

# Yang-Baxter deformations of Minkowski spacetime

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**ABSTRACT:** We study Yang-Baxter deformations of 4D Minkowski spacetime. The Yang-Baxter sigma model description was originally developed for principal chiral models based on a modified classical Yang-Baxter equation. It has been extended to coset curved spaces and models based on the usual classical Yang-Baxter equation. On the other hand, for flat space, there is the obvious problem that the standard bilinear form degenerates if we employ the familiar coset Poincaré group/Lorentz group. Instead we consider a slice of AdS<sub>5</sub> by embedding the 4D Poincaré group into the 4D conformal group SO(2,4). With this procedure we obtain metrics and  $B$ -fields as Yang-Baxter deformations which correspond to well-known configurations such as T-duals of Melvin backgrounds, Hashimoto-Sethi and Spradlin-Takayanagi-Volovich backgrounds, the T-dual of Grant space, pp-waves, and T-duals of dS<sub>4</sub> and AdS<sub>4</sub>. Finally we consider a deformation with a classical  $r$ -matrix of Drinfeld-Jimbo type and explicitly derive the associated metric and  $B$ -field which we conjecture to correspond to a new integrable system.

**KEYWORDS:** Sigma Models, AdS-CFT Correspondence, Integrable Field Theories, String Duality

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## 1 Introduction

The integrability of 2D non-linear sigma models such as  $O(N)$ -invariant relativistic field theory [1, 2] has been extensively studied in various contexts (for a comprehensive book, see [3]). A prominent example is the AdS/CFT correspondence [4]. The Green-Schwarz action of type IIB string theory on AdS<sub>5</sub> × S<sup>5</sup>, which is often abbreviated as the AdS<sub>5</sub> × S<sup>5</sup> superstring, was constructed in [5] and has been shown to be classically integrable in the sense that the Lax pair exists (i.e. kinematical integrability) [6]. For related topics, see the review [7].

The study of integrable deformations of integrable non-linear sigma models is an interesting topic. Deformations of  $S^3$  and  $\text{AdS}_3$  have been well investigated [8–14, 17–34].<sup>1</sup> A systematic way is the Yang-Baxter sigma model approach proposed by Klimcik [11–13]. It was originally invented for principal chiral models based on the modified classical Yang-Baxter equation (mCYBE). It was generalized to coset spaces [22] and then extended to the standard classical Yang-Baxter equation (CYBE). In particular, squashed  $S^3$  [8] and 3D Schrödinger spacetime [35] are associated with the mCYBE and the CYBE, respectively.<sup>2</sup>

An important application of the Yang-Baxter sigma model description is integrable deformations of the  $\text{AdS}_5 \times S^5$  superstring. A  $q$ -deformation with a classical  $r$ -matrix satisfying the mCYBE has been studied in [36, 37]. Jordanian deformations based on the CYBE have been proposed in [38]. In the latter case, there are a lot of classical  $r$ -matrices satisfying the CYBE and some of them are associated with well-known gravitational backgrounds, such as Lunin-Maldacena-Frolov backgrounds [39, 40], gravity duals for non-commutative gauge theories [41, 42], Schrödinger spacetimes [43–47] and gravity duals for dipole theories [48–52], as shown in a series of works [53–57]. Very recently, the reality of the classical action has been revisited in [58] and a unified picture of deformed integrable sigma models has been provided in [59].

It should be remarked that the Yang-Baxter sigma model approach works well even for non-integrable deformations. The case of  $\text{AdS}_5 \times T^{1,1}$  is known to be non-integrable [60]. However, deformations of this background [39, 61] can be reproduced as Yang-Baxter deformations [62]. The above relations between gravity solutions and classical  $r$ -matrices may be called the gravity/CYBE correspondence (for a brief summary, see [63]).

It would be very interesting to generalize a correspondence of this type to the case of flat space, instead of  $\text{AdS}_5 \times S^5$ . In particular, one expects a relationship between certain classical  $r$ -matrices and Melvin twists [64]. Therefore, in this paper, we consider Yang-Baxter deformations of 4D Minkowski spacetime. For flat space, we however immediately encounter a problem if we employ the familiar coset Poincaré group/Lorentz group, namely the degeneracy of the bilinear form. A possible resolution<sup>3</sup> is to consider instead a slice of  $\text{AdS}_5$  by embedding the 4D Poincaré group into the 4D conformal group  $\text{SO}(2, 4)$ , making a truncation and then deforming the resulting theory. This embedding appears to work well, as we can reproduce the deformed metric and  $B$ -field associated to well-known backgrounds such as T-duals of Melvin backgrounds [67–69], Hashimoto-Sethi backgrounds [70], time-dependent backgrounds of Spradlin-Takayanagi-Volovich [71], the T-dual of Grant space [72], pp-wave backgrounds, and T-duals of  $\text{dS}_4$  and  $\text{AdS}_4$ , as Yang-Baxter deformations. Finally we consider a deformation based on a classical  $r$ -matrix of Drinfeld-Jimbo type [73–75] and explicitly derive the associated metric and  $B$ -field.

In the following, we embed most of these 4D  $\sigma$ -models into string theory. In doing so, one can hope to understand the integrability of said models from a string theory perspective. We start with the simplest ansatz, in which we assume the extra six dimensions to be flat

<sup>1</sup>Recent work on  $\lambda$ -deformations can be found in [14–16].

<sup>2</sup>Both backgrounds can also be obtained via  $TsT$ -transformations [23, 82].

<sup>3</sup>Another way is to employ a generalized symmetric two-form [65]. For a generalization to Schrödinger spacetimes, see [66].

(as resulting from a  $T^6$ -compactification). Here, we will be looking for solutions which contain only a dilaton and no Ramond-Ramond-fields. It turns out that for a large class of the models studied, such a string theory embedding can be found. They all turn out to be  $TsT$  transformations of flat space.<sup>4</sup> As we are starting from an integrable background (i.e. flat space with identifications) and T-duality preserves integrability [76–82], this fact explains the integrability of these models from a string theory point of view.

For other classes of  $r$ -matrices, this is not the case, but they can instead be related to known integrable models with  $dS_4$  or  $AdS_4$  target space via T-duality. In order to complete them to full string theory solutions, the introduction of RR-fields is necessary (in the case of  $dS_4$ , even imaginary RR-form fields are needed), so these models cannot be captured with the simplest possible ansatz. A similar approach in the case of the  $AdS_2 \times S^2$  geometry has been discussed in [83], where type II theories are reduced to 4D theories plus a  $T^6$ .

We would like to stress that there are two distinct parts to this article. The first part treats deformations that are clearly integrable, which can be verified independently from the method presented here. From section 4 on, however, the general statement of integrability of the presented class of examples is a conjecture, which albeit plausible, does at this point not have a formal proof. However, some of our main examples indeed can be verified independently to be integrable. As discussed in the conclusions, a formal proof of the integrability for the class of non-twist deformations is desirable, but exceeds the scope of this work.

The plan of this paper is as follows. In section 2, we introduce Yang-Baxter deformations of 4D Minkowski spacetime. In sections 3–5, we will provide some examples of classical  $r$ -matrices and the associated metrics and two-form  $B$ -fields. There is in particular the class of models that correspond to  $TsT$ -transformations of flat space (section 3). In section 4, we consider non-twist cases including the S-dual of the pp-wave background, T-duals of  $dS_4$  and  $AdS_4$ , and two-parameter deformations. In section 5, we extend the formulation from the CYBE to the mCYBE and then study a deformation with a classical  $r$ -matrix of Drinfeld-Jimbo type. Section 6 is devoted to conclusion and discussion.

## 2 Deformations of 4D Minkowski spacetime

In this section we consider Yang-Baxter deformations of 2D sigma-model actions whose target space is given by 4D Minkowski spacetime.

### 2.1 Coset construction of $AdS_5$ revisited

Let us remind ourselves of the coset construction of  $AdS_5$  with Poincaré coordinates.

It is well known that  $AdS_5$  can be represented by a symmetric coset as

$$AdS_5 = \frac{SO(2,4)}{SO(1,4)} \quad (2.1)$$

and its metric can be computed via a coset construction. To express the generators of the Lie algebras  $\mathfrak{so}(2,4)$  and  $\mathfrak{so}(1,4)$  it is necessary to first introduce some quantities.

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<sup>4</sup>The correspondence between  $TsT$ -transformations and Abelian  $r$ -matrices has been indicated in a series of works [53–55, 57, 58, 62, 63].

We first introduce the gamma matrices  $\gamma_\mu$  and  $\gamma_5$  defined as

$$\begin{aligned} \gamma_1 &= \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, & \gamma_2 &= \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, & \gamma_3 &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \\ \gamma_0 &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, & \gamma_5 &= -i\gamma_0\gamma_1\gamma_2\gamma_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \end{aligned} \quad (2.2)$$

One can introduce  $n_{\mu\nu}$  and  $n_{\mu 5}$  defined as

$$n_{\mu\nu} \equiv \frac{1}{4}[\gamma_\mu, \gamma_\nu], \quad n_{\mu 5} \equiv \frac{1}{4}[\gamma_\mu, \gamma_5]. \quad (2.3)$$

Then the Lie algebra  $\mathfrak{so}(2, 4)$  is spanned by the above quantities,

$$\mathfrak{so}(2, 4) = \text{span}_{\mathbb{R}}\{ \gamma_\mu, \gamma_5, n_{\mu\nu}, n_{\mu 5} \mid \mu, \nu = 0, 1, 2, 3 \}, \quad (2.4)$$

and the subalgebra  $\mathfrak{so}(1, 4)$  is generated by

$$\mathfrak{so}(1, 4) = \text{span}_{\mathbb{R}}\{ n_{\mu\nu}, n_{\mu 5} \mid \mu, \nu = 0, 1, 2, 3 \}. \quad (2.5)$$

Now we can compute the metric of  $\text{AdS}_5$  with Poincaré coordinates by using a coset representative,

$$g = \exp\left[p_0 x^0 + p_1 x^1 + p_2 x^2 + p_3 x^3\right] \exp\left[\gamma_5 \frac{1}{2} \log z\right], \quad (2.6)$$

where the  $p_\mu$  ( $\mu = 0, 1, 2, 3$ ) are defined as

$$p_\mu \equiv \frac{1}{2}(\gamma_\mu - 2n_{\mu 5}). \quad (2.7)$$

The left-invariant one-form  $A = g^{-1}dg$  can now be evaluated easily. The metric is defined as

$$ds^2 = g_{MN}dx^M dx^N = \text{Tr}(A\bar{P}(A)), \quad (2.8)$$

where  $\bar{P}$  is a coset projector from  $\mathfrak{so}(2, 4)$  to  $\mathfrak{so}(2, 4)/\mathfrak{so}(1, 4)$  and is defined as

$$\begin{aligned} \bar{P}(x) &\equiv \gamma_0 \frac{\text{Tr}(\gamma_0 x)}{\text{Tr}(\gamma_0^2)} + \sum_{i=1}^3 \gamma_i \frac{\text{Tr}(\gamma_i x)}{\text{Tr}(\gamma_i^2)} + \gamma_5 \frac{\text{Tr}(\gamma_5 x)}{\text{Tr}(\gamma_5^2)} \\ &= \frac{1}{4} \left[ -\gamma_0 \text{Tr}(\gamma_0 x) + \sum_{i=1}^3 \gamma_i \text{Tr}(\gamma_i x) + \gamma_5 \text{Tr}(\gamma_5 x) \right] \quad \text{for } x \in \mathfrak{so}(2, 4). \end{aligned} \quad (2.9)$$

The resulting metric is given by

$$ds^2 = \frac{-(dx^0)^2 + \sum_{i=1}^3 (dx^i)^2 + dz^2}{z^2}. \quad (2.10)$$

Note that the AdS radius is set to 1.

## 2.2 A conformal embedding of 4D Minkowski spacetime

Here we are interested in a coset construction of 4D Minkowski spacetime. However, there is the obvious problem that the standard bilinear form degenerates if we naively employ the usual coset  $ISO(1,3)/SO(1,3)$ . To avoid this, it is convenient to represent 4D Minkowski spacetime instead as a slice of  $AdS_5$  in Poincaré coordinates.

A possible representation of the group element  $g$  is the following:

$$g = \exp \left[ p_0 x^0 + p_1 x^1 + p_2 x^2 + p_3 x^3 \right]. \quad (2.11)$$

Unlike in formula (2.6), the radial coordinate  $z$  does not appear in the above expression. In other words, to express 4D Minkowski spacetime, a section at  $z = 1$  of  $AdS_5$  has been taken at this stage:

$$\text{4D Minkowski spacetime} = AdS_5|_{z=1}. \quad (2.12)$$

Note here that the 4D Poincaré algebra  $\mathfrak{iso}(1,3)$  and the 4D Lorentz algebra  $\mathfrak{so}(1,3)$  are generated by the following generators, respectively,

$$\begin{aligned} \mathfrak{iso}(1,3) &= \text{span}_{\mathbb{R}} \{ n_{\mu\nu}, p_\mu \mid \mu, \nu = 0, 1, 2, 3 \}, \\ \mathfrak{so}(1,3) &= \text{span}_{\mathbb{R}} \{ n_{\mu\nu} \mid \mu, \nu = 0, 1, 2, 3 \}. \end{aligned} \quad (2.13)$$

Thus it makes sense to use the generators  $p_\mu$  to parameterize the coset representative of  $ISO(1,3)/SO(1,3)$  as (2.11). Eventually the left-invariant one-form  $A = g^{-1}dg$  is written as a linear combination of  $p_\mu$ .

By dropping  $\gamma_5$  of  $\bar{P}$  in (2.9), we introduce the projector for 4D Minkowski spacetime by

$$P(x) = \frac{1}{4} \left[ -\gamma_0 \text{Tr}(\gamma_0 x) + \sum_{i=1}^3 \gamma_i \text{Tr}(\gamma_i x) \right] \quad \text{for } x \in \mathfrak{so}(2,4). \quad (2.14)$$

Then, it is straightforward to compute the metric,

$$ds^2 = \text{Tr}(AP(A)) = -(dx^0)^2 + \sum_{i=1}^3 (dx^i)^2. \quad (2.15)$$

This result is the starting point of our argument in the following.

The definition of the projector (2.14) is justified as follows. From an algebraic point of view, we are able to realize 4D Minkowski spacetime as an embedded coset by considering the quotient of  $\mathfrak{so}(2,4)/\mathfrak{so}(1,4)$  by  $\gamma_5$ . That is as vector spaces

$$\frac{\mathfrak{iso}(1,3)}{\mathfrak{so}(1,3)} = \frac{\mathfrak{so}(2,4)}{\mathfrak{so}(1,4) \oplus \text{span}_{\mathbb{R}}\{\gamma_5\}}, \quad (2.16)$$

where the bilinear form of  $\mathfrak{so}(2,4)$  is *not* degenerate. On the right-hand side, the appropriate coset projector turns out to be

$$P : \mathfrak{so}(2,4) \longrightarrow \frac{\mathfrak{so}(2,4)}{\mathfrak{so}(1,4) \oplus \text{span}_{\mathbb{R}}\{\gamma_5\}}. \quad (2.17)$$

This is the reason why the 4D Minkowski projector  $P$  is given by (2.14) instead of the AdS<sub>5</sub> coset projector  $\bar{P}$  in (2.9).<sup>5</sup>

The point is that we consider a conformal embedding of 4D Minkowski spacetime as in eq. (2.16) and use the coset projector  $P$  in (2.14) to avoid the degeneracy of the bilinear form of the 4D Poincaré group  $ISO(1,3)$ . After that, the projected one-form  $P(A)$  is expanded in terms of  $\gamma_\mu$  ( $\mu = 0, 1, 2, 3$ ) and the trace operation in the action leads to non-vanishing quantities as we will see in the next subsection.

### 2.3 Yang-Baxter sigma model for 4D Minkowski spacetime

Yang-Baxter deformations have only been discussed for curved backgrounds so far. However, it is possible to generalize the formulation to Minkowski spacetime.

The deformed action is given by<sup>6</sup>

$$S = -\frac{1}{2} \int_{-\infty}^{\infty} d\tau \int_0^{2\pi} d\sigma (\gamma^{\alpha\beta} - \epsilon^{\alpha\beta}) \text{Tr} \left[ A_\alpha P \circ \frac{1}{1 - 2\eta R_g \circ P} (A_\beta) \right], \quad (2.18)$$

where  $A_\alpha = g^{-1} \partial_\alpha g$  and  $g$  is given in eq. (2.11). Here  $\eta$  is a constant parameter and the action (2.18) is reduced to the undeformed one for  $\eta = 0$ . The base space is 2D Minkowski spacetime with the metric  $\gamma_{\alpha\beta} = \text{diag}(-1, 1)$ . The anti-symmetric tensor  $\epsilon^{\alpha\beta}$  is normalized as  $\epsilon^{\tau\sigma} = 1$ . The operator  $R_g$  is defined as

$$R_g \equiv g^{-1} R(gXg^{-1})g, \quad (2.19)$$

where a linear operator  $R : \mathfrak{so}(2,4) \rightarrow \mathfrak{so}(2,4)$  is a solution of the CYBE,

$$[R(M), R(N)] - R([R(M), N] + [M, R(N)]) = 0, \quad M, N \in \mathfrak{so}(2,4). \quad (2.20)$$

The  $R$ -operator is related to the *skew-symmetric* classical  $r$ -matrix in tensorial notation through

$$R(X) = \text{Tr}_2[r(1 \otimes X)] = \sum_i (a_i \text{Tr}(b_i X) - b_i \text{Tr}(a_i X)), \quad (2.21)$$

where the classical  $r$ -matrix is given by

$$r = \sum_i a_i \wedge b_i \equiv \sum_i (a_i \otimes b_i - b_i \otimes a_i) \quad (2.22)$$

satisfying the CYBE,

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0. \quad (2.23)$$

The generators  $a_i, b_i$  are elements of  $\mathfrak{so}(2,4)$ . This means that the Yang-Baxter deformations are investigated within  $\mathfrak{so}(2,4)$ .

It is easy to extend this formulation to the mCYBE, which will be discussed in section 5.

<sup>5</sup>Actually, also the projector  $\bar{P}$  would lead to the Minkowski metric (2.15) but the  $\gamma_5$  should be dropped due to the dimensionality we are concerned with.

<sup>6</sup>Here the string tension  $T = \frac{1}{2\pi\alpha'}$  is set to 1, and the conformal gauge is taken so as to drop the dilaton coupling to the world-sheet scalar curvature.

A crucial point is that the steps of taking the  $z = 1$  slice and performing the Yang-Baxter deformation have to be made in this precise order. The two operations in general do not commute and in particular they cannot if the  $r$ -matrix depends on  $\gamma_5$ , which deforms the  $z$  direction. They do commute on the other hand for all the examples listed in section 3.6. The magnetic case for example corresponds to a slice of the gravity dual for non-commutative gauge theory [41, 42] and in fact the classical  $r$ -matrix is identical to the one found in [54].

## 2.4 Embedding into string theory

Starting from a 2D integrable model, it is natural to ask whether it can be embedded into string theory. This requires in general supplementing the fields we can read off from the action by other fields in order to solve the one-loop beta function equations of string theory. As a result of Klimcik's procedure, we always get a 4D metric and  $B$ -field. For the missing internal directions, we make the simplest ansatz of assuming them to be flat, as resulting from an internal  $T^6$  with vanishing  $B$ -field. Adding only a dilaton will be sufficient to produce a solution for a large class of deformations, which all turn out to be related via T-duality to flat space. The minimal ansatz for the dilaton is that it preserves the same symmetries as the known fields. When the  $r$ -matrix only contains  $p_\mu$  or  $n_{\mu\nu}$ , this is sufficient to find a solution. More general  $r$ -matrices which contain also  $\gamma_5$  require also Ramond-Ramond fields in order to solve the equations. We have not actually been able to find the string theory embedding for all of these cases.

## 3 $TsT$ -duals of flat space

In this section, we have collected a class of models for which the simplest ansatz for a string theory embedding can be used, in which we assume the extra six dimensions to be flat (as resulting from a  $T^6$ -compactification) and we introduce only a dilaton field. They all turn out to be  $TsT$ -transformations of flat space with identifications. In cases in which the  $r$ -matrix contains a  $p_\mu$ , one or both T-dualities act trivially.

We will present the examples in the following order. We start with a rather general example containing both  $p$  and  $n$  in the  $r$ -matrix, which corresponds to the T-dual of Melvin background. The  $r$ -matrices of the next two examples presented in section 3.2.1 and section 3.2.2 have the same structure of  $p \wedge n$  and correspond to the T-dual of Grant space and TsT-duals of Minkowski spacetime. The next example, the pp-wave background in section 3.3, has a more complicated  $r$ -matrix with the structure  $r \sim (p_0 - p_3) \wedge n$ . Next we discuss the Hashimoto-Sethi background which in our parametrization has two  $ns$  appearing in the  $r$ -matrix,  $r \sim p_\mu \wedge (n_{\nu\rho} + n_{\sigma\gamma})$ . The next two examples have again a simpler structure of the  $r$ -matrix, with the Spradlin-Takayanagi-Volovich background, where only two  $n$ 's appear in section 3.5, and with models with only  $p$ 's in the  $r$ -matrix appearing in section 3.6.

All the models considered in this section can be embedded in string theory as  $TsT$ -duals of flat space. Let  $r = a \wedge b$  and let  $\alpha$  and  $\beta$  be the dual coordinates to  $a$  and  $b$  in the sense of the Lie algebra (concretely, if  $a = p_\mu$  then  $\alpha = x^\mu$  and if  $a = n_{\mu\nu}$  then  $\alpha$



is the angle in the plane  $\langle x^\mu, x^\nu \rangle$ ). Impose periodic boundary conditions on  $\alpha$  and  $\beta$  and consider the torus with parameter  $\tau$  generated by  $\langle \alpha, \beta \rangle$ . The effect of the deformation is to transform  $\tau$  into  $\tau_\eta = \tau/(1 + \eta\tau)$ . In string theory this is realized via the following sequence of transformations (*TsT*):

- *T*-dualize  $\alpha \rightarrow \tilde{\alpha}$ ;
- shift  $\beta \rightarrow \eta\tilde{\alpha} + \beta$ ;
- *T*-dualize  $\tilde{\alpha} \rightarrow \alpha$ . [39].

In the special case of  $a = p_\mu$  and  $b = n_{\rho\sigma}$ , this corresponds to the Melvin twist where  $x^\mu$  is the Melvin circle.

### 3.1 T-dual of Melvin background

Here we will consider an explicit example of a Yang-Baxter deformation. Our first example is the classical *r*-matrix

$$r = \frac{1}{2} p_3 \wedge n_{12}. \tag{3.1}$$

The associated geometry is a Melvin twist of 4D Minkowski spacetime.

Let us first compute the explicit form of  $A_\alpha$  and a deformed current  $J_\alpha$  defined as

$$J_\alpha \equiv \frac{1}{1 - 2\eta R_g \circ P} A_\alpha. \tag{3.2}$$

First of all,  $P(A_\alpha)$  is evaluated as

$$P(A_\alpha) = \frac{1}{2} [\gamma_0 \partial_\alpha x^0 + \gamma_1 \partial_\alpha x^1 + \gamma_2 \partial_\alpha x^2 + \gamma_3 \partial_\alpha x^3]. \tag{3.3}$$

Note here that  $P(A_\alpha)$  can also be expressed as

$$\begin{aligned} P(A_\alpha) &= P \circ \left( 1 - 2\eta R_g \circ P \right) (J_\alpha) \\ &= P(J_\alpha) - 2\eta P \circ R_g(P(J_\alpha)). \end{aligned} \tag{3.4}$$

Then, by plugging (3.3) into (3.4) and solving the four equations, the explicit form of  $P(J_\alpha)$  can be determined to be

$$P(J_\alpha) = \gamma_0 j_\alpha^0 + \gamma_1 j_\alpha^1 + \gamma_2 j_\alpha^2 + \gamma_3 j_\alpha^3, \tag{3.5}$$

where the components of  $j_\alpha^\mu$  are given by

$$\begin{aligned} j_\alpha^0 &= \frac{1}{2} \partial_\alpha x^0, \\ j_\alpha^1 &= \frac{\partial_\alpha x^1 + \eta^2 x^1 (x^1 \partial_\alpha x^1 + x^2 \partial_\alpha x^2) - \eta x^2 \partial_\alpha x^3}{2(1 + \eta^2 [(x^1)^2 + (x^2)^2])}, \\ j_\alpha^2 &= \frac{\partial_\alpha x^2 + \eta^2 x^2 (x^1 \partial_\alpha x^1 + x^2 \partial_\alpha x^2) + \eta x^1 \partial_\alpha x^3}{2(1 + \eta^2 [(x^1)^2 + (x^2)^2])}, \\ j_\alpha^3 &= \frac{\partial_\alpha x^3 + \eta (x^2 \partial_\alpha x^1 - x^1 \partial_\alpha x^2)}{2(1 + \eta^2 [(x^1)^2 + (x^2)^2])}. \end{aligned} \tag{3.6}$$

Thus the classical action can be rewritten as

$$\begin{aligned}
 S = & -\frac{1}{2} \int_{-\infty}^{\infty} d\tau \int_0^{2\pi} d\sigma \gamma^{\alpha\beta} \left[ -\partial_\alpha x^0 \partial_\beta x^0 + \partial_\alpha r \partial_\beta r + \frac{r^2 \partial_\alpha \theta \partial_\beta \theta + \partial_\alpha x^3 \partial_\beta x^3}{1 + \eta^2 r^2} \right] \\
 & + \int_{-\infty}^{\infty} d\tau \int_0^{2\pi} d\sigma \epsilon^{\alpha\beta} \frac{\eta r^2}{1 + \eta^2 r^2} \partial_\alpha \theta \partial_\beta x^3,
 \end{aligned}
 \tag{3.7}$$

where we have performed a coordinate transformation,

$$x^1 = r \cos \theta, \qquad x^2 = r \sin \theta.
 \tag{3.8}$$

From this action, one can read off the metric and  $B$ -field,

$$\begin{aligned}
 ds^2 = & -(dx^0)^2 + dr^2 + \frac{r^2 d\theta^2 + (dx^3)^2}{1 + \eta^2 r^2}, \\
 B = & \frac{\eta r^2}{1 + \eta^2 r^2} d\theta \wedge dx^3.
 \end{aligned}
 \tag{3.9}$$

It should be remarked that the Yang-Baxter deformations cannot reproduce the associated dilaton, although it may be possible to perform a supercoset construction in principle. However we can embed the background in string theory observing that the one-loop beta function vanishes by adding a dilaton [67–69]

$$\Phi = -\frac{1}{2} \log(1 + \eta^2 r^2).
 \tag{3.10}$$

The background obtained in this way is a  $TsT$  transformation of flat space on the torus generated by  $x^3$  and  $\theta$ , i.e. the result of the following chain of transformations:

- $T$ -duality in  $x^3$ ;
- shift  $\theta \rightarrow \eta \tilde{x}^3 + \theta$ ;
- $T$ -duality in  $\tilde{x}^3$ .

Consistency requires  $\tilde{x}^3$  to be periodic with period  $\tilde{x}^3 \simeq \tilde{x}^3 + 2\pi/\eta$  and  $x^3$  with period<sup>7</sup>  $x^3 \simeq x^3 + \alpha'\eta/(2\pi)$ .

## 3.2 Generalized Melvin backgrounds

### 3.2.1 T-dual of Grant space

Let us consider the classical  $r$ -matrix

$$r = \frac{1}{2} p_1 \wedge n_{03}.
 \tag{3.11}$$

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<sup>7</sup>We reintroduce the explicit parameter  $\alpha'$  in the identifications to manifestly illustrate the dimensions of the variables.

The derivation is almost the same as in the previous subsection, hence we will not repeat the detailed explanation but simply present the deformed metric and  $B$ -field. The associated metric and  $B$ -field are given by

$$\begin{aligned}
 ds^2 &= -\frac{2dx^-dx^+ + \eta^2(x^+dx^- + x^-dx^+)^2}{1 + 2\eta^2 x^-x^+} + \frac{(dx^1)^2}{1 + 2\eta^2 x^-x^+} + (dx^2)^2 \\
 &= -dt^2 + (dx^2)^2 + \frac{1}{1 + \eta^2 t^2} [(dx^1)^2 + t^2 d\phi^2], \\
 B &= \frac{\eta}{1 + 2\eta^2 x^-x^+} (x^+dx^- - x^-dx^+) \wedge dx^1 \\
 &= \frac{\eta t^2}{1 + \eta^2 t^2} dx^1 \wedge d\phi.
 \end{aligned} \tag{3.12}$$

Here the light-cone coordinates are given by

$$x^\pm \equiv \frac{x^0 \pm x^3}{\sqrt{2}}, \tag{3.13}$$

and we have introduced new coordinates given by

$$x^0 = t \cosh \phi, \quad x^3 = t \sinh \phi. \tag{3.14}$$

The metric and  $B$ -field in (3.12) agree with (2.7) and (2.8) in [72] for  $x^2 = 0$ .

The background can be embedded in string theory by adding a dilaton

$$\Phi = -\frac{1}{2} \log(1 + \eta^2 t^2), \tag{3.15}$$

which solves the beta function equations. In fact this is an exact string theory, resulting from a  $TsT$  transformation of flat space on the torus generated by  $x^1$  (which has periodicity  $\alpha'\eta/(2\pi)$ ) and  $\phi$ .

### 3.2.2 Time-like Melvin circle

Let us consider the classical  $r$ -matrix

$$r = \frac{1}{2} p_0 \wedge n_{12}. \tag{3.16}$$

This  $r$ -matrix is Abelian and the associated metric and  $B$ -field are given by

$$\begin{aligned}
 ds^2 &= -\frac{(dx^0)^2}{1 - \eta^2 r^2} + dr^2 + \frac{r^2 d\theta^2}{1 - \eta^2 r^2} + (dx^3)^2, \\
 B &= \frac{\eta r^2}{1 - \eta^2 r^2} dx^0 \wedge d\theta.
 \end{aligned} \tag{3.17}$$

The Ricci scalar curvature is negative,

$$R = -\frac{2\eta^2(5 + 2r^2\eta^2)}{(1 - \eta^2 r^2)^2}, \tag{3.18}$$

and this background has a curvature singularity at  $r = 1/\eta$ . This background is related to the previous one by analytic continuation

$$(t, x^1, \phi) \rightarrow (ir, ix^0, i\theta). \tag{3.19}$$

By supplementing the fields by a dilaton

$$\Phi = -\frac{1}{2} \log(1 - \eta^2 r^2) \tag{3.20}$$

which solves the beta function equations, we obtain a  $TsT$ -dual of flat space on the torus  $\langle x^0, \theta \rangle$ . Note that the Melvin circle  $x^0$  has periodicity  $x^0 \simeq x^0 + \alpha'\eta/(2\pi)$ .

### 3.3 PP-wave background

Let us consider the classical  $r$ -matrix

$$r = \frac{1}{2\sqrt{2}}(p_0 - p_3) \wedge n_{12}. \tag{3.21}$$

The associated deformed background is

$$\begin{aligned} ds^2 &= -2dx^+ dx^- - \eta^2 r^2 (dx^+)^2 + (dr)^2 + r^2 d\theta^2, \\ B &= \eta r^2 dx^+ \wedge d\theta, \end{aligned} \tag{3.22}$$

where we have introduced the polar coordinate system for  $x^1$  and  $x^2$  given by

$$x^1 = r \cos \theta, \quad x^2 = r \sin \theta. \tag{3.23}$$

This is a pp-wave background which can also be understood as a generalization of a (null)  $TsT$  transformation obtained as

- a T-duality from  $\theta$  to  $\tilde{\theta}$ , followed by
- the shifts  $x^0 \rightarrow \eta\tilde{\theta} + x^0$ ,  $x^3 \rightarrow -\eta\tilde{\theta} + x^3$  and the final
- T-duality from  $\tilde{\theta}$  to  $\theta$ .

Note that this requires the identifications  $x^0 \simeq x^0 + \alpha'\eta/(2\pi)$  and  $x^3 \simeq x^3 + \alpha'\eta/(2\pi)$ .

### 3.4 Hashimoto-Sethi background

We next consider the classical  $r$ -matrix

$$r = \frac{1}{2\sqrt{2}} p_2 \wedge (n_{01} + n_{13}). \tag{3.24}$$

The resulting metric and  $B$ -field are given by

$$\begin{aligned} ds^2 &= -2dx^- dx^+ + \frac{1}{1 + \eta^2(x^+)^2} ((dx^1)^2 + (dx^2)^2 + \eta^2 x^1 dx^+ (2x^+ dx^1 - x^1 dx^+)), \\ B &= \frac{\eta}{1 + \eta^2(x^+)^2} (x^1 dx^+ - x^+ dx^1) \wedge dx^2. \end{aligned} \tag{3.25}$$

Note that this background depends on the light-cone time  $x^+$  explicitly. The associated dilaton to complete the string embedding is taken to be

$$\Phi = -\frac{1}{2} \log(1 + \eta^2(x^+)^2). \quad (3.26)$$

The metric and  $B$ -field (3.25) agree with those of the Hashimoto-Sethi background. To show this agreement, one has to introduce new coordinates,

$$x^+ = \frac{1}{\eta} y^+, \quad x^1 = y^+ \tilde{y}, \quad x^- = \eta y^- + \frac{\eta}{2} y^+ \tilde{y}^2, \quad x^2 = -\tilde{z}. \quad (3.27)$$

This reproduces the expression in eq. (25) of [70] where the background is shown to be the result of a  $TsT$  transformation.

### 3.5 Spradlin-Takayanagi-Volovich background

Let us here consider the classical  $r$ -matrix

$$r = \frac{1}{2} n_{12} \wedge n_{03}. \quad (3.28)$$

Then the associated metric and  $B$ -field are given by

$$ds^2 = -2dx^+ dx^- + dr^2 + \frac{1}{1 + 2\eta^2 r^2 x^- x^+} (r^2 d\theta^2 - r^2 \eta^2 (x^+ dx^- - x^- dx^+)^2),$$

$$B = \frac{\eta r^2}{1 + 2\eta^2 r^2 x^- x^+} (x^- dx^+ - x^+ dx^-) \wedge d\theta. \quad (3.29)$$

The light-cone coordinates are given in (3.13). This background (3.29) is really time-dependent and has a curvature singularity.

By using the coordinates in (3.14), one can rewrite the expressions in (3.29) as

$$ds^2 = -dt^2 + dr^2 + \frac{r^2 d\theta^2 + t^2 d\phi^2}{1 + \eta^2 r^2 t^2},$$

$$B = \frac{\eta r^2 t^2}{1 + \eta^2 r^2 t^2} d\phi \wedge d\theta. \quad (3.30)$$

Note that the coordinates in (3.14) do not cover the whole  $x^0$ - $x^3$  plane and the background (3.30) contains no singularity. Then the metric and  $B$ -field in (3.30) agree with those of (6.1) in [71]. This is a time-dependent background realized by a  $TsT$  transformation of Minkowski spacetime on the torus generated by  $\phi$  and  $\theta$ . The associated dilaton is

$$\Phi = -\frac{1}{2} \log(1 + \eta^2 r^2 t^2). \quad (3.31)$$

### 3.6 Locally flat spaces

In this section, we consider classical  $r$ -matrices of the type

$$r = \frac{1}{2} p_\mu \wedge p_\nu \quad (\mu, \nu = 0, 1, 2, 3). \quad (3.32)$$

In these cases the associated geometries are locally flat and the  $B$ -field is closed. The backgrounds are nevertheless non-trivial and are associated to non-commutative field theories [84]. The following is a list of possible  $r$ -matrices.

**(1) Magnetic flux**  $r = \frac{1}{2} p_i \wedge p_j$  ( $i \neq j$ ,  $i, j = 1, 2, 3$ ) (Melvin Shift Twist)

With this  $r$ -matrix, the following metric and  $B$ -field are obtained:

$$ds^2 = -(dx^0)^2 + \frac{(dx^i)^2 + (dx^j)^2}{1 + \eta^2} + \sum_{k=1}^3 \epsilon^{ijk} (dx^k)^2, \quad (3.33)$$

$$B = \frac{\eta}{1 + \eta^2} dx^i \wedge dx^j.$$

This background is obtained via a  $TsT$  transformation on the torus  $\langle x^i, x^j \rangle$ .

A multi-parameter generalization of this type is obtained using the  $r$ -matrix

$$r = \frac{1}{2} (a_3 p_1 \wedge p_2 + a_1 p_2 \wedge p_3 + a_2 p_3 \wedge p_1), \quad (3.34)$$

where  $a_1, a_2, a_3 \in \mathbb{R}$  are deformation parameters. The resulting metric and  $B$ -field are given by

$$ds^2 = -(dx^0)^2 + \frac{(dx^1)^2 + (dx^2)^2 + (dx^3)^2 + \eta^2 (a_1 dx^1 + a_2 dx^2 + a_3 dx^3)^2}{1 + \eta^2 (a_1^2 + a_2^2 + a_3^2)}, \quad (3.35)$$

$$B = \frac{\eta}{1 + \eta^2 (a_1^2 + a_2^2 + a_3^2)} (a_3 dx^1 \wedge dx^2 + a_1 dx^2 \wedge dx^3 + a_2 dx^3 \wedge dx^1).$$

**(2) Electric flux**  $r = \frac{1}{2} p_0 \wedge p_i$  ( $i = 1, 2, 3$ ) (Melvin Shift Twist)

This  $r$ -matrix leads to the metric and  $B$ -field

$$ds^2 = \frac{-(dx^0)^2 + (dx^i)^2}{1 - \eta^2} + \sum_{j \neq i} (dx^j)^2, \quad (3.36)$$

$$B = -\frac{\eta}{1 - \eta^2} dx^0 \wedge dx^i.$$

This corresponds to turning on an electric flux and the range of  $\eta$  is restricted to  $-1 < \eta < 1$ . Once more we can obtain this background via a  $TsT$  transformation on the torus  $\langle x^0, x^i \rangle$ . The background is related to the non-commutative open string theory (NCOS) [85].

A multi-parameter generalization of this type is obtained by the  $r$ -matrix

$$r = \frac{1}{2} p_0 \wedge (a_1 p_1 + a_2 p_2 + a_3 p_3), \quad (3.37)$$

where  $a_1, a_2, a_3 \in \mathbb{R}$  are deformation parameters. The resulting metric and  $B$ -field are given by

$$ds^2 = \frac{-(dx^0)^2 + \eta^2 (a_1 dx^1 + a_2 dx^2 + a_3 dx^3)^2}{1 - \eta^2 (a_1^2 + a_2^2 + a_3^2)} + (dx^1)^2 + (dx^2)^2 + (dx^3)^2,$$

$$B = \frac{\eta}{1 - \eta^2 (a_1^2 + a_2^2 + a_3^2)} (a_1 dx^1 + a_2 dx^2 + a_3 dx^3) \wedge dx^0. \quad (3.38)$$

**(3) Light-like flux**  $r = \frac{1}{2\sqrt{2}}(p_0 - p_3) \wedge p_2$  (Null Melvin Shift Twist)

In this case, we obtain the following metric and  $B$ -field

$$\begin{aligned} ds^2 &= -2dx^+ dx^- - \eta^2(dx^+)^2 + (dx^1)^2 + (dx^2)^2, \\ B &= -\eta dx^+ \wedge dx^2. \end{aligned} \tag{3.39}$$

This corresponds to turning on a light-like flux resulting from a  $T$ -duality in  $x^2$ , opposite shifts in  $x^0$  and  $x^3$  and a final  $T$ -duality in  $\tilde{x}^2$ . The background is related to the light-like non-commutativity argued in [86].

#### 4 Non-twist cases

So far, we have discussed the cases associated with  $TsT$  transformations. In this section, we shall consider the other cases, for instance, including a single T-duality, or a chain of  $TsT$ -transformations followed by an S-duality. We will refer to the cases as non-twist cases.

As the first example, we revisit the pp-wave case and argue its relation to S-duality. Then we will reproduce T-duals of  $dS_4$  and  $AdS_4$  respectively, as Yang-Baxter deformations. These examples may be regarded as non-trivial examples of our procedure. Finally we provide more complicated examples including two-parameter deformations.

Let us remark that the presence of  $\gamma_5$  in the all of the  $r$ -matrices appearing in this section plays a crucial role. It can intuitively be interpreted as a deformation of the radial direction of  $AdS_5$  and indicates that the location of the slice is moved. This is also the reason why the integrability of the  $TsT$ -cases of the previous section is obvious, while in general it is not automatic in the non-twist cases, except for special cases such as the T-duals of  $dS_4$  and  $AdS_4$ .

##### 4.1 PP-wave revisited — S-duality

In the previous section, we have considered the pp-wave background (3.22). We will show that the same metric can be reproduced by considering the  $r$ -matrix

$$r = \frac{1}{4\sqrt{2}}(\gamma_5 - 2n_{03}) \wedge (p_0 - p_3). \tag{4.1}$$

The resulting metric and  $B$ -field are given by

$$\begin{aligned} ds^2 &= -2dx^+ dx^- - \eta^2 r^2 (dx^+)^2 + (dr)^2 + r^2 d\theta^2, \\ B &= \eta r dx^+ \wedge dr. \end{aligned} \tag{4.2}$$

As noted above, the metric is the same as in (3.22), but the  $B$ -field is different. In particular, the  $B$ -field carries an index for the radial direction  $r$ , which is not a  $U(1)$ -direction. Hence it does not seem likely that the background (4.2) can be obtained via a simple twisting procedure, as opposed to (3.22). This difference comes from the fact that the second  $r$ -matrix (4.1) is non-Abelian, while the first one (3.21) is Abelian. This non-Abelian  $r$ -matrix appears to be related to a chain of dualities. In fact, the  $r$ -matrix (4.1) is identical

to the one in (2.13) of [56] that has been employed to study a deformation of  $\text{AdS}_5 \times S^5$ . It has been shown in [57] that this  $r$ -matrix corresponds to a duality chain of three TsT transformations and an S-duality.<sup>8</sup> This is an example of a non-Abelian  $r$ -matrix being related to an intricate duality chain. This indicates that general non-Abelian  $r$ -matrices can be interpreted in a similar way.

As another possibility, non-Abelian  $r$ -matrices may indicate deformations beyond the standard twists or dualities. To elaborate this statement, it would be interesting to study more non-trivial examples of classical  $r$ -matrices, as we will show in the next subsection.

## 4.2 T-dual of $\text{dS}_4$

Now we consider the classical  $r$ -matrix

$$r = \frac{1}{4} \gamma_5 \wedge p_0. \tag{4.3}$$

Note that this  $r$ -matrix contains  $\gamma_5$  and leads to the metric and  $B$ -field given by

$$ds^2 = \frac{-(dx^0)^2 + dr^2}{1 - \eta^2 r^2} + r^2 \sin^2 \theta d\phi^2 + r^2 d\theta^2, \tag{4.4}$$

$$B = \frac{\eta r}{1 - \eta^2 r^2} dx^0 \wedge dr.$$

where we have introduced new coordinates  $r$ ,  $\theta$  and  $\phi$  through

$$x^1 = r \cos \phi \sin \theta, \quad x^2 = r \sin \phi \sin \theta, \quad x^3 = r \cos \theta. \tag{4.5}$$

Note here that the above  $B$ -field can be rewritten in the form of a total derivative.

The deformed background (4.4) is already simple. However, in order to understand the background well, we can perform a T-duality along the  $x^0$ -direction.<sup>9</sup> The resulting background is given by

$$ds^2 = (dr + \eta r dx^0)^2 - (dx^0)^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2). \tag{4.6}$$

Note that the  $B$ -field has disappeared now. Then, by performing a coordinate transformation from  $x^0$  to  $t$ ,

$$x^0 = t - \frac{1}{2\eta} \log(\eta^2 r^2 - 1), \tag{4.7}$$

one can reproduce the well-known metric of  $\text{dS}_4$  in static coordinates,

$$ds^2 = -(1 - \eta^2 r^2) dt^2 + \frac{dr^2}{1 - \eta^2 r^2} + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \tag{4.8}$$

with a cosmological horizon at  $r = 1/\eta$ . Thus we have shown that the background (4.4) is nothing but a T-dual of  $\text{dS}_4$ . The  $\text{dS}_4$  geometry cannot be realized as a twist of 4D Minkowski spacetime, hence it is quite remarkable that the T-dual of  $\text{dS}_4$  has been obtained as a Yang-Baxter deformation.

It is worth to mention that the  $r$ -matrix in eq.(4.3) has already appeared in [88] where it was related to four-dimensional de Sitter space in a different context. This supports our argument from another perspective

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<sup>8</sup>In this spirit, we can understand the background in eq. (4.2) as the S-dual of the one in eq.(3.22).

<sup>9</sup>This is a time-like T-duality. For the detail, see the original argument [87].



### 4.3 T-dual of AdS<sub>4</sub>

As another example, let us consider the classical  $r$ -matrix

$$r = \frac{1}{4} \gamma_5 \wedge p_1. \tag{4.9}$$

This  $r$ -matrix also contains  $\gamma_5$ . The associated metric and  $B$ -field are given by

$$ds^2 = \frac{dt^2 + (dx^1)^2}{1 + \eta^2 t^2} + t^2 \cosh^2 \phi d\theta^2 - t^2 d\phi^2, \tag{4.10}$$

$$B = \frac{\eta t}{1 + \eta^2 t^2} dt \wedge dx^1,$$

where we have introduced new coordinates  $t$ ,  $\theta$  and  $\phi$  through

$$x^0 = t \sinh \phi, \quad x^2 = t \cos \theta \cosh \phi, \quad x^3 = t \sin \theta \cosh \phi. \tag{4.11}$$

Note here that the  $B$ -field can be recast as a total derivative.

As in the previous case, it is nice to perform a T-duality along the  $x^1$ -direction. Then the resulting background is given by<sup>10</sup>

$$ds^2 = (dt - \eta t dx^1)^2 + (dx^1)^2 + t^2 (-d\phi^2 + \cosh^2 \phi d\theta^2). \tag{4.12}$$

Now the  $B$ -field has disappeared. Let us perform a coordinate transformation,

$$x^1 = y + \frac{1}{2\eta} \log(\eta^2 t^2 + 1). \tag{4.13}$$

Then the resulting metric is given by

$$ds^2 = (1 + \eta^2 t^2) dy^2 + \frac{dt^2}{1 + \eta^2 t^2} + t^2 (-d\phi^2 + \cosh^2 \phi d\theta^2). \tag{4.14}$$

By replacing the coordinates (with a double Wick rotation) by

$$y \rightarrow it, \quad t \rightarrow r, \quad \phi \rightarrow i\theta, \quad \theta \rightarrow \phi, \tag{4.15}$$

one can obtain the standard metric of AdS<sub>4</sub> with the global coordinates

$$ds^2 = -(1 + \eta^2 r^2) dt^2 + \frac{dr^2}{1 + \eta^2 r^2} + r^2 (d\theta^2 + \cos^2 \theta d\phi^2). \tag{4.16}$$

Note that  $\eta^2$  measures the curvature.

### 4.4 More complicated backgrounds

Here, we give a list of classical  $r$ -matrices for which the corresponding backgrounds have not been identified yet. All of the classical  $r$ -matrices we consider here satisfy the CYBE given in (2.23). Nevertheless, this does not guarantee integrability without explicitly constructing the Lax pair. In the previous cases, the string embedding assured integrability via T-duality to integrable backgrounds. We conjecture that also the following examples are new integrable models, but at this point have no explicit proof.

In the following, we assume that the light-cone coordinates are defined in (3.13) and  $x^1$ ,  $x^2$  are rewritten with the polar coordinates given in (3.23).

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<sup>10</sup>Note that, at this stage, one can see that this metric describes AdS<sub>4</sub> by explicitly computing the scalar curvature and the Ricci tensor.

### 4.4.1 One-parameter deformations

Let us consider more complicated one-parameter deformations.

The first example is

$$r = \frac{1}{4\sqrt{2}} (\gamma_5 - 2n_{12}) \wedge (p_0 - p_3), \quad (4.17)$$

and the resulting metric and  $B$ -field are given by

$$ds^2 = \frac{-2dx^- dx^+ - 2\eta^2 r dx^+ [r dx^+ - x^+ (dr + r d\theta)]}{1 - \eta^2 (x^+)^2} + dr^2 + r^2 d\theta^2, \quad (4.18)$$

$$B = \frac{\eta}{1 - \eta^2 (x^+)^2} dx^+ \wedge (r dr - x^+ dx^- + r^2 d\theta).$$

This  $r$ -matrix is non-Abelian. The Ricci scalar and Riemann square vanish. This background has a coordinate singularity at  $x^+ = \pm 1/\eta$ .

The second example is

$$r = \frac{1}{4} (n_{05} - n_{35}) \wedge (p_0 - p_3), \quad (4.19)$$

and the metric and  $B$ -field are given by

$$ds^2 = \frac{-2dx^+ dx^- + dr^2 - \eta^2 (x^+)^2 (x^+ dr - r dx^+)^2}{1 - \eta^2 (x^+)^4} + r^2 d\theta^2 \quad (4.20)$$

$$B = \frac{\eta x^+}{1 - \eta^2 (x^+)^4} dx^+ \wedge (r dr - x^+ dx^-).$$

This is an Abelian  $r$ -matrix in which  $\gamma_5$  is not contained but  $n_{05}$  and  $n_{35}$  carry the index 5. Note that the Ricci scalar and Riemann square vanish and  $x^+ = \pm |\eta|^{-1/2}$  are just coordinate singularities.

### 4.4.2 Two-parameter deformations

It may be interesting to consider two-parameter deformations.

The first example is given by

$$r = \frac{s_1}{2} E_{24} \wedge (E_{22} - E_{44}) + \frac{s_2}{2} E_{13} \wedge (E_{11} - E_{33}) \quad (4.21)$$

$$= \frac{s_1}{8} (p_0 - p_3) \wedge (2n_{03} - \gamma_5) + \frac{s_2}{8} (p_0 + p_3) \wedge (2n_{03} + \gamma_5),$$

where  $s_1$  and  $s_2$  are arbitrary constant parameters. This  $r$ -matrix has been used to study a two-parameter deformation of AdS<sub>5</sub> in [56]. The metric and  $B$ -field associated to (4.21) are given by

$$ds^2 = \frac{-2dx^+ dx^- + dr^2}{1 + s_1 s_2 \eta^2 r^2} + r^2 d\theta^2 - \frac{\eta^2 r^2 (s_1 dx^+ + s_2 dx^-)^2}{2(1 + s_1 s_2 \eta^2 r^2)}, \quad (4.22)$$

$$B = \frac{\eta r}{\sqrt{2}(1 + s_1 s_2 \eta^2 r^2)} (s_1 dx^+ - s_2 dx^-) \wedge dr.$$

When  $s_1 = -s_2 = s$ , the metric simplifies to

$$\begin{aligned} ds^2 &= \frac{-(dx^0)^2 + dr^2}{1 - s^2\eta^2 r^2} + r^2 d\theta^2 + (dx^3)^2, \\ B &= \frac{s\eta r}{1 - s^2\eta^2 r^2} dx^0 \wedge dr. \end{aligned} \tag{4.23}$$

A T-duality along the  $x^0$ -direction leads to the geometry of  $dS_3 \times \mathbb{R}$ .

The second example is given by

$$\begin{aligned} r &= \frac{1}{\sqrt{2}} E_{24} \wedge \left[ (a + ib) \left( E_{22} - \frac{1}{4} E \right) - (a - ib) \left( E_{44} - \frac{1}{4} E \right) \right] \\ &= \frac{1}{4\sqrt{2}} (p_0 - p_3) \wedge [a(2n_{03} - \gamma_5) + 2bn_{12}], \end{aligned} \tag{4.24}$$

where  $a, b \in \mathbb{R}$  are deformation parameters and  $E \equiv \sum_{i=1}^4 E_{ii}$ . The resulting metric and  $B$ -field are given by

$$\begin{aligned} ds^2 &= -2dx^+ dx^- + dr^2 + r^2 d\theta^2 - \eta^2 (a^2 + b^2) r^2 (dx^+)^2, \\ B &= \eta r dx^+ \wedge (adr + brd\theta). \end{aligned} \tag{4.25}$$

Thus this background is regarded as an interpolation between the plane waves in (3.22) and (4.2).<sup>11</sup>

The third example is given by

$$\begin{aligned} r &= \frac{i}{\sqrt{2}} \left[ E_{24} \wedge \left( a_1 \left( E_{22} - \frac{1}{4} E \right) - a_2 \left( E_{44} - \frac{1}{4} E \right) \right) - (a_1 + a_2) E_{23} \wedge E_{34} \right] \\ &= -\frac{i}{4\sqrt{2}} \left[ (p_0 - p_3) \wedge \left( a_1 \left( in_{12} - n_{03} + \frac{\gamma_5}{2} \right) - a_2 \left( in_{12} + n_{03} - \frac{\gamma_5}{2} \right) \right) \right. \\ &\quad \left. + (a_1 + a_2) (p_1 - ip_2) \wedge (n_{01} + n_{13} + i(n_{02} + n_{23})) \right], \end{aligned} \tag{4.26}$$

where  $a_1, a_2 \in \mathbb{R}$  are deformation parameters. The resulting metric and  $B$ -field are given by

$$\begin{aligned} ds^2 &= -2dx^+ dx^- + dr^2 + r^2 d\theta^2 \\ &\quad - \eta^2 \frac{[(a_1 + a_2)x^+ dr - a_2 r dx^+]^2 + (a_1 + a_2)^2 r^2 (x^+)^2 d\theta^2}{1 + \eta^2 (a_1 + a_2)^2 (x^+)^2}, \\ B &= \frac{\eta r}{1 + \eta^2 (a_1 + a_2)^2 (x^+)^2} ((a_1 + a_2)x^+ dr - a_2 r dx^+) \wedge d\theta. \end{aligned} \tag{4.27}$$

When  $a_1 = -a_2$ , the background in (3.22) is reproduced. Thus this background can be regarded as a deformation of the pp-wave background.

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<sup>11</sup>It would be interesting to see if this background can be understood as the result of an  $SL_2(\mathbb{R})$  transformation acting on the pp wave of eq. (3.22).

## 5 A deformation of Drinfeld-Jimbo type

So far, we have considered classical  $r$ -matrices satisfying the classical Yang-Baxter equation (2.23) (or (2.20)). Here, as an exceptional case, let us consider a classical  $r$ -matrix of Drinfeld-Jimbo (DJ) type,

$$r_{\text{DJ}} = -i \sum_{1 \leq i < j \leq 4} E_{ij} \wedge E_{ji}, \quad (E_{ij})_{kl} \equiv \delta_{ik} \delta_{jl}, \quad (5.1)$$

which satisfies the modified Yang-Baxter equation,

$$[R(M), R(N)] - R([R(M), N] + [M, R(N)]) = [M, N], \quad (5.2)$$

where  $M, N \in \mathfrak{so}(2, 4)$ . In comparison to the CYBE in (2.20), the right-hand side of (5.2) is modified.<sup>12</sup> The  $r$ -matrix (5.1) was utilized to construct an integrable deformation of the  $\text{AdS}_5 \times S^5$  superstring [36, 37]. The deformed metric and  $B$ -field were explicitly computed in [89].

The computation scheme is essentially identical because we do not mention the associated Lax pair and the kappa transformation related to the Green-Schwarz string action.

The resulting metric and  $B$ -field are given by

$$ds^2 = -r^2 \sin^2 \theta dt^2 + dr^2 + \frac{r^2}{1 + \eta^2 r^4 \sin^2 \theta} (d\theta^2 + \cos^2 \theta d\phi^2), \quad (5.3)$$

$$B = -\frac{\eta r^4 \sin \theta \cos \theta}{1 + \eta^2 r^4 \sin^2 \theta} d\theta \wedge d\phi.$$

Here we have performed a coordinate transformation,

$$\begin{aligned} x^0 &= r \sin \theta \sinh t, & x^1 &= r \cos \theta \cos \phi, \\ x^2 &= r \cos \theta \sin \phi, & x^3 &= r \sin \theta \cosh t, \end{aligned} \quad (5.4)$$

and rescaled  $\eta \rightarrow \eta/2$ . It is worth noting that the metric in (5.3) would be regular as opposed to the case of the deformation of  $\text{AdS}_5$ . The scalar curvature has no singularity, while the singular surface is identified in the deformed  $\text{AdS}_5$  [89]. This may be due to the fact that we are now working on a slice of  $\text{AdS}_5$  at  $z = 1$ .

## 6 Conclusion and discussion

In this paper, we have discussed Yang-Baxter deformations of 4D Minkowski spacetime using a conformal embedding of the spacetime into  $\text{AdS}_5$ . Via this procedure we have succeeded in reproducing the metric and  $B$ -field of well-known backgrounds such as T-duals of Melvin backgrounds, Hashimoto-Sethi backgrounds, time-dependent backgrounds

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<sup>12</sup>The  $r$ -matrices satisfying the mCYBE in (5.2) are said to be *non-split type*, as opposed to the *split type* obtained when the right-hand side of (5.2) is negative. Our choice of sign ensures the reality of the  $B$ -field as in [22, 89].

$r$ -matrix	Type of Twist	Background
$p_i \wedge p_j$ ( $i, j = 1, 2, 3$ )	Melvin Shift Twist	Seiberg-Witten
$p_0 \wedge p_i$	Melvin Shift Twist	NCOS
$(p_0 + p_i) \wedge p_j$ ( $i \neq j$ )	Null Melvin Shift Twist	light-like NC
$\frac{1}{2}p_3 \wedge n_{12}$	Melvin Twist	T-dual Melvin
$\frac{1}{2\sqrt{2}}p_2 \wedge (n_{01} + n_{13})$	Melvin Null Twist	Hashimoto-Sethi
$\frac{1}{2}n_{12} \wedge n_{03}$	R Melvin R Twist	Spradlin-Takayanagi-Volovich
$\frac{1}{2}p_1 \wedge n_{03}$	Melvin Boost Twist	T-dual of Grant space
$\frac{1}{2\sqrt{2}}(p_0 - p_3) \wedge n_{12}$	Null Melvin Twist	pp-wave
$\frac{1}{4\sqrt{2}}(\gamma_5 - 2n_{03}) \wedge (p_0 - p_3)$	Non-Twist	pp-wave
$\frac{1}{4}\gamma_5 \wedge p_0$	Non-Twist	T-dual of dS <sub>4</sub>
$\frac{1}{4}\gamma_5 \wedge p_1$	Non-Twist	T-dual of AdS <sub>4</sub>
DJ-type (mCYBE)	Non-Twist	$q$ -deformation (?)

**Table 1.** A catalog of classical  $r$ -matrices and the associated backgrounds.

of Spradlin-Takayanagi-Volovich, the T-dual of Grant space, pp-wave backgrounds, and T-duals of dS<sub>4</sub> and AdS<sub>4</sub>. For a large class of them, an embedding into string theory exists as a  $TsT$ -transformation of flat space with identifications, where the extra six dimensions are flat and only a dilaton is introduced. Finally we have studied a deformation by using a classical  $r$ -matrix of Drinfeld-Jimbo type and have derived the associated metric and  $B$ -field explicitly. The backgrounds that we have identified are summarized in table 1.

There are many open questions. The undeformed case is trivially integrable and hence the integrability should be preserved under Yang-Baxter deformations. It would be interesting to consider what happens to the Lax pairs and to study the associated algebras. In particular, deformed Poincaré algebras are studied in [90–92] in terms of classical  $r$ -matrices. It would be very interesting to clarify the correspondence between the list in [90–92] and our results.<sup>13</sup>

It would be most important to generalize our argument to 10D Minkowski spacetime in order to extend our argument to a consistent string theory. For this purpose, we have to consider the 10D conformal group SO(2, 10) and realize 10D Minkowski spacetime as a slice of 11D AdS space. We expect that in this case 10D supersymmetric configurations like the fluxtrap backgrounds [93–96] could be reproduced as Yang-Baxter deformations.

As opposed to deformations of AdS<sub>5</sub> × S<sup>5</sup>, our computations have the advantage of being very simple and hence make it easier to generalize the Yang-Baxter deformations. Another interesting possible direction would be a supercoset construction including spacetime fermions. Furthermore, it would be very interesting to investigate some quantum aspects such as the relation between string spectra and Yang-Baxter deformations.<sup>14</sup>

<sup>13</sup>See also the discussion in [97] which has appeared subsequently to this article.

<sup>14</sup>For some advances regarding the quantum aspects of the most studied example of  $\beta$  and  $\gamma$  deformations, see the review [98].

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