SMALL DATA BLOW-UP OF L^2 OR H^1-SOLUTION FOR THE SEMILINEAR SCHRÖDINGER EQUATION WITHOUT GAUGE INVARIANCE

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Abstract. We consider the initial value problem for the semilinear Schrödinger equation:

$$\text{(NLS)} \qquad i\partial_t u + \Delta u = \lambda |u|^p, \quad (t,x) \in [0,T) \times \mathbb{R}^n,$$

where $p > 1$, $\lambda \in \mathbb{C}\setminus\{0\}$. In this paper, we will prove a small data blow-up result of L^2 and H^1-solution for (NLS) in $1 < p < 1 + 4/n$. Also, an upper bound of the lifespan will be given (Theorem 2.2).

1. Introduction

We study the initial value problem for the nonlinear Schrödinger equation (NLS) with a non-gauge invariant power nonlinearity:

$$\begin{cases} i\partial_t u + \Delta u = \lambda |u|^p, & (t,x) \in [0,T) \times \mathbb{R}^n, \\ u(0,x) = \varepsilon f(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.1}$$

where u is a complex-valued unknown function of (t,x), $p > 1$, $\lambda \in \mathbb{C}\setminus\{0\}$, $T > 0$, f is a prescribed complex-valued function and $\varepsilon > 0$ is a small parameter.

Our aim in this paper is to improve the result obtained in [13] and determine the critical exponent p_c for (1.1). Here critical exponent means the number with the following properties:

$$\begin{cases} \text{If } p > p_c, & \text{a small data global existence (SDGE) result holds.} \\ \text{If } 1 < p < p_c, & \text{SDGE does not hold.} \end{cases} \tag{1.2}$$

This type problem has been studied extensively for the corresponding nonlinear heat equation, the wave equation and the damped wave equation (see e.g. [4, 17, 26, 28] and their references therein). It is well known that the critical exponent for the heat equation and the damped wave equation is $p_F = 1 + 2/n$ and that for the wave equation is the Strauss exponent $p_S(n-1)$ (defined later). However, that of (1.1) has not been well studied so far (see e.g. [7]).

On the other hand, there are many papers for NLS with a gauge invariant power nonlinearity:

$$i\partial_t \varphi + \Delta \varphi = \mu |\varphi|^{p-1} \varphi, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^n, \tag{1.3}$$

where $\mu \in \mathbb{R}$.

It is well known that large data local well-posedness holds for (1.3) in H^m-sense $(m = 0,1)$ under $1 < p < p_m$, where $p_m = 1 + 4/(n-2m)$ is called H^m-critical

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exponent (if $m = 1$ and $n = 1$ or 2, then p_1 stands ∞) (see e.g. [1, 8, 15, 27]). Moreover, the L^2-norm of those solutions for (1.3) conserves

$$\|\varphi(t)\|_{L^2} = \|\varphi(0)\|_{L^2}, \text{ for any } t \in \mathbb{R}. \tag{1.4}$$

Thus L^2-conservation law and the local well-posedness imply large data global well-posedness of (1.3) in the L^2-sense, in the L^2-subcritical range, i.e. $1 < p < p_0$. In the L^2-critical case, i.e. $p = p_0$, SDGE is well known.

Even if the nonlinearity do not satisfy the gauge invariant property such as (1.1), large data local well-posedness also holds in the L^2-sense under $1 < p \le p_0$. The proof is the same as (1.3). And SDGE also can be obtained for (1.1) in L^2-critical case. However, unlike the gauge invariant nonlinearilty $|u|^{p-1}u$, it is not trivial whether the L^2-conservation law (1.4) holds or not. Thus global well-posedness results for (1.1) are not obvious in the L^2-subcritical case. Especially, in [13], a small data blow-up result was obtained in the case $1 < p \le p_F$. More precisely, it was shown that local solution in the L^2-sense can not be extended globally for some f, no matter how small ε is. Then the following natural question arises: *What happens for the local L^2-solution in the case $p_F \le p < p_0$?* In this paper, we will give an answer of a small data blow-up result in $1 < p < p_0$. This implies that p_F is not critical to L^2-solution for (1.1).

In [10], an upper bound of the lifespan of L^2-solutions was obtained in the case $1 < p < p_F$, though it has not been known in the case $p_F \le p < p_0$. In this paper, we will extend the range of exponents to $1 < p < p_0$ and improve the upper estimate. As the result, it can be seen that the upper estimate of the lifespan is sharp (see below Remark 2.1). The proof of our theorem in this paper looks like previous ones in [13] and [10]. However, there are some different points from their papers. So we should compare the proofs of the results obtained in [13] and [10] with ours. We explain the details in Section 4 (Concluding Remarks).

We are also interested in H^1-solution. For H^1-solution φ to (1.3), the energy conservation law is well known, and so H^1-solution is important from the physical viewpoints. There are many results about global behavior of H^1-solution to (1.3). For example, some blow-up results for H^1-solution to (1.3) were obtained (see [6, 18, 19] etc.). However, the blow-up results obtained in [6, 18, 19] requires the assumption that the data is large. So these results should be distinguished from our small data blow-up results (Theorem 2.2).

Even if the nonlinearity do not satisfy the gauge invariant property such as (1.1), SDGE of H^1-solution is well known in the case $p_0 \le p \le p_1$, (see e.g. Theorem 6.2.1 in [1]). There are no results about global behavior of H^1-solution to (1.1) in the opposite case $p < p_0$. Then we will prove a small data blow-up result of H^1-solution for (1.1) in the L^2-subcritical case. This means that the exponent p_0 is the threshold between SDGE and Blow-up of H^1-solution.

Our main result (Theorem 2.2) can be extended to more general H^s-solution for $s \ge 0$, though we do not pursue this problem for simplicity.

We summarize SDGE and Blow-up results about (1.1) at the table:

$f \setminus p$	1		p_0		p_1	
L^2		Blow-up	SDGE	?		
H^1		Blow-up		SDGE		?

Our proof is based on a test-function method used in [16] to obtain an upper bound of lifespan for some parabolic equation. This method was applied to other equations (see [24, 10, 12]). However, their argument does not work for L^2-data. Then we modify this method to fit (1.1) in L^2-setting.

2. Main Result

Hereafter m stands for 0 or 1 for simplicity. In Subsection 2.2, we state our main result, namely, a small data blow-up result for H^m-solution. In Subsection 2.3, we recall lower bounds of the lifespan.

2.1. Local Well-Posedness in H^m.

In this subsection, we state our main result in this paper, which gives an upper bound of the lifespan T_ε for H^m-solution in the L^2-subcritical case.

At first, we define the H^m-solution and the lifespan. We rewrite (1.1) into the integral equation

$$u(t) = \varepsilon U(t) f - i\lambda \int_0^t U(t-s) |u|^p \, ds, \tag{2.1}$$

where $U(t) = \exp(it\Delta)$. We consider the following function space:

$$X_T^m = C([0,T); H^m) \cap L_t^\gamma \left(0,T; W_x^{m,\rho}\right),$$

where $\rho = p+1$, $\gamma = 4(p+1)/\{n(p-1)\}$. The function $u \in X_T^m$ which satisfies (2.1) is called H^m-solution. We remember $p_m = 1 + 4/(n-2m)$. The local well-posedness of H^m-solution for large data is well known:

Proposition 2.1. *Let $1 < p < p_m$, $\lambda \in \mathbb{C}$, $\varepsilon > 0$ and $f \in H^m$. Then there exist a positive time $T = T(\varepsilon, \|f\|_{H^m}) > 0$ and a unique solution $u \in X_T^m$ of (2.1).*

For the proof, see [14, 27]. Let T_ε be the maximal existence time (lifespan):

$$T_\varepsilon = T(m,\varepsilon,f) \equiv \sup\left\{T \in (0,\infty]\,;\ \text{there exists a unique solution } u \in X_T^m \text{ to (2.1)}\right\}.$$

2.2. Main Result (small data blow-up).

Next, we state our small data blow-up result for the H^m-solution. We denote $\lambda_1 = \mathrm{Re}\lambda$, $\lambda_2 = \mathrm{Im}\lambda$, $f_1 = \mathrm{Re}f$ and $f_2 = \mathrm{Im}f$. We choose the data as follows:

$$\lambda_2 f_1(x) \ \text{ or } \ -\lambda_1 f_2(x) \geq \begin{cases} |x|^{-k}, & \text{if } |x| > 1, \\ 0, & \text{if } |x| \leq 1, \end{cases} \tag{2.2}$$

where $n/2 < k < 2/(p-1)$. We note that such k exists if and only if $1 < p < p_0$. The following is valid:

Theorem 2.2. *Let $1 < p < p_0$, $\lambda \in \mathbb{C} \setminus \{0\}$ and $f \in H^m$. If f satisfies (2.2), then there exist $\varepsilon_0 > 0$ and $C = C(k,p,\lambda) > 0$ such that*

$$T_\varepsilon \leq C\varepsilon^{1/\kappa}$$

for any $\varepsilon \in (0,\varepsilon_0)$, where $\kappa \equiv k/2 - 1/(p-1)$. Moreover, the H^m-norm of solutions blows up in finite time:

$$\lim_{t \to T_\varepsilon - 0} \|u(t)\|_{H^m} = \infty. \tag{2.3}$$

This is an improvement of Theorem 2.2 of [13] and Theorem 2.4 of [10].

Remark 2.1. *Theorem 2.2 and the lower bound of the lifespan (obtained in Corollary 2.3 and Corollary 2.4 below) imply*

$$\varepsilon^{1/\omega_0} \lesssim T_\varepsilon \lesssim \varepsilon^{1/\kappa}$$

where $\omega_0 = n/4 - 1/(p-1)$ *for small* $\varepsilon > 0$. *The optimality of estimates to the lifespan is still open since it is not allowed to take* k *as* $n/2$. *Nevertheless, our theorem implies that the optimal order is* $1/\omega_0$, *since* $\lim_{k\to n/2+0}\kappa = \omega_0$.

Remark 2.2. *(2.3) with* $m = 0$ *means that the* L^2*-conservation law does not hold for (1.1).*

Remark 2.3. *By the same argument of the main result, we can prove the same small data blow-up result of* L^2*-solution for* $L^2 \cap L^r$*-data in the case* $1 < p < 1+2r/n$ *for* $r \in [1,2]$. *Indeed, if* $1 < p < 1+2r/n$, *then we can choose* $k \in (n/r, 2/(p-1))$ *and so we can take the initial function* f *belonging to* $L^2 \cap L^r$ *and satisfying (2.2) for* $n/r < k < 2/(p-1)$.

In particular, if we choose $r = 1+1/p \in [1,2]$, *then we get the small data blow-up result for* $1 < p < p_S$, *where* p_S *is the Strauss exponent which is the positive root of*

$$nx^2 - (n+2)\,x - 2 = 0. \tag{2.4}$$

We note that SDGE result holds in the case $p_S < p < p_0$ *for* $L^2 \cap L^{1+1/p}$*-data.*

Remark 2.4. *For the critical case* $p = p_m$ *and the supercritical case* $p > p_m$, *some non-existence results to (1.1) were also obtained in the recent paper* [11].

2.3. **Lower Bounds of the lifespan.** In this subsection, we recall some results for lower bounds of the lifespan. These lower bounds follows from the Proposition 2.1 immediately:

Corollary 2.3. *Under the same assumptions as in Proposition 2.1, the following estimate is valid:*

$$T_\varepsilon \geq C\varepsilon^{1/\omega_m},$$

where

$$\omega_m = \begin{cases} n/4 - 1/(p-1), & \text{if } m = 0, \\ n/2\,(p+1) - 1/(p-1), & \text{if } m = 1, \end{cases}$$

and $C = C(n, p, \|f\|_{H^m})$ *is a positive constant.*

If p is restricted to $1 < p < p_0$ and $\varepsilon \ll 1$, then the lower estimate of H^1-solution can be improved as follows:

Corollary 2.4. *Let* $1 < p < p_0$, $\lambda \in \mathbb{C}$, $\varepsilon > 0$ *and* $f \in H^1$. *Then the following estimate is valid:*

$$T_\varepsilon \geq C\varepsilon^{1/\omega_0},$$

where $C = C(n, p, \|f\|_{H^1})$ *is a positive constant.*

Moreover, SDGE is well known in H^m-critical:

Corollary 2.5. *Let* $p = p_m$, $\lambda \in \mathbb{C}$ *and* $f \in H^m$. *Then there exists* $\varepsilon_0 > 0$ *such that* $T_\varepsilon = \infty$ *for* $\varepsilon \in (0, \varepsilon_0)$.

For the proof, see e.g. [1, 2, 15].

Here we remark a SDGE result of H^1-solution for (1.1) in $p_0 \leq p < p_1$:

Proposition 2.6. *Let $p_0 \le p < p_1$, $\lambda \in \mathbb{C}$ and $f \in H^1$. Then there exists $\varepsilon_0 > 0$ such that $T_\varepsilon = \infty$ for $\varepsilon \in (0, \varepsilon_0)$.*

For the proof, see Theorem 6.2.1 in [1].

3. Proof of Theorem 2.2

3.1. **Test-function method.** In this subsection, we prepare some integral inequalities by using some test-functions. We introduce a non-negative smooth function ϕ as follows, which was constructed in the papers [3, 5]:

$$\phi(x) = \phi(|x|), \ \phi(0) = 1, \ 0 < \phi(x) \le 1 \text{ for } |x| > 0,$$

where $\phi(|x|)$ is decreasing of $|x|$ and $\phi(|x|) \to 0$ as $|x| \to \infty$ sufficiently fast. Moreover, there exists $\mu > 0$ such that

$$|\Delta\phi| \le \mu\phi, \quad x \in \mathbb{R}^n, \tag{3.1}$$

and $\|\phi\|_{L^1} = 1$. This can be done by letting $\phi(r) = e^{-r^\nu}$ for $r \gg 1$ with $\nu \in (0,1]$ and extending ϕ to $[0,\infty)$ by a smooth approximation. Let θ be sufficiently large and

$$\eta(t) = \eta_T(t) = \begin{cases} 0, & \text{if } t > T, \\ (1 - t/T)^\theta, & \text{if } 0 \le t \le T, \end{cases}$$

where $T > 0$. Furthermore, set $\eta_R(t) = \eta\left(t/R^2\right)$, $\phi_R(x) = \phi(x/R)$ and $\psi_R(t,x) = \eta_R(t)\phi_R(x)$ for $R > 0$. We also denote $L_R = \left[0, TR^2\right)$.

First, we reduce the integral equation (2.1) into the weak form. Hereafter, solution stands for L^2 or H^1-solution.

Lemma 3.1. *Let u be a solution of (1.1) on $[0, T_\varepsilon)$. Then u satisfies*

$$\begin{aligned} &\int_{L_R\times\mathbb{R}^n} u(-i\partial_t(\psi_R) + \Delta(\psi_R))dxdt \\ &= i\varepsilon\int_{\mathbb{R}^n} f(x)\phi_R(x)\,dx + \lambda\int_{L_R\times\mathbb{R}^n} |u|^p\psi_R dxdt, \end{aligned} \tag{3.2}$$

for any $T, R > 0$ with $TR^2 < T_\varepsilon$.

This lemma can be proved in the standard manner (see Proposition 3.1 in [13]).

Next, we will lead an integral inequality. Hereafter we only consider the case of $\lambda_1 > 0$ for simplicity. The other cases can be treated in the almost same way.

We introduce some notations:

$$\begin{aligned} I_R(T) &= \int_{[0,TR^2)\times\mathbb{R}^n} |u|^p\psi_R dxdt, \\ J_R &= \varepsilon\int_{\mathbb{R}^n} -f_2(x)\phi(x/R)\,dx, \\ A(T) &= \left(\int_{[0,T)\times\mathbb{R}^n} |\partial_t\eta(t)|^q\eta(t)^{-q/p}\phi(x)\,dxdt\right)^{1/q} \end{aligned}$$

and

$$B(T) = \mu\left(\int_{[0,T)\times\mathbb{R}^n} \eta_T(t)\phi(x)\,dxdt\right)^{1/q},$$

where $q = p/(p-1)$. By the direct computation, we have

$$A(T) = a_p T^{-1/p},\ B(T) = \mu b_p T^{1/q}, \tag{3.3}$$

where $a_p = \theta\{\theta - 1/(p-1)\}^{-1/q}$, $b_p = (\theta+1)^{-1/q}$.We also denote $D(T) = A(T) + B(T)$. Then we have the following:

Lemma 3.2. *Let u be a solution of (1.1) on $[0, T_\varepsilon)$. Then the following inequality holds:*

$$\lambda_1 I_R(T) + J_R \le R^s I_R(T)^{1/p} D(T) \tag{3.4}$$

for any $T, R > 0$ with $TR^2 < T_\varepsilon$, where $s = -2 + (2+n)/q$.

Proof. Since u is a solution on $[0, T_\varepsilon)$ and $TR^2 < T_\varepsilon$, we get (3.2). Moreover, note that $\lambda_1 > 0$, by taking real part as (3.2), we obtain

$$\begin{aligned} \lambda_1 I_R(T) + J_R &= \int_{L_R \times \mathbb{R}^n} \mathrm{Re}\{u(-i\partial_t(\psi_R) + \Delta(\psi_R))\} dxdt \\ &\le \int_{L_R \times \mathbb{R}^n} |u| \{|\partial_t(\psi_R)| + |\Delta(\psi_R)|\} dxdt \\ &\equiv K_R^1 + K_R^2 \end{aligned} \tag{3.5}$$

We note that $(\partial_t \eta)(t) = 0$ except on $(0, T)$. By using the identity

$$\partial_t \psi_R(t, x) = R^{-2} \phi_R(x) (\partial_t \eta)(t/R^2)$$

and the Hölder inequality, we can get

$$\begin{aligned} K_R^1 &= R^{-2} \int_{L_R \times \mathbb{R}^n} |u| \psi_R^{1/p} |(\partial_t \eta)(t/R^2)| \eta_R^{-1/p} \phi_R^{1/q} dxdt \\ &\le R^{-2} I_R(T)^{1/p} \left(\int_{L_R \times \mathbb{R}^n} |(\partial_t \eta)(t/R^2)|^q \eta_R^{-q/p} \phi_R dxdt \right)^{1/q} \\ &= I_R(T)^{1/p} A(T) R^s, \end{aligned} \tag{3.6}$$

where we have used the changing variables with $t/R^2 = t'$ and $x/R = x'$ to obtain the last identity. Next, by the identity $\Delta(\phi(x/R)) = R^{-2}(\Delta\phi)(x/R)$, the Hölder inequality and the estimate (3.1), we have

$$\begin{aligned} K_R^2 &= R^{-2} \int_{L_R \times \mathbb{R}^n} |u| \eta(t/R^2) |(\Delta\phi)(x/R)| dxdt \\ &\le \mu R^{-2} \int_{L_R \times \mathbb{R}^n} |u| \psi_R dxdt \\ &\le \mu R^{-2} I_R(T)^{1/p} \left(\int_{L_R \times \mathbb{R}^n} \psi_R dxdt \right)^{1/q} \\ &= I_R(T)^{1/p} B(T) R^s, \end{aligned} \tag{3.7}$$

where we have used the changing variables again. By combining the estimates (3.5)-(3.7), we have the conclusion. □

3.2. **Upper estimate of J_R.** Next, we give an upper bound of J_R. Let $\sigma > 0$ and $0 < \omega < 1$. We introduce the function

$$\Psi(\sigma, \omega) \equiv \max_{x \geq 0} (\sigma x^{\omega} - x) = (1 - \omega) \omega^{\frac{\omega}{1-\omega}} \sigma^{\frac{1}{1-\omega}}. \tag{3.8}$$

The following estimate is valid:

Lemma 3.3. *Let u be a solution of (1.1) on $[0, T_\varepsilon)$. Then the estimate*

$$J_R \leq C_1 R^{sq} D(T)^q \tag{3.9}$$

holds for any $T, R > 0$ with $TR^2 < T_\varepsilon$, where $C_1 = \lambda_1^{1-q} (p-1)(1/p)^q$.

The proof of this lemma is based on that of Theorem 3.3 in [16] and Theorem 2.2 in [24].

Proof. Since u is a solution on $[0, T_\varepsilon)$, by using (3.4), we obtain

$$J_R \leq R^s D(T) I_R(T)^{1/p} - \lambda_1 I_R(T) \leq \lambda_1 \Psi(D(T) R^s / \lambda_1, 1/p).$$

By (3.8), we have (3.9), which completes the proof of the lemma. □

3.3. **Proof of Theorem 2.2.** In this subsection, we give a proof of Theorem 2.2. When $\lambda_1 > 0$, we may assume that f_2 satisfies

$$-f_2(x) \geq \begin{cases} \lambda_1^{-1} |x|^{-k}, & \text{if } |x| > 1, \\ 0, & \text{if } |x| \leq 1, \end{cases} \tag{3.10}$$

where $n/2 < k < 2/(p-1)$.

Proof. First, we note that by Corollary 2.3, there exists $\varepsilon_0 > 0$ such that $T_\varepsilon > 1$ for any $\varepsilon \in (0, \varepsilon_0)$. Next, we consider the lower bound of J_R. By changing variables and (3.10), we have

$$\begin{aligned} J_R &= \varepsilon R^n \int_{\mathbb{R}^n} -f_2(Rx) \phi(x)\, dx \\ &\geq \lambda_1^{-1} \varepsilon R^{n-k} \int_{|x| \geq 1/R} |x|^{-k} \phi(x)\, dx \\ &\geq \lambda_1^{-1} \varepsilon R^{n-k} \int_{|x| \geq 1/R_0} |x|^{-k} \phi(x)\, dx = C_k \varepsilon R^{n-k}. \end{aligned}$$

for any $R > R_0$, where $0 < R_0 < \left(a_p^{-1} b_p\right)^{1/2}$ is a constant.

Next, let $\tau \in (1, T_\varepsilon)$ and $R > R_0$. By using (3.9) with $T = \tau R^{-2}$, we have

$$\varepsilon \leq C_k^{-1} C_1 \left\{ R^s D\left(\tau R^{-2}\right) \right\}^q R^{-n+k} \equiv C_2 H(\tau, R), \tag{3.11}$$

where $C_2 = C_k^{-1} C_1$. By (3.3), we can rewrite H as

$$H(\tau, R) \equiv R^{-n+k} \left\{ D\left(\tau R^{-2}\right) R^s \right\}^q = \left(a_p \tau^{-1/p} R^{\alpha_1} + \mu b_p \tau^{1/q} R^{-\alpha_2} \right)^q, \tag{3.12}$$

where $\alpha_1 = k/q$, $\alpha_2 = 2 - k/q$.

For any $\tau \in (1, T_\varepsilon)$, setting $R_\tau = \left(a_p^{-1} b_p \tau\right)^{1/2} > R_0$, we have

$$H(\tau, R_\tau) = C_3 \tau^{\kappa}, \tag{3.13}$$

where $\kappa = k/2 - 1/(p-1)$ and $C_3 = (1+\mu)^q a_p^{(q/2)\alpha_2} b_p^{(q/2)\alpha_1}$. By combining (3.11) and (3.13), we have $\varepsilon \le C_4 \tau^\kappa$, with $C_4 = C_2 C_3$. From the assumption $k < 2/(p-1)$, we obtain $\kappa < 0$. Therefore, we can get

$$\tau \le C\varepsilon^{1/\kappa}$$

for any $\tau \in (1, T_\varepsilon)$, with some $C > 0$. Finally, we obtain $T_\varepsilon \le C\varepsilon^{1/\kappa}$.

Blow-up of L^2 or H^1-norm can be proved in the standard way (see e.g. [1, 13]). We will omit the detail, which completes the proof of the theorem. □

4. Concluding Remarks

- **(The differences of the proofs.)** We explain the differences of the proofs among Theorem 2.2 in [13], Theorem 2.4 in [10] and Theorem 2.2 in the present paper.

  There are two major different points between Theorem 2.2 in [13] and Theorem 2.2 in this paper. Firstly the proof of Theorem 2.2 in [13] is based on a contradiction argument. More precisely, the first author and Wakasugi [13] assume that $T_\varepsilon = \infty$ and they derive $u \equiv 0$ on $[0,\infty) \times \mathbb{R}^n$ and lead a contradiction to the sign condition on the data f. However, we do not use the contradiction argument in the proof of Theorem 2.2 in this paper, and so we can derive the upper estimate (2.2) of the lifespan. Secondly, in the proof of Theorem 2.2 in [13], they used the Lebesgue convergence theorem to obtain (4.5) in their paper. To do so, they needed the assumption that the data f belongs to L^1. For the proof of Theorem 2.2 in this paper, we choose the data f more concretely, however in our argument, the data f may not belong to L^1.

  There are also two remarkably different points between Theorem 2.4 in [10] and Theorem 2.2 in this paper. Firstly the proof of Theorem 2.2 is much simpler and shorter than that of Theorem 2.4 in [10]. Secondly, in the proof of Theorem 2.4 in [10], the result of Theorem 2.2 in [13] was used, and so it was assumed that the data f belongs to L^1-space. We emphasize that we do not need the result of Theorem 2.2 in [13] in order to get our theorem.
- **(Faster decay case.)** We should consider the global behavior of the solution in the case where the function f decays faster near infinity than ones in our main theorem (Theorem 2.2). If $1 < p \le 1 + 2/n$ and the function f decays faster near infinity as f belongs to L^1, then the corresponding solution to (1.1) blows up in finite time (See [13]). When $1 + 2/n < p < p_0$, $\varepsilon \ll 1$, and the function f decays faster, we do not know how the solution to (1.1) behaves.

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