On the crispness of ω and arithmetic with a bisimulation in a constructive naive set theory

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Abstract

We show that the crispness of ω is not provable in a constructive naive set theory **CONS** in $\mathbf{FL}_{ew}\forall$, intuitionistic predicate logic minus the contraction rule. In the proof, we construct a circularly defined object **fix**, a fixed point of the successor function **suc**, by using a fixed-point theorem.

1 Introduction

Without the contraction rule, the comprehension principle does not imply a contradiction on substructural logics. This is well known. A significant use of these naive set theories is proving a fixed-point theorem for a *general recursive definition*: the comprehension principle implies that for any formula $\varphi(x, \cdots, y)$, there is a term θ within many substructural logics such that

$$(\forall x)[x \in \theta \equiv \varphi(x, \cdots, \theta)].$$

See, for example, Cantini [C03] or Terui [Tr04]. This fact allows us to define the set of natural numbers ω seemingly inductively as follows:

$$(\forall x)[x \in \omega \equiv [x = \bar{0} \vee (\exists y)[y \in \omega \otimes x = \mathbf{suc}(y)]]]$$

where $\bar{0}$ is a set representing 0 and **suc** is a successor function (for more details, see section 2.2). To know how much arithmetic can be developed in these naive set theories is to know the limit of the power of general recursive definitions.

However, it is not known whether ω is a crisp set. That is, whether *tertium non datur*—$(\forall x)[(x \in \omega) \vee (x \notin \omega)]$—holds for ω. This is an important problem because it concerns the nature of sets arising from a general recursive definition. In this paper we give a partial answer to the question of crispness. The framework for this paper is *Constructive Naive Set Theory* **CONS**, a naive set theory within the full Lambek predicate calculus with exchange and weakening rules $\mathbf{FL}_{ew}\forall$ (intuitionistic predicate logic minus the contraction rule); **CONS** is very *constructive*. We prove the following.

THEOREM 1.1
CONS does not prove the crispness of ω.

In the proof, a circularly defined object **fix**, finitely generated and potentially infinite, plays a key role. This theorem means that, quite contrary to classical theories, the distinction between finiteness and non-finiteness in **CONS** is indefinite because of the existence of potentially infinite objects.

The structure of this paper is as follows. We introduce $\mathbf{FL}_{ew}\forall$ and **CONS** in section 2. Next, we introduce a secondary motivation of this paper in section 3: to stress how difficult it is to apply Leibniz equality to potentially infinite objects. The difficulty in proving crispness seems to be due to this. Therefore, in section 3.1 we define a bisimulation relation $\sim$ which is easy to handle in the spirit of [BM96] and define $\sim$-equivalent classes $\tilde{\omega}$ in section 3.2. We prove theorem 1.1 in section 4. We begin by proving an analogue of theorem 1.1 for $\tilde{\omega}$ in section 4.1: we construct a non-terminating automaton **fix** as a fixed point of **suc** with respect to $\sim$, that is, $\mathbf{fix} \sim \mathbf{suc}(\mathbf{fix})$, and prove that **fix** is a counterexample of *tertium non datur* for $\tilde{\omega}$. We then modify **fix** to prove the theorem: we *unfold* **fix** to contradict provability of *tertium non datur* for ω. Section 4.2 contains this proof.

2 Preliminaries

2.1 A constructive naive set theory

Our framework in this paper is full Lambek predicate calculus with exchange (e) and weakening (w) $\mathbf{FL}_{ew}\forall$ (intuitionistic predicate logic minus the contraction rule). Let s, t be arbitrary terms and let Γ, Σ be finite multisets (possibly empty) of formulae. Brackets of the form $[\alpha]$ in the right-hand side of sequents are so called $\in$-*levels*: these are used to prove theorem 2.3.

Definition 2.1
$\mathbf{FL}_{ew}\forall$ consists of the following rules.

$$t \in s \vdash t \in s \quad [0] \qquad \bot \vdash \quad [0]$$

$$\frac{\Gamma \vdash A \quad [\alpha] \quad A, \Sigma \vdash B \quad [\beta]}{\Gamma, \Sigma \vdash B \quad [\alpha + \beta]} \, cut$$

Structural rules:

$$\frac{\Gamma, B, A, \Sigma \vdash C \quad [\alpha]}{\Gamma, A, B, \Sigma \vdash C \quad [\alpha]} \, e, \qquad \frac{\Gamma \vdash C \quad [\alpha]}{\Gamma, A, \Sigma \vdash C \quad [\alpha]} \, w$$

Implication:

$$\frac{\Gamma \vdash A \quad [\alpha] \quad B, \Sigma \vdash C \quad [\beta]}{A \to B, \Gamma, \Sigma \vdash C \quad [\alpha + \beta]}, \qquad \frac{\Gamma, A \vdash B \quad [\alpha]}{\Gamma \vdash A \to B \quad [\alpha]}$$

Multiplicative connectives (fusion):

$$\frac{\Gamma, A, B, \Sigma \vdash C \quad [\alpha]}{\Gamma, A \otimes B, \Sigma \vdash C \quad [\alpha]}, \qquad \frac{\Gamma \vdash A \quad [\alpha] \quad \Sigma \vdash B \quad [\beta]}{\Gamma, \Sigma \vdash A \otimes B \quad [\alpha + \beta]}$$

Additive connectives:

$$\frac{\Gamma, A_i, \Sigma \vdash C[\alpha]}{\Gamma, A_1 \wedge A_2, \Sigma \vdash C \quad [\alpha]}, \qquad \frac{\Gamma \vdash A \quad [\alpha] \quad \Gamma \vdash B \quad [\beta]}{\Gamma \vdash A \wedge B \quad [\alpha + \beta]}$$

$$\frac{\Gamma, A, \Sigma \vdash C \quad [\alpha] \quad \Gamma, B, \Sigma \vdash C \quad [\beta]}{\Gamma, A \vee B, \Sigma \vdash C \quad [\alpha + \beta]} \qquad \frac{\Gamma \vdash A_i \quad [\alpha]}{, \Gamma \vdash A_1 \vee A_2 \quad [\alpha]}$$

Quantifiers (a is a variable not free in $\Gamma \vdash \forall x A$ and $\Gamma, \exists x A \vdash B$):

$$\frac{\Gamma, A[x := s] \vdash B \quad [\alpha]}{\Gamma, \forall x A \vdash B \quad [\alpha]}, \qquad \frac{\Gamma \vdash A[x := a] \quad [\alpha]}{\Gamma \vdash \forall x A \quad [\alpha]}, \qquad \frac{\Gamma, A[x := a] \vdash B \quad [\alpha]}{\Gamma, \exists x A \vdash B \quad [\alpha]}, \qquad \frac{\Gamma \vdash A[x := s] \quad [\alpha]}{\Gamma \vdash \exists x A \quad [\alpha]}$$

Here the negation $\neg A$ is an abbreviation of $A \to \bot$. We note that **BCK**-logic is a $\to$-fragmentof $\mathbf{FL}_{ew}\forall$ from the first 5 rules and the 2 implication rules.

$\mathbf{FL}_{ew}\forall$ is very *constructive* in the sense that $\mathbf{FL}_{ew}\forall$ satisfies the disjunction property and the existence property. This is crucial for the proof of the main theorem.

Next we introduce a particular naive set theory.

Definition 2.2
Let **CONS** be a set theory within $\mathbf{FL}_{ew}\forall$ with a binary predicate $\in$, terms of the form $\{x : \varphi(x)\}$, and the following two $\in$-rules:

$$\frac{A[x := s], \Gamma \vdash B \quad [\alpha]}{s \in \{x : A\}, \Gamma \vdash B \quad [\alpha + 1]}, \qquad \frac{\Gamma \vdash A[x := a] \quad [\alpha]}{\Gamma \vdash a \in \{x : A\} \quad [\alpha + 1]}.$$

We note that **CONS** is similar to Uwe Petersen's $\mathbf{L}^i\mathbf{D}_\lambda$ [P00]. $\mathbf{L}^i\mathbf{D}_\lambda$ has only one connective $\to$ (its introduction/elimination rule is the same as ours); other connectives are simulated by using sets. Kazushige Terui's **LAST** [Tr04] is similar to, but more complex than, $\mathbf{L}^i\mathbf{D}_\lambda$. **LAST** has only one connective, linear implication, and only one quantifier, $\forall$. Other connectives are simulated by implication, $\forall$, and set terms. **LAST** also has the modal operators ! and §: $!A$ allows the contraction rule to be applied to $!A$, and this makes **LAST** stronger than $\mathbf{L}^i\mathbf{D}_\lambda$. Comparing the theories, Cantini's **GL** is closer to **CONS**. **CONS** is a subsystem of **GL**, which is itself a naive set theory in Grisin logic (classical logic minus the contraction rule). The main difference between **GL** and **CONS** is that **GL** is a multiple conclusion logic (so in this sense, it is classical) and **CONS** is a single conclusion logic.

It is easy to see that the cut elimination theorem is provable in **CONS**.

Theorem 2.3 (Cut elimination)
If A is provable in **CONS**, then it has a cut-free proof in **CONS**.

The proof of this is essentially the one given in [C03]: since **CONS** is "highly selfreferential and impredicative, it is not possible to eliminate cuts by progressively decreasing the complexity of the cut formulas", but "*lack of contraction* still allows to apply a standard elimination procedure, by use of $\in$-level". Actually, the cut elimination here is easier than for classical or intuitionistic logic. Let us state the following essential lemma:

Lemma 2.4
For any deduction with cut

$$\frac{\dfrac{\mathcal{S}_1}{\Gamma_1 \vdash \varphi_1 \quad [\alpha]} \quad \dfrac{\mathcal{S}_2}{\Gamma_2, \varphi_1 \vdash \varphi \quad [\beta]}}{\Gamma_1, \Gamma_2 \vdash \varphi \quad [\alpha + \beta]}$$

there is a cut-free deduction of $\Gamma_1, \Gamma_2 \vdash \varphi$ whose $\in$-level is $\leq \alpha + \beta$.

Proof. We briefly sketch the proof. The proof uses triple induction: the main induction is on the sum of $\in$-levels, and sub-inductions are on the logical complexity of the cut formula and on the sum of the height of $\mathcal{S}_1$ and $\mathcal{S}_2$. For example, let us consider the following case:

$$\frac{\Gamma_1 \vdash t \in \{x : A\} \quad [\alpha+1] \quad \Gamma_2, t \in \{x : A\} \vdash B \quad [\beta+1]}{\Gamma_1, \Gamma_2 \vdash B \quad [\alpha+\beta+2]}.$$

This can be written as follows:

$$\frac{\Gamma_1 \vdash A[x := t] \quad [\alpha] \quad \Gamma_2, A[x := t] \vdash B \quad [\beta]}{\Gamma_1, \Gamma_2 \vdash B \quad [\alpha+\beta]}.$$

Here, the complexity of the cut formula is increased, but its $\in$-level is decreased. The induction hypothesis shows that we can rewrite this to a cut-free proof. We note that this proof is simpler than that of **GL**: we do not have to consider cases where $C(\in, \rightarrow)$ in **GL** because **CONS** is a single conclusion calculus. $\Box$

This cut elimination procedure makes proofs normal and that this proves the normalization theorem. Let us compare non-normal proofs and normal proofs. For non-normal proofs, the proof might have detours and the connectives and set terms, which have been introduced already, might disappear as the proof proceeds, that is, as the $\in$-level increases. For normal proofs, in contrast, there are no detours and the number of the nested boxes never decreases when $\in$-levels increase:

$$\frac{\dfrac{\vdash s \in t \quad [\alpha]}{\vdash s \in \{x_0 : x_0 \in t\} \quad [\alpha+1]}}{\vdash s \in \{x_1 : x_1 \in \{x_0 : x_0 \in t\}\} \quad [\alpha+2]}.$$

Here, set terms, $\{x_0 : x_0 \in t\}$ etc. are boxes, and applying the $\in$-introduction rule to a formula introduces a new box. Logical connectives work similarly: once the connective is introduced, it persists unless it is put in the new box. Conversely, the back calculation to construct the proof from the consequence is very easy for normal proofs: we can estimate what introduction rule was applied in the previous deduction step by examining the logical connectives of the consequence (this plays a very important role in the proof of lemma 4.2).

Next, we define many standard set-theoretic relations.

Definition 2.5
- Leibniz equality: $a = b \equiv (\forall z)[a \in z \equiv b \in z]$,
- Extensional equality: $a =_{ext} b \equiv (\forall z)[z \in a \equiv z \in b]$.

Lemma 2.6
- $\vdash (\forall x)[x = x]$,
- $s = t, \varphi(s) \vdash \varphi(t)$ for any term s, t,
- $\vdash (\forall x, y)[x = y \rightarrow x =_{ext} y]$.

Definition 2.7
- The empty set: $\emptyset = \{x : \bot\}$,
- The universal set: $\mathbf{V} = \{x : x = x\}$.

V is a set in **CONS**, and this makes **CONS** very different from **ZFC**. Similarly, we can define the following relations [C03]:

Definition 2.8
For any term s, t,

- singleton: $\{s\} = \{x : x = s\}$,
- pair: $\{s, t\} = \{x : x = s \vee x = t\}$,
- ordered pair: $\langle s, t\rangle = \{s, \{t\}\}$.

We note that writing an ordered n-tuples in the form $\langle a_0, \cdots, a_{n-1}\rangle$ is an abbreviation for the iteration of ordered pairs such as $\langle a_0, a_1, a_2\rangle = \langle a_0, \langle a_1, a_2\rangle\rangle$.

Definition 2.9

$$\mathbf{dom}(f) = \{y : (\exists x)\langle x, y\rangle \in f\}$$

Finally, we introduce the crispness[1].

Definition 2.10
- A set X is crisp iff $\vdash (\forall x)[x \in X \vee x \notin X]$,
- A relation R is crisp iff $\vdash (\forall x, y)[xRy \vee \neg xRy]$.

2.2 The fixed point theorem and arithmetic

As we saw, one of the most important properties of **CONS** is that it allows the fixed point theorem, or the *general recursion form* [C03][P00]: we can define a set z which is defined by using z itself (strictly speaking, we can construct a term θ such that $\theta =_{\text{ext}} \{u : \varphi(u, \cdots, \theta)\}$) by just a diagonalization argument.

Theorem 2.11 (The fixed point theorem)
For any formula $\varphi(x, \cdots, y)$,

$$\vdash (\exists z)(\forall x)[x \in z \equiv \varphi(x, \cdots, z)]$$

Proof. We construct a term θ such that $\theta =_{\text{ext}} \{u : \varphi(u, \cdots, \theta)\}$. Before that, we introduce the sketch of the proof of the following lemma along the line of [C03].

Lemma 2.12
For any relation f, there is a term I_f such that $f^{-1}(I_f) =_{ext} I_f$ where $f^{-1}(a) = \{x : \langle x, a\rangle \in f\}$.

Proof. Fix any f.

Let D_f be such that $D_f = \{z : (\exists x, g)[z = \langle x, g\rangle \otimes x \in f^{-1}(g^{-1}(g))]\}$ and $I_f = D_f^{-1}(D_f)$. Then the following chain of equivalence is provable in **CONS**:

$$\begin{aligned} x \in I_f &\equiv \langle x, D_f\rangle \in D_f \\ &\equiv x \in f^{-1}(D_f^{-1}(D_f)) \\ &\equiv x \in f^{-1}(I_f). \end{aligned}$$

[1] There are different definitions of crispness. The definition Hajek uses in [H05] is that ω is crisp if $t \in \omega \rightarrow (t \in \omega) \otimes (t \in \omega)$ for any t (we can apply contraction-like rule to ω). It is easy to see that tertium non datur implies this in Łukasiewicz logic.

$\Box$

Let $f_\varphi = \{z : (\exists x, y)[z = \langle x, y\rangle \otimes \varphi(x, \cdots, y)]\}$. It is enough to apply lemma 2.12 to f_φ, from which $\theta = I_{f_\varphi}$. $\Box$

Next, we develop arithmetic by using theorem 2.11. We note that many definitions definition of $\bar{0}$ and $\mathbf{suc}(y)$ are possible; $\bar{0}$ is defined as $\emptyset = \{x : \bot\}$ and $\mathbf{suc}(y)$ is defined as $\{y\}$ in Zermelo style, for example, in [C03]. For any natural number n in the metalanguage, let us write the corresponding *numeral* $\bar{n}$ which has the form $\underbrace{\mathbf{suc}(\mathbf{suc}(\cdots \mathbf{suc}(}_{n \text{ times}}\bar{0})\cdots))$. The set of natural numbers, ω, is defined as in section 1. Similarly, we can define arithmetical operations as relations.

Definition 2.13
Addition **Plus** is defined as follows: a tuple $\langle x, y, z\rangle$ is in **Plus** iff the additive disjunction of the following clauses holds.

- $\langle x, \bar{0}, x\rangle \in \mathbf{Plus}$,
- $\langle x, \mathbf{suc}(y), \mathbf{suc}(z)\rangle \in \mathbf{Plus}$ if $\langle x, y, z\rangle \in \mathbf{Plus}$.

Formally:

$$\begin{aligned}\mathbf{Plus} \;=\; & \{v : (\exists x, y, z)[v = \langle x, y, z\rangle \otimes \\ & \quad [(y = \bar{0} \otimes z = x) \vee (\exists y', z')[\langle x, y', z'\rangle \in \mathbf{Plus} \otimes y = \mathbf{suc}(y') \otimes z = \mathbf{suc}(z')]]]\}\end{aligned}$$

We can define multiplication **Times** as follows:

$$\begin{aligned}\mathbf{Times} \;=\; & \{v : (\exists x, y, z \in \omega)[v = \langle x, y, z\rangle \otimes [(y = \bar{0} \otimes z = \bar{0}) \\ & \quad \vee(\exists y', z')[\langle x, y', z'\rangle \in \mathbf{Times} \otimes y = \mathbf{suc}(y') \otimes \mathbf{Plus}(z', x, z)]]]\}\end{aligned}$$

However, it is difficult to show whether we can define **Plus** as a function: in other words, is there a unique z for any $x, y \in \omega$ such that $\langle x, y, z\rangle \in \mathbf{Plus}$. Petr Hájek showed the following in a naive set theory in Łukasiewicz infinite valued predicate logic [H05]:

- mathematical induction on ω implies the crispness of ω and that **Plus** defines a function and
- mathematical induction implies a contradiction.

Since then, it has been an open question whether **plus** can be a function.

Lastly, we show that the fixed point theorem, which implies fixed points for all monotone operators. For example,

- power set operator $\mathcal{P}$: $X =_{\text{ext}} \mathcal{P}X$ for $X =_{ext} \{x : (\forall y)[y \in x \to y \in X] \to x \in X\}$,
- stream operator ${}^\infty$: ${}^\infty A =_{ext} A$ for $A =_{ext} \{x : (\exists y, z \in A)x = \langle y, z\rangle\}$,
- successor operator **Suc**: $\mathbf{Suc}\,\omega =_{ext} \omega$ for $\omega =_{ext} \{x : x \in \omega \vee (\exists y \in \omega)[x = \mathbf{suc}(y)]\}$.

However we do not know the *size* of these points: we do not know whether they are the *least* fixed points or the *largest* fixed points or even something else. We will discuss this in section 4.2 again.

3 Non-extensionality and bisimulation

In this section, we introduce a relatively new concept, *bisimulation*, which makes it easy to develop arithmetic in **CONS**. A secondary motive for this paper is to stress how difficult Leibniz equality is to handle and introduce a suitable alternative. The root of the difficulty seems to be the potentially infinite character of **CONS**; this character will be introduced in the proof of theorem 1.1. It is difficult to define the identity criteria for potential infinite objects as **fix**. Therefore we define a bisimulation relation, a very natural identity criteria for circularly defined objects, in section 3.1. After defining the bisimulation relation we define its equivalent classes $\tilde{\omega}$ and develop its arithmetic in section 3.2.

3.1 Bisimulation

In this section, let us see how difficult Leibniz equality is to handle and define the bisimulation relation $\sim$ to surmount these difficulties.

Identity is a key to developing mathematics if identity is not decidable. Leibniz equality, $a = b \equiv (\forall z)[a \in z \equiv b \in z]$, has been widely used in the study of naive set theories. One of the most important properties of Leibniz equality is that it is *contractive*.

Lemma 3.1
CONS proves that Leibniz equality is contractive. For any s, t,

$$\vdash s = t \to (s = t) \otimes (s = t).$$

For our aim, $=$ is too strict to use because of its contractiveness. For example, it is well-known that Leibniz equality is different from extensional equality, given by $a =_{ext} b \equiv (\forall z)[z \in a \equiv z \in b]$. The axiom of extensionality implies a contradiction in many naive set theories [G82]. Leibniz equality acts more like syntactical identity.

Lemma 3.2 (The literal identity property)
If **CONS** proves $t = u$ then t and u are syntactically identical.

Proof. The proof in [C03] carries to **CONS**. For simplicity, we consider the proof of $(\forall z)[t \in z \to u \in z]$. Its normal proof must be of the following form:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ t \in z \vdash u \in z\end{array}}{\vdash t \in z \to u \in z}}{\vdash (\forall z)[t \in z \to u \in z]}$$

However, $t \in z \vdash u \in z$ is provable only if t is syntactically equal to u. $\Box$

We note that Terui proved a similar theorem for **LAST** [Tr04]. This syntactical quality seems to prevent a straightforward arithmetic. For example, the fixed point theorem only proves the existence of θ; it does not guarantee the uniqueness of such a term. Therefore, we cannot prove the uniqueness of sets, such as ω, defined by the fixed point theorem. Thus, if we use $=$, we cannot take full advantage of the power of general recursive forms.

What about extensional equality $=_{ext}$ instead? Even $=_{ext}$ is not sufficient to develop arithmetic: it is still too strict. For example, even if $a =_{ext} \bar{0}$, we still have $\{a\} \neq_{ext} \bar{1}$ when $a \neq \bar{0}$. This means that the two series,

- $\bar{0}, \mathbf{suc}(\bar{0}), \mathbf{suc}(\mathbf{suc}(\bar{0})), \mathbf{suc}(\mathbf{suc}(\mathbf{suc}(\bar{0}))), \cdots$
- $a, \mathbf{suc}(a), \mathbf{suc}(\mathbf{suc}(a)), \mathbf{suc}(\mathbf{suc}(\mathbf{suc}(a))), \cdots$

differ completely with respect to $=_{ext}$. Therefore $=_{ext}$ is not sufficient to develop arithmetic.

We define an identity relation in the naive set theory so that we can take full advantage of the fixed point theorem. For this identity relation, we introduce bisimulation, $\sim$ [BM96]. The motivation for introducing $\sim$ is to express *hereditary* extensional equality with respect to iterations of the successor function.

Definition 3.3
- Relation $R^{\sim}$ is defined as follows:

$$\begin{aligned}(\forall x, y)[\langle x, y\rangle \in R^{\sim} \quad \equiv \quad & [x =_{ext} y \vee \\ & (\exists v, w)[\langle v, w\rangle \in R^{\sim} \otimes x = \mathbf{suc}(v) \otimes y = \mathbf{suc}(w)]],\end{aligned}$$

- $x \sim y$ is an abbreviation of $\langle x, y\rangle \in R^{\sim}$.

The existence of $R^{\sim}$ is guaranteed by theorem 2.11.

Lemma 3.4
- $(\forall x, y)[x =_{ext} y \rightarrow x \sim y]$,
- $(\forall x, y)[\mathbf{suc}(x) \sim \mathbf{suc}(y) \equiv x \sim y]$.

Let us remark here on the view of sets as automata (mathematically, directed graphs with only one kind of directed edge, $\in$) [A88]. For example, $\{\emptyset, \{\emptyset\}\}$ gives an automata of the following form:

$$\begin{tikzcd} & \{\emptyset, \{\emptyset\}\} \arrow[dl, "\in"'] \arrow[dr, "\in"] & \\ \emptyset & & \{\emptyset\} \arrow[d, "\in"] \\ & & \emptyset \end{tikzcd}$$

We see that all sets form well-founded trees in **ZF**. As automata, they terminate eventually. The axiom of foundation guarantees that after starting from any node in a tree we will reach $\emptyset$ in finite steps; $\emptyset$ can be seen as a terminal state with respect to $\in$ since $\vdash (\forall x)[x \notin \emptyset]$. However, non-terminating automata can be defined in **ZFA**. For example, we can define a set $a = \{a\}$ which represents a non-terminating automaton with graph

$$\begin{tikzcd} a \arrow[loop right, "\in"] \end{tikzcd}$$

Similarly, in **CONS**, theorem 2.11 guarantees the existence of a term θ such that $\theta =_{ext} \{\theta\}$. We can generalize this framework by regarding natural numbers as automata, or directed graphs, with only one label, **suc**, on edges. Theorem 2.11 says that any finite automata with edges labeled $\in$, **suc**, or as other functions or relations can be represented in **CONS** in this way.

For identity on such automata, it is common in computer science to use bisimulation to identify observable behaviors. Here $\sim$ satisfies all conditions of bisimulation [S11] with respect to the successor function **suc**: $a \sim b$ means that the behavior of a and b are similar with respect to **suc**.

3.2 Arithmetic with bisimulation

For each $x \in \omega$ we define an equivalence class $[x]$ and the set of equivalence classes$\langle\tilde{\omega}, \sim\rangle$ together with an arithmetic developed from $\sim$.

Definition 3.5
- For any a, $[a] = \{x : x \sim a\}$,
- $\tilde{\omega}$ is a set of $\sim$-equivalence classes whose representative element is a natural number:
$$\tilde{\omega} = \{x : (\exists y)[y \in \omega \otimes x = [y]]\}.$$

Hereafter, we write $a \in \tilde{\omega}$ when $(\exists x \in \omega)a \in [x]$.

We can develop arithmetic over $\langle\tilde{\omega}, \sim\rangle$ by using a general recursive form. For example, we may define**PLUS**, an analogue of **Plus** on $\tilde{\omega}$.

Definition 3.6
- **PLUS**, a relation such that $\vdash \mathbf{PLUS} \subseteq \tilde{\omega} \times \tilde{\omega} \times \tilde{\omega}$, is defined as

$$\begin{aligned}\mathbf{PLUS} \;=\; \{v : (\exists x, y, z \in \tilde{\omega})[v = \langle x, y, z\rangle \otimes [(y \sim \bar{0} \otimes z \sim x) \\ \vee(\exists y', z')[\langle x, y', z'\rangle \in \mathbf{PLUS} \otimes y \sim \mathbf{suc}(y') \otimes z \sim \mathbf{suc}(z')]]]\}.\end{aligned}$$

- **TIMES**, a relation such that $\vdash \mathbf{TIMES} \subseteq \tilde{\omega} \times \tilde{\omega} \times \tilde{\omega}$ is defined as

$$\begin{aligned}\mathbf{TIMES} \;=\; \{v : (\exists x, y, z \in \tilde{\omega})[v = \langle x, y, z\rangle \otimes [(y \sim \bar{0} \otimes z \sim \bar{0}) \\ \vee(\exists y', z')[\langle x, y', z'\rangle \in \mathbf{TIMES} \otimes y \sim \mathbf{suc}(y') \otimes \mathbf{PLUS}(z', x, z)]]]\}.\end{aligned}$$

We can develop arithmetic over $\tilde{\omega}$ by using $\sim$, **PLUS** and **TIMES**. It is easy to see that the arithmetic over $\tilde{\omega}$ is a *conservative* extension of arithmetic over ω in the following sense.

Lemma 3.7
For any $a, b, c \in \omega$,

- $\langle a, b, c\rangle \in$ **Plus** implies $\langle a, b, c\rangle \in$ **PLUS**,
- $\langle a, b, c\rangle \in$ **Times** implies $\langle a, b, c\rangle \in$ **TIMES**.

Let us extend this result.

Definition 3.8 (Arithmetical formulae)
- **PLUS**(x, y, z), **TIMES**(x, y, z) are arithmetical formulae.
- If φ_0, φ_1 are arithmetical formulae, then so is $\varphi_0 \circ \varphi_1$ where $\circ$ is any logical connective.
- If $\varphi[x]$ is arithmetical, then so is $(Qx \in \tilde{\omega})\varphi[x]$ where Q is any quantifier.

Lemma 3.9

Assume that $s \sim \bar{n}$ for some numeral $\bar{n} \in \omega$. Then, for any arithmetical formula φ,

$$s \sim t \vdash \varphi(s) \equiv \varphi(t)$$

The proof is by induction on n.

4 The non-crispness of ω

We prove theorem 1.1 in this section. First, we prove an analogue of theorem 1.1 for $\tilde{\omega}$ in section 4.1: we implement a non-terminating automaton **fix**, a counterexample to the crispness for $\tilde{\omega}$, by using the fixed point lemma and bisimulation. Second, we modify **fix** to show theorem 1.1 in section 4.2: we *unfold* **fix** to serve as a counterexample in the proof of theorem 1.1.

4.1 The non-crispness of $\tilde{\omega}$

In this section, we prove the unprovability of the crispness of $\tilde{\omega}$ in **CONS**.

First, let us introduce a simple automaton **fix** such that $\mathbf{fix} \circlearrowright^{\mathbf{suc}}$

Definition 4.1

fix is a fixed point of the successor function **suc** with respect to $\sim$:

$$\mathbf{suc}(\mathbf{fix}) \sim \mathbf{fix}.$$

More precisely, **fix** is defined by the fixed point theorem:

$$(\forall x)[(x \in \mathbf{fix}) \equiv (x \in \mathbf{suc}(\mathbf{fix}))].$$

From a behavioral viewpoint, all numerals (determinate members of $\tilde{\omega}$) are terminating automata. However, **fix** never terminates. Therefore, we can see that **fix** is a non-standard element of $\tilde{\omega}$ in the following sense.

Lemma 4.2

(1) **CONS** does not prove $\mathbf{fix} \in \tilde{\omega}$.

(2) **CONS** does not prove $\mathbf{fix} \notin \tilde{\omega}$.

proof Let us prove (1). Assume otherwise: $\vdash$ "$\mathbf{fix} \in \tilde{\omega}$". Then, an easy back-calculation shows that the normal proof of this assumption should be of the following form:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{\vdash t_1 = \bar{0} \vee (\exists y \in \omega) t_1 = \mathbf{suc}(y)}}{\vdash t_1 \in \omega} \qquad \dfrac{\vdots}{\vdash t_0 = \mathbf{suc}(t_1)}}{\vdash t_1 \in \omega \otimes t_0 = \mathbf{suc}(t_1)}}{\vdash (\exists x \in \omega) t_0 = \mathbf{suc}(x)}}{\vdash t_0 = \bar{0} \vee (\exists x \in \omega) t_0 = \mathbf{suc}(x)} \qquad \dfrac{\vdots}{\vdash t_0 \sim \mathbf{fix}}}{\dfrac{\vdash t_0 \in \omega \otimes t_0 \sim \mathbf{fix}}{\dfrac{\vdash (\exists x \in \omega)\mathbf{fix} \sim x}{\vdash \mathbf{fix} \in \tilde{\omega}}}}$$

Since $\vdash \mathbf{fix} \sim \mathbf{suc}(\mathbf{fix})$, this proof never terminates; the proof never achieves the bottom case, $\vdash t_n \in \omega$ and $\mathbf{suc}(\mathbf{suc}(\cdots(t_n)\cdots)) \sim \mathbf{fix}$ for some t_n, in finite steps. Therefore, there is no finite proof of $\vdash$ "$\mathbf{fix} \in \tilde{\omega}$".

As for (2), assume otherwise: $\vdash$ "$\mathbf{fix} \notin \tilde{\omega}$". Its normal proof should be of the following form:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{x_0 \sim \bar{0}, \mathbf{fix} \sim \mathbf{suc}(x_0) \vdash \bot} \quad \dfrac{\dfrac{\vdots}{x_1 \in \tilde{\omega}, x_0 \sim \mathbf{suc}(x_1), \mathbf{fix} \sim \mathbf{suc}(x_0) \vdash \bot}}{(\exists x_1 \in \omega) x_0 \sim \mathbf{suc}(x_1)], \mathbf{fix} \sim \mathbf{suc}(x_0) \vdash \bot}}{[x_0 \sim \bar{0} \vee (\exists x_1 \in \omega) x_0 \sim \mathbf{suc}(x_1)], \mathbf{fix} \sim \mathbf{suc}(x_0) \vdash \bot}}{x_0 \in \omega, \mathbf{fix} \sim \mathbf{suc}(x_0) \vdash \bot}}{(\exists x_0 \in \omega)\mathbf{fix} \sim \mathbf{suc}(x_0) \vdash \bot} \quad \dfrac{\vdots}{\mathbf{fix} \sim \bar{0} \vdash \bot}}{\dfrac{\dfrac{\mathbf{fix} \sim \bar{0} \vee (\exists x_0 \in \omega)\mathbf{fix} \sim \mathbf{suc}(x_0) \vdash \bot}{\mathbf{fix} \in \tilde{\omega} \vdash \bot}}{\vdash \mathbf{fix} \notin \tilde{\omega}}}
$$

In this way, to show $x_1 \in \tilde{\omega}, x_0 \sim \mathbf{suc}(x_1), \mathbf{fix} \sim \mathbf{suc}(x_0) \vdash \bot$, we should prove $x_1 \sim \bar{0}, x_0 \sim \mathbf{suc}(x_1), \mathbf{fix} \sim \mathbf{suc}(x_0) \vdash \bot$ and $(\exists x_2) x_1 \sim \mathbf{suc}(x_2), x_0 \sim \mathbf{suc}(x_1), \mathbf{fix} \sim \mathbf{suc}(x_0) \vdash \bot$. The former is obvious, but the latter is problematic: to show this we need to show $(\exists x_3) x_2 \sim \mathbf{suc}(x_3), x_1 \sim \mathbf{suc}(x_2), x_0 \sim \mathbf{suc}(x_1), \mathbf{fix} \sim \mathbf{suc}(x_0) \vdash \bot$. However, this regress continues unless we reach the bottom case; in other words, there is a term δ such that δ is not of the form $\mathbf{suc}(\gamma)$ and $\vdash \delta \not\sim \bar{0}$ and $\mathbf{suc}(\mathbf{suc}(\cdots(\delta)\cdots)) \sim \mathbf{fix}$. However, this is impossible because $\vdash \mathbf{fix} \sim \mathbf{suc}(\mathbf{fix})$. This means that there is no finite proof of $\vdash \mathbf{fix} \notin \tilde{\omega}$. $\Box$

This proof shows that neither the proposition asserting that the fixed point **fix** on **suc** (unique up to $\sim$) is a member of $\tilde{\omega}$ nor its negation can be proved in finite length in **CONS**.

Therefore, we have

Lemma 4.3
CONS does not prove $(\mathbf{fix} \in \tilde{\omega}) \vee (\mathbf{fix} \notin \tilde{\omega})$.

Proof. Otherwise, since **CONS** satisfies the disjunction property, $\mathbf{fix} \in \tilde{\omega}$ or $\mathbf{fix} \notin \tilde{\omega}$ is a theorem of **CONS**. $\Box$

4.2 The non-crispness of ω: unfolding **fix**

In this section, we prove theorem 1.1. The difficulty for this is that **fix** is clearly not a member of ω because it is not of the form $\{x : x = a\}$ due to it being constructed by using the fixed point theorem. We therefore define a (possibly partial) function which *unfolds* **fix**.

Definition 4.4 (rank)
rk is a relation over sets and (possibly) natural numbers:

$$\langle x, y\rangle \in \mathbf{rk} \equiv [(x \sim \bar{0} \otimes y = \bar{0}) \vee (\exists z_0, z_1)[\langle z_0, z_1\rangle \in \mathbf{rk} \otimes x =_{ext} \mathbf{suc}(z_0) \otimes y = \mathbf{suc}(z_1)]]$$

Roughly speaking, **rk** *unfolds* nested boxes and counts how many singletons are nested. We note that **CONS** proves any numerals are in the range of **rk**.

We do not know whether **CONS** proves that **rk** can unfold **fix**. That is, whether $\mathbf{fix} \in \mathbf{dom}(\mathbf{rk})$, **CONS** proves that **rk** cannot unfold **fix** or if **CONS** can prove neither that **rk** can unfold **fix** nor that **rk** cannot unfold **fix**. However, at least we can say that **CONS** does not reject $\mathbf{fix} \in \mathbf{dom}(\mathbf{rk})$; in other words, **CONS** does not prove there is no s such that $\langle \mathbf{fix}, s\rangle \in \mathbf{rk}$. For, if so, it proves $\neg(\exists x \in \omega) x \sim \mathbf{fix}$, which contradicts lemma 4.2. So, let us extend the **CONS** by adding the following axiom:

Definition 4.5
$\underline{\mathbf{CONS}}$ is an extension of **CONS** by adding

- the new axiom "$\mathbf{fix} \in \mathbf{dom}(\mathbf{rk})$",
- the new constant c which satisfies $\langle \mathbf{fix}, c\rangle \in \mathbf{rk}$.

The intuitive image of c is $\mathbf{suc}(\mathbf{suc}(\mathbf{suc}(\cdots)))$, an infinite stream of **suc** (i.e. $\{\{\{\cdots\}\}\}$ in Zermelo style). Therefore, we can prove the analogue of lemma 4.2 as follows.

Lemma 4.6
(1) $\underline{\mathbf{CONS}}$ does not prove $c \in \omega$,
(2) $\underline{\mathbf{CONS}}$ does not prove $c \notin \omega$, and
(3) $\underline{\mathbf{CONS}}$ does not prove $c \in \omega \vee c \notin \omega$.

Proof. Otherwise, we can prove the negation of lemma 4.2. $\square$

This gives a proof of theorem 1.1: this theorem is proved because $\underline{\mathbf{CONS}}$, an extension of **CONS**, cannot prove $(\forall x)[x \in \omega \vee x \notin \omega]$. $\square$

The non-crispness of ω implies some corollaries; we prove one of them here.

Corollary 4.7
Arithmetic developed in the theory **CONS** does not prove the axioms of Robinson's minimal arithmetic **Q**.

Proof. $\underline{\mathbf{CONS}}$ does not prove $(\forall x)[x \not\sim \mathbf{suc}(x)]$ since $\mathbf{suc}(c) \sim c$. $\square$

5 Conclusion

We proved that the constructive naive set theory **CONS** does not prove the crispness of ω. Formally, $(\forall x)[(x \in \omega) \vee (x \notin \omega)]$.

The crispness of ω is an important problem because it concerns the nature of sets defined by a general recursive form. In the proof, a circularly defined object **fix**, which is finitely generated and potentially infinite, plays a key role. We remark that we introduced $\langle \tilde{\omega}, \sim\rangle$ to show the above theorem, and this system seems to have many interesting aspects in its own right. For example, our theorem shows that we can never prove the negation of the statement that ω, which is *seemingly* defined inductively, contains an infinite object **fix**. This highlights the *potentially infinite* character of **CONS**. Quite contrary to classical theories, the distinction between finiteness and non-finiteness in **CONS** is indefinite because of the existence of potentially infinite objects.

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