

Kähler Cones on Calabi-Yau threefolds

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The purpose of this note is to briefly explain some recent work which develops further some ideas from [8] concerning Calabi-Yau threefolds. Details of this theory will appear at least in preprint form in the near future.

Recall that a Calabi-Yau threefold is a complex projective smooth threefold X with canonical class $K_X = 0$ and with $h^1(\mathcal{O}_X) = h^2(\mathcal{O}_X) = 0$. Usually we shall assume that X is not the quotient of an abelian threefold, and then by Theorem 1.5 of [5] the linear form on $H^2(X, \mathbb{Z})$ given by the second Chern class $c_2(X)$ is non-trivial. We recall from [1] that a Calabi-Yau threefold has finite fundamental group unless some finite cover is either abelian or decomposable as a product of an elliptic curve and a K3 surface.

If the Calabi-Yau threefold X is simply connected, its diffeomorphism class is determined up to a finite number of possibilities by the information :

(1) The cubic form $\mu : H^2(X, \mathbb{Z}) \to \mathbb{Z}$ given by cup-product.

(2) The linear form $c_2 : H^2(X, \mathbb{Z}) \to \mathbb{Z}$ given by cup-product with $c_2(X) \in H^4(X, \mathbb{Z})$.

(3) The middle cohomology $H^3(X, \mathbb{Z})$.

If moreover $H^3(X, \mathbb{Z})$ is torsion-free, the above information characterizes the diffeomorphism class of X uniquely [7].

Since $H^2(X, \mathbb{Z})$ is canonically isomorphic to $\mathrm{Pic}(X)$, we shall usually think of its elements as line bundles or divisor classes on X, and denote $\mu(D)$ by D^3. In [8], we considered the real vector space $H^2(X, \mathbb{R}) = H^2(X, \mathbb{Z}) \otimes \mathbb{R}$ of dimension ρ. In this space there are two cones which are of particular interest :

THE KAHLER CONE. This is the open cone $\mathcal{K} \subset H^2(X, \mathbb{R})$ consisting of the Kähler classes ;

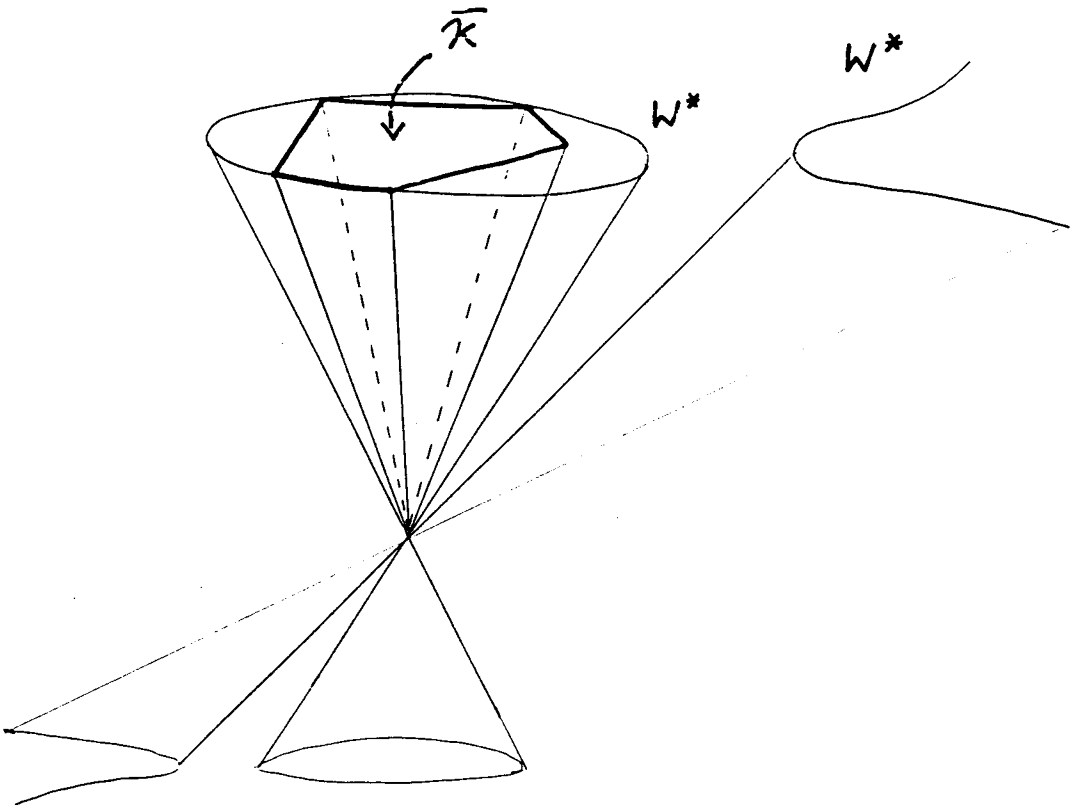
from the point of view of divisors, it is the open cone generated by the ample divisor classes. The closure $\overline{\mathcal{K}} \subset H^2(X,\mathbb{R})$ is then the cone consisting of numerical classes of nef divisors, that is real divisor classes D s.t. $D.C \geqslant 0$ for all curves C on X. We denote the boundary of $\overline{\mathcal{K}}$ by $\partial\overline{\mathcal{K}}$. The cone $\overline{\mathcal{K}}$ is the dual of the Mori cone $\overline{NE}(X)$. In our case however, it is more convenient to work with the cone $\overline{\mathcal{K}}$ because of its relationship to the second cone we study.

THE CUBIC CONE. The Cubic Cone $W^* \subset H^2(X,\mathbb{R})$ is the cone defined by the given cubic form above, i.e. $W^* = \{ D \in H^2(X,\mathbb{R}) \ ; \ D^3 = 0 \}$.
This in turn determines a cubic hypersurface $W \subset \mathbb{P}^{\rho-1}(\mathbb{R})$.

Remarks. (1) The rational points of $\mathcal{K}$ correspond to ample divisor classes on X.
(2) Observe that $D^3 \geqslant 0$ for all $D \in \overline{\mathcal{K}}$; furthermore the linear form c_2 is non-negative on $\overline{\mathcal{K}}$ — this latter claim follows from Theorem 1.1 of [6].
(3) A non-zero element $D \in H^2(X,\mathbb{R})$ has D^2

(numerically) trivial if and only if the point of $\mathbb{P}^{\rho-1}(\mathbb{R})$ corresponding to D is a singular point of W.

A picture of what the geometry of the above two cones in $H^2(X, \mathbb{R}) \cong \mathbb{R}^\rho$ might look like (when $\rho = 3$) is provided by the figure below. In order not to complicate the picture, the hyperplane $c_2 = 0$ has not been included. The reader should however bear in mind that $\overline{K}$ is on the positive side of this hyperplane, although possibly touching it.



Almost immediate from the theory of [8] are the following two facts concerning the relationship between these two cones. We say that a birational contraction morphism is <u>primitive</u> if it cannot be further factored into birational morphisms, or equivalently that it contracts just a 1-dimensional space of numerical classes of 1-cycles.

FACT 1. Away from W^*, the cone $\overline{\mathcal{K}}$ is locally rational polyhedral; the codimension 1 faces corresponding to primitive birational contractions of X.

FACT 2. Non-singular rational points of W^* which are on $\overline{\mathcal{K}}$ but not on the hyperplane $c_2 = 0$ give rise to elliptic fibre space structures on X.

In both cases, the morphism determined by a rational point D is just ϕ_{nD} for n sufficiently large — in the first case $\phi_{nD} : X \to \overline{X}$ is birational and in the second case $\phi_{nD} : X \to S$ an elliptic fibre space.

For X a Calabi-Yau threefold, it is known from results of Tian, Todorov and Ran that the first order deformations are unobstructed, and so the versal deformation (Kuranishi) space of X can be regarded as an open polydisc in $H^1(X, T_X) \cong H^1(X, \Omega^2_X)$. In fact, by recent theory of Viehweg, there exists a global quasi-projective moduli space of polarized Calabi-Yau threefolds with given Hilbert polynomial. Moreover, Todorov has announced a result that the space of all deformations of a given X has the structure of a quasi-projective variety.

For a given Calabi-Yau threefold, we know from the well-known theorem of Yau that the Kähler classes correspond bijectively with the Calabi-Yau metrics on X. In [3], the local moduli space parametrizing complex structure plus Calabi-Yau metric is studied under the assumption that the Kähler cone is locally independent of the complex structure. This is clearly true of many examples, which for instance are embedded as complete intersections in some rigid ambient space, the Kähler cone of which restricts to the Kähler cone on X. One motivation for the present investigation was to clarify any dependence the Kähler cone might have on the complex structure; it transpires that this in turn has implications as far as the existence of elliptic fibre space structures on X and its deformations.

If $\pi : \mathcal{X} \rightarrow B$ is a smooth family of Calabi-Yau threefolds over say a polydisc B, then we can identify the cohomology groups $H^2(X_b, \mathbb{Z}) \cong H^2(\mathcal{X}, \mathbb{Z})$, where X_b denotes the fibre of π over $b \in B$. We can therefore consider the Kähler cones $\mathcal{K}(b)$ of X_b for $b \in B$ to be cones in some fixed vector space $H^2(X, \mathbb{R})$. The question raised by [3] is whether these cones are all the same, or more generally in the case of an arbitrary base manifold B, whether the Kähler cone is locally constant.

THEOREM 1. For a smooth family of Calabi-Yau threefolds $\pi : \mathcal{X} \rightarrow B$ in which none of the threefolds X_b contains a geometrically ruled surface over a curve of positive genus, the Kähler cone is locally constant. More generally, the Kähler cone is locally constant over a dense subset B_0 of B; the maximal such subset B_0 being locally the complement of a countable collection of closed analytic subvarieties.

A geometrically ruled surface over a non-singular curve C is a projective surface E (with at worst Du Val singularities) which admits a morphism $h : E \to C$ whose reduced fibres are all $\mathbb{P}^1$'s. The reader will observe that the non-smooth fibres will in fact be double fibres on which E will have two A_1 type singularities.

If in Theorem 1 we take B to be for instance a polydisc and B_0 maximal, then the complement of B_0 is a countable union of analytic subvarieties of B. Each such subvariety will correspond to some cohomology class, and will be the locus (assumed not to be the whole of B) over which the class represents an effective divisor (which will be geometrically ruled over a curve of positive genus). The interest therefore lies in the case of Calabi-Yau threefolds containing geometrically ruled surfaces of positive genus, and in particular the behaviour of such surfaces and their fibres under deformations of the threefold.

The basic idea behind Theorem 1 is to show that if X contains no geometrically ruled surfaces of positive genus and if $\phi : X \to \overline{X}$ is a contraction morphism on X, then the exceptional locus of ϕ will extend 'sideways' under any small 1-parameter deformation $\pi : \mathcal{X} \to \Delta$ for X. This together with Fact 1 above giving rationality of faces of $\overline{\mathcal{K}}$ away from W^*, ensures that $\overline{\mathcal{K}}$ is locally invariant in the family. The main tool here is the theory of simultaneous resolutions of Du Val singularities of surfaces.

Consider now the case of X a Calabi-Yau threefold containing a geometrically ruled surface E over a curve C of genus $g > 0$. By taking the rational point where the line $\{E + xH \,;\, x \in \mathbb{R}\}$ (H any ample divisor) cuts $\partial\overline{\mathcal{K}}$ for the last time, the morphism ϕ_D corresponding to D (some multiple of this rational class) is a birational contraction morphism which contracts E down to C and is an isomorphism elsewhere. By taking different ample divisors H, we see that there is a codimension one face of $\overline{\mathcal{K}}$ corresponding to E.

Using Hodge theory, we show that there is a natural map $H^1(T_X) \longrightarrow H^1(\mathcal{O}_E(E)) = H^1(\omega_E)$ which is surjective, and in particular that $g \leq h^{1,2}$.

If now $\pi : \mathcal{X} \to B$ is the Kuranishi family for $X = X_0$, it can be shown that the locus Γ in B over which E deforms is a closed analytic subvariety of codimension $\geqslant g$, with equality if E is smooth. Moreover, we show that if $g = 1$, then the Kähler cone is not invariant in the family, and so is not invariant under a general small deformation of X. As we see below, such examples do certainly exist with E ruled over an elliptic curve, and so it is not true in general that the Kähler cone is locally constant. There is evidence to support a conjecture however that it is only surfaces ruled over a curve of genus one which will cause the Kähler cone not to be constant under small deformations. This evidence includes a 1st order deformation calculation which shows that if $g > 1$, then at least to 1st order some fibres (in fact $2g-2$ of them) will extend sideways in any given 1-parameter small deformation. The conjecture is certainly true in the case when the rational points of the associated cubic hypersurface $W \subset \mathbb{P}^{\rho-1}(\mathbb{R})$ are dense. As was shown in [8], this arithmetical condition on W is automatically satisfied when ρ is large enough (for instance $\rho > 19$ will do). If this conjecture holds, then with $\pi : \mathcal{X} \to B$ the Kuranishi family and B_0 as in Theorem 1, the complement of B_0 would be a countable union of codimension ≥ 1 subvarieties, corresponding precisely to classes E which represent a geometrically ruled surface of genus one at some point of B. The author would conjecture further that unless X has a decomposable cover (product of elliptic curve and K3 surface), then there should locally be only finitely many such subvarieties.

If the Kähler cone is not locally constant in families, this has important consequences concerning the cubic cone W^* and the Kähler cone $\mathcal{K}$ at the points $b \in B_0$, with B_0 the dense subset of B over which the cone is locally constant. In particular, by means of a Hilbert scheme argument, we can show that if $\pi : \mathcal{X} \to B$ is a smooth family of Calabi-Yau threefolds (over say a polydisc B) for which the Kähler cone is not constant, then there exists an open subset W_0 in the smooth locus of W, disjoint from the hyperplane $c_2 = 0$, and with the property that a corresponding half-cone W_0^+ on W_0 is contained in $\overline{\mathcal{K}}(b)$ for all $b \in B_0$. When we know that the rational points in W are dense, the Fact 2 above can be applied.

THEOREM 2. If X is a Calabi-Yau threefold for which the rational points on the associated cubic hypersurface W are dense, then either the Kähler cone is locally invariant in smooth families containing X, or a generic small deformation of X will have an elliptic fibre space structure.

We noted above that the arithmetical condition is satisfied when $\rho > 19$; it is however satisfied in many other cases; for instance if say W has a singularity at a rational point (which certainly happens when for instance some deformation of X has the structure of a fibre space over $\mathbb{P}^1$ or an elliptic fibre space over a surface S with $\rho(S) > 1$). Recalling the fact from [4] that the Euler number of elliptic Calabi-Yau threefolds is bounded, we deduce for sufficiently large Euler number that the Kähler cone will always be locally invariant.

The theory described above can be applied to the question of whether some deformation of a Calabi-Yau threefold can have an elliptic fibre space structure. It had been observed in [4] that the condition $b_2(X) > 1$ was clearly necessary for there to be an affirmative answer. We observe first that this condition is certainly not sufficient.

Example 1. Let $\overline{X} \subset \mathbb{P}(3,2,2,2,2,1)$ be the weighted complete intersection cut out by two general equations of degree 6. The normal threefold $\overline{X}$ is a Calabi-Yau model in the sense of [8]; it has a genus 10 curve of A_1 singularities. Blowing this curve up, we get a Calabi-Yau threefold X with $\rho(X) = 2$. The cubic hypersurface $W \subset \mathbb{P}^1$ consists of three irrational points; and this means that it is impossible for any deformation of X to have an elliptic fibre space structure, since such a structure implies that W has a rational point.

This type of arithmetical obstruction to the existence of elliptic structures on deformations ceases to occur when ρ is large enough (for instance if $\rho > 19$). For more fundamental obstructions to the above question having an affirmative answer are provided by the geometry of the two cones $\overline{\mathcal{K}}$ and W^*. If for instance X contains no geometrically ruled surface of of positive genus and $\overline{\mathcal{K}}$ and W^* are disjoint (so from Fact 1 above, $\overline{\mathcal{K}}$ is in fact rational polyhedral), then Theorem 1 together with the result stated before Theorem 2 imply that the Kähler cone is locally constant in all families containing X; in particular $\overline{\mathcal{K}}$ and W^* remain disjoint, and so no elliptic fibre space structure can occur. One suspects that this kind of situation might not be too uncommon, and unlike the arithmetical type of obstruction, there seems to be no obvious restriction on ρ for it to occur. An example due to Borcea [2] with $\rho = 12$ and $h^{1,2} = 15$ is of the above type, its codimension 1 faces of $\overline{\mathcal{K}}$ corresponding to small contractions (of finitely many $\mathbb{P}^1$'s).

The situation for Calabi-Yau manifolds therefore can be contrasted with the case of K3 surfaces, where any K3 surface is the deformation of an elliptic one. Another striking difference from the case of K3 surfaces is provided by the existence of Calabi-Yau threefolds with infinitely many automorphism classes of elliptic structures. For K3 surfaces, this doesn't occur [9].

Example 2 We start with a normal threefold $\overline{X}$ listed in [10], that is

$$\overline{X} = \left(\begin{array}{l\|ccc} \mathbb{P}^2 & 2 & 0 & 1 \\ \mathbb{P}^2 & 0 & 2 & 1 \\ \mathbb{P}(2,1,1) & 0 & 0 & 4 \end{array} \right)$$

This represents a weighted complete intersection in the product $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}(2,1,1)$ given by general polynomials homogeneous of degrees $(2,0,0)$, etc. in each set of variables. As $\mathbb{P}(2,1,1)$ is just a cone in $\mathbb{P}^3$, we can write $\overline{X}$ if we wish as a complete intersection in $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^3$. It is easily checked that $\overline{X}$ is a Calabi-Yau model with an elliptic curve C of A_1 singularities. If we blow C up, we get a Calabi-Yau threefold X with $\rho(X) = 4$, $e(X) = -128$. An easy calculation reveals that $W \subset \mathbb{P}^3$ is a cubic with 4 nodes (at rational points).

It follows automatically that the rational points in W are dense. We now make a generic deformation X_1 of X. Because X contains a ruled surface over an elliptic curve, the Kähler cone must change. The result stated before Theorem 2 provides us with an open subset $W_0 \subset W$, the rational points of which will all provide elliptic fibre space structures on X_1. Since W is irreducible, points on W corresponding to the same fibre space structure will be collinear — hence we have infinitely many elliptic structures on X_1. Since the automorphism group of W is finite, we also have infinitely many automorphism classes of elliptic fibre space structures.

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