Summary of the thesis "Stochastic Stability of Partially Expanding Maps via Spectral Approaches"

Yushi Nakano

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Abstract

We consider stochastic stability of expanding maps and partially expanding maps under quenched random perturbations. In the first part, we consider quenched random perturbations of skew products of rotations on the unit circle over uniformly expanding maps on the unit circle. It is known that if the skew product satisfies a certain condition (shown to be generic in the case of linear expanding maps), then the transfer operator of the skew product has a spectral gap. Using semiclassical analysis we show that the spectral gap is preserved under small random perturbations. This implies exponential decay of quenched random correlation functions for smooth observables at small noise levels. In the second part, adapting another previously developed spectral approach, we also show stochastic stability of expanding maps under quenched random perturbations whose base dynamics are not invertible necessarily.

In dynamical systems theory, two questions have been studied for a long time; one is how the typical orbits asymptotically behave as time goes to infinity, and the other is how stable the dynamical behaviour is under perturbations of the systems. In this thesis, we consider a stability of statistical behaviours of two broad classes of dynamical systems — *expanding maps* and *partially expanding maps*, under small noise perturbations, which is called *stochastic stability*.

It is typical in dynamical systems theory to find dynamics whose orbits exhibit an extremely complex behaviour despite of its simple evolution law. This is true also in our case of partially expanding maps. A successful approach in ergodic theory to understanding such dynamical systems with complex behaviour is through establishing the existence of an *absolutely continuous ergodic invariant probability measure* (abbreviated *aceip*). We recall that given a dynamical system $f : X \to X$ in a measurable space (X, Σ), a probability measure μ on X is said to be invariant with respect to f when $\mu(f^{-1}A) = \mu(A)$ for all $A \in \Sigma$. Furthermore, we say that μ is ergodic when $\mu(A)$ is 0 or 1 for any f-invariant $A \in \Sigma$ (i.e., $f^{-1}A = A$). By Birkhoff 's ergodic theorem, for each L^1 observable

$\varphi : X \to \mathbb{R}$, we get that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^j(x)) = \int \varphi(x) d\mu$$

if x is in a μ-full measure set X_0. This implies that the time average of observables along the orbits of f issued from the point x coincides with the space average of the observables, if x is in a typical set X_0 in the sense of μ. Moreover, if X is a compact smooth Riemannian manifold, the absolute continuity of μ (with respect to the normalized Lebesgue measure m on X) provides a physical meaning since it implies this relation holds for any initial point x in the Lebesgue positive measure set X_0. For more detailed background of aceip (such as the relation with the border theory of thermodynamical formalism) and its history in dynamical systems theory, the reader is referred to, e.g., [7, 15].

Stochastic stability was introduced as the stability of the time average, the most basic statistical data, under small perturbations by Kolmogorov and Sinai [44]. However, stochastic stability is a rather vague notion depending on the nature of the dynamical systems under consideration and subjects of interest. In the context of random perturbations of uniformly hyperbolic maps by Markov chains, the theory was basically completed by Kifer [36, 37], see also Young [40] for the case of Axiom A diffeomorphisms. A substantial progress after Kifer's early work was established by Katok and Kifer [35] for a real quadratic map with Misiurewicz parameter, which has been further developed by several authors to prove stochastic stability of non-uniformly hyperbolic maps. See for example [12], [1] and reference therein for recent development in this area. Another extension was done by Kifer [38] in the case of perturbations induced by skew-product mappings or random dynamical systems, which we will adopt as our perturbation scheme throughout this paper (refer to [15, Appendix D], [6] for the relation between random perturbations by Markov chains and random dynamical systems; see also [2] for general description of random dynamical systems). Precise definition of our perturbation models will appear in (2) and (5)

In this thesis, we consider stochastic stabilities in two different contexts. One is stochastic stability of partially expanding maps on the torus. In contrast to the (non-uniformly) hyperbolic case, only a few stochastic stability results are known for dynamical systems with nonhyperbolic directions (such as hyperbolic flows or partially hyperbolic maps). Stochastic stability for Anosov flow was first proved by Butterley and Liverani [18, 19] by showing the spectral stability of the generator of the transfer operator. Although their stability results may give us more information than stochastic stability, their results require the perturbation to be deterministic. In Theorem 1 of Part I, we show stochastic stability of partially expanding maps on the torus satisfying a certain condition (shown to be generic in the case of linear expanding maps in Chapter 3) under perturbation induced by skew-product mappings. To the best of our knowledge, this is the first result for stochastic stability of partially expanding maps. Our main technique to analyse the dynamics is through adapting microlocal and semiclassical analysis of transfer operators, which is a technique developed in the last decade (see the introduction in Part I for more historical comments). As a result of this spectral technique, we also show that the *exponential decay of correlation functions* of these partially expanding maps is preserved under small noise perturbations, which will be precisely stated in Theorem 2. After completing this thesis, we learned that Dyatolov and Zworski [25] also showed stochastic stability of Anosov flow by some microlocal technique, although their technique seems quite different from ours.

A significant restriction on our perturbations in Part I is that the base transformations of the skew-product mappings inducing the perturbations are required to be invertible. This is partly because the multiplicative ergodic theorem to cocycles of linear operators which we shall employ for the spectral analysis of random transfer operators is now only available for invertible base transformations, compare with [33, Section 1]. Thus, as an attempt to remove the invertibility, we show stochastic stability of expanding maps under non-invertible perturbations by adapting another spectral approach in Part II. Although only expanding maps are considered in Part II to keep our presentation transparent, this approach is expected to be applicable to partially expanding maps on the torus exhibiting the exponential decay of correlation functions.

Notes

This article is a summary of the thesis "Stochastic Stability of Partially Expanding Maps via Spectral Approaches", which has been submitted in fulfilment of the requirements for the degree of doctor of Human and Environmental Studies at Kyoto university. The result in Part I is a modified version of the paper "On the Spectra of Quenched Random Perturbations of Partially Expanding Maps on the Torus", which is a joint work with Jens Wittsten and will be published in *Nonlinearity*.

Part I: On the Spectra of Quenched Random Perturbations of Partially Expanding Maps on the Torus

Let X be a compact smooth Riemannian manifold. Recall that a dynamical system $f : X \to X$ is said to be mixing with respect to an invariant measure μ when $\mu(A \cap f^{-n}B)$ converges to $\mu(A)\mu(B)$ as time n goes to infinity for any Borel sets A and B. This means that the events A and $f^{-n}B$ are asymptotically independent, so mixing indicates a certain amount of complexity of the dynamical system. For mixing dynamical systems, a fundamental question is how fast the correlation functions decay (see Section for definitions). In fact, if the correlation functions of a mixing system decay exponentially fast, then several other statistical properties of the dynamical system also hold. For an extensive background on such matters we refer to Bonatti, Díaz and Viana [15, Appendix

E] and the references therein.

For a hyperbolic dynamical system f, individual trajectories tend to have chaotic behavior. Statistical properties of the system, such as exponential decay of correlations, are therefore preferably obtained by instead studying how densities of points evolve under a so-called transfer operator M_f induced by f. The typical approach to proving exponential decay of correlations is through the construction of a functional space $\mathcal{H}$ adapted to the dynamics such that the transfer operator $M_f : \mathcal{H} \to \mathcal{H}$ has a *spectral gap*, that is, there exists a disc of strictly smaller radius than the spectral radius of M_f outside of which the spectrum of M_f consists only of discrete eigenvalues of finite multiplicity. Essentially, exponential decay of correlations is equivalent to the existence of a gap in the spectrum between the eigenvalue 1 and the second largest eigenvalue of M_f, counting multiplicities. (In the presence of noise, spectral stability thus becomes a natural object of study.) Early work in this area was done by Bowen, Ruelle and Sinai, see for example the newly revised edition of Bowen's book [16] for a historical account. A celebrated construction of anisotropic Banach spaces was later established by Blank, Keller and Liverani [13], whose approach has been further developed by several authors. Recently, these techniques have been shown to also be applicable to dynamical systems with a one dimensional nonhyperbolic direction (such as hyperbolic flow), see for example Liverani [41], Butterley and Liverani [18,19] and Tsujii [45,46]; see also Baladi and Liverani [9] for a historical account. As evidenced by the mentioned articles, the spectral analysis becomes more delicate for systems with a nonhyperbolic direction.

Let the two dimensional torus be denoted $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$, where $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$. Consider skew products on $\mathbb{T}^2$ of the form

$$(x, s) \mapsto (E(x), s + \tau(x) \bmod 1)$$

where $E : \mathbb{S}^1 \to \mathbb{S}^1$ is a hyperbolic system, and τ is a real valued function on $\mathbb{S}^1$. Known as *compact group extensions*, these were studied by Dolgopyat [23] who proved superpolynomial decay of correlations under Diophantine conditions in the case when E is an Anosov diffeomorphism. Tsujii [45] considered the closely related model given by the semi-flow obtained by suspending a uniformly expanding map E under a ceiling function τ, and obtained a precise description of the spectra of the corresponding transfer operators by imposing a *transversality* condition on the dynamics. Tsujii [45] showed that this condition is generic for linear maps E, and that it fails precisely when the ceiling τ is cohomologous to a constant. The corresponding smooth compact group extension was studied by Faure [27] who introduced a similar condition on the dynamics, using the terminology *partially captive* for such systems. (These satisfy the transversality condition, and the conditions are comparable when the expanding map E is linear. In particular, if f_0 is partially captive then the function τ_0 in (1) cannot be cohomologous to a constant.) Here we mention the recent preprints by Butterley and Eslami [17] and Eslami [26], wherein the same dynamics is studied under much weaker regularity assumptions, utilizing an extension of Tsujii's transversality condition.

The map studied by Faure [27], which throughout this article shall be referred to as our unperturbed dynamical system, is the simplest model of a hyperbolic system with a central direction. It is defined as follows: Let $g_0 : \mathbb{S}^1 \to \mathbb{S}^1$ be a $\mathscr{C}^\infty$ diffeomorphism and let $\tau_0 : \mathbb{S}^1 \to \mathbb{R}$ be a $\mathscr{C}^\infty$ function. Let $k \geq 2$ be a positive integer, and consider the skew product $f_0 : \mathbb{T}^2 \to \mathbb{T}^2$ of class $\mathscr{C}^\infty$ given by

$$f_0 : \begin{pmatrix} x \\ s \end{pmatrix} \mapsto \begin{pmatrix} kg_0(x) & \mod 1 \\ s + \frac{1}{2\pi}\tau_0(x) & \mod 1 \end{pmatrix} \tag{1}$$

on the torus. The map $E_0 : x \mapsto kg_0(x) \mod 1$ is assumed to be an *expanding map* on $\mathbb{S}^1$ in the sense that $\min_x E_0'(x) > 1$, and we then say that f_0 is a *partially expanding map* on $\mathbb{T}^2$. (Since the differential of $x \mapsto E_0(x)$ is linear and $T_x\mathbb{S}^1 \simeq \mathbb{R}$, it follows that $(dE_0)_x : T_x\mathbb{S}^1 \to T_{E_0(x)}\mathbb{S}^1$ is a scalar, which we denote by $E_0'(x)$.) Note that with this terminology, the derivative of an expanding map is always positive, and that g_0 is orientation-preserving by assumption.

Faure [27] establishes the existence of a spectral gap through semiclassical analysis, which is an asymptotic theory in which the Planck constant appearing in the Shrödinger equation is regarded as a small parameter $h > 0$. It is a fairly recent discovery that spectral properties of transfer operators of (partially) hyperbolic maps are naturally studied within this framework, the ideas having appeared in Baladi and Tsujii [10, 11] (see also Avila, Gouëzel and Tsujii [5]), and formalized in a series of papers primarily by Faure, Roy and Sjöstrand [28–30]. This approach has been getting traction lately with contributions in this and related areas also by Arnoldi [3], Arnoldi, Faure and Weich [4], Dyatlov and Zworski [24], Faure and Tsujii [31, 32], and Tsujii [47], among others. For partially expanding maps, the first two references are particularly relevant. So far, the focus seems to have been on deterministic systems, and one of our goals is to show that the semiclassical approach is also applicable in the case of random perturbations.

To circumvent the lack of hyperbolicity of f_0 in the s direction, Faure [27] uses Fourier analysis in the s direction to decompose the transfer operator induced by f_0 into a collection of (weighted) transfer operators of the expanding map $E_0 : \mathbb{S}_x^1 \to \mathbb{S}_x^1$, indexed by a Fourier parameter $\nu \in \mathbb{Z}$. The resulting operators are examples of Fourier integral operators, and thus naturally studied using microlocal analysis (when $\nu \in \mathbb{Z}$ is fixed) and semiclassical analysis (with a semiclassical parameter of size $h \sim 1/|\nu|$, tending to 0). Roughly speaking, if f_0 is partially captive, then the spectral radius decreases in the *semiclassical limit* $|\nu| \to \infty$. On the other hand, outside a small disc, the spectrum of each transfer operator (for fixed $\nu \in \mathbb{Z}$) consists of discrete eigenvalues of finite multiplicity (the so-called *Ruelle resonances*), resulting in a spectral gap for the collection. This (and an additional assumption on the peripheral spectrum) is known to give exponential decay of operational correlations for smooth observables (Faure [27, Theorem 5]).

In this thesis we show that the presence of the spectral gap observed in the deterministic case (as described above) is preserved under quenched random perturbations at small noise levels. For random transfer operators, the notion of spectrum needs clarification; in particular, the notion of discrete spectrum should be understood in terms of Lyapunov exponents and invariant subspaces instead of eigenvalues and eigenfunctions, see Section . We also show existence and strong stability of random measures, see Theorem 1. Using the spectral results we then establish our main theorem: if f_0 is partially captive then the quenched random correlations for $\mathscr{C}^\infty$ observables decay exponentially fast, see Theorem 2.

The perturbation model

Let $(\Omega, \mathcal{F})$ be a Lebesgue space with a probability measure $\mathbb{P}$. Let $\theta : \Omega \to \Omega$ be an ergodic $\mathbb{P}$-preserving bi-measurable bijection. Let $\mathscr{C}^\infty(\mathbb{T}^2, \mathbb{T}^2)$ be the space of smooth endomorphisms on $\mathbb{T}^2$ endowed with a $\mathscr{C}^\infty$ metric,

$$d_{\mathscr{C}^\infty}(f,g) = \sum_{j=0}^{\infty} 2^{-j} \frac{d_{\mathscr{C}^j}(f,g)}{1 + d_{\mathscr{C}^j}(f,g)},$$

where $d_{\mathscr{C}^j}(f,g)$ is the usual $\mathscr{C}^j$ distance between f and g. We endow $\mathscr{C}^\infty(\mathbb{T}^2, \mathbb{T}^2)$ with the Borel σ-algebra.

Let $\{f_\epsilon\}_{\epsilon>0}$ be a family of measurable mappings $f_\epsilon : \Omega \to \mathscr{C}^\infty(\mathbb{T}^2, \mathbb{T}^2)$ such that for each $\epsilon > 0$, $f_\epsilon(\omega)$ is for $\mathbb{P}$-almost every $\omega \in \Omega$ of the form

$$f_\epsilon(\omega) : \begin{pmatrix} x \\ s \end{pmatrix} \mapsto \begin{pmatrix} kg_\epsilon(\omega, x) & \mod 1 \\ s + \frac{1}{2\pi}\tau_\epsilon(\omega, x) & \mod 1 \end{pmatrix}, \tag{2}$$

where $\omega \mapsto f_\epsilon(\omega)(z)$ is a measurable mapping from Ω to $\mathbb{T}^2$ for each $z = (x,s) \in \mathbb{T}^2$. Here $g_\epsilon(\omega) = g_\epsilon(\omega, \cdot) : \mathbb{S}^1 \to \mathbb{S}^1$ is a $\mathscr{C}^\infty$ diffeomorphism and $\tau_\epsilon(\omega) = \tau_\epsilon(\omega, \cdot) : \mathbb{S}^1 \to \mathbb{R}$ is a $\mathscr{C}^\infty$ function, $\mathbb{P}$-almost surely. We also assume that

$$\operatorname*{ess\,sup}_{\omega} d_{\mathscr{C}^\infty}(f_\epsilon(\omega), f_0) \to 0 \quad \text{as } \epsilon \to 0, \tag{3}$$

where f_0 is the partially expanding map given by (1). The value $f_\epsilon(\omega)(z)$ is denoted simply by $f_\epsilon(\omega, z)$. For each $\epsilon > 0$, it follows that $(\omega, z) \mapsto f_\epsilon(\omega, z)$ is a measurable mapping from $\Omega \times \mathbb{T}^2$ to $\mathbb{T}^2$, see Castaing and Valadier [21, Lemma 3.14]. When convenient, we will identify $f_0 : \mathbb{T}^2 \to \mathbb{T}^2$ with the constant map $\Omega \ni \omega \mapsto f_0$.

For each $\epsilon \geq 0$ and $\omega \in \Omega$ we let $E_\epsilon(\omega)$ denote the map $E_\epsilon(\omega) : x \mapsto kg_\epsilon(\omega, x)$ mod 1, interpreted for $\epsilon = 0$ to mean $E_{\epsilon=0}(\omega) \equiv E_0$ for all ω. The value $E_\epsilon(\omega)(x)$ is denoted simply by $E_\epsilon(\omega, x)$. In view of (3) it then follows that $E_\epsilon(\omega)$ is an expanding map $\mathbb{P}$-almost surely if ϵ is sufficiently small. In fact, if $\lambda_0 = \min_x E_0'(x)$ and we set $\lambda = (\lambda_0 + 1)/2$, then $\lambda > 1$ and we can find an $\epsilon_0 > 0$ such that

$$\operatorname*{ess\,inf}_{\omega} \min_{x} \frac{dE_\epsilon(\omega, x)}{dx} \geq \lambda, \quad 0 \leq \epsilon < \epsilon_0. \tag{4}$$

In the sequel, the quantity $dE_\epsilon(\omega, x)/dx$ will sometimes be denoted simply by $E_\epsilon'(\omega, x)$.

Remark. When there is no ambiguity, the noise level ϵ will sometimes be omitted from the notation, in particular when the dependence on the noise parameter $\omega \in \Omega$ is already displayed. In fact, with the exception of the underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and the map $\theta : \Omega \to \Omega$, dependence on the noise parameter $\omega \in \Omega$ will always be taken to imply dependence on the noise level ϵ. Throughout the rest of the paper we will also permit us to use ϵ_0 as a way to denote the upper bound of a range $0 \leq \epsilon < \epsilon_0$ for which (4) holds, even if ϵ_0 may change between occurrences. This will mostly be showcased only in the statements of our results; we shall in fact always assume that ϵ belongs to such a range.

Given $\epsilon > 0$ and $n \geq 1$, let $f_\epsilon^{(n)}(\omega, z)$ be the fiber component of the nth iteration of the (double) skew product mapping

$$\Theta_\epsilon(\omega, z) = (\theta\omega, f_\epsilon(\omega, z)), \quad \omega \in \Omega, \quad z \in \mathbb{T}^2,$$

and let $f_\epsilon^{(0)}(\omega) = \mathrm{Id}_{\mathbb{T}^2}$ for all $\omega \in \Omega$. With the notation $f_\epsilon^{(n)}(\omega) = f_\epsilon^{(n)}(\omega, \cdot)$ we explicitly have

$$f_\epsilon^{(n)}(\omega) = f_\epsilon(\theta^{n-1}\omega) \circ f_\epsilon(\theta^{n-2}\omega) \circ \cdots \circ f_\epsilon(\omega).$$

The mapping given by $(n, \omega, z) \mapsto f_\epsilon^{(n)}(\omega, z)$ is an RDS on $\mathbb{T}^2$ over $\theta : \Omega \to \Omega$, which we call the RDS induced by f_ϵ. (Naturally, this RDS depends also on θ, but since θ will be fixed throughout, mention of this map will be omitted; For definition of RDS, see [42, Subsection 1.1].) For convenience we introduce the notation

$$E_\epsilon^{(n)}(\omega, x) = E_\epsilon(\theta^{n-1}\omega) \circ \ldots \circ E_\epsilon(\omega)(x), \quad n \geq 1,$$
$$\tau_\epsilon^{(n)}(\omega, x) = \sum_{j=0}^{n-1} \tau_\epsilon(\theta^j\omega, E_\epsilon^{(j)}(\omega, x)), \quad n \geq 1.$$

In the last sum, θ^0 and $E_\epsilon^{(0)}(\omega, \cdot)$ are to be interpreted as the identity maps on Ω and $\mathbb{S}^1$, respectively, so that $E_\epsilon^{(1)}(\omega, \cdot) = E_\epsilon(\omega)$ and $\tau_\epsilon^{(1)}(\omega, \cdot) = \tau_\epsilon(\omega)$. Then

$$f_\epsilon^{(n)}(\omega) : \begin{pmatrix} x \\ s \end{pmatrix} \mapsto \begin{pmatrix} E_\epsilon^{(n)}(\omega, x) \\ s + \frac{1}{2\pi}\tau_\epsilon^{(n)}(\omega, x) \mod 1 \end{pmatrix}, \quad n \geq 1.$$

The Perron-Frobenius transfer operator $M^*_{f_\epsilon^{(n)}(\omega)} : \mathscr{C}^\infty(\mathbb{T}^2) \to \mathscr{C}^\infty(\mathbb{T}^2)$ corresponding to $f_\epsilon^{(n)}(\omega)$ is defined as the random operator cocycle

$$M^*_{f_\epsilon^{(n)}(\omega)}\psi(z) = \sum_{f_\epsilon^{(n)}(\omega, z')=z} \frac{\psi(z')}{|\det \partial f_\epsilon^{(n)}(\omega, z')/\partial z|}, \quad \psi \in \mathscr{C}^\infty(\mathbb{T}^2),$$

where $\partial f_\epsilon^{(n)}(\omega, z')/\partial z$ is the Jacobian matrix of $z \mapsto f_\epsilon^{(n)}(\omega, z)$ at $z' \in \mathbb{T}^2$, and $\mathscr{C}^\infty(\mathbb{T}^2)$ is the space of complex valued functions on $\mathbb{T}^2$ of class $\mathscr{C}^\infty$. Note that by (2) and (4) we $\mathbb{P}$-almost surely have that

$$\det(\partial f_\epsilon^{(n)}(\omega, z)/\partial z) = dE_\epsilon^{(n)}(\omega, x)/dx \geq \lambda^n, \quad z = (x, s).$$

Thus, the operator $M^*_{f_\epsilon^{(n)}(\omega)}$ extends to a bounded operator on $L^2(\mathbb{T}^2)$, $\mathbb{P}$-almost surely. The extension will also be denoted by $M^*_{f_\epsilon^{(n)}(\omega)}$. Its adjoint $M_{f_\epsilon^{(n)}(\omega)}$ with respect to the usual scalar product on $L^2(\mathbb{T}^2)$ is the Ruelle transfer operator given by $M_{f_\epsilon^{(n)}(\omega)}\psi(z) = \psi(f_\epsilon^{(n)}(\omega, z))$.

Invariant measures

Let $f : \Omega \to \mathscr{C}^\infty(\mathbb{T}^2, \mathbb{T}^2)$ be a measurable mapping. Let $\mathcal{B}(\mathbb{T}^2)$ be the Borel σ-field of $\mathbb{T}^2$. Recall that a measure μ on $\Omega \times \mathbb{T}^2$ is called f-invariant when μ is invariant with respect to the skew product mapping $\Theta(\omega, z) = (\theta\omega, f(\omega, z))$ and the marginal $\pi_\Omega\mu$ of μ coincides with $\mathbb{P}$, where $\pi_\Omega : \Omega \times \mathbb{T}^2 \to \Omega$ is given by $\pi_\Omega(\omega, z) = \omega$. It is known that when μ is an f-invariant probability measure, there is a unique function $\mu_\cdot(\cdot) : \Omega \times \mathcal{B}(\mathbb{T}^2) \to [0, 1]$, $(\omega, B) \mapsto \mu_\omega(B)$, such that

(i) $\omega \mapsto \mu_\omega(B)$ is measurable for each $B \in \mathcal{B}(\mathbb{T}^2)$,

(ii) μ_ω is $\mathbb{P}$-almost surely a probability measure on $\mathbb{T}^2$,

(iii) $\int u d\mu = \int u d\mu_\omega d\mathbb{P}$ for each $u \in L^1(\mu)$.

Moreover, since we assume that θ is measurably invertible, the pushforward $f(\omega)_*\mu_\omega$ of μ_ω by $f(\omega)$ $\mathbb{P}$-almost surely coincides with $\mu_{\theta\omega}$, see Arnold [2, Chapter 1]. We call the function $\mu_\cdot(\cdot)$ the disintegration of μ (with respect to $\mathbb{P}$).

For the perturbation scheme f_ϵ given by (2), the existence of an absolutely continuous invariant probability measure on $\Omega \times \mathbb{T}^2$ is an immediate consequence of established results concerning the existence of such measures for uniformly expanding maps.

Theorem 1. *For each $0 \leq \epsilon < \epsilon_0$, there exists an f_ϵ-invariant probability measure μ^ϵ on $\Omega \times \mathbb{T}^2$ such that if $\mu^\epsilon_\cdot(\cdot)$ is the disintegration of μ^ϵ then μ^ϵ_ω is $\mathbb{P}$-almost surely equivalent to normalized Lebesgue measure on $\mathbb{T}^2$, and $d\mu^\epsilon_\omega = h_\epsilon(\omega, x)dxds$. Each density $h_\epsilon(\omega)$ is the uniquely defined positive function in $\mathscr{C}^\infty(\mathbb{S}^1)$ such that $\int_{\mathbb{S}^1} h_\epsilon(\omega, x)dx = 1$ and $M^*_{f_\epsilon^{(n)}(\omega)} h_\epsilon(\omega) = h_\epsilon(\theta^n\omega)$ for $n \geq 1$. Moreover,*

$$\operatorname*{ess\,sup}_\omega \|h_\epsilon(\omega) - h_0\|_{(m)} \to 0 \quad \text{as } \epsilon \to 0$$

for all $m \in \mathbb{N}$, where $h_0 \equiv h_{\epsilon=0}(\omega)$ is independent of ω.

Decay of random correlation functions

Let μ be an f-invariant measure, and $\mu_\cdot(\cdot)$ the disintegration of μ. For each $\phi, \psi \in \mathscr{C}^\infty(\mathbb{T}^2)$ let us define the *quenched operational correlation function*

$\mathrm{Cor}^{\mathrm{op}}_{\phi,\psi}(\omega,n)$ of (f,μ) by

$$\mathrm{Cor}^{\mathrm{op}}_{\phi,\psi}(\omega,n) = \int \phi \circ f^{(n)}(\omega) \cdot \bar{\psi} dxds - \int \phi d\mu_{\theta^n \omega} \int \bar{\psi} dxds.$$

We say that the operational correlation functions of (f,μ) decay exponentially fast when there exists a number $0 < \rho < 1$ (independent of ω) and a set $\tilde{\Omega}$ of full measure such that for any $\omega \in \tilde{\Omega}$ and $\phi, \psi \in \mathscr{C}^\infty(\mathbb{T}^2)$ there is a constant $c(\omega)$ (depending on ϕ and ψ) such that

$$|\mathrm{Cor}^{\mathrm{op}}_{\phi,\psi}(\omega,n)| \le c(\omega)\rho^n.$$

Similarly, we define the *quenched classical correlation function* $\mathrm{Cor}^{\mathrm{cl}}_{\phi,\psi}(\omega,n)$ of (f,μ) by

$$\mathrm{Cor}^{\mathrm{cl}}_{\phi,\psi}(\omega,n) = \int \phi \circ f^{(n)}(\omega) \cdot \bar{\psi} d\mu_\omega - \int \phi d\mu_{\theta^n \omega} \int \bar{\psi} d\mu_\omega,$$

and define exponential decay in the same way. We can now state our main result.

Theorem 2. *Let $\{f_\epsilon\}_{\epsilon>0}$ be a family of $\mathscr{C}^\infty(\mathbb{T}^2,\mathbb{T}^2)$ valued random variables satisfying* (2) *and* (3)*, and let μ^ϵ be the f_ϵ-invariant measure provided by Theorem 1 with disintegration $\mu^\epsilon_\omega(dxds) = h_\epsilon(\omega,x)dxds$. Assume that f_0 is partially captive (see* [42, Definition 2.3]*). Then there is an ϵ_0 such that if $0 \le \epsilon < \epsilon_0$ then the quenched random (operational and classical) correlation functions of $(f_\epsilon, \mu^\epsilon)$ decay exponentially fast.*

Part II: Stochastic Stability for Expanding Maps via a Perturbative Spectral Approach

The typical approach to proving statistical properties of expanding maps (such as the existence of SRB measures, the exponential decay of correlations, and the central limit theorem) is through demonstrating the spectral gap of the transfer operator of expanding maps in a suitable Banach space. In addition, these statistical properties and quantities are expected to be stable if "the spectrum of the transfer operator" is also stable. This perturbative spectral approach was developed by Baladi and Young and their contemporaries, who sought a simple proof that a (piecewise) expanding map is stochastically stable (i.e., the densities of the unique absolutely continuous invariant probability measures for the dynamics are stable) under independent and identically distributed perturbations, and that its related statistical quantities, such as the rate of the exponential decay of correlations, are also stable (see [7] and references therein). This approach was extended by Baladi [6] and independently by Bogenschütz [14], to the case of perturbations induced by skew-product mappings. However, these extensions are restricted to mixing or invertible base dynamics. In this appendix, an alternative perturbative spectral approach based on the Baladi-Young perturbation lemmas is presented, in which the base dynamics need not be mixing or invertible. Consequently, stochastic stability and upper bounds of the exponential decay of correlations for expanding maps under perturbations induced by skew-product mappings whose base dynamics are not invertible necessarily are demonstrated. Our result extends the result established by Baladi, Kondah, and Schmitt in [8].

Definitions and results

Let $\mathscr{C}^r(X,X)$ be the space of all $\mathscr{C}^r$ endomorphisms on a compact smooth Riemannian manifold X, endowed with the usual $\mathscr{C}^r$ metric $d_{\mathscr{C}^r}(\cdot,\cdot)$ with $r>1$. (Given that $r=k+\gamma$ for some $k\in\mathbb{N}$, $k\geq 1$ and $0\leq\gamma\leq 1$, $f\in\mathscr{C}^r(X,X)$ denotes the k-th derivative of f is γ-Hölder.) f in $\mathscr{C}^r(X,X)$ is said to be an expanding map when there exist constants $C>0$ and $\lambda>1$ such that

$$\|Df^n(x)v\|\geq C\lambda^n\|v\|, \quad n\geq 1$$

for each $x\in X$ and $v\in T_xM$. For the properties of expanding maps, the reader is referred to [34]. The expanding constant $\Lambda_r(f)$ of an expanding map $f:X\to X$ is defined by

$$\Lambda_r(f)=\limsup_{m\to\infty}\left(\sup_{x\in X}\sum_{f^m(y)=x}\frac{\|D(f_y^{-m})(x)\|^r}{|\det Df^m(y)|}\right)^{1/m},$$

which is strictly smaller than 1 (see (2.16) in [8]). Here, f_y^{-m} is the corresponding local inverse branch in a neighborhood of x for each $y\in f^{-m}(\{x\})$.

Let Ω be a separable complete metric space endowed with the Borel σ-field $\mathcal{B}(\Omega)$ with complete probability measure $\mathbb{P}$. Given an expanding map $f_0:X\to X$ of class $\mathscr{C}^r$, let $\{f_\epsilon\}_{\epsilon>0}$ be a family of continuous mappings defined on Ω with values in $\mathscr{C}^r(X,X)$ such that

$$\operatorname*{ess\,sup}_{\omega\in\Omega} d_{\mathscr{C}^r}(f_\epsilon(\omega),f_0)\to 0 \quad \text{as } \epsilon\to 0. \tag{5}$$

For each $\epsilon>0$, adopting the notation $f_\epsilon(\omega,\cdot)=f_\epsilon(\omega)$, the distance between $f_\epsilon(\omega,x)$ and $f_\epsilon(\omega',x)$ is bounded by $d_{\mathscr{C}^r}(f_\epsilon(\omega),f_\epsilon(\omega'))$ for each $x\in X$ and each $\omega,\omega'\in\Omega$. Thus, it is straightforward to realize that $f_\epsilon:\Omega\times X\to X$ is a continuous (in particular, measurable) mapping. Note also that if $\epsilon>0$ is sufficiently small, $f_\epsilon(\omega)$ is $\mathbb{P}$-almost surely an expanding map of class $\mathscr{C}^r$.

Let $\theta:\Omega\to\Omega$ be a measure-preserving measurable transformation on $(\Omega,\mathbb{P})$. For each $\epsilon>0$ and $n\geq 1$, let $f_\epsilon^{(n)}(\omega,x)$ be the fiber component in the n-th iteration of the skew product mapping

$$\Theta_\epsilon(\omega,x)=(\theta\omega,f_\epsilon(\omega,x)), \quad (\omega,x)\in\Omega\times X,$$

where we simply write $\theta\omega$ for $\theta(\omega)$. Setting the notation $f_\epsilon^{(n)}(\omega) = f_\epsilon^{(n)}(\omega, \cdot)$,the explicit form of $f_\epsilon^{(n)}(\omega)$ is

$$f_\epsilon^{(n)}(\omega) = f_\epsilon(\theta^{n-1}\omega) \circ f_\epsilon(\theta^{n-2}\omega) \circ \cdots \circ f_\epsilon(\omega).$$

In [8] and other articles on fiber dynamics, θ is required to be a bimeasurable transformation, i.e., an invertible measurable transformation whose inverse mapping is also measurable (see, for example, [14, 20, 33]; a significant exception is described in Baladi [6]). However, some framework accommodates important examples that are not generally invertible, as shown in Example 6. Let $L^p_\nu(S)$ be the usual L^p space on a measurable space (S, Σ, ν) endowed with the L^p norm $\|\cdot\|_{L^p}$ where $1 \le p \le \infty$. For each $u \in L^\infty_{\mathbb{P}}(\Omega)$, a functional $\ell_\theta u : L^1_{\mathbb{P}}(\Omega) \to \mathbb{C}$ is defined as by $\ell_\theta u(\varphi) = \int u(\omega)\cdot\varphi(\theta\omega)d\mathbb{P}$ for each $\varphi \in L^1_{\mathbb{P}}(\Omega)$. Since $\mathbb{P}$ is an invariant measure, $|\ell_\theta u(\varphi)| \le \|u\|_{L^\infty}\|\varphi\circ\theta\|_{L^1} = \|u\|_{L^\infty}\|\varphi\|_{L^1}$, i.e., $\|\ell_\theta u\|_{(L^1_{\mathbb{P}}(\Omega))^*} \le \|u\|_{L^\infty}$. Thus, by the Riesz representation theorem, $\ell_\theta u \in L^\infty_{\mathbb{P}}(\Omega) \cong (L^1_{\mathbb{P}}(\Omega))^*$ and $\ell_\theta : L^\infty_{\mathbb{P}}(\Omega) \to L^\infty_{\mathbb{P}}(\Omega)$ is a bounded operator on $L^\infty_{\mathbb{P}}(\Omega)$ such that

$$\int \ell_\theta u(\omega)\cdot\varphi(\omega)d\mathbb{P} = \int u(\omega)\cdot\varphi(\theta\omega)d\mathbb{P}, \quad \varphi \in L^1_{\mathbb{P}}(\Omega).$$

(ℓ_θ is called the transfer operator of θ with respect to $\mathbb{P}$.)

Let $\mathscr{C}^{r-1}(X)$ be the space of all complex-valued functions on X of class $\mathscr{C}^{r-1}$ endowed with the usual $\mathscr{C}^{r-1}$ norm $\|\cdot\|_{\mathscr{C}^{r-1}}$, and let m be the normalized Lebesgue measure on X. Let $L^\infty_{\mathbb{P}}(\Omega, \mathscr{C}^{r-1}(X))$ be the Lebesgue-Bochner space of mappings defined on Ω taking values in the Banach space $\mathscr{C}^{r-1}(X)$ endowed with the L^∞ norm $\|u\|_{L^\infty} := \operatorname{ess\,sup}_{\omega\in\Omega}\|u(\omega)\|_{\mathscr{C}^{r-1}}$. Here the usual abuse of notation is adopted (where an L^∞ mapping is identified by its equivalence class). The definition and properties of this space are provided in [22]. Here it is merely stated that if $u \in L^\infty_{\mathbb{P}}(\Omega, \mathscr{C}^{r-1}(X))$, then u is Bochner measurable, i.e., $u = \lim_{n\to\infty} u_n$ $\mathbb{P}$-almost surely, where $u_n : \Omega \to \mathscr{C}^{r-1}(X)$ is a simple function of each $n \ge 1$. Setting $u(\omega, \cdot) = u(\omega)$, for each $x \in X$ the mapping $\omega \mapsto u(\omega, x)$ is $\mathbb{P}$-almost surely the limit of the sequence $\{u_n(\cdot, x)\}_{n\ge1}$ of simple functions, and is thus measurable because $\mathbb{P}$ is a complete probability measure. Furthermore, $\|u(\cdot, x)\|_{L^\infty} \le \|u\|_{L^\infty}$; that is, $u(\cdot, x) \in L^\infty_{\mathbb{P}}(\Omega)$ for each $x \in X$. It is supposed that for ℓ_θ (and therefore θ), there exists a bounded operator $\tilde{\ell}_\theta$ on $L^\infty_{\mathbb{P}}(\Omega, \mathscr{C}^{r-1}(X))$ such that the following holds for each $u \in L^\infty_{\mathbb{P}}(\Omega, \mathscr{C}^{r-1}(X))$, each bounded linear functional $A : \mathscr{C}^{r-1}(X) \to \mathbb{C}$, each bounded operator $\mathcal{A} : \mathscr{C}^{r-1}(X) \to \mathscr{C}^{r-1}(X)$, each $x \in X$ and $\mathbb{P}$-almost every $\omega \in \Omega$:

$$\tilde{\ell}_\theta u(\omega, x) = \ell_\theta[u(\cdot, x)](\omega) \tag{6}$$

$$\ell_\theta[Au(\cdot)](\omega) = A\tilde{\ell}_\theta u(\omega), \quad \tilde{\ell}_\theta[\mathcal{A}u(\cdot)](\omega) = \mathcal{A}\tilde{\ell}_\theta u(\omega), \tag{7}$$

and

$$\|\tilde{\ell}_\theta u\|_{L^\infty} \le \|u\|_{L^\infty}. \tag{8}$$

Now, some definitions are provided on measure-preserving skew-product transformations. Let $\mathcal{B}(X)$ be the Borel σ-field of X. It is known that for each probability measure μ on $\Omega \times X$ with marginal $\mathbb{P}$ on Ω, there exists a function $\mu.(\cdot) : \Omega \times \mathcal{B}(X) \to [0,1]$, $(\omega, B) \mapsto \mu_\omega(B)$ that satisfies the following three conditions: $\omega \mapsto \mu_\omega(B)$ is measurable for each $B \in \mathcal{B}(X)$; μ_ω is $\mathbb{P}$-almost surely a probability measure on X; $\int \varphi d\mu = \int \varphi d\mu_\omega d\mathbb{P}$ for each $\varphi \in L^1_\mu(\Omega \times X)$. This function, which is $\mathbb{P}$-almost surely unique, is called the disintegration of μ [2, Chapter 1]. Let $f : \Omega \times X \to X$ be a measurable mapping. A measure μ on $\Omega \times X$ is called invariant under f when μ is invariant under the skew-product mapping $\Theta(\omega, x) = (\theta\omega, f(\omega, x))$ and the marginal measure of μ coincides with $\mathbb{P}$. [1] Given an absolutely continuous invariant probability measure μ of a measurable mapping $f : \Omega \times X \to X$, the operational forward quenched correlation function $\mathrm{Cor}_{\varphi,u}(\omega, n)$ of $\varphi \in L^1_m(X)$ and $u \in L^\infty_m(X)$ at $\omega \in \Omega$ is defined by

$$\mathrm{Cor}_{\varphi,u}(\omega, n) = \int \varphi \circ f^{(n)}(\omega) \cdot u dm - \int \varphi d\mu_{\theta^n\omega} \int u dm,$$

and we call $\ell^n_\theta \,\mathrm{Cor}_{\varphi,u}(\omega, n)$ the operational backward quenched correlation function of φ and u at $\omega \in \Omega$. (Since μ_ω is $\mathbb{P}$-almost surely absolutely continuous, $\mathrm{Cor}_{\varphi,u}(\cdot, n)$ is in $L^\infty_{\mathbb{P}}(\Omega)$ and $\ell^n_\theta \,\mathrm{Cor}_{\varphi,u}(\cdot, n)$ is well defined.) The backward quenched correlation functions of (f, μ) are said to decay exponentially fast in a Banach space $E \subset L^\infty_m(X)$ when there exist constants $C > 0$ and $0 < \rho < 1$ (independent of ω) such that for any $\varphi \in L^1_m(X)$ and $u \in E$,

$$|\ell^n_\theta \,\mathrm{Cor}_{\varphi,u}(\omega, n)| \le C\rho^n \|\varphi\|_{L^1} \|u\|_E \quad \mathbb{P}\text{-a.s.}, \tag{9}$$

where $\|\cdot\|_E$ is the norm of E. Similarly, the operational integrated correlation functions of (f, μ) decay exponentially fast in a Banach space $E \subset L^\infty_{\mathbb{P}\times m}(\Omega\times X)$ when there exist constants $C > 0$ and $0 < \rho < 1$ (independent of ω) such that for any $\varphi \in L^1_{\mathbb{P}\times m}(\Omega \times X)$ and $u \in E$, the mapping $\Omega \ni \omega \mapsto \mathrm{Cor}_{\varphi(\theta^n\omega),u(\omega)}(\omega, n)$ is integrable for each $n \ge 1$. Setting $\varphi(\omega) = \varphi(\omega, \cdot)$,

$$\left| \int \mathrm{Cor}_{\varphi(\theta^n\cdot),u(\cdot)}(\cdot, n) d\mathbb{P} \right| \le C\rho^n \|\varphi\|_{L^1} \|u\|_E. \tag{10}$$

The smallest number $\bar\rho$ such that (9) (or(10)) holds for any $\rho > \bar\rho$ is called the rate of exponential decay of backward quenched correlation functions (resp. integrated correlation functions) in E. When θ is bimeasurable, since $\ell_\theta u = u \circ \theta^{-1}$ (see Example 6), then $\ell^n_\theta[\mathrm{Cor}_{\varphi(\theta^n\cdot),u(\cdot)}(\cdot, n)](\omega) = \ell^n_\theta[\mathrm{Cor}_{\varphi(\omega),u(\theta^{-n}\omega)}(\cdot, n)](\omega)$ $\mathbb{P}$-almost surely. Thus, the exponential decay of backward quenched correlations in $\mathscr{C}^{r-1}(X)$ yields the exponential decay of forward quenched correlations in $\mathscr{C}^{r-1}(X)$ (i.e., (9) holds, where $\ell^n_\theta \,\mathrm{Cor}_{\varphi,u}(\omega, n)$ is replaced by $\mathrm{Cor}_{\varphi,u}(\omega, n)$) and also the exponential decay of integrated correlations in $L^\infty_{\mathbb{P}}(\Omega, \mathscr{C}^{r-1}(X))$. Under these conditions, the mixing of the skew-product mapping is equivalent to the mixing of the base dynamics (see comments in [20, Subsection 0.2]). As is well known, any expanding map $f : X \to X$ admits a unique absolutely continuous

[1]When θ is a bimeasurable transformation, it follows from Theorem 1.4.5 in [2] which the pushforward measure of μ_ω by $f(\omega)$ coincides with $\mu_{\theta\omega}$ $\mathbb{P}$-almost surely if and only if μ is invariant under f. Such measures μ_ω where $\omega \in \Omega$ are called stationary measures in [8].

ergodic invariant probability measure (abbreviated to aceip) on X with a density function of class $\mathscr{C}^{r-1}$. In addition, the correlations decay exponentially fast in $\mathscr{C}^{r-1}(X)$ (see e.g. [43]). The aceip of the expanding map $f_0 : X \to X$ is denoted by μ^0. Let $h_0 : X \to \mathbb{C}$ be the density function of μ^0. The rate of exponential decay of correlations of (f_0, μ^0) is denoted by ρ_0.

Finally, a Banach space $K_{\mathbb{P}}(\Omega, \mathscr{C}^{r-1}(X))$ of random observables is introduced as the Kolmogorov quotient (by equality $\mathbb{P}$-almost everywhere) of the space

$$\mathcal{K}_{\mathbb{P}}(\Omega, \mathscr{C}^{r-1}(X)) = \left\{ u \in \mathcal{L}^{\infty}_{\mathbb{P}}(\Omega, \mathscr{C}^{r-1}(X)) : \omega \mapsto \int u(\omega) dm \text{ is constant } \mathbb{P}\text{-a.s.} \right\}$$

endowed with the L^∞ norm. Here, $\mathcal{L}^{\infty}_{\mathbb{P}}(\Omega, \mathscr{C}^{r-1}(X))$ is the space of all Bochner measurable mappings $u : \Omega \to \mathscr{C}^{r-1}(X)$ with finite L^∞ norm. (It shall be proved that $K_{\mathbb{P}}(\Omega, \mathscr{C}^{r-1}(X))$ is a Banach space and that $\int u(\cdot) dm$ is measurable.) As before, a mapping in $\mathcal{K}_{\mathbb{P}}(\Omega, \mathscr{C}^{r-1}(X))$ is identified by its equivalence class in $K_{\mathbb{P}}(\Omega, \mathscr{C}^{r-1}(X))$.

The following theorem is essentially an extension of Theorems A, B and C in [8] to perturbations induced by skew-product mappings whose base dynamics satisfy (6), (7) and (8).

Theorem 3. *Let $f_0 : X \to X$ be an expanding map, and $\{f_\epsilon\}_{\epsilon > 0}$ be a family of continuous mappings on $(\Omega, \mathbb{P})$ with values in $\mathscr{C}^r(X, X)$ satisfying* (5)*. Suppose that $\theta : \Omega \to \Omega$ is a measure-preserving transformation satisfying* (6)*,* (7) *and* (8)*. Then, for any sufficiently small $\epsilon > 0$, there exists a unique absolutely continuous invariant probability measure μ^ϵ on $\Omega \times X$ whose density function $h_\epsilon = \frac{d\mu^\epsilon}{d(\mathbb{P} \times m)}$ is in $K_{\mathbb{P}}(\Omega, \mathscr{C}^{r-1}(X))$ in the notation $h_\epsilon(\omega) = h_\epsilon(\omega, \cdot)$, and we have*

$$\operatorname*{ess\,sup}_{\omega \in \Omega} \| h_\epsilon(\omega) - h_0 \|_{\mathscr{C}^{r-1}} \to 0 \quad \text{as } \epsilon \to 0.$$

Moreover, for each sufficiently small $\epsilon > 0$, the backward quenched correlation functions and the integrated correlation functions of $(f_\epsilon, \mu^\epsilon)$ decay exponentially fast with rate $0 < \rho_\epsilon < 1$ in $\mathscr{C}^{r-1}(X)$ and in $K_{\mathbb{P}}(\Omega, \mathscr{C}^{r-1}(X))$, respectively, and we have

$$\varlimsup_{\epsilon \to 0} \rho_\epsilon \le \max\{\rho_0, \Lambda_r(f_0)\}.$$

Remark 4. Bogenschütz [14] and Baladi [6] also investigated stability problems of expanding maps using perturbative spectral approaches. Apart from the invertibility of the base dynamics, Theorem 3 differs from Bogenschütz's result in which he postulated a perturbation lemma for linear cocycles. Therefore, in his result, the "coefficient" C in (9) may depend on ω, and the integrated correlations may not decay exponentially fast, as demonstrated by Buzzi in [20, Appendix A]. Within the setting of mixing base dynamics, Baladi obtained a sharper spectral stability, which yields a more satisfactory result for the decay rate stability; compare [6, Theorem 5 and Proposition 3.1] and her Banach space $\mathcal{B}(\alpha)$ with Theorem 3 and $K_{\mathbb{P}}(\Omega, \mathscr{C}^{r-1}(X))$. However, the quasi-compactness of the transfer operator of the skew-product mapping in the Banach space $\mathcal{B}(\alpha)$ implies the mixing in the skew-product mapping (in particular, the mixing in the base dynamics). Thus, Baladi's Banach space $\mathcal{B}(\alpha)$ is not applicable to setting used in this study, in which the base dynamics are not necessarily mixing.

Remark 5. It follows from Theorem 3 that if $(\theta, \mathbb{P})$ is ergodic, then $(\Theta_\epsilon, \mu^\epsilon)$ is ergodic for any sufficiently small $\epsilon > 0$. Indeed, let $A \in \mathcal{B}(\Omega) \times \mathcal{B}(M)$ be invariant under Θ_ϵ, and suppose that $0 < \mu^\epsilon(A) < 1$. Then, it follows from Theorem 3 and the invariance of A that for each $B \in \mathcal{B}(\Omega) \times \mathcal{B}(M)$, if the length of $B^\omega = \{x \in X : (x, \omega) \in B\}$ is $\mathbb{P}$-almost surely constant (where the constant is denoted as $\ell(B)$), then

$$(\mathbb{P} \times m)(A \cap B) = \mu^\epsilon(A) \cdot (\mathbb{P} \times m)(B). \tag{11}$$

Let $\Gamma_1 = \{\omega \in \Omega : m(A^\omega) = 0\}$. Then, noting that $A^\omega = (f(\omega))^{-1} A^{\theta\omega}$ by the invariance of A and that $f(\omega)$ is non-singular with respect to m for each $\omega \in \Omega$, $\theta^{-1}\Gamma_1 = \Gamma_1$. Since $(\theta, \mathbb{P})$ is ergodic and $\mathbb{P}(\Gamma_1) \neq 1$ (otherwise, $\mu^\epsilon(A) = 0$ by the absolute continuity of μ^ϵ), $\mathbb{P}(\Gamma_1) = 0$. On the other hand, $\Gamma_2 = \{\omega \in \Omega : m(A^\omega) = 1\}$ is not a full measure set since $\mu^\epsilon(A) < 1$. Thus, the set $\Gamma_3 = \{\omega \in \Omega : 0 < m(A^\omega) < 1\}$ is a positive measure set, and we can find a positive measure set $\Gamma \subset \Gamma_3$ and $B_1, B_2 \in \mathcal{B}(\Omega) \times \mathcal{B}(M)$ such that $m(B_1^\omega)$ and $m(B_2^\omega)$ are $\mathbb{P}$-almost surely constant, $\ell(B_1) = \ell(B_2) \neq 0$, $B_1^\omega \cap A^\omega = \emptyset$ and $B_2^\omega \subset A^\omega$ for each $\omega \in \Gamma$, and $B_1^\omega = B_2^\omega$ for each $\omega \in \Omega \backslash \Gamma$. Since these results contradict (11), $(\Theta_\epsilon, \mu^\epsilon)$ is ergodic.

Example 6. We consider examples of measure-preserving transformations satisfying conditions (6), (7), and (8). The most trivial example is a bimeasurable transformation. When $\theta : \Omega \to \Omega$ is bimeasurable, $\ell_\theta u(\omega) = u(\theta^{-1}\omega)$ for each $u \in L^\infty_{\mathbb{P}}(\Omega)$ and $\mathbb{P}$-almost every $\omega \in \Omega$ since $u(\omega) = u(\theta(\theta^{-1}\omega))$. For each $u \in L^\infty_{\mathbb{P}}(\Omega, \mathscr{C}^{r-1}(X))$, let us define $\tilde{\ell}_\theta u : \Omega \to \mathscr{C}^{r-1}(X)$ by $\tilde{\ell}_\theta u = u \circ \theta^{-1}$. Then, $\tilde{\ell}_\theta u$ is Bochner measurable since $\tilde{\ell}_\theta u$ is the composition of the Bochner measurable mapping $u : \Omega \to \mathscr{C}^{r-1}(X)$ and the measurable mapping $\theta^{-1} : \Omega \to \Omega$. It is straightforward to verify that $\tilde{\ell}_\theta u \in L^\infty_{\mathbb{P}}(\Omega, \mathscr{C}^{r-1}(X))$ and that $\tilde{\ell}_\theta$ is a bounded operator on $L^\infty_{\mathbb{P}}(\Omega, \mathscr{C}^{r-1}(X))$ satisfying (6), (7) and (8).

Now we consider a piecewise smooth mapping $\theta : \Omega \to \Omega$ of class $\mathscr{C}^1$ on a compact region $\Omega \subset \mathbb{R}^d$, i.e., Ω is the disjoint union of connected and open subsets $\Gamma_1, \ldots, \Gamma_k$ up to a set of Lebesgue measures 0 such that $\theta|_{\Gamma_j}$ agrees with a $\mathscr{C}^1$ map θ_j defined on a neighborhood of $\overline{\Gamma}_j$ and θ_j is a diffeomorphism on the mapped image for each $1 \leq j \leq k$. For a detailed study of these mappings, the reader is referred to [33]. Let V be the normalized Lebesgue measure on Ω and define the transfer operator $\ell_{\theta,V} : L^1_V(\Omega) \to L^1_V(\Omega)$ of θ with respect to V as

$$\ell_{\theta,V} u = \sum_{j=1}^{k} \frac{1_{\Gamma_j} \cdot u}{|\det D\theta_j|} \circ \theta_j^{-1}, \quad u \in L^1_V(\Omega).$$

From the change of variables formula, it follows that $\int \ell_{\theta,V} u \cdot \varphi dV = \int u \cdot \varphi \circ \theta dV$ for each $u, \varphi \in L^1_V(\Omega)$ satisfying $u \cdot \varphi \circ \theta \in L^1_V(\Omega)$ (in particular, $\varphi \in L^\infty_V(\Omega)$).

Thus, if $\mathbb{P}$ is an absolutely continuous invariant measure of θ, then the density function $p \in L^1_V(\Omega)$ of $\mathbb{P}$ is a fixed point of $\ell_{\theta,V}$. It is assumed that $\mathbb{P}$ is an absolutely continuous invariant probability measure whose density function p is strictly positive V-almost everywhere. Extensive examples of such measure-preserving transformations $(\theta, \mathbb{P})$ are given in [7]. Then, for each $u \in L^\infty_{\mathbb{P}}(\Omega)$ and $\varphi \in L^1_{\mathbb{P}}(\Omega)$, we have

$$\int u \cdot \varphi \circ \theta d\mathbb{P} = \int \ell_{\theta,V}(u \cdot p) \cdot \varphi dV = \int \frac{\ell_{\theta,V}(u \cdot p)}{p} \cdot \varphi d\mathbb{P}.$$

Thus, for each $u \in L^\infty_{\mathbb{P}}(\Omega)$, $\ell_\theta u = \ell_{\theta,V}(u \cdot p)/p$ $\mathbb{P}$-almost surely. For each $u \in L^\infty_{\mathbb{P}}(\Omega, \mathscr{C}^{r-1}(X))$, a mapping $\tilde{\ell}_\theta u : \Omega \to \mathscr{C}^{r-1}(X)$ is defined as $\tilde{\ell}_\theta u = [\sum_{j=1}^k (1_{\Gamma_j} \cdot u \cdot p \cdot |\det D\theta_j|^{-1}) \circ \theta_j^{-1}]/p$. Since every subspaces of a separable metric space $\mathscr{C}^{r-1}(X)$ is itself a separable space (see e.g. [48, Theorem 16.2.b and 16.11]), the (weakly) measurable mappings 1_{Γ_j}, p, $|\det D\theta|^{-1}$ $(1 \le j \le k)$, and therefore $\tilde{\ell}_\theta u$, are Bochner measurable by the Pettis measurability theorem. Note that $\|\tilde{\ell}_\theta u(\omega)\|_{\mathscr{C}^{r-1}} \le \|u\|_{L^\infty} |\ell_\theta 1_\Omega(\omega)|$ $\mathbb{P}$-almost surely, since all of 1_{Γ_j}, p, $|\det D\theta|^{-1}$ $(1 \le j \le k)$ are independent of x. It follows from this and the fact $\ell_\theta 1_\Omega = 1_\Omega$ (note that $\ell_{\theta,V} p = p$) that $\tilde{\ell}_\theta$ is a bounded operator on $L^\infty_{\mathbb{P}}(\Omega, \mathscr{C}^{r-1}(X))$ satisfying (8). It is straightforward to check by construction that $\tilde{\ell}_\theta$ satisfies (6) and (7).

Finally, the one-sided shift $\theta : \Omega \to \Omega$ is considered: $(\Omega, \mathbb{P}) = (\tilde{\Omega}^{\mathbb{N}}, \tilde{\mathbb{P}}^{\mathbb{N}})$ is the product space of a probability separable metric space $(\tilde{\Omega}, \tilde{\mathbb{P}})$, in which $(\theta\omega)_j = \omega_{j+1}$ for each $j \in \mathbb{N} = \{0, 1, \ldots\}$ and each $\omega = (\omega_0\omega_1 \ldots) \in \Omega$. We note that for each $u \in L^\infty_{\mathbb{P}}(\Omega)$ and $\varphi \in L^1_{\mathbb{P}}(\Omega)$,

$$\int \left(\int u(\tilde{\omega}\omega) d\tilde{\mathbb{P}}(\tilde{\omega}) \right) \cdot \varphi(\omega) d\mathbb{P} = \int u(\tilde{\omega}\omega_0\omega_1 \ldots) \cdot \varphi(\theta(\tilde{\omega}\omega_0\omega_1 \ldots)) d\tilde{\mathbb{P}}(\tilde{\omega}) d\mathbb{P}(\omega).$$

Thus, $\ell_\theta u(\omega) = \int u(\tilde{\omega}\omega) d\tilde{\mathbb{P}}(\tilde{\omega})$ for $\mathbb{P}$-almost every $\omega \in \Omega$. By Fubini's theorem (consider the equivalence between the weak measurability and the Bochner measurability of a mapping $u : \Omega \to \mathscr{C}^{r-1}(X)$), for any $u \in L^\infty_{\mathbb{P}}(\Omega, \mathscr{C}^{r-1}(X))$, there exists a Bochner measurable mapping $\tilde{\ell}_\theta u : \Omega \to \mathscr{C}^{r-1}(X)$ given by

$$\tilde{\ell}_\theta u(\omega) = \int u(\tilde{\omega}\omega) d\tilde{\mathbb{P}}(\tilde{\omega}), \quad \omega \in \Omega.$$

Furthermore, (8) for this bounded operator $\tilde{\ell}_\theta$ on $L^\infty_{\mathbb{P}}(\Omega, \mathscr{C}^{r-1}(X))$ follows from the Bochner integrability of $\tilde{\Omega} \ni \tilde{\omega} \mapsto u(\tilde{\omega}\omega)$ for $\mathbb{P}$-almost every $\omega \in \Omega$ (by Fubini's theorem) and the triangle inequality. (6) and (7) are immediately obtained by construction.

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