))\, d\tau_2\, d\tau_1 (\tilde{Y}^{(0)}_t - \tilde{Y}^{(0)}_s) \tilde{Y}^{(1)}_s \tilde{X}^1_{s,t} \\
&= \int_0^1 \int_0^{\tau_1} (\nabla^2\varphi(Y^{(0)}_s + \tau_2(Y^{(0)}_t - Y^{(0)}_s)) - \nabla^2\varphi(\tilde{Y}^{(0)}_s + \tau_2(\tilde{Y}^{(0)}_t - \tilde{Y}^{(0)}_s)))\, d\tau_2\, d\tau_1 \\
&\quad \times (Y^{(0)}_t - Y^{(0)}_s) Y^{(1)}_s X^1_{s,t} \\
&\quad + \int_0^1 \int_0^{\tau_1} \nabla^2\varphi(\tilde{Y}^{(0)}_s + \tau_2(\tilde{Y}^{(0)}_t - \tilde{Y}^{(0)}_s))\, d\tau_2\, d\tau_1 ((Y^{(0)}_t - Y^{(0)}_s) - (\tilde{Y}^{(0)}_t - \tilde{Y}^{(0)}_s)) Y^{(1)}_s X^1_{s,t} \\
&\quad + \int_0^1 \int_0^{\tau_1} \nabla^2\varphi(\tilde{Y}^{(0)}_s + \tau_2(\tilde{Y}^{(0)}_t - \tilde{Y}^{(0)}_s))\, d\tau_2\, d\tau_1 (\tilde{Y}^{(0)}_t - \tilde{Y}^{(0)}_s)(Y^{(1)}_s - \tilde{Y}^{(1)}_s) X^1_{s,t} \\
&\quad + \int_0^1 \int_0^{\tau_1} \nabla^2\varphi(\tilde{Y}^{(0)}_s + \tau_2(\tilde{Y}^{(0)}_t - \tilde{Y}^{(0)}_s))\, d\tau_2\, d\tau_1 (\tilde{Y}^{(0)}_t - \tilde{Y}^{(0)}_s) \tilde{Y}^{(1)}_s (X^1_{s,t} - \tilde{X}^1_{s,t}).
\end{aligned}
$$

So, we get

$$
\begin{aligned}
&|A^1_{s,t}|/(t-s)^{2\beta} \\
&\le \|\nabla^2\varphi\|_{1\text{-Höl}} (|Y^{(0)}_0 - \tilde{Y}^{(0)}_0| + 2\|Y^{(0)} - \tilde{Y}^{(0)}\|_{\beta\text{-Höl}} T^\beta) \|Y^{(0)}\|_{\beta\text{-Höl}} (|Y^{(1)}_0| + \|Y^{(1)}\|_{\beta\text{-Höl}} T^\beta) \|\!|X^1|\!\|_\beta \\
&\quad + \|\nabla^2\varphi\|_\infty \|Y^{(0)} - \tilde{Y}^{(0)}\|_{\beta\text{-Höl}} (|Y^{(1)}_0| + \|Y^{(1)}\|_{\beta\text{-Höl}} T^\beta) \|\!|X^1|\!\|_\beta \\
&\quad + \|\nabla^2\varphi\|_\infty \|\tilde{Y}^{(0)}\|_{\beta\text{-Höl}} (|Y^{(1)}_0 - \tilde{Y}^{(1)}_0| + \|Y^{(1)} - \tilde{Y}^{(1)}\|_{\beta\text{-Höl}} T^\beta) \|\!|X^1|\!\|_\beta \\
&\quad + \|\nabla^2\varphi\|_\infty \|\tilde{Y}^{(0)}\|_{\beta\text{-Höl}} (|\tilde{Y}^{(1)}_0| + \|\tilde{Y}^{(1)}\|_{\beta\text{-Höl}} T^\beta) \|\!|X^1 - \tilde{X}^1|\!\|_\beta \\
&\le C\|\nabla^2\varphi\|_{C^{0,1}} \big(\|Y^{(0)}\|_{\beta\text{-Höl}} (|Y^{(1)}_0| + \|Y^{(1)}\|_{\beta\text{-Höl}}) \|\!|X^1|\!\|_\beta \\
&\quad + (|Y^{(1)}_0| + \|Y^{(1)}\|_{\beta\text{-Höl}}) \|\!|X^1|\!\|_\beta + \|\tilde{Y}^{(0)}\|_{\beta\text{-Höl}} \|\!|X^1|\!\|_\beta + \|\tilde{Y}^{(0)}\|_{\beta\text{-Höl}} (|\tilde{Y}^{(1)}_0| + \|\tilde{Y}^{(1)}\|_{\beta\text{-Höl}}) \big) \delta \\
&\le C\|\nabla^2\varphi\|_{C^{0,1}} C^2_{X^1} C^2_Y \delta,
\end{aligned}
$$

where in the last inequality we used

$$
\|Y^{(0)}\|_{\beta\text{-Höl}} \le (1 + \|\!|X^1|\!\|_\beta)(|Y^{(1)}_0| + \|Y\|_{X,\beta} T^\beta) \le C C_{X^1} C_Y.
$$

We also decompose $A^2_{s,t}$ as follows:

$$
\begin{aligned}
A^2_{s,t} &= \int_0^1 (\nabla\varphi(Y^{(0)}_s + \tau(Y^{(0)}_t - Y^{(0)}_s)) - \nabla\varphi(\tilde{Y}^{(0)}_s + \tau(\tilde{Y}^{(0)}_t - \tilde{Y}^{(0)}_s)))\, d\tau R^1_0(X,Y)_{s,t} \\
&\quad + \int_0^1 \nabla\varphi(\tilde{Y}^{(0)}_s + \tau(\tilde{Y}^{(0)}_t - \tilde{Y}^{(0)}_s))\, d\tau (R^1_0(X,Y)_{s,t} - R^1_0(\tilde{X},\tilde{Y})_{s,t}).
\end{aligned}
$$

So, we get

$$
\begin{aligned}
|A^2_{s,t}|/(t-s)^{2\beta} &\le \|\nabla\varphi\|_{1\text{-Höl}} (|Y^{(0)}_0 - \tilde{Y}^{(0)}_0| + 2\|Y^{(0)} - \tilde{Y}^{(0)}\|_{\beta\text{-Höl}} T^\beta) \|\!|R^1_0(X,Y)|\!\|_{2\beta} \\
&\quad + \|\nabla\varphi\|_\infty \|\!|R^1_0(X,Y) - R^1_0(\tilde{X},\tilde{Y})|\!\|_{2\beta} \\
&\le C\|\nabla\varphi\|_{C^{0,1}} C_Y \delta.
\end{aligned}
$$

Hence, from the estimates of $A^1_{s,t}$ and $A^2_{s,t}$ above, we have

$$\|R^1_0(X,\varphi(Y)) - R^1_0(\tilde{X},\varphi(\tilde{Y}))\|_{2\beta} \le C\|\nabla\varphi\|_{C^{1,1}} C^2_{X^1} C^2_Y \delta. \tag{4.6}$$

We next provide an estimate of $\|R^0_1(X,\varphi(Y)) - R^0_1(\tilde{X},\varphi(\tilde{Y}))\|_\beta$. Set $(s,t) \in \Delta_T$ with $s < t$. Then,

$$\begin{aligned}
&R^0_1(X,\varphi(Y))_{s,t} - R^0_1(\tilde{X},\varphi(\tilde{Y}))_{s,t} \\
&= (\nabla\varphi(Y^{(0)}_t)Y^{(1)}_t - \nabla\varphi(Y^{(0)}_s)Y^{(1)}_s) - (\nabla\varphi(\tilde{Y}^{(0)}_t)\tilde{Y}^{(1)}_t - \nabla\varphi(\tilde{Y}^{(0)}_s)\tilde{Y}^{(1)}_s) \\
&= (\nabla\varphi(Y^{(0)}_t) - \nabla\varphi(Y^{(0)}_s))Y^{(1)}_t + \nabla\varphi(Y^{(0)}_s)(Y^{(1)}_t - Y^{(1)}_s) \\
&\quad - (\nabla\varphi(\tilde{Y}^{(0)}_t) - \nabla\varphi(\tilde{Y}^{(0)}_s))\tilde{Y}^{(1)}_t - \nabla\varphi(\tilde{Y}^{(0)}_s)(\tilde{Y}^{(1)}_t - \tilde{Y}^{(1)}_s) \\
&= ((\nabla\varphi(Y^{(0)}_t) - \nabla\varphi(Y^{(0)}_s)) - (\nabla\varphi(\tilde{Y}^{(0)}_t) - \nabla\varphi(\tilde{Y}^{(0)}_s)))Y^{(1)}_t \\
&\quad + (\nabla\varphi(\tilde{Y}^{(0)}_t) - \nabla\varphi(\tilde{Y}^{(0)}_s))(Y^{(1)}_t - \tilde{Y}^{(1)}_t) \\
&\quad + (\nabla\varphi(Y^{(0)}_s) - \nabla\varphi(\tilde{Y}^{(0)}_s))(Y^{(1)}_t - Y^{(1)}_s) \\
&\quad + \nabla\varphi(\tilde{Y}^{(0)}_s)((Y^{(1)}_t - Y^{(1)}_s) - (\tilde{Y}^{(1)}_t - \tilde{Y}^{(1)}_s)).
\end{aligned}$$

So, we get

$$\begin{aligned}
&|R^0_1(X,\varphi(Y))_{s,t} - R^0_1(\tilde{X},\varphi(\tilde{Y}))_{s,t}|/(t-s)^\beta \\
&\le \|\nabla\varphi(Y^{(0)}) - \nabla\varphi(\tilde{Y}^{(0)})\|_{\beta\text{-Höl}}(|Y^{(1)}_0| + \|Y^{(1)}\|_{\beta\text{-Höl}}T^\beta) \\
&\quad + \|\nabla\varphi\|_{1\text{-Höl}}\|\tilde{Y}^{(0)}\|_{\beta\text{-Höl}}(|Y^{(1)}_0 - \tilde{Y}^{(1)}_0| + \|Y^{(1)} - \tilde{Y}^{(1)}\|_{\beta\text{-Höl}}T^\beta) \\
&\quad + \|\nabla\varphi\|_{1\text{-Höl}}(|Y^{(0)}_0 - \tilde{Y}^{(0)}_0| + \|Y^{(0)} - \tilde{Y}^{(0)}\|_{\beta\text{-Höl}}T^\beta)\|Y^{(1)}\|_{\beta\text{-Höl}} \\
&\quad + \|\nabla\varphi\|_\infty\|Y^{(1)} - \tilde{Y}^{(1)}\|_{\beta\text{-Höl}}.
\end{aligned}$$

Using Eq. (4.4), we have

$$\|R^0_1(X,\varphi(Y)) - R^0_1(\tilde{X},\varphi(\tilde{Y}))\|_\beta \le C\|\nabla\varphi\|_{C^{1,1}} C_{X^1} C^2_Y \delta. \tag{4.7}$$

Finally, from Eqs. (4.6), (4.7), and

$$\begin{aligned}
\|Y^{(0)} - \tilde{Y}^{(0)}\|_{\beta\text{-Höl}} &\le (1 + \|X^1\|_\beta)(|Y^{(1)}_0 - \tilde{Y}^{(1)}_0| + d_{X,\tilde{X},\beta}(Y,\tilde{Y})T^\beta) \\
&\quad + (|\tilde{Y}^{(1)}_0| + \|\tilde{Y}^{(1)}\|_{\beta\text{-Höl}}T^\beta)\|X^1 - \tilde{X}^1\|_\beta \\
&\le C(C_Y\|X^1 - \tilde{X}^1\|_\beta + C_{X^1}(|Y^{(1)}_0 - \tilde{Y}^{(1)}_0| + d_{X,\tilde{X},\beta}(Y,\tilde{Y}))),
\end{aligned}$$

we get

$$\begin{aligned}
d_{X,\tilde{X},\beta}(\varphi(Y),\varphi(\tilde{Y})) &= \|R^1_0(X,\varphi(Y)) - R^1_0(\tilde{X},\varphi(\tilde{Y}))\|_{2\beta} + \|R^0_1(X,\varphi(Y)) - R^0_1(\tilde{X},\varphi(\tilde{Y}))\|_\beta \\
&\le C\|\nabla\varphi\|_{C^{1,1}} C^2_{X^1} C^2_Y \delta \\
&\le C\|\nabla\varphi\|_{C^{1,1}} C^2_{X^1} C^2_Y \big(|Y^{(0)}_0 - \tilde{Y}^{(0)}_0| + C_Y\|X^1 - \tilde{X}^1\|_\beta \\
&\qquad + C_{X^1}(|Y^{(1)}_0 - \tilde{Y}^{(1)}_0| + d_{X,\tilde{X},\beta}(Y,\tilde{Y}))\big).
\end{aligned}$$

Thus we obtain Eq. (4.5). □

Lemmas 4.3, 4.6, and Remark 4.4 yield the following proposition immediately.

Proposition 4.7. *Let $Y \in \Pi_X(\overline{S_\beta}(E,F))$ for some $X \in G\Omega_\beta(E)$ and $\varphi \in C^{2,1}(F, L(E,F))$ such that $\nabla\varphi$ and $\nabla^2\varphi$ are bounded on F. Then, $\varphi(Y)$ belongs to $\Pi_X(\overline{S_\beta}(E, L(E,F)))$.*

Proof. From Lemma 4.3, $(X, \varphi(Y))$ belongs to $M_\beta(E, L(E,F))$. By the definition of $Y = (Y^{(0)}, Y^{(1)}) \in \Pi_X(\overline{S_\beta}(E,F))$, there exists a sequence $\{(X(n), Y(n))\}_{n=1}^\infty \subset S_\beta(E,F)$ which converges to (X,Y) with respect to the distance m_β. From Remark 4.4, $(X(n), \varphi(Y(n)))$ belongs to $S_\beta(E, L(E,F))$. Then, from Lemma 4.6, $(X(n), \varphi(Y(n)))$ converges to $(X, \varphi(Y))$ with respect to the distance m_β. Thus, $(X, \varphi(Y))$ belongs to $\overline{S_\beta}(E, L(E,F))$. □

4.2.2 Estimates of $I(X, \varphi(Y))$

Let $(X,Y) \in M_\beta(E,F)$ and $\varphi \in C^{1,1}(F, L(E,F))$ such that $\nabla\varphi$ is bounded on F. We set $I(X, \varphi(Y)) = (I(X,\varphi(Y))^{(0)}, I(X,\varphi(Y))^{(1)})$ as

$$I(X,\varphi(Y))^{(0)}_t := I^\gamma(X,\varphi(Y))_{0,t} \quad \text{and} \quad I(X,\varphi(Y))^{(1)}_t := \varphi(Y^{(0)}_t) \quad \text{for } t \in [0,T].$$

For the proof of Theorem 4.2, we provide several estimates of $I(X, \varphi(Y))$.

Lemma 4.8. *Let $(X,Y) \in M_\beta(E,F)$ and $\varphi \in C^{1,1}(F, L(E,F))$ such that $\nabla\varphi$ is bounded on F. Suppose that $I^\gamma(X,\varphi(Y))$ is additive on Δ_T. Then, $I(X,\varphi(Y))$ belongs to $\mathcal{Q}^{\beta,2}_X(F)$ and there exists a positive constant C which depends only on β, γ, and T such that*

$$\begin{aligned}\|I(X,\varphi(Y))\|_{X,\beta} &\le C\|\nabla\varphi\|_{C^{0,1}}(1 + \|X^1\|_\beta)^3(1 + \|X^1\|_\beta + \|X^2\|_{2\beta}) \\ &\quad \times \big(|Y^{(1)}_0| + \|Y\|_{X,\beta}T^\beta + (1 + |Y^{(1)}_0| + \|Y^{(1)}\|_{\beta\text{-Höl}})(|Y^{(1)}_0| + \|Y\|_{X,\beta})T^\beta\big).\end{aligned} \tag{4.8}$$

Moreover, C can be taken independently with respect to T in each finite interval.

Proof. We first see from Lemmas 3.24, 4.3, and the additivity of $I^\gamma(X,\varphi(Y))$ that $I(X,\varphi(Y))$ belongs to $\mathcal{Q}^{\beta,2}_X(F)$. We prove Eq. (4.8). Set $(s,t) \in \Delta_T$ with $s < t$. Then, from the additivity of $I^\gamma(X,\varphi(Y))$, Eqs. (2.23), and (4.3), we get

$$\begin{aligned}|R^1_0(X, I(X,\varphi(Y)))_{s,t}| &\le |\nabla\varphi(Y^{(0)}_s)Y^{(1)}_s X^2_{s,t}| + |\mathcal{I}^{1,\gamma}_X(R^1_0(X,\varphi(Y)))_{s,t}| + |\mathcal{I}^{2,\gamma}_X(R^0_1(X,\varphi(Y)))_{s,t}| \\
&\le \|\nabla\varphi\|_\infty(|Y^{(1)}_0| + \|Y^{(1)}\|_{\beta\text{-Höl}}T^\beta)\|X^2\|_{2\beta}(t-s)^{2\beta} \\
&\quad + C\|R^1_0(X,\varphi(Y))\|_{2\beta}\|X^1\|_\beta(t-s)^{3\beta} \\
&\quad + C\|R^0_1(X,\varphi(Y))\|_\beta(1 + \|X^1\|_\beta)\max_{1\le i\le 2}\|X^i\|_{i\beta}(t-s)^{3\beta} \\
&\le \|\nabla\varphi\|_\infty(|Y^{(1)}_0| + \|Y^{(1)}\|_{\beta\text{-Höl}}T^\beta)\|X^2\|_{2\beta}(t-s)^{2\beta} \\
&\quad + C\|\varphi(Y)\|_{X,\beta}(\|X^1\|_\beta + (1 + \|X^1\|_\beta)\max_{1\le i\le 2}\|X^i\|_{i\beta})(t-s)^{3\beta} \\
&\le \|\nabla\varphi\|_\infty(|Y^{(1)}_0| + \|Y^{(1)}\|_{\beta\text{-Höl}}T^\beta)\|X^2\|_{2\beta}(t-s)^{2\beta} \\
&\quad + C\|\nabla\varphi\|_{C^{0,1}}(1 + \|X^1\|_\beta)^2(1 + |Y^{(1)}_0| + \|Y^{(1)}\|_{\beta\text{-Höl}})(|Y^{(1)}_0| + \|Y\|_{X,\beta}) \\
&\qquad \times (\|X^1\|_\beta + (1 + \|X^1\|_\beta)\max_{1\le i\le 2}\|X^i\|_{i\beta})(t-s)^{3\beta}.\end{aligned}$$

Also, by the definitions of $I(X,\varphi(Y))$ and controlled paths, we get

$$\begin{aligned}
\|R_1^0(X,I(X,\varphi(Y)))\|_\beta &= \|\varphi(Y^{(0)})\|_{\beta\text{-Höl}} \\
&\le \|\nabla\varphi\|_\infty \|Y^{(0)}\|_{\beta\text{-Höl}} \\
&\le \|\nabla\varphi\|_\infty \big((|Y_0^{(1)}| + \|Y^{(1)}\|_{\beta\text{-Höl}}T^\beta)\|X^1\|_\beta + \|R_0^1(X,Y)\|_{2\beta}T^\beta\big) \\
&\le \|\nabla\varphi\|_\infty (1+\|X^1\|_\beta)(|Y_0^{(1)}| + \|Y\|_{X,\beta}T^\beta).
\end{aligned}$$

Combining these estimates, we have

$$\begin{aligned}
\|I(X,\varphi(Y))\|_{X,\beta} &\le \|\nabla\varphi\|_\infty (1+\|X^1\|_\beta + \|X^2\|_{2\beta})(|Y_0^{(1)}| + \|Y\|_{X,\beta}T^\beta) \\
&\quad + C\|\nabla\varphi\|_{C^{0,1}}(1+\|X^1\|_\beta)^3(1+\|X^1\|_\beta+\|X^2\|_{2\beta}) \\
&\quad\ \times (1+|Y_0^{(1)}| + \|Y^{(1)}\|_{\beta\text{-Höl}})(|Y_0^{(1)}| + \|Y\|_{X,\beta})T^\beta.
\end{aligned}$$

This yields Eq. (4.8) immediately. □

Lemma 4.9. *Let $(X,Y),(\tilde X,\tilde Y) \in M_\beta(E,F)$, and $\varphi \in C^{2,1}(F,L(E,F))$ such that $\nabla\varphi$ and $\nabla^2\varphi$ are bounded on F. Suppose that $I^\gamma(X,\varphi(Y))$ and $I^\gamma(\tilde X,\varphi(\tilde Y))$ are additive on Δ_T. Then, there exists a positive constant C which depends only on β, γ, and T such that*

$$\begin{aligned}
&d_{X,\tilde X,\beta}(I(X,\varphi(Y)), I(\tilde X,\varphi(\tilde Y))) \\
&\le C\|\nabla\varphi\|_{C^{1,1}}(C_{X^1}^3 C_X C_Y^3(|Y_0^{(0)} - \tilde Y_0^{(0)}| + \|X^1 - \tilde X^1\|_\beta + \|X^2 - \tilde X^2\|_{2\beta}) \\
&\quad + C_{X^1}^4 C_X C_Y^2(|Y_0^{(1)} - \tilde Y_0^{(1)}| + d_{X,\tilde X,\beta}(Y,\tilde Y)T^\beta)).
\end{aligned} \tag{4.9}$$

Moreover, C can be taken independently with respect to T in each finite interval.

Proof. Set

$$\rho := |Y_0^{(0)} - \tilde Y_0^{(0)}| + d_{\beta,2}(X,\tilde X)$$

and denote $I(X,\varphi(Y))$ and $I(\tilde X,\varphi(\tilde Y))$ by $Z=(Z^{(0)},Z^{(1)})$ and $\tilde Z = (\tilde Z^{(0)},\tilde Z^{(1)})$, respectively, that is, for each $t\in[0,T]$, we set

$$\begin{aligned}
Z_t^{(0)} &:= I^\gamma(X,\varphi(Y))_{0,t} \quad\text{and}\quad Z_t^{(1)} := \varphi(Y_t^{(0)}), \\
\tilde Z_t^{(0)} &:= I^\gamma(\tilde X,\varphi(\tilde Y))_{0,t} \quad\text{and}\quad \tilde Z_t^{(1)} := \varphi(\tilde Y_t^{(0)}).
\end{aligned}$$

First, from Eq. (4.4) and

$$\begin{aligned}
|Y_0^{(0)} - \tilde Y_0^{(0)}| + \|Y^{(0)} - \tilde Y^{(0)}\|_{\beta\text{-Höl}} &\le (1+\|X^1\|_\beta)(|Y_0^{(1)} - \tilde Y_0^{(1)}| + d_{X,\tilde X,\beta}(Y,\tilde Y)T^\beta) \\
&\quad + (1+|\tilde Y_0^{(1)}| + \|\tilde Y^{(1)}\|_{\beta\text{-Höl}}T^\beta)(|Y_0^{(0)} - \tilde Y_0^{(0)}| + \|X^1 - \tilde X^1\|_\beta) \\
&\le C_{X^1}(|Y_0^{(1)} - \tilde Y_0^{(1)}| + d_{X,\tilde X,\beta}(Y,\tilde Y)T^\beta) + CC_Y\rho,
\end{aligned} \tag{4.10}$$

we have

$$\begin{aligned}
\|R_1^0(X,Z) - R_1^0(\tilde X,\tilde Z)\|_\beta &= \|\varphi(Y^{(0)}) - \varphi(\tilde Y^{(0)})\|_{\beta\text{-Höl}} \\
&\le C\|\nabla\varphi\|_{C^{0,1}} C_{X^1}C_Y(|Y_0^{(0)} - \tilde Y_0^{(0)}| + \|Y^{(0)} - \tilde Y^{(0)}\|_{\beta\text{-Höl}}) \\
&\le C\|\nabla\varphi\|_{C^{0,1}}\big(C_{X^1}C_Y^2\rho + C_{X^1}^2 C_Y(|Y_0^{(1)} - \tilde Y_0^{(1)}| + d_{X,\tilde X,\beta}(Y,\tilde Y)T^\beta)\big).
\end{aligned} \tag{4.11}$$

We next provide an estimate of $\|R_0^1(X,Z)-R_0^1(\tilde X,\tilde Z)\|_{2\beta}$. Set $(s,t)\in\Delta_T$ with $s<t$. From the additivity of $I^\gamma(X,\varphi(Y))$ and $I^\gamma(\tilde X,\varphi(\tilde Y))$,

$$
\begin{aligned}
R_0^1(X,Z)_{s,t}-R_0^1(\tilde X,\tilde Z)_{s,t} &= (\nabla\varphi(Y_s^{(0)})Y_s^{(1)}X_{s,t}^2+\mathcal{I}_X^{1,\gamma}(R_0^1(X,\varphi(Y))_{s,t}+\mathcal{I}_X^{2,\gamma}(R_1^0(X,\varphi(Y))_{s,t})\\
&\quad -(\nabla\varphi(\tilde Y_s^{(0)})\tilde Y_s^{(1)}\tilde X_{s,t}^2+\mathcal{I}_{\tilde X}^{1,\gamma}(R_0^1(\tilde X,\varphi(\tilde Y))_{s,t}+\mathcal{I}_{\tilde X}^{2,\gamma}(R_1^0(\tilde X,\varphi(\tilde Y))_{s,t})\\
&= (\nabla\varphi(Y_s^{(0)})Y_s^{(1)}X_{s,t}^2-\nabla\varphi(\tilde Y_s^{(0)})\tilde Y_s^{(1)}\tilde X_{s,t}^2)\\
&\quad +(\mathcal{I}_X^{1,\gamma}(R_0^1(X,\varphi(Y))_{s,t}-\mathcal{I}_{\tilde X}^{1,\gamma}(R_0^1(\tilde X,\varphi(\tilde Y))_{s,t})\\
&\quad +(\mathcal{I}_X^{2,\gamma}(R_1^0(X,\varphi(Y))_{s,t})-\mathcal{I}_{\tilde X}^{2,\gamma}(R_1^0(\tilde X,\varphi(\tilde Y))_{s,t})\\
&=: A_{s,t}^0+A_{s,t}^1+A_{s,t}^2.
\end{aligned}
$$

By inequalities of the form $|abc-\tilde a\tilde b\tilde c|\le|a-\tilde a||b||c|+|\tilde a||b-\tilde b||c|+|\tilde a||\tilde b||c-\tilde c|$,

$$
\begin{aligned}
|A_{s,t}^0| &\le |\nabla\varphi(Y_s^{(0)})-\nabla\varphi(\tilde Y_s^{(0)})||Y_s^{(1)}||X_{s,t}^2|\\
&\quad +|\nabla\varphi(\tilde Y_s^{(0)})||Y_s^{(1)}-\tilde Y_s^{(1)}||X_{s,t}^2|\\
&\quad +|\nabla\varphi(\tilde Y_s^{(0)})||\tilde Y_s^{(1)}||X_{s,t}^2-\tilde X_{s,t}^2|\\
&\le \|\nabla\varphi\|_{1\text{-Höl}}(|Y_0^{(0)}-\tilde Y_0^{(0)}|+\|Y^{(0)}-\tilde Y^{(0)}\|_{\beta\text{-Höl}}T^\beta)(|Y_0^{(1)}|+\|Y^{(1)}\|_{\beta\text{-Höl}}T^\beta)\|X^2\|_{2\beta}(t-s)^{2\beta}\\
&\quad +\|\nabla\varphi\|_\infty(|Y_0^{(1)}-\tilde Y_0^{(1)}|+\|Y^{(1)}-\tilde Y^{(1)}\|_{\beta\text{-Höl}}T^\beta)\|X^2\|_{2\beta}(t-s)^{2\beta}\\
&\quad +\|\nabla\varphi\|_\infty(|\tilde Y_0^{(1)}|+\|\tilde Y^{(1)}\|_{\beta\text{-Höl}}T^\beta)\|X^2-\tilde X^2\|_{2\beta}(t-s)^{2\beta}.
\end{aligned}
$$

Then, using Eq. (4.10), we get

$$
|A_{s,t}^0|/(t-s)^{2\beta}\le C\|\nabla\varphi\|_{C^{0,1}}\big(C_XC_Y^2\rho+C_{X^1}C_XC_Y(|Y_0^{(1)}-\tilde Y_0^{(1)}|+d_{X,\tilde X,\beta}(Y,\tilde Y)T^\beta)\big).
$$

Also, from Eqs. (2.23) and (2.25),

$$
\begin{aligned}
|A_{s,t}^1| &\le |\mathcal{I}_X^{1,\gamma}(R_0^1(X,\varphi(Y)-R_0^1(\tilde X,\varphi(\tilde Y)))_{s,t}|\\
&\quad +|\mathcal{I}_X^{1,\gamma}(R_0^1(\tilde X,\varphi(\tilde Y))_{s,t}-\mathcal{I}_{\tilde X}^{1,\gamma}(R_0^1(\tilde X,\varphi(\tilde Y))_{s,t}|\\
&\le C\|R_0^1(X,\varphi(Y)-R_0^1(\tilde X,\varphi(\tilde Y))\|_{2\beta}\|X^1\|_\beta(t-s)^{3\beta}\\
&\quad +C\|R_0^1(\tilde X,\varphi(\tilde Y))\|_{2\beta}\|X^1-\tilde X^1\|_\beta(t-s)^{3\beta}
\end{aligned}
$$

and

$$
\begin{aligned}
|A_{s,t}^2| &\le |\mathcal{I}_X^{2,\gamma}(R_1^0(X,\varphi(Y))-R_1^0(\tilde X,\varphi(\tilde Y)))_{s,t}|\\
&\quad +|\mathcal{I}_X^{2,\gamma}(R_1^0(\tilde X,\varphi(\tilde Y))_{s,t}-\mathcal{I}_{\tilde X}^{2,\gamma}(R_1^0(\tilde X,\varphi(\tilde Y)))_{s,t}|\\
&\le C\|R_1^0(X,\varphi(Y))-R_1^0(\tilde X,\varphi(\tilde Y))\|_\beta(1+\|X^1\|_\beta)\max_{1\le i\le 2}\|X^i\|_{i\beta}(t-s)^{3\beta}\\
&\quad +C\|R_1^0(\tilde X,\varphi(\tilde Y))\|_\beta(1+\|X^1\|_\beta+\|\tilde X^1\|_\beta)d_{\beta,2}(X,\tilde X)(t-s)^{3\beta}.
\end{aligned}
$$

Combining these estimates, we get

$$
\begin{aligned}
&(|A^1_{s,t}|+|A^2_{s,t}|)/(t-s)^{3\beta}\\
&\le C\Big(d_{X,\tilde{X},\beta}(\varphi(Y),\varphi(\tilde{Y}))\big(\|X^1\|_\beta+(1+\|X^1\|_\beta)\max_{1\le i\le 2}\|X^i\|_{i\beta}\big)\\
&\quad+\|\varphi(\tilde{Y})\|_{\tilde{X},\beta}\big(\|X^1-\tilde{X}^1\|_\beta+(1+\|X^1\|_\beta+\|\tilde{X}^1\|_\beta)d_{\beta,2}(X,\tilde{X})\big)\Big)\\
&\le C\big(d_{X,\tilde{X},\beta}(\varphi(Y),\varphi(\tilde{Y}))C_{X^1}C_X+\|\varphi(\tilde{Y})\|_{\tilde{X},\beta}C_{X^1}d_{\beta,2}(X,\tilde{X})\big)\\
&\le C\|\nabla\varphi\|_{C^{1,1}}\big(C^3_{X^1}C_XC^3_Y\rho+C^4_{X^1}C_XC^2_Y(|Y^{(1)}_0-\tilde{Y}^{(1)}_0|+d_{X,\tilde{X},\beta}(Y,\tilde{Y}))+C^3_{X^1}C^2_Y\rho\big),
\end{aligned}
$$

where in the last inequality we used Eqs. (4.3) and (4.5). Hence, from the estimates of $A^0_{s,t}$, $A^1_{s,t}$, and $A^2_{s,t}$ above, we have

$$
\begin{aligned}
&\|R^1_0(X,Z)-R^1_0(\tilde{X},\tilde{Z})\|_{2\beta}\\
&\le C\|\nabla\varphi\|_{C^{1,1}}(C^3_{X^1}C_XC^3_Y\rho+C^4_{X^1}C_XC^2_Y(|Y^{(1)}_0-\tilde{Y}^{(1)}_0|+d_{X,\tilde{X},\beta}(Y,\tilde{Y})T^\beta)).
\end{aligned}
\tag{4.12}
$$

Finally, from Eqs. (4.11) and (4.12), we thus obtain Eq. (4.9). □

4.2.3 A priori estimates

We now derive a priori estimates for solutions to rough differential equations, which will be used in the proof of Proposition 4.17. For this, we first introduce the following two lemmas.

Lemma 4.10. *Let $\xi \in F$, $\varphi \in C^{1,1}(F, L(E,F))$ such that φ and $\nabla\varphi$ are bounded on F, $X \in G\Omega_\beta(E)$, and $Y \in \Pi_X(\overline{S_\beta}(E,F))$. Assume that Y is a solution to Eq. (4.1). Furthermore, we take $(a,b) \in \Delta_T$ such that $a < b$ and*

$$
C_{\beta,\gamma}\|\varphi\|_{C^{1,1}}(1+\max_{1\le i\le 2}\|X^i\|_{i\beta;[0,T]})^2(b-a)^\beta=\kappa<1, \tag{4.13}
$$

where $C_{\beta,\gamma} := (\beta/(\beta-\gamma))^2$. Then, we have the following bound

$$
\|Y\|_{X,\beta;[a,b]}\le(1-\kappa)^{-1}\|\varphi\|_\infty\|\nabla\varphi\|_{C^{0,1}}(1+\|X^1\|_{\beta;[a,b]})(\|X^1\|_{\beta;[a,b]}+\|X^2\|_{2\beta;[a,b]}).
$$

Proof. Set $(s,t) \in \Delta_T$ with $a \le s < t \le b$. From Eq. (4.2) and the definition of controlled paths,

$$
|R^0_1(X,Y)_{s,t}|\le\|\nabla\varphi\|_\infty|Y^{(0)}_t-Y^{(0)}_s|\le\|\nabla\varphi\|_\infty(|Y^{(1)}_s||X^1_{s,t}|+|R^1_0(X,Y)_{s,t}|).
$$

Then, we get

$$
\|R^0_1(X,Y)\|_{\beta;[a,b]}-\|\nabla\varphi\|_\infty\|\varphi\|_\infty\|X^1\|_{\beta;[a,b]}\le\|\nabla\varphi\|_\infty\|R^1_0(X,Y)\|_{2\beta;[a,b]}(b-a)^\beta.
$$

From Eq. (4.2) and the additivity of $I^\gamma(X,\varphi(Y))$,

$$
R^1_0(X,Y)_{s,t}=\nabla\varphi(Y^{(0)}_s)Y^{(1)}_sX^2_{s,t}+\mathcal{I}^{1,\gamma}_X(R^1_0(X,\varphi(Y)))_{s,t}+\mathcal{I}^{2,\gamma}_X(R^0_1(X,\varphi(Y)))_{s,t}.
$$

Then, from Eqs. (4.2) and (2.23),

$$
\begin{aligned}
&|||R_0^1(X,Y)|||_{2\beta;[a,b]} - \|\nabla\varphi\|_\infty\|\varphi\|_\infty |||X^2|||_{2\beta;[a,b]} \\
&\le C_{\beta,\gamma} |||R_0^1(X,\varphi(Y))|||_{2\beta;[a,b]} |||X^1|||_{\beta;[a,b]}(b-a)^\beta \\
&\quad + C_{\beta,\gamma} |||R_1^0(X,\varphi(Y))|||_{\beta;[a,b]}(1+|||X^1|||_{\beta;[a,b]}) \max_{1\le i\le 2} |||X^i|||_{i\beta;[a,b]}(b-a)^\beta \\
&\le C_{\beta,\gamma}(1+|||X^1|||_{\beta;[a,b]}) \max_{1\le i\le 2} |||X^i|||_{i\beta;[a,b]} \|\varphi(Y)\|_{X,\beta;[a,b]}(b-a)^\beta.
\end{aligned}
$$

Furthermore, in the same way as in the proof of Lemma 4.3, we get

$$
\begin{aligned}
&|R_0^1(X,\varphi(Y))_{s,t}|/(t-s)^{2\beta} \\
&\le (\|\nabla\varphi\|_{1\text{-Höl}}|Y_t^{(0)}-Y_s^{(0)}||Y_s^{(1)}||X_{s,t}^1| + \|\nabla\varphi\|_\infty |R_0^1(X,Y)_{s,t}|)/(t-s)^{2\beta} \\
&\le \|\nabla\varphi\|_{1\text{-Höl}}\|\varphi\|_\infty \|Y^{(0)}\|_{\beta\text{-Höl};[a,b]} |||X^1|||_{\beta;[a,b]} + \|\nabla\varphi\|_\infty |||R_0^1(X,Y)|||_{2\beta;[a,b]}
\end{aligned}
$$

and

$$
\begin{aligned}
&|R_1^0(X,\varphi(Y))_{s,t}|/(t-s)^{\beta} \\
&\le (|\nabla\varphi(Y_t^{(0)})-\nabla\varphi(Y_s^{(0)})||Y_s^{(1)}| + |\nabla\varphi(Y_s^{(0)})||Y_t^{(1)}-Y_s^{(1)}|)/(t-s)^{\beta} \\
&\le \|\nabla\varphi\|_{1\text{-Höl}}\|\varphi\|_\infty \|Y^{(0)}\|_{\beta\text{-Höl};[a,b]} + \|\nabla\varphi\|_\infty |||R_1^0(X,Y)|||_{\beta;[a,b]}.
\end{aligned}
$$

Then, we have

$$
\begin{aligned}
\|\varphi(Y)\|_{X,\beta;[a,b]} &= |||R_0^1(X,\varphi(Y))|||_{2\beta} + |||R_1^0(X,\varphi(Y))|||_{\beta} \\
&\le \|\nabla\varphi\|_{1\text{-Höl}}\|\varphi\|_\infty(1+|||X^1|||_{\beta;[a,b]})\|Y^{(0)}\|_{\beta\text{-Höl};[a,b]} + \|\nabla\varphi\|_\infty\|Y\|_{X,\beta;[a,b]} \\
&\le \|\nabla\varphi\|_{1\text{-Höl}}\|\varphi\|_\infty(1+|||X^1|||_{\beta;[a,b]})(\|\varphi\|_\infty |||X^1|||_{\beta;[a,b]} + |||R_0^1(X,Y)|||_{2\beta;[a,b]}(b-a)^\beta) \\
&\quad + \|\nabla\varphi\|_\infty\|Y\|_{X,\beta;[a,b]}.
\end{aligned}
$$

Combining these estimates yields

$$
\begin{aligned}
&|||R_0^1(X,Y)|||_{2\beta;[a,b]} - \|\nabla\varphi\|_\infty\|\varphi\|_\infty |||X^2|||_{2\beta;[a,b]} \\
&\le C_{\beta,\gamma}(1+|||X^1|||_{\beta;[a,b]})^2 \max_{1\le i\le 2} |||X^i|||_{i\beta;[a,b]} \\
&\quad \times \|\nabla\varphi\|_{1\text{-Höl}}\|\varphi\|_\infty(\|\varphi\|_\infty |||X^1|||_{\beta;[a,b]} + |||R_0^1(X,Y)|||_{2\beta;[a,b]}(b-a)^\beta)(b-a)^\beta \\
&\quad + C_{\beta,\gamma}(1+|||X^1|||_{\beta;[a,b]}) \max_{1\le i\le 2} |||X^i|||_{i\beta;[a,b]} \|\nabla\varphi\|_\infty \|Y\|_{X,\beta;[a,b]}(b-a)^\beta \\
&\le \|\nabla\varphi\|_{1\text{-Höl}} \max_{1\le i\le 2} |||X^i|||_{i\beta;[a,b]}(\|\varphi\|_\infty |||X^1|||_{\beta;[a,b]} + |||R_0^1(X,Y)|||_{2\beta;[a,b]}(b-a)^\beta) \\
&\quad + C_{\beta,\gamma}\|\nabla\varphi\|_\infty(1+|||X^1|||_{\beta;[a,b]}) \max_{1\le i\le 2} |||X^i|||_{i\beta;[a,b]} \|Y\|_{X,\beta;[a,b]}(b-a)^\beta,
\end{aligned}
$$

where in the last inequality we used Eq. (4.13). Therefore, we get

$$\begin{aligned}
\|Y\|_{X,\beta;[a,b]} &= \|R_0^1(X,Y)\|_{2\beta;[a,b]} + \|R_1^0(X,Y)\|_{\beta;[a,b]} \\
&\leq \|\nabla\varphi\|_\infty \|\varphi\|_\infty (\|X^1\|_{\beta;[a,b]} + \|X^2\|_{2\beta;[a,b]}) \\
&\quad + \|\nabla\varphi\|_\infty \|R_0^1(X,Y)\|_{2\beta;[a,b]}(b-a)^\beta \\
&\quad + \|\nabla\varphi\|_{1\text{-Höl}} \|\varphi\|_\infty \max_{1\leq i\leq 2} \|X^i\|_{i\beta;[a,b]} \|X^1\|_{\beta;[a,b]} \\
&\quad + \|\nabla\varphi\|_{1\text{-Höl}} \max_{1\leq i\leq 2} \|X^i\|_{i\beta;[a,b]} \|R_0^1(X,Y)\|_{2\beta;[a,b]}(b-a)^\beta \\
&\quad + C_{\beta,\gamma}\|\nabla\varphi\|_\infty (1+\|X^1\|_{\beta;[a,b]}) \max_{1\leq i\leq 2} \|X^i\|_{i\beta;[a,b]} \|Y\|_{X,\beta;[a,b]}(b-a)^\beta \\
&\leq \|\varphi\|_\infty \|\nabla\varphi\|_{C^{0,1}} (1+\|X^1\|_{\beta;[a,b]})(\|X^1\|_{\beta;[a,b]} + \|X^2\|_{2\beta;[a,b]}) \\
&\quad + \|\nabla\varphi\|_{C^{0,1}} (1+\max_{1\leq i\leq 2} \|X^i\|_{i\beta;[a,b]}) \|R_0^1(X,Y)\|_{2\beta;[a,b]}(b-a)^\beta \\
&\quad + C_{\beta,\gamma}\|\nabla\varphi\|_\infty (1+\|X^1\|_{\beta;[a,b]}) \max_{1\leq i\leq 2} \|X^i\|_{i\beta;[a,b]} \|Y\|_{X,\beta;[a,b]}(b-a)^\beta \\
&\leq \|\varphi\|_\infty \|\nabla\varphi\|_{C^{0,1}} (1+\|X^1\|_{\beta;[a,b]})(\|X^1\|_{\beta;[a,b]} + \|X^2\|_{2\beta;[a,b]}) \\
&\quad + C_{\beta,\gamma}\|\nabla\varphi\|_{C^{0,1}} (1+\max_{1\leq i\leq 2} \|X^i\|_{i\beta;[a,b]})^2 \|Y\|_{X,\beta;[a,b]}(b-a)^\beta \\
&\leq \|\varphi\|_\infty \|\nabla\varphi\|_{C^{0,1}} (1+\|X^1\|_{\beta;[a,b]})(\|X^1\|_{\beta;[a,b]} + \|X^2\|_{2\beta;[a,b]}) + \kappa \|Y\|_{X,\beta;[a,b]}.
\end{aligned}$$

Thus we obtain the claim of the lemma. □

Lemma 4.11. *Let $X \in \Omega_\beta(E)$ and $\{u_k\}_{k=0}^m$ be a set of positive numbers such that $0 = u_0 < u_1 < \cdots < u_m = T$. Take continuous functions $Y^{(0)} \in \mathcal{C}_1(F)$ and $Y^{(1)} \in \mathcal{C}_1(L(E,F))$ and suppose that $Y := (Y^{(0)}, Y^{(1)})$ satisfies*

$$\|Y^{(l)}\|_{\beta\text{-Höl};[u_k,u_{k+1}]} < \infty \quad \text{and} \quad \|R_0^1(X,Y)\|_{2\beta;[u_k,u_{k+1}]} < \infty$$

for each $l = 0, 1$ and $k = 0, \ldots, m-1$. Then, Y belongs to $\mathcal{Q}_X^{\beta,2}(F)$ and the following bounds hold true:

$$\|R_0^1(X,Y)\|_{2\beta;[0,T]} \leq \sum_{k=0}^{m-1} \|R_0^1(X,Y)\|_{2\beta;[u_k,u_{k+1}]} + \sum_{k=1}^{m-1} \|R_1^0(X,Y)\|_{\beta;[0,u_k]} \|X^1\|_{\beta;[u_k,u_{k+1}]} \tag{4.14}$$

and

$$\|R_1^0(X,Y)\|_{2\beta;[0,T]} \leq \sum_{k=0}^{m-1} \|R_1^0(X,Y)\|_{\beta;[u_k,u_{k+1}]}. \tag{4.15}$$

Proof. Set $(s,t) \in \Delta_T$ with $s < t$ and positive integers i and j such that $0 \leq i \leq j \leq m-1$,

$u_i \leq s \leq u_{i+1}$, and $u_j \leq t \leq u_{j+1}$. For each $l = 0, 1$, we get

$$
\begin{aligned}
|Y_t^{(l)} - Y_s^{(l)}| &\leq |Y_t^{(l)} - Y_{u_j}^{(l)}| + \sum_{k=i+1}^{j-1} |Y_{u_{k+1}}^{(l)} - Y_{u_k}^{(l)}| + |Y_{u_{i+1}}^{(l)} - Y_s^{(l)}| \\
&\leq \|Y^{(l)}\|_{\beta\text{-Höl};[u_j,u_{j+1}]}(t-u_j)^\beta \\
&\quad + \sum_{k=i+1}^{j-1} \|Y^{(l)}\|_{\beta\text{-Höl};[u_k,u_{k+1}]}(u_{k+1}-u_k)^\beta \\
&\quad + \|Y^{(l)}\|_{\beta\text{-Höl};[u_i,u_{i+1}]}(u_{i+1}-s)^\beta \\
&\leq (t-s)^\beta \sum_{k=0}^{m-1} \|Y^{(l)}\|_{\beta\text{-Höl};[u_k,u_{k+1}]}.
\end{aligned}
$$

Hence, $Y^{(0)}$ and $Y^{(1)}$ are β-Hölder continuous on $[0,T]$ and also Eq. (4.15) holds. We next prove Eq. (4.14). In the same way, we decompose $R_0^1(X,Y)$ as follows:

$$
\begin{aligned}
R_0^1(X,Y)_{s,t} &= Y_t^{(0)} - Y_s^{(0)} - Y_s^{(1)} X_{s,t}^1 \\
&= (Y_t^{(0)} - Y_{u_j}^{(0)}) + \sum_{k=i+1}^{j-1} (Y_{u_{k+1}}^{(0)} - Y_{u_k}^{(0)}) + (Y_{u_{i+1}}^{(0)} - Y_s^{(0)}) - Y_s^{(1)} X_{s,t}^1 \\
&= Y_{u_j}^{(1)} X_{u_j,t}^1 + R_0^1(X,Y)_{u_j,t} \\
&\quad + \sum_{k=i+1}^{j-1} \left\{ Y_{u_k}^{(1)} X_{u_k,u_{k+1}}^1 + R_0^1(X,Y)_{u_k,u_{k+1}} \right\} \\
&\quad + Y_s^{(1)} X_{s,u_{i+1}}^1 + R_0^1(X,Y)_{s,u_{i+1}} \\
&\quad - Y_s^{(1)} \left(X_{u_j,t}^1 + \sum_{k=i+1}^{j-1} X_{u_k,u_{k+1}}^1 + X_{s,u_{i+1}}^1 \right) \quad \text{(from (2.3) and (3.4))} \\
&= R_0^1(X,Y)_{u_j,t} + \sum_{k=i+1}^{j-1} R_0^1(X,Y)_{u_k,u_{k+1}} + R_0^1(X,Y)_{s,u_{i+1}} \\
&\quad + (Y_{u_j}^{(1)} - Y_s^{(1)}) X_{u_j,t}^1 + \sum_{k=i+1}^{j-1} (Y_{u_k}^{(1)} - Y_s^{(1)}) X_{u_k,u_{k+1}}^1. \qquad (4.16)
\end{aligned}
$$

So, we get

$$
\begin{aligned}
|R_0^1(X,Y)_{s,t}| &\leq \|R_0^1(X,Y)\|_{2\beta;[u_j,u_{j+1}]}(t-u_j)^{2\beta} \\
&\quad + \sum_{k=i+1}^{j-1} \|R_0^1(X,Y)\|_{2\beta;[u_k,u_{k+1}]}(u_{k+1}-u_k)^{2\beta} \\
&\quad + \|R_0^1(X,Y)\|_{2\beta;[u_i,u_{i+1}]}(u_{i+1}-s)^\beta \\
&\quad + \|Y^{(1)}\|_{\beta\text{-Höl};[u_i,u_j]} \|X^1\|_{\beta;[u_j,u_{j+1}]}(u_j-s)^\beta (t-u_j)^\beta
\end{aligned}
$$

$$+\sum_{k=i+1}^{j-1}\|Y^{(1)}\|_{\beta\text{-Höl};[u_i,u_k]}\|\!|X^1|\!\|_{\beta;[u_k,u_{k+1}]}(u_k-s)^\beta(u_{k+1}-u_k)^\beta$$

$$\leq (t-s)^{2\beta}\bigg\{\sum_{k=0}^{m-1}\|\!|R_0^1(X,Y)|\!\|_{2\beta;[u_k,u_{k+1}]}+\sum_{k=1}^{m-1}\|Y^{(1)}\|_{\beta\text{-Höl};[0,u_k]}\|\!|X^1|\!\|_{\beta;[u_k,u_{k+1}]}\bigg\}.$$

Hence, Eq. (4.14) holds and so $R_0^1(X,Y)\in\mathcal{C}_2^{2\beta}(F)$. Thus we obtain the claim of the lemma. □

Lemmas 4.10 and 4.11 yield the following proposition.

Proposition 4.12. *Under the notation and assumptions of Lemma 4.10, we take positive numbers $\{u_j\}_{j=0}^m$ such that $0=u_0<u_1<\cdots<u_m=T$ and*

$$C_{\beta,\gamma}\|\varphi\|_{C^{1,1}}(1+\max_{1\leq i\leq 2}\|\!|X^i|\!\|_{i\beta;[0,T]})^2(u_{j+1}-u_j)^\beta=\kappa_j<1 \tag{4.17}$$

for each $j=0,1,\dots,m-1$. Then, we have the following bound

$$\|Y\|_{X,\beta;[0,T]}\leq m^2(1-\kappa)^{-1}\|\varphi\|_\infty\|\nabla\varphi\|_{C^{0,1}}(1+\|\!|X^1|\!\|_{\beta;[0,T]})^2(\|\!|X^1|\!\|_{\beta;[0,T]}+\|\!|X^2|\!\|_{2\beta;[0,T]}),$$

where $\kappa:=\max_{0\leq j\leq m-1}\kappa_j$.

Proof. From Eqs. (4.14) and (4.15), we have

$$\begin{aligned}\|Y\|_{X,\beta;[0,T]}&=\|\!|R_0^1(X,Y)|\!\|_{2\beta;[0,T]}+\|\!|R_1^0(X,Y)|\!\|_{\beta;[0,T]}\\&\leq\sum_{j=0}^{m-1}\|\!|R_0^1(X,Y)|\!\|_{2\beta;[u_j,u_{j+1}]}+\sum_{j=1}^{m-1}\|\!|R_1^0(X,Y)|\!\|_{\beta;[0,u_j]}\|\!|X^1|\!\|_{\beta;[u_j,u_{j+1}]}\\&\quad+\sum_{j=0}^{m-1}\|\!|R_1^0(X,Y)|\!\|_{\beta;[u_j,u_{j+1}]}\\&\leq\sum_{j=0}^{m-1}\|Y\|_{X,\beta;[u_j,u_{j+1}]}+\|\!|R_1^0(X,Y)|\!\|_{\beta;[0,T]}\sum_{j=1}^{m-1}\|\!|X^1|\!\|_{\beta;[u_j,u_{j+1}]}\\&\leq(1+\sum_{i=1}^{m-1}\|\!|X^1|\!\|_{\beta;[u_j,u_{j+1}]})\sum_{j=0}^{m-1}\|Y\|_{X,\beta;[u_j,u_{j+1}]}.\end{aligned}$$

Then, using Lemma 4.10 with $a=u_j$ and $b=u_{j+1}$, we get

$$\begin{aligned}\|Y\|_{X,\beta;[0,T]}&\leq(1+\sum_{i=1}^{m-1}\|\!|X^1|\!\|_{\beta;[u_j,u_{j+1}]})\|\varphi\|_\infty\|\nabla\varphi\|_{C^{0,1}}\\&\quad\times\sum_{j=0}^{m-1}(1-\kappa_j)^{-1}(1+\|\!|X^1|\!\|_{\beta;[u_j,u_{j+1}]})(\|\!|X^1|\!\|_{\beta;[u_j,u_{j+1}]}+\|\!|X^2|\!\|_{2\beta;[u_j,u_{j+1}]})\\&\leq m(1-\kappa)^{-1}\|\varphi\|_\infty\|\nabla\varphi\|_{C^{0,1}}(1+\|\!|X^1|\!\|_{\beta;[0,T]})^2\sum_{j=0}^{m-1}(\|\!|X^1|\!\|_{\beta;[u_j,u_{j+1}]}+\|\!|X^2|\!\|_{2\beta;[u_j,u_{j+1}]})\\&\leq m^2(1-\kappa)^{-1}\|\varphi\|_\infty\|\nabla\varphi\|_{C^{0,1}}(1+\|\!|X^1|\!\|_{\beta;[0,T]})^2(\|\!|X^1|\!\|_{\beta;[0,T]}+\|\!|X^2|\!\|_{2\beta;[0,T]}),\end{aligned}$$

as desired. Thus we obtain the claim of the proposition. □

The readers may find in [6, Proposition 8.3] sharper estimates than those provided above.

4.2.4 Basic lemmas for shift operators

We provide several lemmas which will be used later. Let V be a finite-dimensional normed space and let $\psi \in \mathcal{C}_1(V)$ and $\Psi \in \mathcal{C}_2(V)$. For each $u \in [0,T]$, we define $\theta_u(\psi)$ and $\theta_u(\Psi)$ by $\theta_u(\psi)_t := \psi_{u+t}$ for $t \in [0, T-u]$ and $\theta_u(\Psi)_{s,t} := \Psi_{u+s,u+t}$ for $(s,t) \in \Delta_{T-u}$, respectively.

Lemma 4.13. *Let $X = (1, X^1, X^2) \in \Omega_{\beta,T}(E)$ and $Y = (Y^{(0)}, Y^{(1)}) \in \mathcal{Q}^{\beta,2}_{X,T}(F)$. Then, for each $u \in [0,T]$, the following statements hold true:*

1. *$\theta_u(X) := (1, \theta_u(X^1), \theta_u(X^2))$ belongs to $\Omega_{\beta,T-u}(E)$;*
2. *If $X \in S\Omega_{\beta,T}(E)$, then $\theta_u(X) \in S\Omega_{\beta,T-u}(E)$;*
3. *If $X \in G\Omega_{\beta,T}(E)$, then $\theta_u(X) \in G\Omega_{\beta,T-u}(E)$;*
4. *$\theta_u(Y) := (\theta_u(Y^{(0)}), \theta_u(Y^{(1)}))$ belongs to $\mathcal{Q}^{\beta,2}_{\theta_u(X),T-u}(F)$;*
5. *If $X \in G\Omega_{\beta,T}(E)$ and $\theta_u(Y) \in \Pi_X(\overline{S_{\beta,T}}(E,F))$, then $\theta_u(Y) \in \Pi_{\theta_u(X)}(\overline{S_{\beta,T-u}}(E,F))$.*

Lemma 4.13 follows immediately from the definition of θ_u. The following lemma is used in the proof of Theorem 4.2 when we prove the existence of global solutions to rough differential equations.

Lemma 4.14. *Let $\varphi \in C^{2,1}(F, L(E,F))$ such that $\nabla\varphi$ and $\nabla^2\varphi$ are bounded on F, $X \in G\Omega_\beta(E)$, and $Y \in \Pi_X(\overline{S_\beta}(E,F))$. Then, for each $u \in [0,T]$ and $v \in [0, T-u]$,*

$$I^\gamma(\theta_u(X), \varphi(\theta_u(Y)))_{0,v} = I^\gamma(X, \varphi(Y))_{u,u+v}. \tag{4.18}$$

Proof. From Lemma 4.13 (5) and Proposition 4.7, $(\theta_u(X), \varphi(\theta_u(Y)))$ belongs to $\overline{S_{\beta,T-u}}(E, L(E, F))$. Then, from Proposition 3.14,

$$\begin{aligned}
&(\text{left-hand side of } (4.18)) \\
&= \lim_{|\mathcal{P}_{0,v}|\to 0} \sum_{i=0}^{m-1} \left\{ \varphi(\theta_u(Y^{(0)})_{u_i})\theta_u(X^1)_{u_i,u_{i+1}} + \nabla\varphi(\theta_u(Y^{(0)})_{u_i})\theta_u(Y^{(1)})_{u_i}\theta_u(X^2)_{u_i,u_{i+1}} \right\} \\
&= \lim_{|\mathcal{P}_{0,v}|\to 0} \sum_{i=0}^{m-1} \left\{ \varphi(Y^{(0)}_{u+u_i})X^1_{u+u_i,u+u_{i+1}} + \nabla\varphi(Y^{(0)}_{u+u_i})Y^{(1)}_{u+u_i}X^2_{u+u_i,u+u_{i+1}} \right\} \\
&= (\text{right-hand side of } (4.18)),
\end{aligned}$$

where the limits are taken over all finite partitions $\mathcal{P}_{0,v} = \{u_0, u_1, \dots, u_m\}$ of the interval $[0,v]$ such that $0 = u_0 \le u_1 \le \cdots \le u_m = v$ and $|\mathcal{P}_{0,v}| := \max_{0\le i\le m-1}|u_{i+1} - u_i|$. Thus we obtain the claim of the lemma. □

Lemma 4.14 yields the following lemma, which is used in the proof of Theorem 4.2 when we prove the uniqueness of global solutions to rough differential equations.

Lemma 4.15. *Let $\xi \in F$, $\varphi \in C^{2,1}(F, L(E,F))$ such that $\nabla\varphi$ and $\nabla^2\varphi$ are bounded on F, $X = (1, X^1, X^2) \in G\Omega_{\beta,T}(E)$, and $Y = (Y^{(0)}, Y^{(1)}) \in \Pi_X(\overline{S_{\beta,T}}(E,F))$. Assume that Y is a solution on $[0,T]$ to the rough differential equation driven by X along φ and starting at ξ. Then, for each $u \in [0,T]$, $\theta_u(Y) := (\theta_u(Y^{(0)}), \theta_u(Y^{(1)})) \in \Pi_X(\overline{S_{\beta,T-u}}(E,F))$ is a solution on $[0, T-u]$ to the rough differential equation driven by $\theta_u(X) := (1, \theta_u(X^1), \theta_u(X^2)) \in G\Omega_{\beta,T-u}(E)$ along φ and starting at $Y^{(0)}_u$.*

Proof. For the claim of the lemma, it suffices to show that, for each $t \in [0, T-u]$,

$$\theta_u(Y^{(0)})_t = Y_u^{(0)} + I^\gamma(\theta_u(X), \varphi(\theta_u(Y)))_{0,t} \quad \text{and} \quad \theta_u(Y^{(1)})_t = \varphi(\theta_u(Y^{(0)})_t). \tag{4.19}$$

Since Y is the solution on $[0, T]$, the second identity of Eq. (4.19) is obvious from the definition of θ_u. Also, the first identity of Eq. (4.19) follows immediately from the assumptions that Y is the solution on $[0, T]$ and Lemma 4.14; indeed, we get

$$\begin{aligned}\theta_u(Y^{(0)})_t &= Y_{u+t}^{(0)} \\ &= \xi + I^\gamma(X, \varphi(Y))_{0,u+t} \\ &= \xi + I^\gamma(X, \varphi(Y))_{0,u} + I^\gamma(X, \varphi(Y))_{u,u+t} \\ &= Y_u^{(0)} + I^\gamma(\theta_u(X), \varphi(\theta_u(Y)))_{0,t}, \quad \text{(from (4.18))}\end{aligned}$$

as desired. Here, we used the additivity of $I^\gamma(X, \varphi(Y))$, which follows from Proposition 4.7 and the assumptions that $X \in G\Omega_{\beta,T}(E)$ and $Y \in \Pi_X(\overline{S_{\beta,T}}(E, F))$. Thus the claim of the lemma holds true. □

4.3 Proof of Theorem 4.2

In this subsection, we provide a proof of Theorem 4.2, which is inspired by those of [8, Theorem 8.4] and [9, Propositions 7 and 8]. Although a part of the proof is given along the same lines of them, some further discussions are needed when we construct the global solution by concatenating the local solutions as in the proof of Proposition 4.17 stated below. This is because it is uncertain whether $I^\gamma(X, \varphi(Y))$ is additive on Δ_T even for every $X \in G\Omega_\beta(E)$ and $Y \in \mathcal{Q}_X^{\beta,2}(F)$. For this reason, we have to discuss the construction of the global solution in $\Pi_X(\overline{S_\beta}(E, F)) \subset \mathcal{Q}_X^{\beta,2}(F)$ for $X \in G\Omega_\beta(E)$.

4.3.1 Main part of the proof of Theorem 4.2

First, we show the local existence and uniqueness of solutions to rough differential equations.

Proposition 4.16. *Under the assumptions of Theorem 4.2, there exists $T_0 \in (0, 1]$ and a unique element $Y = (Y^{(0)}, Y^{(1)})$ of $\Pi_X(\overline{S_{\beta,T_0}}(E, F))$ which satisfies Eq. (4.2) for any $T \le T_0$. Here, T_0 can be taken independently with respect to ξ.*

Proof. For $Y = (Y^{(0)}, Y^{(1)}) \in \Pi_X(\overline{S_\beta}(E, F))$, we define $G_T(Y) = (G_T(Y)^{(0)}, G_T(Y)^{(1)})$ as

$$G_T(Y)_t^{(0)} := \xi + I^\gamma(X, \varphi(Y))_{0,t} \quad \text{and} \quad G_T(Y)_t^{(1)} := \varphi(Y_t^{(0)}) \quad \text{for } t \in [0, T].$$

From Propositions 3.15 and 4.7, $G_T(Y)$ belongs to $\Pi_X(\overline{S_\beta}(E, F))$. Thus, G_T leaves $\Pi_X(\overline{S_\beta}(E, F))$ invariant, that is, $G_T \colon \Pi_X(\overline{S_\beta}(E, F)) \to \Pi_X(\overline{S_\beta}(E, F))$. For $r > 0$, we define $B_T(r)$ as

$$B_T(r) := \{Y \in \Pi_X(\overline{S_\beta}(E, F)) : Y_0^{(0)} = \xi, Y_0^{(1)} = \varphi(\xi), \|G_T(Y)\|_{X,\beta} \le r\}.$$

It is straightforward to show that the subset $\{Y \in \Pi_X(\overline{S_\beta}(E, F)) : Y_0^{(0)} = \xi, Y_0^{(1)} = \varphi(\xi)\}$ of $\Pi_X(\overline{S_\beta}(E, F))$ is a complete metric space under the distance $m_{X,\beta}$ and $B_T(r)$ is a closed ball of radius r centered at $\Psi = (\Psi^{(0)}, \Psi^{(1)})$ in the subspace, where

$$\Psi_t^{(0)} := \xi + \varphi(\xi) X_{0,t}^1 \quad \text{and} \quad \Psi_t^{(1)} := \varphi(\xi) \quad \text{for } t \in [0, T].$$

Let $Y \in \Pi_X(\overline{S_\beta}(E,F))$ such that $Y_0^{(1)} = \varphi(\xi)$. From Eq. (4.8), we have

$$\|G_T(Y)\|_{X,\beta} \le K\|\nabla\varphi\|_{C^{0,1}}(\|\varphi\|_\infty + \|Y\|_{X,\beta}T^\beta + (1+\|\varphi\|_\infty + \|Y\|_{X,\beta})(\|\varphi\|_\infty + \|Y\|_{X,\beta})T^\beta),$$

where K is a constant which depends only on β, γ, and $X = (1, X^1, X^2)$, as long as $T \le 1$. We define $r := K\|\nabla\varphi\|_{C^{0,1}}(\|\varphi\|_\infty + 1)$. If $T \in (0,1]$ is sufficiently small,

$$K\|\nabla\varphi\|_{C^{0,1}}(\|\varphi\|_\infty + rT^\beta + (1+\|\varphi\|_\infty + r)(\|\varphi\|_\infty + r)T^\beta) \le r.$$

Then, G_T leaves $B_T(r)$ invariant, that is, $G_T\colon B_T(r) \to B_T(r)$. Moreover, for each $Y, \tilde{Y} \in B_T(r)$,

$$m_{X,\beta}(G_T(Y), G_T(\tilde{Y})) \le C\|\nabla\varphi\|_{C^{1,1}} C_{X^1}^4 C_X C_r^2 m_{X,\beta}(Y,\tilde{Y})T^\beta,$$

where $C_r := 1 + 2(\|\varphi\|_\infty + r)$ and C is a constant which depends only on β and γ. This follows from Eq. (4.9) with $X = \tilde{X}$, $Y_0^{(0)} = \tilde{Y}_0^{(0)}$, and $Y_0^{(1)} = \tilde{Y}_0^{(1)} = \varphi(\xi)$ and $T \le 1$. Furthermore, by choosing $T = T_0$ smaller such that

$$C\|\nabla\varphi\|_{C^{1,1}} C_{X^1}^4 C_X C_r^2 T_0^\beta = \kappa_0 < 1,$$

we obtain $m_{X,\beta}(G_{T_0}(Y), G_{T_0}(\tilde{Y})) \le \kappa_0 m_{X,\beta}(Y,\tilde{Y})$. Hence, G_{T_0} is a strict contraction in $B_{T_0}(r)$. Therefore, G_{T_0} admits a unique fixed point $Y \in B_{T_0}(r)$. This is the unique solution on the small interval $[0,T_0]$ as desired. Thus we obtain the claim of the proposition. □

Next, we construct a global solution on the whole interval $[0,T]$ by concatenating the local solutions. For this, we introduce a few more notations. Let $S_{\varphi,\xi}(X) = (S_{\varphi,\xi}(X)^{(0)}, S_{\varphi,\xi}(X)^{(1)})$ denote the local solution on $[0,T_0]$ constructed in Proposition 4.16. We define $N_0 := \lceil T/T_0 \rceil$ and $t_i := \min\{iT_0, T\}$ for $i = 0,1,\dots,N_0$. We note that $t_0 = 0$, $t_{N_0-1} < T$, $t_{N_0} = T$, and the obvious relations $t_{N_0} - t_{N_0-1} \le T_0$ and $t_{i+1} - t_i = T_0$ for each $i = 0,\dots,N_0-2$. Then, concatenating the local solutions, we define $\hat{Y}^{(0)} \in \mathcal{C}_1(F)$ and $\hat{Y}^{(1)} \in \mathcal{C}_1(L(E,F))$ as follows: for each $l = 0,1$,

$$\hat{Y}_t^{(l)} := S_{\varphi,\xi_0}(\theta_{t_0}(X))_{t-t_0}^{(l)} \quad \text{for } t \in [t_0,t_1]$$

with $\xi_0 := \xi$ and

$$\hat{Y}_t^{(l)} := S_{\varphi,\xi_i}(\theta_{t_i}(X))_{t-t_i}^{(l)} \quad \text{for } t \in [t_i,t_{i+1}]$$

with $\xi_i := S_{\varphi,\xi_{i-1}}(\theta_{t_{i-1}}(X))_{T_0}^{(0)}$ for $i = 1,\dots,N_0-1$, inductively. It follows from Proposition 4.16 that T_0 does not change when the starting point ξ is replaced by $\xi_1,\dots,\xi_{N_0-1}$. Our candidate of the global solution is defined by $\hat{Y} := (\hat{Y}^{(0)}, \hat{Y}^{(1)})$. Furthermore, from the definition of $\hat{Y}$,

$$\|\hat{Y}^{(l)}\|_{\beta\text{-Höl};[t_i,t_{i+1}]} = \|S_{\varphi,\xi_i}(\theta_{t_i}(X))^{(l)}\|_{\beta\text{-Höl};[0,t_{i+1}-t_i]} < \infty, \quad l = 0,1,$$

and

$$\|R_0^1(X,\hat{Y})\|_{2\beta;[t_i,t_{i+1}]} = \|R_0^1(\theta_{t_i}(X), S_{\varphi,\xi_i}(\theta_{t_i}(X)))\|_{\beta;[0,t_{i+1}-t_i]} < \infty$$

hold for each $i = 0,1,\dots,N_0-1$. Then we see from Lemma 4.11 that $\hat{Y}$ belongs to $\mathcal{Q}_X^{\beta,2}(F)$. Moreover, for $\hat{Y}$ to be a global solution, the following proposition should be valid.

Proposition 4.17. *Under the above notation, $\hat{Y}$ belongs to $\Pi_X(\overline{S_\beta}(E,F))$.*

We prove this later in this subsection. Using Proposition 4.17, we now prove Theorem 4.2.

Proof of Theorem 4.2. We first prove that $\hat{Y} = (\hat{Y}^{(0)}, \hat{Y}^{(1)})$ defined above is a solution on the whole interval $[0,T]$, that is, the following identities hold true: for each $t \in [0,T]$,

$$\hat{Y}^{(0)}_t = \xi + I^\gamma(X, \varphi(\hat{Y}))_{0,t} \quad \text{and} \quad \hat{Y}^{(1)}_t = \varphi(\hat{Y}^{(0)}_t). \tag{4.20}$$

The second identity of Eq. (4.20) follows immediately from the definition of $\hat{Y}$. Indeed, by the definition of t_i, there exists $i = 0, 1, \dots, N_0 - 1$ such that $t_i \le t \le t_{i+1}$ and then

$$\hat{Y}^{(1)}_t = S_{\varphi,\xi_i}(\theta_{t_i}(X))^{(1)}_{t-t_i} = \varphi(S_{\varphi,\xi_i}(\theta_{t_i}(X))^{(1)}_{t-t_i}) = \varphi(\hat{Y}^{(0)}_t).$$

We now prove the first identity of Eq. (4.20). First of all, we note that $I^\gamma(X, \varphi(\hat{Y}))$ is additive on Δ_T. This follows from Theorems 3.11, 3.12, Propositions 4.7, and 4.17. By using this property and Lemma 4.14, we can show that

$$\xi + I^\gamma(X, \varphi(\hat{Y}))_{0,t} = \xi_i + I^\gamma(X, \varphi(\hat{Y}))_{t_i,t}. \tag{4.21}$$

Indeed, the left-hand side of Eq. (4.21) is decomposed as follows:

$$\begin{aligned}
\xi + I^\gamma(X, \varphi(\hat{Y}))_{0,t} &= \xi + I^\gamma(X, \varphi(\hat{Y}))_{0,t_1} + I^\gamma(X, \varphi(\hat{Y}))_{t_1,t} \\
&= S_{\varphi,\xi}(X)^{(0)}_{T_0} + I^\gamma(X, \varphi(\hat{Y}))_{t_1,t} \\
&= \xi_1 + I^\gamma(X, \varphi(\hat{Y}))_{t_1,t}
\end{aligned}$$

and moreover

$$\begin{aligned}
\xi_1 + I^\gamma(X, \varphi(\hat{Y}))_{t_1,t} &= \xi_1 + I^\gamma(X, \varphi(\hat{Y}))_{t_1,t_2} + I^\gamma(X, \varphi(\hat{Y}))_{t_2,t} \\
&= \xi_1 + I^\gamma(\theta_{t_1}(X), \theta_{t_1}(\hat{Y}))_{0,T_0} + I^\gamma(X, \varphi(\hat{Y}))_{t_2,t} \\
&\qquad \text{(from (4.18) with } Y = \hat{Y},\ u = t_1, \text{ and } v = t_2 - t_1 = T_0) \\
&= \xi_1 + I^\gamma(\theta_{t_1}(X), S_{\varphi,\xi_1}(\theta_{t_1}(X)))_{0,T_0} + I^\gamma(X, \varphi(\hat{Y}))_{t_2,t} \\
&\qquad \text{(from } \theta_{t_1}(\hat{Y}) = S_{\varphi,\xi_1}(\theta_{t_1}(X)) \text{ on } [0,T_0]) \\
&= S_{\varphi,\xi_1}(\theta_{t_1}(X))^{(0)}_{T_0} + I^\gamma(X, \varphi(\hat{Y}))_{t_2,t} \\
&= \xi_2 + I^\gamma(X, \varphi(\hat{Y}))_{t_2,t}.
\end{aligned}$$

By repeating this argument with $t_3, t_4, \dots, t_i$, Eq. (4.21) holds true. Then, from Eq. (4.21), Lemma 4.14, and the definition of $\hat{Y}$, we get

$$\begin{aligned}
\hat{Y}^{(0)}_t &= S_{\varphi,\xi_i}(\theta_{t_i}(X))_{t-t_i} \\
&= \xi_i + I^\gamma(\theta_{t_i}(X), \varphi(S_{\varphi,\xi_i}(\theta_{t_i}(X))))_{0,t-t_i} \\
&= \xi_i + I^\gamma(\theta_{t_i}(X), \varphi(\theta_{t_i}(\hat{Y})))_{0,t-t_i} \quad \text{(from } \theta_{t_i}(\hat{Y}) = S_{\varphi,\xi_i}(\theta_{t_i}(X)) \text{ on } [0,T_0]) \\
&= \xi_i + I^\gamma(X, \varphi(\hat{Y}))_{t_i,t} \quad \text{(from (4.18) with } Y = \hat{Y},\ u = t_i, \text{ and } v = t - t_i) \\
&= \xi + I^\gamma(X, \varphi(\hat{Y}))_{0,t}, \quad \text{(from (4.21))}
\end{aligned}$$

as desired. Hence, the first identity of Eq. (4.20) holds.

We next prove the uniqueness of global solutions to Eq. (4.2). Let $Y = (Y^{(0)}, Y^{(1)}), \tilde{Y} = (\tilde{Y}^{(0)}, \tilde{Y}^{(1)}) \in \Pi_X(\overline{S_\beta}(E,F))$ be solutions on the whole interval $[0,T]$, driven by $X = (1, X^1, X^2) \in G\Omega_\beta(E)$ along φ and starting at ξ. We define $\tau := \inf\{t \in [0,T] : Y_t^{(0)} \neq \tilde{Y}_t^{(0)} \text{ or } Y_t^{(1)} \neq \tilde{Y}_t^{(1)}\}$ and assume that $\tau < T$. From Lemma 4.15, $\theta_\tau(Y) := (\theta_\tau(Y^{(0)}), \theta_\tau(Y^{(1)}))$ and $\theta_\tau(\tilde{Y}) := (\theta_\tau(\tilde{Y}^{(0)}), \theta_\tau(\tilde{Y}^{(1)}))$ are solutions on the interval $[0, T-\tau]$, driven by $\theta_\tau(X) = (1, \theta_\tau(X^1), \theta_\tau(X^2)) \in G\Omega_{\beta,T-\tau}(E)$ along φ and starting at $Y_\tau^{(0)} = \tilde{Y}_\tau^{(0)}$. Since we already know local uniqueness of the solution from Proposition 4.16, we see that $\theta_\tau(Y) = \theta_\tau(\tilde{Y})$ on $[0, T_0 \wedge (T-\tau)]$. Hence, it follows that $Y = \tilde{Y}$ on $[0, (\tau + T_0) \wedge T]$ and so $(\tau + T_0) \wedge T \leq \tau$. This contradicts the assumption $\tau < T$. Thus we establish the uniqueness of the global solutions. $\square$

4.3.2 Proof of Proposition 4.17

In the remainder of this subsection, we will prove Proposition 4.17.

Lemma 4.18. *Let $X \in G\Omega_\beta(E)$ and $\{X(n)\}_{n=1}^\infty \subset S\Omega_\beta(E)$ such that $\lim_{n\to\infty} d_{\beta,2}(X(n), X) = 0$. Take $Y \in \mathcal{Q}_X^{\beta,2}(F)$ and positive numbers $\{u_k\}_{k=0}^m$ such that $0 = u_0 < u_1 < \cdots < u_m = T$ and suppose that, for each n, there exists $Y(n) \in \mathcal{Q}_{X(n)}^{1,2}(F)$ that satisfies*

$$\lim_{n\to\infty} \| R_l^{1-l}(X,Y) - R_l^{1-l}(X(n), Y(n)) \|_{(2-l)\beta;[u_k,u_{k+1}]} = 0$$

for each $l = 0,1$ and $k = 0,1,\dots,m-1$. Then, for each $l = 0,1$

$$\lim_{n\to\infty} \| R_l^{1-l}(X,Y) - R_l^{1-l}(X(n), Y(n)) \|_{(2-l)\beta;[0,T]} = 0.$$

Proof. Set $(s,t) \in \Delta_T$ with $s < t$ and positive integers i and j such that $0 \leq i \leq j \leq m-1$, $u_i \leq s \leq u_{i+1}$, and $u_j \leq t \leq u_{j+1}$. In the same way as in the proof of Lemma 4.11, for each $n = 1, 2, \dots$, we get

$$\begin{aligned}
&|R_1^0(X,Y)_{s,t} - R_1^0(X(n),Y(n))_{s,t}| \\
&= |(Y_t^{(1)} - Y_s^{(1)}) - (Y(n)_t^{(1)} - Y(n)_s^{(1)})| \\
&\leq |(Y_t^{(1)} - Y_{u_j}^{(1)}) - (Y(n)_t^{(1)} - Y(n)_{u_j}^{(1)})| + \sum_{k=i+1}^{j-1} |(Y_{u_{k+1}}^{(1)} - Y_{u_k}^{(1)}) - (Y(n)_{u_{k+1}}^{(1)} - Y(n)_{u_k}^{(1)})| \\
&\quad + |(Y_{u_{i+1}}^{(1)} - Y_s^{(1)}) - (Y(n)_{u_{i+1}}^{(1)} - Y(n)_s^{(1)})| \\
&\leq \|Y^{(1)} - Y(n)^{(1)}\|_{\beta\text{-Höl};[u_j,u_{j+1}]} (t-u_j)^\beta + \sum_{k=i+1}^{j-1} \|Y^{(1)} - Y(n)^{(1)}\|_{\beta\text{-Höl};[u_k,u_{k+1}]} (u_{k+1}-u_k)^\beta \\
&\quad + \|Y^{(1)} - Y(n)^{(1)}\|_{\beta\text{-Höl};[u_i,u_{i+1}]} (u_{i+1}-s)^\beta \\
&\leq (t-s)^\beta \sum_{k=0}^{m-1} \|Y^{(1)} - Y(n)^{(1)}\|_{\beta\text{-Höl};[u_k,u_{k+1}]}.
\end{aligned}$$

This yields $\lim_{n\to\infty} \|R^0_1(X,Y) - R^0_1(X(n),Y(n))\|_{\beta;[0,T]} = 0$. Also, from Eq. (4.16), we have

$$
\begin{aligned}
&R^1_0(X,Y)_{s,t} - R^1_0(X(n),Y(n))_{s,t} \\
&= (R^1_0(X,Y)_{u_j,t} - R^1_0(X(n),Y(n))_{u_j,t}) \\
&\quad + \sum_{k=i+1}^{j-1} (R^1_0(X,Y)_{u_k,u_{k+1}} - R^1_0(X(n),Y(n))_{u_k,u_{k+1}}) \\
&\quad + (R^1_0(X,Y)_{s,u_{i+1}} - R^1_0(X,Y)_{s,u_{i+1}}) \\
&\quad + (Y^{(1)}_{u_j} - Y^{(1)}_s)X^1_{u_j,t} - (Y(n)^{(1)}_{u_j} - Y(n)^{(1)}_s)X(n)^1_{u_j,t} \\
&\quad + \sum_{k=i+1}^{j-1} \Big\{ (Y^{(1)}_{u_k} - Y^{(1)}_s)X^1_{u_k,u_{k+1}} - (Y(n)^{(1)}_{u_k} - Y(n)^{(1)}_s)X(n)^1_{u_k,u_{k+1}} \Big\}.
\end{aligned}
$$

So, by inequalities of the form $|ab - \tilde{a}\tilde{b}| \leq |a-\tilde{a}||b| + |\tilde{a}||b-\tilde{b}|$, we get

$$
\begin{aligned}
&|R^1_0(X,Y)_{s,t} - R^1_0(X(n),Y(n))_{s,t}| \\
&= \|R^1_0(X,Y) - R^1_0(X(n),Y(n))\|_{2\beta;[u_j,u_{j+1}]}(t-u_j)^{2\beta} \\
&\quad + \sum_{k=i+1}^{j-1} \|R^1_0(X,Y) - R^1_0(X(n),Y(n))\|_{2\beta;[u_k,u_{k+1}]}(u_{k+1}-u_k)^{2\beta} \\
&\quad + \|R^1_0(X,Y) - R^1_0(X(n),Y(n))\|_{2\beta;[u_i,u_{i+1}]}(u_{i+1}-s)^{2\beta} \\
&\quad + (\|Y^{(1)} - Y(n)^{(1)}\|_{\beta\text{-Höl};[u_i,u_j]}\|X^1\|_{\beta;[u_j,u_{j+1}]} \\
&\qquad + \|Y(n)^{(1)}\|_{\beta\text{-Höl};[u_i,u_j]}\|X^1 - X(n)^1\|_{\beta;[u_j,u_{j+1}]})(u_j-s)^\beta(t-u_j)^\beta \\
&\quad + \sum_{k=i+1}^{j-1} \Big\{ \|Y^{(1)} - Y(n)^{(1)}\|_{\beta\text{-Höl};[u_i,u_k]}\|X^1\|_{\beta;[u_k,u_{k+1}]} \\
&\qquad + \|Y(n)^{(1)}\|_{\beta\text{-Höl};[u_i,u_k]}\|X^1 - X(n)^1\|_{\beta;[u_k,u_{k+1}]} \Big\}(u_k-s)^\beta(u_{k+1}-u_k)^\beta \\
&\leq (t-s)^{2\beta} \sum_{k=0}^{m-1} \|R^1_0(X,Y) - R^1_0(X(n),Y(n))\|_{2\beta;[u_k,u_{k+1}]} \\
&\quad + (t-s)^{2\beta} M \sum_{k=1}^{m-1} \Big\{ \|R^0_1(X,Y) - R^0_1(X(n),Y(n))\|_{\beta;[0,u_k]} + \|X^1 - X(n)^1\|_{\beta;[u_k,u_{k+1}]} \Big\},
\end{aligned}
$$

where

$$
M := \|R^0_1(X,Y)\|_{\beta;[0,T]} \vee \|X^1\|_{\beta;[0,T]} \vee \sup_{n\geq 1}\{\|R^0_1(X(n),Y(n))\|_{\beta;[0,T]} \vee \|X(n)^1\|_{\beta;[0,T]}\}.
$$

This yields $\lim_{n\to\infty} \|R^1_0(X,Y) - R^1_0(X(n),Y(n))\|_{2\beta;[0,T]} = 0$. Thus the claim of the lemma holds true. $\square$

We are now ready to prove Proposition 4.17.

Proof of Proposition 4.17. By the definition of $X \in G\Omega_\beta(E)$, there exists a sequence of smooth rough paths $\{X(n)\}_{n=1}^\infty \subset S\Omega_\beta(E)$ which converges to X with respect to the distance $d_{\beta,2}$. Then, for each $n = 1, 2, \dots$, there exists a unique element $y(n) \in \mathcal{C}_1^1(F)$ such that

$$y(n)_t = \xi + \int_0^t \varphi(y(n)_u)\, dX(n)^1_{0,u} \quad \text{for all } t \in [0,T].$$

Here, the right-hand side is the Riemann–Stieltjes integral of $\varphi(y(n))$ along $X(n)^1_{0,\cdot}$. This follows from the usual theory of ordinary differential equations; for example, see [21, Theorem 2.3.1]. Set

$$Y(n)^{(0)}_t := y(n)_t \quad \text{and} \quad Y(n)^{(1)}_t := \varphi(y(n)_t) \quad \text{for } t \in [0,T].$$

We first show that $Y(n) := (Y(n)^{(0)}, Y(n)^{(1)})$ belongs to $\mathcal{Q}^{1,2}_{X(n)}(F)$. From the definition of $Y(n)$, we can easily verify that $Y(n)^{(0)}$ and $Y(n)^{(1)}$ are Lipschitz continuous on $[0,T]$. We now prove that $R^1_0(X(n), Y(n)) \in \mathcal{C}_2^2(F)$. Set $(s,t) \in \Delta_T$ with $s < t$. Then,

$$\begin{aligned} R^1_0(X(n), Y(n))_{s,t} &= \int_s^t (\varphi(y(n)_u) - \varphi(y(n)_s))\, dX(n)^1_{0,u} \\ &= \lim_{|\mathcal{P}_{s,t}| \to 0} \sum_{i=0}^{m-1} (\varphi(y(n)_{v_i}) - \varphi(y(n)_s)) X(n)^1_{v_i, v_{i+1}}, \end{aligned}$$

where the limit is taken over all finite partitions $\mathcal{P}_{s,t} = \{v_0, v_1, \dots, v_m\}$ of the interval $[s,t]$ such that $s = v_0 \le v_1 \le \dots \le v_m = t$ and $|\mathcal{P}_{s,t}| := \max_{0 \le i \le m-1} |v_{i+1} - v_i|$. So, we get

$$\begin{aligned} |R^1_0(X(n), Y(n))_{s,t}| &\le \lim_{|\mathcal{P}_{s,t}| \to 0} \sum_{i=0}^{m-1} \|\varphi\|_{1\text{-Höl}} \|y(n)\|_{1\text{-Höl}} (v_i - s) \|X(n)^1\|_1 (v_{i+1} - v_i) \\ &\le \|\varphi\|_{1\text{-Höl}} \|y(n)\|_{1\text{-Höl}} \|X(n)^1\|_1 (t-s)^2. \end{aligned}$$

Thus, $R^1_0(X(n), Y(n)) \in \mathcal{C}_2^2(F)$ and so $Y(n) \in \mathcal{Q}^{1,2}_{X(n)}(F)$. We also note that $Y(n)$ is a solution to the rough differential equation driven by $X(n)$ along φ and starting at ξ. This fact is used frequently in this proof without being explicitly noted. For the claim of the proposition, it remains to prove that

$$\lim_{n \to \infty} d_{X, X(n), \beta}(\hat{Y}, Y(n)) = 0 \tag{4.22}$$

since we already know that $\lim_{n\to\infty} d_{\beta,2}(X(n), X) = 0$, $\hat{Y}^{(0)}_0 = Y(n)^{(0)}_0 = \xi$, $\hat{Y}^{(1)}_0 = Y(n)^{(1)}_0 = \varphi(\xi)$, $Y(n) \in \mathcal{Q}^{1,2}_{X(n)}(F)$, and $\hat{Y} \in \mathcal{Q}^{\beta,2}_X(F)$. For this, we introduce the following symbols:

$$\begin{aligned} \hat{C}_{X^1} &:= 1 + \|X^1\|_{\beta;[0,T]} + \sup_{n \ge 1} \|X(n)^1\|_{\beta;[0,T]}, \\ \hat{C}_X &:= \hat{C}_{X^1} + \|X^2\|_{2\beta;[0,T]} + \sup_{n \ge 1} \|X(n)^2\|_{2\beta;[0,T]}, \end{aligned}$$

and

$$\hat{C}_{\hat{Y}} := 1 + \|\hat{Y}\|_{X,\beta;[0,T]} + \sup_{n \ge 1} \|Y(n)\|_{X(n),\beta;[0,T]}.$$

Here, from Proposition 4.12 with $Y = Y(n)$, we can easily verify that $\sup_{n\geq 1} \|\!|Y(n)|\!\|_{X(n),\beta;[0,T]}$ is finite. Furthermore, we take $T_1 \in (0, T_0]$ such that

$$C\|\nabla\varphi\|_{C^{1,1}}\hat{C}^4_{X^1}\hat{C}_X\hat{C}^2_{\hat{Y}}T_1^\beta = \kappa_1 < 1, \tag{4.23}$$

where constant C is the same as in Eq. (4.9). We define $N_1 := \lceil T/T_1 \rceil$ and $u_i := \min\{iT_1, T\}$ for $i = 0, 1, \dots, N_1$. We arrange $\{t_i\}_{i=0}^{N_0} \cup \{u_i\}_{i=0}^{N_1}$ in the ascending order and denote it by $\{s_k\}_{k=0}^{N_*}$, namely, $0 = s_0 < s_1 < \cdots < s_{N_*-1} < s_{N_*} = T$. From Lemma 4.18, Eq. (4.22) is proven if we show that

$$\lim_{n\to\infty} \|\!|R_l^{1-l}(X, \hat{Y}) - R_l^{1-l}(X(n), Y(n))|\!\|_{(2-l)\beta;[s_k,s_{k+1}]} = 0 \tag{4.24}$$

holds for each $l = 0, 1$ and $k = 0, 1, \dots, N_* - 1$. We will prove Eq. (4.24) by induction on k. Set $(s, t) \in \Delta_T$ with $0 = s_0 \leq s < t \leq s_1$. Then, for each $l = 0, 1$,

$$\begin{aligned}
&\|\!|R_l^{1-l}(X, \hat{Y})_{s,t} - R_l^{1-l}(X(n), Y(n))|\!\|_{(2-l)\beta;[0,s_1]} \\
&= \|\!|R_l^{1-l}(X, I(X, \varphi(\hat{Y}))) - R_l^{1-l}(X(n), I(X(n), \varphi(Y(n))))|\!\|_{(2-l)\beta;[0,s_1]} \\
&\qquad \text{(since } \hat{Y} \text{ is the solution on } [0, T_0]) \\
&\leq d_{X,X(n),\beta}(I(X, \varphi(\hat{Y})), I(X(n), \varphi(Y(n)))) \\
&\leq (1 - \kappa_1)^{-1}C\|\nabla\varphi\|_{C^{1,1}}\big\{\hat{C}^3_{X^1}\hat{C}_X\hat{C}^3_{\hat{Y}}\big(|\hat{Y}^{(0)}_{s_k} - Y(n)^{(0)}_{s_k}| \\
&\quad + \|\!|X^1 - X(n)^1|\!\|_{\beta;[0,s_1]} + \|\!|X^2 - X(n)^2|\!\|_{2\beta;[0,s_1]}\big) + \hat{C}^4_{X^1}\hat{C}_X\hat{C}^2_{\hat{Y}}|\hat{Y}^{(1)}_0 - Y(n)^{(1)}_0|\big\} \\
&\qquad \text{(from (4.9) and (4.23))} \\
&= (1 - \kappa_1)^{-1}C\|\nabla\varphi\|_{C^{1,1}}\hat{C}^3_{X^1}\hat{C}_X\hat{C}^3_{\hat{Y}}(\|\!|X^1 - X(n)^1|\!\|_{\beta;[0,s_1]} + \|\!|X^2 - X(n)^2|\!\|_{2\beta;[0,s_1]}) \to 0 \\
&\qquad \text{(from } \hat{Y}^{(0)}_0 = Y(n)^{(0)}_0 \text{ and } \hat{Y}^{(1)}_0 = Y(n)^{(1)}_0)
\end{aligned}$$

as n tends to infinity. Hence, Eq. (4.24) holds for $k = 0$. Suppose that Eq. (4.24) holds for each $k = 0, 1, \dots, K$ with $0 \leq K \leq N_* - 2$. Set $(s, t) \in \Delta_T$ with $s_{K+1} \leq s < t \leq s_{K+2}$. By the definition of $\{s_k\}_{k=0}^{N_*}$, there exists $i = 0, 1, \dots, N_0 - 1$ such that $t_i \leq s_{K+1} \leq s < t \leq s_{K+2} \leq t_{i+1}$. Then, for each $l = 0, 1$ and $u \in [0, s_{K+2} - s_{K+1}]$,

$$\theta_{s_{K+1}}(\hat{Y}^{(l)})_u = \theta_{s_{K+1}-t_i}(\theta_{t_i}(\hat{Y}^{(l)}))_u = \theta_{s_{K+1}-t_i}(S_{\varphi,\xi_i}(\theta_{t_i}(X))^{(l)})_u.$$

Since $S_{\varphi,\xi_i}(\theta_{t_i}(X))$ is the solution on $[0, t_{i+1} - t_i]$ driven by $\theta_{t_i}(X)$ and starting at $\xi_i = \hat{Y}^{(0)}_{t_i}$, we see from Lemma 4.15 that $\theta_{s_{K+1}-t_i}(S_{\varphi,\xi_i}(\theta_{t_i}(X)))$ is the solution on $[0, (t_{i+1} - t_i) - (s_{K+1} - t_i)] = [0, t_{i+1} - s_{K+1}]$ driven by $\theta_{s_{K+1}-t_i}(\theta_{t_i}(X)) = \theta_{s_{K+1}}(X)$ starting at $\theta_{s_{K+1}-t_i}(\hat{Y}^{(0)})_{t_i} = \hat{Y}^{(0)}_{s_{K+1}}$. In particular, $\theta_{s_{K+1}}(\hat{Y})$ is the solution on $[0, s_{K+2} - s_{K+1}] \subset [0, t_{i+1} - s_{K+1}]$ driven by $\theta_{s_{K+1}}(X)$ starting at $\hat{Y}^{(0)}_{s_{K+1}}$. Letting

$$Z_{s_{K+1}} := I(\theta_{s_{K+1}}(X), \varphi(\theta_{s_{K+1}}(\hat{Y})) \quad \text{and} \quad Z_{s_{K+1}}(n) := I(\theta_{s_{K+1}}(X(n)), \varphi(\theta_{s_{K+1}}(Y(n))),$$

we have

$$\begin{aligned}
&R_l^{1-l}(X,\hat{Y})_{s,t} - R_l^{1-l}(X(n),Y(n))_{s,t}\\
&= R_l^{1-l}(\theta_{s_{K+1}}(X),\theta_{s_{K+1}}(\hat{Y}))_{s-s_{K+1},t-s_{K+1}} - R_l^{1-l}(\theta_{s_{K+1}}(X(n)),\theta_{s_{K+1}}(Y(n)))_{s-s_{K+1},t-s_{K+1}}\\
&\qquad \text{(from the definition of } \theta_{s_{K+1}})\\
&= R_l^{1-l}(\theta_{s_{K+1}}(X),Z_{s_{K+1}})_{s-s_{K+1},t-s_{K+1}} - R_l^{1-l}(\theta_{s_{K+1}}(X(n)),Z_{s_{K+1}}(n))_{s-s_{K+1},t-s_{K+1}}\\
&\qquad \text{(since } \theta_{s_{K+1}}(\hat{Y}) \text{ is the solution on } [0,s_{K+2}-s_{K+1}])
\end{aligned}$$

for each $l=0,1$. Hence, using Eqs. (4.9) and (4.23), we get

$$\begin{aligned}
&\|R_l^{1-l}(X,\hat{Y})_{s,t} - R_l^{1-l}(X(n),Y(n))\|_{(2-l)\beta;[s_{K+1},s_{K+2}]}\\
&= \|R_l^{1-l}(\theta_{s_{K+1}}(X),Z_{s_{K+1}}) - R_l^{1-l}(\theta_{s_{K+1}}(X),Z_{s_{K+1}}(n))\|_{(2-l)\beta;[0,s_{K+2}-s_{K+1}]}\\
&\le d_{\theta_{s_{K+1}}(X),\theta_{s_{K+1}}(X(n)),\beta}(Z_{s_{K+1}},Z_{s_{K+1}}(n))\\
&\le (1-\kappa_1)^{-1}C\|\nabla\varphi\|_{C^{1,1}}\{\hat{C}_{X^1}^3\hat{C}_X\hat{C}_{\hat{Y}}^3\big(|\hat{Y}^{(0)}_{s_{K+1}} - Y(n)^{(0)}_{s_{K+1}}|\\
&\quad + \|\theta_{s_{K+1}}(X^1)-\theta_{s_{K+1}}(X(n)^1)\|_{\beta;[0,s_{K+2}-s_{K+1}]} + \|\theta_{s_{K+1}}(X^2)-\theta_{s_{K+1}}(X(n)^2)\|_{2\beta;[0,s_{K+2}-s_{K+1}]}\big)\\
&\quad + \hat{C}_{X^1}^4\hat{C}_X\hat{C}_{\hat{Y}}^2|\hat{Y}^{(1)}_{s_{K+1}} - Y(n)^{(1)}_{s_{K+1}}|\}.
\end{aligned}$$

Then, for the proof of Eq. (4.24) with $k=K+1$, it suffices to prove that $\lim_{n\to\infty}|\hat{Y}^{(l)}_{s_{K+1}} - Y(n)^{(l)}_{s_{K+1}}| = 0$ for each $l=0,1$. First of all, from the induction hypothesis and Lemma 4.18 with $Y=\hat{Y}$, $u_k=s_k$, and $T=s_{K+1}$, we already know that

$$\lim_{n\to\infty}\|R_l^{1-l}(X,\hat{Y}) - R_l^{1-l}(X(n),Y(n))\|_{(2-l)\beta;[0,s_{K+1}]} = 0 \tag{4.25}$$

holds for each $l=0,1$. Furthermore, from $\hat{Y}_0^{(0)} = Y(n)_0^{(0)} = \xi$ and $\hat{Y}_0^{(1)} = Y(n)_0^{(1)} = \varphi(\xi)$,

$$\begin{aligned}
&|\hat{Y}^{(0)}_{s_{K+1}} - Y(n)^{(0)}_{s_{K+1}}|\\
&= |(\hat{Y}^{(0)}_{s_{K+1}} - \hat{Y}_0^{(0)}) - (Y(n)^{(0)}_{s_{K+1}} - Y(n)_0^{(0)})|\\
&= |(\hat{Y}_0^{(1)}X^1_{0,s_{K+1}} - Y(n)_0^{(1)}X(n)^1_{0,s_{K+1}}) + (R_0^1(X,\hat{Y})_{0,s_{K+1}} - R_0^1(X(n),Y(n))_{0,s_{K+1}})|\\
&\le |\varphi(\xi)|\|X^1 - X(n)^1\|_{\beta;[0,s_{K+1}]}s_{K+1}^{\beta} + \|R_0^1(X,\hat{Y}) - R_0^1(X(n),Y(n))\|_{2\beta;[0,s_{K+1}]}s_{K+1}^{2\beta}
\end{aligned}$$

and

$$\begin{aligned}
|\hat{Y}^{(1)}_{s_{K+1}} - Y(n)^{(1)}_{s_{K+1}}| &= |(\hat{Y}^{(1)}_{s_{K+1}} - \hat{Y}_0^{(1)}) - (Y(n)^{(1)}_{s_{K+1}} - Y(n)_0^{(1)})|\\
&= |R_1^0(X,\hat{Y})_{0,s_{K+1}} - R_1^0(X(n),Y(n))_{0,s_{K+1}}|\\
&\le \|R_1^0(X,\hat{Y}) - R_1^0(X(n),Y(n))\|_{\beta;[0,s_{K+1}]}s_{K+1}^{\beta}.
\end{aligned}$$

Then, from Eq. (4.25), we have $\lim_{n\to\infty}|\hat{Y}^{(l)}_{s_{K+1}} - Y(n)^{(l)}_{s_{K+1}}| = 0$ for each $l=0,1$. Thus, Eq. (4.24) holds for $k=K+1$. Consequently, we obtain the claim of the proposition. □

Remark 4.19. The above proof requires the result of global existence of solutions $y(n)$ using the basic theory of ordinary differential equations. It is uncertain whether there are more direct proofs of Proposition 4.17 without using the approximate solutions $Y(n) = (y(n),\varphi(y(n)))$ of $\hat{Y}$.

References

[1] M. Besalú, D. Márquez-Carreras, and C. Rovira, Delay equations with non-negativity constraints driven by a Hölder continuous function of order $\beta \in (\frac{1}{3}, \frac{1}{2})$, *Potential Anal.* **41**, no. 1, (2014), 117–141.

[2] M. Besalú and D. Nualart, Estimates for the solution to stochastic differential equations driven by a fractional Brownian motion with Hurst parameter $H \in (\frac{1}{3}, \frac{1}{2})$, *Stochastic and Dynamics* **11** (2011), 243–263.

[3] M. Besalú and C. Rovira, Stochastic delay equations with non-negativity constraints driven by fractional Brownian motion, *Bernoulli* **18**, no. 1, (2012), 24–45.

[4] A. M. Davie, Differential equations driven by rough paths: an approach via discrete approximation, *Appl. Math. Res. Express. AMRX* **2008**, no. 2, (2008), 1–40.

[5] D. Feyel and A. de La Pradelle, Curvilinear integrals along enriched paths, *Electron. J. Probab.* **11**, no. 34, (2006), 860–892.

[6] P. K. Friz and M. Hairer, *A course on rough paths, with an introduction to regularity structures*, Universitext, Springer, 2014.

[7] P. K. Friz and N. Victoir, Euler estimates for rough differential equations, *J. Differential Equations* **244**, no. 2, (2008), 388–412.

[8] P. K. Friz and N. V. Victoir, *Multidimensional stochastic processes as rough paths, theory and applications*, Cambridge Studies in Advanced Mathematics **120**, Cambridge University Press, Cambridge, 2010.

[9] M. Gubinelli, Controlling rough paths, *J. Funct. Anal.* **216** (2004), 86–140.

[10] M. Gubinelli, Ramification of rough paths, *J. Differential Equations* **248**, no. 4, (2010), 693–721.

[11] M. Gubinelli, P. Imkeller, and N. Perkowski, A Fourier approach to pathwise stochastic integration, preprint, 2014.

[12] Y. Hu and D. Nualart, Differential equations driven by Hölder continuous functions of order greater than 1/2, *Stochastic Analysis and Applications*, Abel Symp., 2, Springer, Berlin (2007), 399–413.

[13] Y. Hu and D. Nualart, Rough path analysis via fractional calculus, *Trans. Amer. Math. Soc.* **361** (2009), 2689–2718.

[14] Y. Ito, Integrals along rough paths via fractional calculus, *Potential Anal.* **42**, no. 1, (2015), 155–174.

[15] Y. Ito, Extension theorem for rough paths via fractional calculus, preprint, 2014.

[16] Y. Ito, Differential equations driven by rough paths: an approach via fractional calculus, in preparation.

[17] A. Lejay, *An introduction to rough paths*, Séminaire de Probabilités XXXVII, 1–59, Lecture Notes in Math. **1832**, Springer, Berlin, 2003.

[18] A. Lejay, *Yet another introduction to rough paths*, Séminaire de Probabilités XLII, 1–101, Lecture Notes in Math. **1979**, Springer, Berlin, 2009.

[19] T. J. Lyons, Differential equations driven by rough signals, *Rev. Math. Iberoamericana* **14** (1998), 215–310.

[20] T. J. Lyons, M. J. Caruana, and T. Lévy, *Differential equations driven by rough paths*, Lecture Notes in Math. **1908**, Springer, Berlin, 2007. Lectures from the 34th Summer School on Probability Theory held in Saint-Flour, July 6–24, 2004. With an introduction concerning the Summer School by Jean Picard.

[21] T. J. Lyons and Z. Qian, *System control and rough paths*, Oxford Mathematical Monographs, Oxford Science Publications, Oxford University Press, Oxford, 2002.

[22] D. Nualart and A. Răşcanu, Differential equations driven by fractional Brownian motion, *Collect. Math.* **53** (2001), 55–81.

[23] S. G. Samko, A. A. Kilbas, and O. I. Marichev, *Fractional integrals and derivatives, theory and applications*, Gordon and Breach Science Publishers, Yvendon, 1993.

[24] L. C. Young, An inequality of Hölder type connected with Stieltjes integration, *Acta Math.* **67** (1936), 251–282.

[25] M. Zähle, Integration with respect to fractal functions and stochastic calculus. I, *Probab. Theory Related Fields* **111** (1998), 333–374.