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Collective dynamics from stochastic thermodynamics

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Abstract

From a viewpoint of stochastic thermodynamics, we derive equations that describe the collective dynamics near the order-disorder transition in the globally coupled XY model and near the synchronization–desynchronization transition in the Kuramoto model. A new way of thinking is to interpret the deterministic time evolution of a macroscopic variable as an external operation to a thermodynamic system. We then find that the irreversible work determines the equation for the collective dynamics. When analyzing the Kuramoto model, we employ a generalized concept of irreversible work which originates from a non-equilibrium identity associated with steady state thermodynamics.

1. Introduction

Since the discovery of the fluctuation theorem [1–3], non-equilibrium statistical mechanics, which aims at connecting microscopic mechanics with macroscopic properties under non-equilibrium conditions, has been intensively studied. In particular, thermodynamic concepts such as heat, work, and entropy production are seriously re-considered so as to have a consistent thermodynamics framework for each realization of fluctuating quantities [4, 5]. This framework has been referred to as *stochastic thermodynamics*. Owing to much effort, nowadays, it can be said that the foundation of stochastic thermodynamics has been established, and we should consider a next challenge based on the development of stochastic thermodynamics.

In the present paper, we discuss collective dynamics in systems consisting of many elements. This topic is of course one of important problems in non-equilibrium physics, but one may wonder how this problem is related to stochastic thermodynamics. Here, the first purpose of this paper is to shed light on the connection between collective dynamics and stochastic thermodynamics. A key point is that the deterministic time evolution of a macroscopic variable is interpreted as an external operation to a thermodynamic system, and the weak irreversible work is ascribed to a macroscopic friction force for the external system. The last phrase is taken from page 192 in [4]. The combination of the friction force and the thermodynamic force gives rise to the total force. When the total force is expressed in terms of the order parameter, a differential equation of the order parameter is determined.

In section 2, we shall explain basic notions by analyzing the globally coupled XY model subjected to thermal noise. According to equilibrium statistical mechanics, the order-disorder transition point in this model is determined by a self-consistent equation for the order parameter characterizing the phase order. We then consider the time evolution of the order parameter near the transition point. Since its characteristic time scale is much longer than other variables, we interpret the time dependence of the parameter as a nearly quasi-static operation to the system. In the quasi-static limit, the so-called adiabatic theorem holds, which claims that the work is equal to the free energy change. We find that this relation is equivalent to the self-consistent equation for determining the transition point. Then, in nearly quasi-static processes, the irreversible work, which is defined as the difference between the work and the free energy change, appears slightly. Here, the irreversible work is characterized by a macroscopic friction constant. Since the irreversible work in nearly quasi-static processes is

connected to fluctuations of irreversible work in the quasi-static processes, the friction constant is determined from the time correlation of a thermodynamic force at the trivial state. By calculating the friction constant, we obtain a differential equation of the order parameter.

This method is elegant but seems applicable to only thermodynamic systems. As another example of collective dynamics, in section 3, we study the Kuramoto model which is the simplest model that describes the collective synchronization [7, 8]. However, there are neither thermodynamics, equilibrium statistical mechanics, nor Hamiltonian in the Kuramoto model. The situation is rather different from the globally coupled XY model. Nevertheless, when we add a noise term to the Kuramoto model, the Langevin equation for each element is similar to that of the globally coupled XY model [9]. Only difference is that there exists a non-equilibrium driving force in the Kuramoto model. Thus, from a viewpoint of stochastic thermodynamics, the analysis of the noisy Kuramoto model requires an extension of the irreversible work and the fluctuation-dissipation relation. Here comes the steady state thermodynamics of Langevin equations [10]. We already found the generalization of the irreversible work in transitions between two steady states by extending the Jarzynski equality [11] to that valid in non-equilibrium systems. By using this extended equality, we derive a formula of the friction constant in terms of time correlation functions at the trivial state. As a result, we obtain a differential equation of the order parameter near the transition point of the noisy Kuramoto model. Furthermore, we can take the noiseless limit of the equation.

It should be noted that the collective dynamics of globally coupled XY model and the Kuramoto model were studied by the so-called bifurcation analysis using a center manifold theory [7, 12, 13]. That is, in this paper, we do not derive new equations of the order parameters, but we present a simpler derivation method than previously known ones. In particular, if we already know the self-consistent equation, we have only to calculate the friction constant in terms of time correlation functions. The calculation is quite elementary. Furthermore, by distinguishing 'static quantities' such as the free energy and 'dynamic quantities' such as the friction constant, we can gasp the problem in a clear manner. Therefore, we expect that the method will be applied to systems for which the collective dynamics are not studied yet. In the last section, we argue such future problems to be studied. Throughout this paper, the Boltzmann constant is set to unity, and β is always identified with $1/T$.

2. Globally coupled XY model

2.1. Equilibrium statistical mechanics

Let θ_i $(1 \leqslant i \leqslant N)$ be a phase variable of ith element. We denote a collection of phases $(\theta_i)_{i=1}^{N}$ by $\boldsymbol{\theta}$ and define the Hamiltonian as

$$H(\boldsymbol{\theta}) = -\frac{K}{N} \sum_{i,j} \cos\left(\theta_i - \theta_j\right). \tag{1}$$

The canonical ensemble of the system is given by

$$p^{\mathrm{can}}(\boldsymbol{\theta}) = \frac{1}{Z} \mathrm{e}^{-\beta H(\boldsymbol{\theta})}. \tag{2}$$

We want to derive the equilibrium value of the order parameter defined by

$$r \mathrm{e}^{\mathrm{i}\varphi} \equiv \frac{1}{N} \sum_{j=1}^{N} \mathrm{e}^{\mathrm{i}\theta_j} \tag{3}$$

with $r \geqslant 0$.

We first notice that the Hamiltonian is expressed as

$$H(\boldsymbol{\theta}) = -Kr \sum_{i} \cos\left(\theta_i - \varphi\right). \tag{4}$$

Although r and φ depend on $\boldsymbol{\theta}$, we can assume that they take the equilibrium values (with probability one) in the limit $N \to \infty$ owing to the law of large numbers. We fix r and φ to these values. We then write

$$p^{\mathrm{can}}(\boldsymbol{\theta}) = \prod_{i} p_{\mathrm{one}}^{\mathrm{can}}\left(\theta_i; r, \varphi\right), \tag{5}$$

where

$$p_{\mathrm{one}}^{\mathrm{can}}\left(\theta_i; r, \varphi\right) = \frac{1}{Z_{\mathrm{one}}(r)} \mathrm{e}^{-\beta H_{\mathrm{one}}\left(\theta_i; r, \varphi\right)} \tag{6}$$

with

$$H_{\text{one}}(\theta; r, \varphi) = -Kr\cos(\theta - \varphi). \tag{7}$$

$Z_{\text{one}}(r)$ is the normalization constant given by

$$Z_{\text{one}}(r) = \int_0^{2\pi} \mathrm{d}\theta e^{\beta Kr\cos(\theta)}. \tag{8}$$

The equilibrium value of r is then determined by

$$
\begin{aligned}
r &= \lim_{N \to \infty} \frac{1}{N} \sum_{j=1}^{N} \cos\left(\theta_j - \varphi\right) \\
&= \int_0^{2\pi} \mathrm{d}\theta p_{\text{one}}^{\text{can}}(\theta; r, \varphi) \cos(\theta - \varphi) \\
&= \frac{1}{Z_{\text{one}}(r)} \int_0^{2\pi} \mathrm{d}\theta e^{\beta Kr\cos(\theta)} \cos(\theta) \\
&= \frac{1}{\beta K} \frac{\partial}{\partial r} \log Z_{\text{one}}(r).
\end{aligned}
\tag{9}
$$

By expanding (8) in r, we obtain

$$\log Z_{\text{one}}(r) = \log(2\pi) + \frac{1}{4}(\beta Kr)^2 - \frac{1}{64}(\beta Kr)^4 + O\left(r^6\right). \tag{10}$$

The self-consistent equation (9) becomes

$$r = \frac{1}{2}(\beta Kr) - \frac{1}{16}(\beta Kr)^3 + O\left(r^5\right). \tag{11}$$

This indicates that the transition inverse temperature β_c for fixed K is given by

$$\beta_c K = 2. \tag{12}$$

Indeed, there are no other solutions than the trivial solution $r = 0$ for $\beta < \beta_c$, while there is another solution for $\beta > \beta_c$.

2.2. Collective dynamics

Next, we consider the collective dynamics of the order parameter. We assume that the time evolution of θ_i is described by the Langevin equation

$$
\begin{aligned}
\frac{\mathrm{d}\theta_i}{\mathrm{d}t} &= -\frac{\partial H}{\partial \theta_i} + \xi_i \\
&= -\frac{K}{N} \sum_{j=1}^{N} \sin\left(\theta_i - \theta_j\right) + \xi_i,
\end{aligned}
\tag{13}
$$

where ξ_i is Gaussian-white noise satisfying

$$\left\langle \xi_i(t)\xi_j(t') \right\rangle = 2T\delta_{ij}\delta(t - t'). \tag{14}$$

The stationary probability density is the canonical distribution (2). The problem we want to solve is to obtain a differential equation of the order parameter $re^{i\varphi}$.

In order to set the problem explicitly, we assume the probability density at the initial time $t = 0$ as

$$p_0(\boldsymbol{\theta}) = \prod_i p_{\text{one}}^{\text{can}}\left(\theta_i; r_0, \varphi_0\right) \tag{15}$$

for a given r_0 and φ_0. The probability density $p(\boldsymbol{\theta}, t)$ at time t is determined uniquely. Then, in the limit $N \to \infty$, $r(t)$ and $\varphi(t)$ for each t take definite values for almost all $\boldsymbol{\theta}$ with respect to $p(\boldsymbol{\theta}, t)$. We fix functional forms of $r(t)$ and $\varphi(t)$ to those. Since we can rewrite (13) as

$$\frac{\mathrm{d}\theta_i}{\mathrm{d}t} = -Kr\sin\left(\theta_i - \varphi\right) + \xi_i, \tag{16}$$

the probability density at time t is expressed as

$$p(\boldsymbol{\theta}, t) = \prod_i p_{\text{one}}\left(\theta_i, t\right), \tag{17}$$

where p_{one} is given by the solution of the Fokker–Planck equation associated with (16):

$$\frac{\partial p_{\mathrm{one}}(\theta, t)}{\partial t} + \frac{\partial}{\partial \theta}\left[-Kr\sin(\theta - \varphi)p_{\mathrm{one}} - T\frac{\partial}{\partial\theta}p_{\mathrm{one}}\right] = 0 \qquad (18)$$

with the initial condition $p_{\mathrm{one}}(\theta, 0) = p_{\mathrm{one}}^{\mathrm{can}}(\theta; r_0, \varphi_0)$. Then, $r(t)$ and $\varphi(t)$ satisfy

$$r(t)\, e^{i\varphi(t)} = \int_0^{2\pi} d\theta p_{\mathrm{one}}(\theta, t)e^{i\theta}, \qquad (19)$$

which is regarded as a self-consistent equation for $r(t)$ and $\varphi(t)$.

Here, without loss of generality, we set $\varphi_0 = 0$. Since $p_{\mathrm{one}}^{\mathrm{can}}(\theta; r_0) = p_{\mathrm{one}}^{\mathrm{can}}(-\theta; r_0)$, we find from (18) that $p_{\mathrm{one}}(\theta, t) = p_{\mathrm{one}}(-\theta, t)$. Then, (19) leads to $\varphi(t) = 0$. This means that $\varphi(t) = 0$ is a solution of the self-consistent equation.

Now, we focus on the collective dynamics near the transition point. Explicitly, we set $\beta K = 2 + \epsilon$ with $|\epsilon| \ll 1$ for fixed K. We then expect that the slow dynamics of $r(t)$ are characterized by a scaling form

$$r(t) = \eta^b \tilde{r}(\eta t), \qquad (20)$$

where $\eta \to 0$ and $t \to \infty$ with $\eta t = \tau$ fixed; and $\tilde{r}$ is a function whose functional form is independent of η. We also expect that η is related to ϵ as

$$\eta = |\epsilon|^a. \qquad (21)$$

The question is to derive an equation for $\tilde{r}$ and to determine the values of a and b.

Mathematically, we have only to analyze p_{one} near the transition point. One can apply a center manifold theory to this system. (See section 5.7 in [7].) Instead, we consider the problem from a viewpoint of stochastic thermodynamics. Hereafter, $\langle\ \rangle$ represents the expectation with respect to this initial distribution and the noise sequence ξ. We also denote the expectation of $A(\theta)$ with respect to $p_{\mathrm{one}}(\theta, t)$ by $\langle A\rangle_t$, and $\langle\ \rangle_r^{\mathrm{can}}$ represents the expectation of the canonical ensemble with the Hamiltonian $H_{\mathrm{one}}(\theta; r)$.

2.3. Stochastic thermodynamics

We study the Langevin equation (16), where r is given as a function of time. We interpret the time dependent parameter as a prefixed protocol representing an operation to the system by a virtual external system. Concretely, the force Φ done by the external system is defined as

$$\Phi(\theta; r) \equiv \frac{\partial H_{\mathrm{one}}(\theta; r)}{\partial r}. \qquad (22)$$

By using (7), we express $r(t)$ determined in (19) as

$$r(t) = -\frac{1}{K}\langle \Phi(r)\rangle_t. \qquad (23)$$

According to equilibrium statistical mechanics, we have

$$\langle \Phi(r)\rangle_r^{\mathrm{can}} = \frac{\partial F(r)}{\partial r}, \qquad (24)$$

where $F(r)$ is the free energy defined by $F(r) = -T\log Z_{\mathrm{one}}(r)$. The self-consistent equation (9) is equivalent to $\langle \Phi(r(t))\rangle_t = \langle \Phi(r(t))\rangle_{r(t)}^{\mathrm{can}}$. This is valid only in the limit $t \to \infty$, and in general cases there should be the irreversible work defined by

$$W_{\mathrm{irr}} = \int_0^t ds \frac{dr}{ds}\left[\Phi(\theta(s); r(s)) - \frac{\partial F(r(s))}{\partial r(s)}\right]. \qquad (25)$$

We then obtain

$$\begin{aligned}\frac{d\langle W_{\mathrm{irr}}\rangle}{dt} &= \frac{dr}{dt}\left[\langle \Phi(r(t))\rangle_t - \frac{\partial F(r(t))}{\partial r(t)}\right] \\ &= \frac{dr}{dt}\left[-Kr(t) - \frac{\partial F(r(t))}{\partial r(t)}\right].\end{aligned} \qquad (26)$$

The problem now becomes to evaluate the irreversible work in the stochastic system. The important property here is that the time scale of r is much longer than the relaxation time of the probability density for the Langevin equation (16) near the transition point. That is, the control is assumed to be performed as a *nearly quasi-static* process, which enables us to develop a perturbation theory. Furthermore, owing to the recent progress on the

stochastic thermodynamics, we have several identities associated with thermodynamic works. By utilizing one of them, we can simplify the calculation of the irreversible work.

Concretely, we start with the Jarzynski equality [11]

$$\langle e^{-\beta W_{irr}} \rangle = 1. \tag{27}$$

See appendix A as for the derivation of a generalized version of (27). By combining $e^{-x} \geqslant 1 - x$ with the identity (27), we derive $\langle W_{irr} \rangle \geqslant 0$, which corresponds to the second law of thermodynamics. Furthermore, from the identity (27), in the nearly quasi-static regime $\eta \to 0$, we have

$$\langle W_{irr} \rangle = \frac{\beta}{2} \langle W_{irr}^2 \rangle + O\left(\eta^2\right), \tag{28}$$

which corresponds to the fluctuation-dissipation relation. By taking the derivative with respect to t, we obtain

$$\frac{d\langle W_{irr} \rangle}{dt} = \beta \frac{dr}{dt} \int_0^t ds \frac{dr}{ds} B(t, s) + O\left(\eta^3\right) \tag{29}$$

with

$$B(t, s) = \left\langle \left[\Phi(\theta(t), r(t)) - \frac{\partial F(r(t))}{\partial r(t)} \right] \left[\Phi(\theta(s), r(s)) - \frac{\partial F(r(s))}{\partial r(s)} \right] \right\rangle. \tag{30}$$

The correlation time of $\Phi(\theta, r(s))$ is controlled by T. Since $dr(s)/ds \simeq O(\eta)$ is much smaller than T, (29) becomes

$$\frac{d\langle W_{irr} \rangle}{dt} = \gamma(r(t)) \left(\frac{dr}{dt} \right)^2 + O\left(\eta^3\right) \tag{31}$$

with a friction constant

$$\gamma(r) = \beta \int_0^\infty ds \left\langle \left[\Phi(\theta(s), r) - \frac{\partial F(r)}{\partial r} \right] \left[\Phi(\theta(0), r) - \frac{\partial F(r)}{\partial r} \right] \right\rangle_r^{can}. \tag{32}$$

Essentially the same expression was obtained in [6]. Here, the expectation is taken over samples in which $\theta(0)$ is chosen obeying the canonical ensemble with r, and $\theta(t)$ is determined from the stochastic time evolution with fixed r. By combining (31) with (26), we have

$$\gamma(r(t)) \frac{dr}{dt} = -Kr(t) - \frac{\partial F(r(t))}{\partial r(t)} + O\left(\eta^2\right). \tag{33}$$

This determines the time evolution of $r(t)$ uniquely. By using the expansion (10), we rewrite (33) as

$$\gamma(r(t)) \frac{dr}{dt} = -Kr(t) + K\left[\frac{1}{2}(\beta Kr(t)) - \frac{1}{16}(\beta Kr(t))^3 + O\left(r(t)^5\right) \right] + O\left(\eta^2\right). \tag{34}$$

Recalling $dr/dt = O(\eta)$ and $\beta K = 2 + \epsilon$, we find that the exponents in (20) and (21) are given by $a = 1$ and $b = 1/2$. The equation for $\bar{r}(\tau)$ with $\tau = \eta t$ is

$$\gamma(0) \frac{d\bar{r}}{d\tau} = sgn(\epsilon) \frac{K}{2} \bar{r} - \frac{K}{2} \bar{r}^3 \tag{35}$$

in the limit $\eta \to 0$ and $t \to \infty$. The positivity of γ ensures the stability of the trivial solution $\bar{r} = 0$ for $\epsilon < 0$ and the non-trivial solution $\bar{r} = 1$ for $\epsilon > 0$, respectively. We also note that $\gamma > 0$ implies the monotonic increment of W_{irr} (see (31)), which is a stronger property than the second law of thermodynamics.

Finally, we calculate $\gamma(0)$. From the definition of γ in (32), we have

$$\gamma(0) = \beta K^2 \int_0^\infty ds \langle \cos \theta(s) \cos \theta(0) \rangle_{r=0}^{can}. \tag{36}$$

Let $C(t)$ be $\langle \cos \theta(t) \cos \theta(0) \rangle$ for the free Brownian motion $d\theta/dt = \xi$ which corresponds to the case $r = 0$ in the Langevin equation (16). We then derive

$$\begin{aligned} \frac{dC}{dt} &= -\langle \sin \theta(t) \circ \xi(t) \cos \theta(0) \rangle_{r=0}^{can} \\ &= -T \langle \cos \theta(t) \cos \theta(0) \rangle_{r=0}^{can} \\ &= -TC, \end{aligned} \tag{37}$$

where the symbol '$\circ$' represents the multiplication in the sense of Stratonovich. Since $C(0) = 1/2$, we obtain $C(t) = \exp(-Tt)/2$. The substitution of this result into (36) yields

$$\gamma(0) = \frac{\beta^2 K^2}{2}, \tag{38}$$

which is evaluated to be 2 at the transition point. In sum, the differential equation for $\bar{r}$ is

$$\frac{d\bar{r}}{d\tau} = \text{sgn}(\epsilon)\frac{K}{4}\bar{r} - \frac{K}{4}\bar{r}^3. \tag{39}$$

We guess that there should be some references reporting this result, say around 1970, but we do not find them. As far as we searched, explicit calculation was presented in [14] using bifurcation analysis. Note however that the numerical coefficient of the nonlinear term in equation (7) of [14] is not correct.

From a viewpoint of non-equilibrium statistical mechanics, the essential step of the derivation is the calculation of the friction constant in terms of the time-correlation function. The formula (32), which is one of standard fluctuation-dissipation theorems, can be obtained by various methods. The advantage of our formulation is that we can obtain the nonlinear differential equation with the formula of the friction constant.

3. Kuramoto model

3.1. Model
We study the Kuramoto model [7, 8]

$$\frac{d\theta_i}{dt} = \omega_i - \frac{K}{N}\sum_{j=1}^{N}\sin\left(\theta_i - \theta_j\right), \tag{40}$$

where $K > 0$ and ω_i is a time-independent stochastic variable obeying the probability density $g(\omega)$. We assume that $g(\omega) = g(-\omega)$, $g(\omega) \leqslant g(0)$, and the second derivative of $g(\omega)$ at $\omega = 0$ is negative. Note that $g(\omega) \to 0$ in $\omega \to \infty$ because $\int d\omega g(\omega) = 1$. The collective synchronization occurs when $K > 2/(\pi g(0))$. This result was obtained by the analysis of the self-consistent equation for the order parameter (3), which corresponds to (9) in the globally coupled XY model. After that, Kuramoto and Nishikawa attempted to derive the equation that describes the collective dynamics in the Kuramoto model [15]. However, it turned out that the problem was hard to be solved. Especially, even in the linear regime around the trivial state ($r = 0$), the analysis was far from trivial, as pointed out in [16, 17]. As one remarkable result, Ott and Antonsen derived the differential equation for the collective dynamics by noting a special solution of the nonlinear equation of the distribution [18]. Note that this method relies on a special property of the model [19, 20] and that it cannot be applied to general cases. Quite recently, Chiba has derived the equation of the order parameter by mathematically developing a center manifold theory with a resonance pole [13].

Since the difficulty originates from the deterministic nature of the dynamics, its noisy version

$$\frac{d\theta_i}{dt} = \omega_i - \frac{K}{N}\sum_{j=1}^{N}\sin\left(\theta_i - \theta_j\right) + \xi_i \tag{41}$$

has also been studied, where ξ_i is Gaussian-white noise satisfying (14). This model was first proposed by Sakaguchi [9]. The self-consistent equation of the order parameter in this model was analyzed and the non-trivial solution corresponding to the synchronized state was derived [9]. Then, based on the linear stability analysis of the self-consistent solutions in the noisy Kuramoto model [16, 17], bifurcation analysis was performed so as to obtain a differential equation of the order parameter near the transition point [12]. (See [21] for a story related to the development.)

In this section, we study the collective dynamics near the transition point for the noisy Kuramoto model from a viewpoint of stochastic thermodynamics. We then consider the noiseless limit $T \to 0$.

3.2. Setup of the problem
We start with re-expressing (41) by

$$\frac{d\theta_i}{dt} = \omega_i - Kr\sin\left(\theta_i - \varphi\right) + \xi_i. \tag{42}$$

We denote by $p_{\text{one}}^{\text{ss}}(\theta_i; r, \varphi, \omega_i)$ the stationary probability density for the Langevin equation (42) with (r, φ) fixed. We follow the analysis in the previous section step by step.

We assume the probability density at the initial time $t = 0$ as

$$p_0(\boldsymbol{\theta}) = \prod_i p_{\text{one}}^{\text{ss}}\left(\theta_i; r_0, \varphi_0, \omega_i\right) \tag{43}$$

for given r_0 and φ_0. The probability density $p(\boldsymbol{\theta}, t)$ at time t is determined uniquely. Then, for each t, in the limit $N \to \infty$, $r(t)$ and $\varphi(t)$ take definite values for almost all $\boldsymbol{\theta}$ with respect to $p(\boldsymbol{\theta}, t)$. We fix functional forms of $r(t)$ and $\varphi(t)$. Then, the probability density at time t is expressed as

$$p(\boldsymbol{\theta}, t) = \prod_i p_{\text{one}}\left(\theta_i, t; \omega_i\right), \tag{44}$$

where p_{one} is given by the solution of the Fokker–Planck equation associated with the Langevin equation (42):

$$\frac{\partial p_{\text{one}}(\theta, t; \omega)}{\partial t} + \frac{\partial}{\partial \theta}\left[(\omega - Kr\sin(\theta - \varphi))p_{\text{one}} - T\frac{\partial}{\partial \theta}p_{\text{one}}\right] = 0 \tag{45}$$

with the initial condition $p_{\text{one}}(\theta, 0; \omega) = p_{\text{one}}^{\text{ss}}(\theta; r_0, \varphi_0, \omega)$. Then, $r(t)$ and $\varphi(t)$ satisfy

$$r(t)e^{i\varphi(t)} = \int d\omega g(\omega) \int_0^{2\pi} d\theta p_{\text{one}}(\theta, t; \omega)e^{i\theta}, \tag{46}$$

which is regarded as a self-consistent equation for $r(t)$ and $\varphi(t)$. Without loss of generality, we assume $\varphi_0 = 0$. Since $p_{\text{one}}^{\text{ss}}(-\theta; r_0, 0, -\omega) = p_{\text{one}}^{\text{ss}}(\theta; r_0, 0, \omega)$, we find from (45) that $p_{\text{one}}(-\theta, t; -\omega) = p_{\text{one}}(\theta, t; \omega)$. Then, (46) leads to $\varphi(t) = 0$. Hereafter, $p_{\text{one}}^{\text{ss}}(\theta; r_0, 0, \omega)$ is abbreviated as $p_{\text{one}}^{\text{ss}}(\theta; r_0, \omega)$, and $\langle \ \rangle_\omega$ represents the expectation over initial conditions and noise sequences in the Langevin equation (42) with the frequency $\omega_i = \omega$. We denote the expectation of $A(\theta)$ with respect to $p_{\text{one}}(\theta, t; \omega)$ by $\langle A \rangle_{t,\omega}$, and $\langle \ \rangle_{r,\omega}^{\text{ss}}$ represents the expectation with respect to $p_{\text{one}}^{\text{ss}}(\theta; r, \omega)$.

Now, let $K_c(T)$ be the transition point of the coupling constant for the model with T fixed. We characterize the distance from the transition point by

$$\epsilon \equiv \frac{K - K_c}{K_c}. \tag{47}$$

Then, in the asymptotic regime $|\epsilon| \ll 1$, we expect that r evolves slowly and this behavior may be characterized by a scaling form (20) with (21). The question is to derive a differential equation for $\bar{r}$ together with determining the values of a and b.

3.3. Useful identity

Differently from the previous section, thermodynamic concepts such as the irreversible work are not established for transitions between non-equilibrium steady states. Indeed, the Jarzynski equality (27) is not available for the Langevin equation (42) due to the existence of the driving force ω_i. We thus need to consider an extension of the Jarzynski equality (27). This was proposed by Hatano and Sasa [10]. By defining

$$\phi(\theta; r, \omega) = -\log p_{\text{one}}^{\text{ss}}(\theta; r, \omega), \tag{48}$$

they derived

$$\left\langle e^{-\int_0^t ds \frac{\partial \phi(r(s), \omega)}{\partial r(s)}\frac{dr}{ds}} \right\rangle_\omega = 1. \tag{49}$$

It should be noted that (49) becomes the Jarzynski equality (27) when the stationary probability density is the canonical one. The quantity

$$Y \equiv T \int_0^t ds \frac{\partial \phi(r(s), \omega)}{\partial r(s)}\frac{dr}{ds} \tag{50}$$

is interpreted as a generalized irreversible work in a process starting from steady state. (See [22] for a review of an extended framework of thermodynamics on the basis of (49)). Next, since there is no Hamiltonian, we consider a different formulation from that using the thermodynamic force (22). Concretely, by defining

$$A \equiv -K\cos\theta, \tag{51}$$

and by recalling (3), we have

$$-Kr(t) = \int d\omega g(\omega)\langle A \rangle_{t,\omega}. \tag{52}$$

The first step of the analysis is to estimate $\langle A \rangle_{t,\omega}$ in the nearly quasi-static regime $\eta \to 0$. Here, as essentially the same identity as (49), we derive

$$\langle A \rangle_{r(t),\omega}^{\text{ss}} = \left\langle Ae^{-\int_0^t ds \frac{\partial \phi(r(s), \omega)}{\partial r(s)}\frac{dr}{ds}} \right\rangle_\omega. \tag{53}$$

(See appendix A for the derivation.) In the limit $\eta \to 0$, this identity yields

$$\langle A \rangle_{r(t),\omega}^{\mathrm{ss}} = \langle A \rangle_{t,\omega} - \int_0^t \mathrm{d}s \frac{\mathrm{d}r}{\mathrm{d}s} \left\langle A(\theta(t)) \frac{\partial \phi(\theta(s); r(s), \omega)}{\partial r(s)} \right\rangle_\omega + O\left(\eta^2\right). \tag{54}$$

A similar relation was proposed by using the identity (49) [23]. Furthermore, by noting the time scale separation, we rewrite it as

$$\Gamma(r(t), \omega) \frac{\mathrm{d}r}{\mathrm{d}t} = \langle A \rangle_{t,\omega} - \langle A \rangle_{r(t),\omega}^{\mathrm{ss}} + O\left(\eta^2\right) \tag{55}$$

with

$$\Gamma(r, \omega) = \int_0^\infty \mathrm{d}s \left\langle A(\theta(s)) \frac{\partial \phi(\theta(0); r, \omega)}{\partial r} \right\rangle_{r,\omega}^{\mathrm{ss}}. \tag{56}$$

This is a generalized fluctuation-dissipation relation claiming that the friction constant is expressed as the time correlation function.

Finally, multiplying $g(\omega)$ with the both hand sides of (55) and integrating them in ω, we obtain

$$\gamma(r(t)) \frac{\mathrm{d}r}{\mathrm{d}t} = -Kr(t) - G(r(t)) + O\left(\eta^2\right), \tag{57}$$

where we have used (52); and $\gamma(r)$ and $G(r)$ are expressed as

$$\gamma(r) = \int \mathrm{d}\omega g(\omega) \Gamma(r, \omega), \tag{58}$$

$$G(r) = -K \int \mathrm{d}\omega g(\omega) \langle \cos \theta \rangle_{r,\omega}^{\mathrm{ss}}. \tag{59}$$

Since $G(r) = -G(-r)$ (see appendix B), $G(r)$ can be expanded as

$$G(r) = -a_1 r - a_3 r^3 + O\left(r^5\right). \tag{60}$$

The transition point $K_c(T)$ is determined by

$$K_c = a_1 \mid_{K=K_c}, \tag{61}$$

which is obtained from the condition that the linear term in the right-hand side of (57) becomes zero. By substituting $K = K_c(1 + \epsilon)$ and $r = \eta^{1/2} \bar{r}(\eta t)$ with $\eta = |\epsilon|$ into (57), we obtain

$$\gamma(0) \frac{\mathrm{d}\bar{r}}{\mathrm{d}\tau} = \mathrm{sgn}(\epsilon) K_c \bar{r} + a_3 \bar{r}^3 \tag{62}$$

in the limit $\epsilon \to 0$ and $t \to \infty$, where $\gamma(0)$ and a_3 are evaluated at $K = K_c(T)$.

3.4. Calculation of a_1, a_3, and $\gamma(0)$

We expand $p_{\mathrm{one}}^{\mathrm{ss}}(\theta; r, \omega)$ as

$$p_{\mathrm{one}}^{\mathrm{ss}}(\theta; r, \omega) = \frac{1}{2\pi} + \sum_{n=1}^\infty q_n(\theta; \omega) r^n. \tag{63}$$

From (59) and (60), we have

$$a_1 = K \int \mathrm{d}\omega g(\omega) \int_0^{2\pi} \mathrm{d}\theta \cos \theta q_1(\theta; \omega), \tag{64}$$

$$a_3 = K \int \mathrm{d}\omega g(\omega) \int_0^{2\pi} \mathrm{d}\theta \cos \theta q_3(\theta; \omega). \tag{65}$$

By using expressions of q_1 and q_3, which are given in appendix B, we obtain

$$a_1 = \frac{K^2}{2} \int \mathrm{d}\omega g(\omega) \frac{T}{\omega^2 + T^2}, \tag{66}$$

$$a_3 = -T \frac{K^4}{4} \int \mathrm{d}\omega g(\omega) \frac{T^2 - 2\omega^2}{\left(\omega^2 + T^2\right)^2 \left(\omega^2 + 4T^2\right)}. \tag{67}$$

The coefficients (66) and (67) in the expansion (60) were already calculated [9], but the last term in equation (25) of [9] involves an error. By recalling (61), we explicitly derive the transition point $K_c(T)$ as

$$1 = \frac{K_c(T)}{2} \int d\omega g(\omega) \frac{T}{\omega^2 + T^2}. \tag{68}$$

Next, we calculate the friction constant $\gamma(0)$. By using the expansion of the stationary probability density (63), we have $\phi = \log 2\pi - 2\pi q_1 r + O(r^2)$. We then obtain

$$\frac{\partial \phi}{\partial r} = -K \frac{1}{\omega^2 + T^2} (T \cos \theta + \omega \sin \theta) + O(r) \tag{69}$$

in small r. (See appendix B for the expression of q_1.) By combining (56) with (69), the frequency dependent friction constant $\Gamma(0, \omega)$ is expressed as

$$\Gamma(0, \omega) = K^2 \frac{1}{\omega^2 + T^2} \int_0^\infty dt \, [TC(t) + \omega D(t)], \tag{70}$$

where $C(t) = \langle \cos \theta(t) \cos \theta(0) \rangle_{r=0,\omega}^{ss}$ and $D(t) = \langle \cos \theta(t) \sin \theta(0) \rangle_{r=0,\omega}^{ss}$. Here, as shown in appendix C, we derive

$$\int_0^\infty dt C(t) = \frac{T}{2(\omega^2 + T^2)}, \tag{71}$$

$$\int_0^\infty dt D(t) = -\frac{\omega}{2(\omega^2 + T^2)}. \tag{72}$$

By substituting (71) and (72) into (70), we obtain

$$\Gamma(0, \omega) = K^2 \frac{T^2 - \omega^2}{2(\omega^2 + T^2)^2}. \tag{73}$$

Therefore, the friction constant (58) with $r = 0$ becomes

$$\gamma(0) = K^2 \int d\omega g(\omega) \frac{T^2 - \omega^2}{2(\omega^2 + T^2)^2}. \tag{74}$$

When $g(\omega) = \delta(\omega)$, the result becomes (38) obtained in the previous section.

3.5. Noiseless limit

We consider the noiseless limit $T \to 0$. We first rewrite a_1 as

$$a_1 = \frac{K^2}{2} \int d\omega g(T\omega) \frac{1}{\omega^2 + 1}. \tag{75}$$

This immediately yields

$$\lim_{T \to 0} a_1 = \frac{\pi K^2}{2} g(0), \tag{76}$$

which leads to $K_c = 2/(\pi g(0))$. Similarly, we obtain

$$\lim_{T \to 0} a_3 = \lim_{T \to 0} \frac{K^4}{4T^2} \int d\omega g(T\omega) \left[\frac{\omega^2}{(\omega^2 + 1)^2} - \frac{1}{\omega^2 + 4} \right]$$

$$= \frac{\pi K^4}{16} g''(0), \tag{77}$$

where the double prime represents the second derivative. Next, we evaluate the noiseless limit of $\gamma(0)$. The method used in the estimation of a_1 and a_3 is not effective here. The heart of the calculation is to note an identity

$$\int d\omega \frac{T^2 - \omega^2}{(\omega^2 + T^2)^2} = 0. \tag{78}$$

By using it, we rewrite (74) as

$$\gamma(0) = K^2 \int d\omega (g(\omega) - g(0)) \frac{T^2 - \omega^2}{2(\omega^2 + T^2)^2}. \tag{79}$$

We then obtain

$$\lim_{T \to 0} \gamma(0) = -K^2 \int d\omega \left(g(\omega) - g(0)\right) \frac{1}{2\omega^2}$$
$$= -K^2 \int d\omega g'(\omega) \frac{1}{2\omega}. \tag{80}$$

By substituting (77) and (80) into (62), the equation of $\bar{r}$ becomes

$$\left[-K_c^2 \int d\omega g'(\omega) \frac{1}{2\omega}\right] \frac{d\bar{r}}{d\tau} = \text{sgn}(\epsilon) K_c \bar{r} + \frac{\pi K_c^4}{16} g''(0) \bar{r}^3, \tag{81}$$

which coincides with equation (7.97) in [13]. The right-hand side corresponds to the self-consistent equation obtained by Kuramoto [7]. Furthermore, we remark

$$\frac{\lim_{T \to 0} a_3}{\lim_{T \to 0} \gamma(0)} = \frac{g''(0)}{4\pi g(0)^2} \left(\int_0^\infty d\omega g'(\omega) \frac{1}{\omega}\right)^{-1}. \tag{82}$$

This expression corresponds to equation (138) in [12]. Although the numerical coefficient of the latter is different from (82), there is no contradiction between the two, because $|\alpha|$ in [12] is equal to $r/2\pi$ in this paper. (This unusual convention can be understood from equations (36) and (95) in [12].)

More explicitly, we study a case that $g(\omega)$ is a Cauchy distribution

$$g(\omega) = \frac{\Delta}{\pi} \frac{1}{\omega^2 + \Delta^2}. \tag{83}$$

By substituting it into the formulas given in (76), (77), and (80), we obtain

$$\lim_{T \to 0} a_1 = \frac{K^2}{2\Delta}, \tag{84}$$

$$\lim_{T \to 0} a_3 = -\frac{K^4}{8\Delta^3}, \tag{85}$$

and

$$\lim_{T \to 0} \gamma(0) = \frac{K^2}{2\Delta^2}. \tag{86}$$

Since $K_c = 2\Delta$ (that comes from (84)), (62) becomes

$$2 \frac{d\bar{r}}{d\tau} = \text{sgn}(\epsilon) K_c \bar{r} - K_c \bar{r}^3. \tag{87}$$

Both the decay rate and the growth rate below and above the transition point are $|K - K_c|/2$ in the original time scale, which are equal to the results of the linear stability analysis [16, 17]. The stationary solution above the transition point is equal to the result by Kuramoto [7]. Finally, the order parameter equation presented in [18] becomes (87) near the critical point.

4. Concluding remarks

In this paper, we have studied collective dynamics from a viewpoint of stochastic thermodynamics. The most important achievement is that we can obtain the order parameter equation (81) quickly. The key step in the derivation is to utilize the fluctuation-dissipation relation (54) that is derived from the non-equilibrium identity (53). Owing to this identity, we have only to calculate time correlation functions for a free Brownian particle driven on a ring, in addition to the previously known self-consistent equations [7, 9].

With regard to the formula of the friction constant (74), there are other methods for obtaining it. For example, one may formulate a linear response theory for non-equilibrium steady states so as to derive (74). We then believe that our formulation using the non-equilibrium identity is the simplest among various approaches. To minimize the number of calculation steps will be useful for considering other related subjects.

As is understood from the derivation method, the noiseless limit $T \to 0$ should be taken after the scaling limit $\epsilon \to 0$ and $t \to \infty$ is considered. When both T are ϵ are finite, our theory provides a good approximation for $\epsilon \ll T$. On the contrary, the calculation method cannot be applied to the noiseless Kuramoto model. Nevertheless, one may expect that the behavior for the case $\epsilon \ll T \ll 1$ is close to that for $\epsilon \ll 1$ and $T = 0$. This expectation is true for some cases, but not always valid. For example, it was pointed out in [17] that when $g(\omega)$ is zero expect for $[-\omega_0, \omega_0]$ with some positive ω_0, the order parameter in the noiseless Kuramoto model relaxes to the trivial state in a power law form for $K < K_c$, which is not observed for the case $\epsilon \ll T \ll 1$. We need to

develop a different formulation if we want to understand the behavior of the noiseless Kuramoto model correctly [24].

Although we focus on the simplest model of coupled oscillators, one can study more general cases such that the interaction includes higher harmonics e.g. $\sin(\theta_i - \theta_j) + h \sin 2(\theta_i - \theta_j)$. See [25] for a self-consistent equation, [26] for the analysis using the center manifold theory of the noisy case, and [27] for the generalized center manifold theory for the noiseless case. According to [27], the early attempts [26, 25] have some mistakes. See [28] for a recent study. It should be noted that the symmetry property that leads to $G(r) = -G(-r)$ and $\varphi = \varphi_0$ is broken for the case $h \neq 0$. This makes the calculation complicated. More importantly, it was shown that the value of the critical exponent b changes discontinuously in the noiseless limit. It would be a good problem to obtain a fresh view of this phenomenon by applying the method in this paper.

So far, we have assumed that $N \to \infty$. In finite but large N cases, we naturally expect that small Gaussian noise is added to the equation for the order parameter. We want to theoretically derive this stochastic equation. For example, one may start with the exact stochastic equation for the distribution

$$p(\theta, t) = \frac{1}{N} \sum_{j=1}^{N} \delta\left(\theta_j(t) - \theta\right), \tag{88}$$

which is referred to as Dean's equation [29]. Writing the path-integral expression for the history of p, one may combine the WKB analysis with the techniques in this paper. It is a challenging problem to complete the formulation. See [14, 31, 30] for arguments on finite size fluctuations. We also mention that the noise intensity may be related to the friction constant. Here, since the standard form of the second fluctuation-dissipation relation is not expected for the Kuramoto model, the relation itself is of great interest. In studying this problem, [32, 33], which present the formula for a probe particle in non-equilibrium baths, may be helpful.

Obviously, the exact determination of the differential equation for the order parameter relies on the mean field nature of the model. When we attempt to study models in finite dimensions, further techniques will be necessary so as to derive the time evolution of a spatially modulated order parameter. Then, a local stationary distribution for given spatial configurations of r and φ should be a reference state or an unperturbed state. Although it is a highly non-trivial problem to derive the equation, we should start this analysis seriously, because we have the simplest derivation of the collective dynamics in the mean-field model. The collective dynamics of coupled oscillators defined on random networks and complex networks are also worthwhile to be studied [34, 35].

Finally, we briefly mention a recent work in which the Navier–Stokes equation is derived from Hamiltonian particles systems using a non-equilibrium identity [36]. This derivation method is formally correct and the most compact in existing approaches. Simplifying calculation enables us to extract the essence of the derivation problem, and thus we can now carefully review previous studies by Mori [37], Mclennan [38], Zubarev [39], and Esposito and Marra [40], which will be reported elsewhere. However, the method in [36] involves some mathematical assumptions such as convergences of time correlation functions. Now, look into the Kuramoto model again. If we set $T = 0$ in the integrand of (79), the friction constant $\gamma(0)$ diverges. Thus, the formal calculation does not make a sense. Similarly, in the argument of the hydrodynamic equation, we should check the well-defined nature of the dissipation constants. Maybe related to this issue, we point out that arbitrarily small noise is introduced even in mathematically deriving the Euler equation [41].

Stochastic thermodynamics formalizes thermodynamic concepts of fluctuating quantities. It is obvious from this definition that the framework is useful for analyzing small machines such as molecular motors. In addition to such direct application, universal formulas found in stochastic thermodynamics may be applied to several non-equilibrium dynamics. We hope that this paper will stimulate many researchers who work on various subjects.

Acknowledgments

Appendix A. Derivation of the identity (53)

We consider a time-dependent Markov chain on a finite set X, where a time series $(x_n)_{n=0}^{N}$ with $x_n \in X$ is generated by a transition matrix $T(x_n \to x_{n+1}; \alpha_n)$ with a time dependent parameter α_n. We denote by $P_{ss}(x; \alpha)$

the stationary probability of the transition matrix $T(x \to y; \alpha)$. That is, $P_{ss}(x; \alpha)$ satisfies

$$\sum_x T(x \to y; \alpha) P_{ss}(x; \alpha) = P_{ss}(y; \alpha). \tag{A.1}$$

We then define the dual transition matrix $T^\star(x \to y; \alpha)$ by

$$P_{ss}(x; \alpha) T(x \to y; \alpha) = P_{ss}(y; \alpha) T^\star(y \to x; \alpha). \tag{A.2}$$

It should be noted that $\sum_x T^\star(y \to x; \alpha) = 1$. We then have a trivial identity

$$
\begin{aligned}
& T^\star(x_1 \to x_0; \alpha_0) \cdots T^\star(x_N \to x_{N-1}; \alpha_{N-1}) \\
& = \frac{P_{ss}(x_0; \alpha_0)}{P_{ss}(x_1; \alpha_0)} T(x_0 \to x_1; \alpha_0) \frac{P_{ss}(x_1; \alpha_1)}{P_{ss}(x_2; \alpha_1)} T(x_1 \to x_2; \alpha_1) \cdots
\end{aligned}
\tag{A.3}
$$

Here, by multiplying $A(x_N) P_{ss}(x_N; \alpha_N)$ to the both hand sides and taking the summation over histories $(x_n)_{n=0}^N$, we obtain

$$\langle A \rangle^{ss} = \left\langle \frac{P_{ss}(x_1; \alpha_1)}{P_{ss}(x_1; \alpha_0)} \cdots \frac{P_{ss}(x_N; \alpha_N)}{P_{ss}(x_N; \alpha_{N-1})} A(x_N) \right\rangle, \tag{A.4}$$

where $\langle A \rangle$ for a trajectory dependent quantity A represents

$$\langle A \rangle = \sum_{(x_n)_{n=0}^N} P_{ss}(x_0; \alpha_0) T(x_0 \to x_1; \alpha_0) \cdots T(x_{N-1} \to x_N; \alpha_{N-1}) A\left[(x_n)_{n=0}^N\right]. \tag{A.5}$$

Note that (A.4) is also valid for Markov chains on real numbers.

Next, we study the following Langevin equation for a phase variable θ:

$$\frac{d\theta}{dt} = f(\theta; \alpha) + \xi, \tag{A.6}$$

where $f(\theta + 2\pi; \alpha) = f(\theta; \alpha)$ and ξ is Gaussian white noise satisfying $\langle \xi(t) \xi(t') \rangle = 2T\delta(t - t')$. We denote the stationary probability density by $p_{ss}(\theta; \alpha)$. We discretize the Langevin equation with a time interval Δt. Since this defines the Markov chain on real numbers, we have the identity (A.4). Then, by taking the limit $\Delta t \to 0$, we obtain the identity (53). When we set $A = 1$, it becomes the identity (49).

Appendix B. Stationary probability density

The stationary distribution of (42) with (r, φ) fixed is determined from

$$(\omega - Kr \sin\theta) p_{one}^{ss}(\theta; r, \omega) - T\partial_\theta p_{one}^{ss}(\theta; r, \omega) = J(r, \omega), \tag{B.1}$$

where $J(r, \omega)$ is a constant independent of θ. By substituting (63) into (B.1), we have

$$\omega q_n - T\partial_\theta q_n = K \sin\theta q_{n-1} + J_n \tag{B.2}$$

with $q_0 = 1/(2\pi)$ and J_n is a constant. We solve this equation iteratively. Concretely, we calculate q_1 and q_2 as

$$q_1 = \frac{K}{2\pi} \frac{1}{\omega^2 + T^2} (T \cos\theta + \omega \sin\theta), \tag{B.3}$$

$$q_2 = \frac{K^2}{2\pi} \frac{1}{2(\omega^2 + T^2)(\omega^2 + 4T^2)} \left[\left(2T^2 - \omega^2\right) \cos 2\theta + 3\omega T \sin 2\theta \right]. \tag{B.4}$$

Noting that q_3 is written as

$$q_3 = b_{33} \cos 3\theta + c_{33} \sin 3\theta + b_{31} \cos\theta + c_{31} \sin\theta, \tag{B.5}$$

we calculate only b_{31} as

$$b_{31} = \frac{K^3}{2\pi} \frac{T\left(2\omega^2 - T^2\right)}{2\left(\omega^2 + T^2\right)^2 \left(\omega^2 + 4T^2\right)}. \tag{B.6}$$

As is understood from these calculation, one can prove

$$\int_0^{2\pi} \mathrm{d}\theta \, \cos \theta q_n(\theta; \omega) = 0 \tag{B.7}$$

for even integer n. This leads to $G(-r) = -G(r)$ from (59).

Appendix C. Derivation of (71) and (72)

We study the simple Langevin equation

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \omega + \xi, \tag{C.1}$$

where ξ is Gaussian noise satisfying $\langle \xi(t)\xi(t') \rangle = 2T\delta(t - t')$. We shall calculate the time integration of the correlation functions $C(t) = \langle \cos \theta(t) \cos \theta(0) \rangle$ and $D(t) = \langle \cos \theta(t) \sin \theta(0) \rangle$. We first consider the time derivative of the correlation functions. By using the equation (C.1), we have

$$\frac{\mathrm{d}C}{\mathrm{d}t} = \omega D(t) - TC(t), \tag{C.2}$$

$$\frac{\mathrm{d}D}{\mathrm{d}t} = -\omega C(t) - TD(t), \tag{C.3}$$

where we used $D(t) = -\langle \sin \theta(t) \cos \theta(0) \rangle$ and $C(t) = \langle \sin \theta(t) \sin \theta(0) \rangle$. We note that $C(0) = 1/2$ and $D(0) = 0$. From (C.2) and (C.3), we obtain

$$\frac{\mathrm{d}^2C}{\mathrm{d}t^2} = -2T\frac{\mathrm{d}C}{\mathrm{d}t} - \left(\omega^2 + T^2\right)C(t), \tag{C.4}$$

where $\mathrm{d}C/\mathrm{d}t \,|_{t=0} = -T/2$. The time integration of (C.4) over the interval $[0, \infty]$ leads to

$$\int_0^\infty \mathrm{d}t C(t) = \frac{T}{2\left(\omega^2 + T^2\right)}. \tag{C.5}$$

The time integration of (C.3) yields

$$\int_0^\infty \mathrm{d}t D(t) = -\frac{\omega}{2\left(\omega^2 + T^2\right)}. \tag{C.6}$$

References

[1] Evans D J, Cohen E G D and Morriss G P 1993 *Phys. Rev. Lett.* **71** 2401
[2] Gallavotti G and Cohen E G D 1995 *Phys. Rev. Lett.* **74** 2694
[3] Kurchan J 1998 *J. Phys. A: Math. Gen.* **31** 3719
[4] Sekimoto K 2010 *Stochastic Energetics* (*Lect. Notes Phys.* vol 799) (Berlin: Springer)
[5] Seifert U 2012 *Rep. Prog. Phys.* **75** 126001
[6] Sekimoto K and Sasa S 1997 *J. Phys. Soc. Japan* **66** 3326
[7] Kuramoto Y 1984 *Chemical Oscillations, Waves, and Turbulence* (Berlin: Springer)
[8] Acebron J A, Bonilla L L, Vicente C J P, Ritort F and Spigler R 2005 *Rev. Mod. Phys.* **7** 137
[9] Sakaguchi H 1988 *Prog. Theor. Phys.* **79** 39
[10] Hatano T and Sasa S 2001 *Phys. Rev. Lett.* **86** 3463
[11] Jarzynski C 1997 *Phys. Rev. Lett.* **78** 2690
[12] Crawford J D 1994 *J. Stat. Phys.* **74** 1047
[13] Chiba H 2013 *Ergon. Theor. Dyn. Syst.* **35** 762
[14] Pikovsky A and Ruffo S 1999 *Phys. Rev.* E **59** 1633
[15] Kuramoto Y and Nishikawa I 1987 *J. Stat. Phys.* **49** 569
[16] Strogatz S H and Mirollo R E 1991 *J. Stat. Phys.* **63** 613
[17] Strogatz S H, Mirollo R E and Matthews P C 1992 *Phys. Rev. Lett.* **68** 2730
[18] Ott E and Antonsen T M 2008 *Chaos* **18** 037113
[19] Pikovsky A and Rosenblum M 2011 *Physica* D **240** 872
[20] Marvel S A, Mirollo R E and Strogatz S H 2009 *Chaos* **19** 043104
[21] Strogatz S H 2000 *Physica* D **143** 1
[22] Sasa S 2014 *J. Stat. Mech.* **2014** P01004
[23] Prost J, Joanny J F and Parrondo J M R 2009 *Phys. Rev. Lett.* **103** 090601
[24] Sasa S 2015 in preparation
[25] Daido H 1996 *Physica* D **91** 24
[26] Crawford J D and Davies K T R 1999 *Physica* D **125** 1
[27] Chiba H and Nishikawa I 2011 *Chaos* **21** 043103
[28] Komarov M and Pikovsky A 2013 *Phys. Rev. Lett.* **111** 204101

[29] Dean D S 1996 *J. Phys. A: Math. Gen.* **29** L613
[30] Hildebrand E J, Buice M A and Chow C C 2007 *Phys. Rev. Lett.* **98** 054101
[31] Daido H 1990 *J. Stat. Phys.* **60** 753
[32] Maes C 2014 *J. Stat. Phys.* **154** 705
[33] Maes C and Steffenoni S 2014 arXiv:1410.5933
[34] Ichinomiya T 2004 *Phys. Rev.* E **70** 026116
[35] Arenas A, Diaz-Guilera A, Kurths J, Moreno Y and Zhou C 2008 *Phys. Rep.* **469** 93
[36] Sasa S 2014 *Phys. Rev. Lett.* **112** 100602
[37] Mori H 1958 *Phys. Rev.* **112** 1829
[38] Mclennan J A 1960 *Phys. Fluids* **3** 493
      Mclennan J A 1988 *Introduction to Non-equilibrium Statistical Mechanics* (Englewood Cliffs, NJ: Prentice-Hall)
[39] Zubarev D N 1974 *Nonequilibrium Statistical Thermodynamics* (New York: Consultants Bureau)
[40] Esposito R and Marra R 1994 *J. Stat. Phys.* **74** 981
[41] Olla S, Varadhan S R S and Yau H T 1993 *Commun. Math. Phys.* **155** 523