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Induced nilpotent orbits and birational geometry

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Introduction.

Let G be a complex simple algebraic group and let $\mathfrak{g}$ be its Lie algebra. A nilpotent orbit $\mathcal{O}$ in $\mathfrak{g}$ is an orbit of a nilpotent element of $\mathfrak{g}$ by the adjoint action of G on $\mathfrak{g}$. Then $\mathcal{O}$ admits a natural symplectic 2-form ω and the nilpotent orbit closure $\bar{\mathcal{O}}$ has symplectic singularities in the sense of [Be] and [Na3] (cf. [Pa], [Hi]). In [Ri], Richardson introduced the notion of so-called the *Richardson orbit*. A nilpotent orbit $\mathcal{O}$ is called Richardson if there is a parabolic subgroup Q of G such that $\mathcal{O} \cap n(\mathfrak{q})$ is an open dense subset of $n(\mathfrak{q})$, where $n(\mathfrak{q})$ is the nil-radical of $\mathfrak{q}$. Later, Lusztig and Spaltenstein [L-S] generalized this notion to the *induced orbit*. A nilpotent orbit $\mathcal{O}$ is an induced orbit if there are a parabolic subgroup Q of G and a nilpotent orbit $\mathcal{O}'$ in the Levi subalgebra $\mathfrak{l}(\mathfrak{q})$ of $\mathfrak{q} := \mathrm{Lie}(Q)$ such that $\mathcal{O}$ meets $n(\mathfrak{q}) + \mathcal{O}'$ in an open dense subset. If $\mathcal{O}$ is an induced orbit, one has a natural map (cf. (1.2))

$$\nu : G \times^Q (n(\mathfrak{q}) + \bar{\mathcal{O}}') \to \bar{\mathcal{O}}.$$

The map ν is a generically finite, projective, surjective map. This map is called the *generalized Springer map*. In this paper, we shall study the induced orbits from the view point of *birational geometry*. For a Richardson orbit $\mathcal{O}$, the Springer map ν is a map from the cotangent bundle $T^*(G/Q)$ of the flag variety G/Q to $\bar{\mathcal{O}}$. In [Fu], Fu proved that, if $\bar{\mathcal{O}}$ has a crepant (projective) resolution, it is a Springer map. Note that Q is not unique (even up to the conjugate) for a Richardson orbit $\mathcal{O}$. This means that $\bar{\mathcal{O}}$ has many different crepant resolutions. In [Na], the author has given a description of all crepant resolutions of $\tilde{O}$ and proved that any two different crepant resolutions are connected by *Mukai flops*. The purpose of this paper is to generalize these to *all* nilpotent orbits O. If O is not Richardson, $\tilde{O}$ has no crepant resolution. The substitute of a crepant resolution, is a $\mathbf{Q}$*-factorial terminalization*. Let X be a complex algebraic variety with rational Gorenstein singularities. A partial resolution $f : Y \to X$ of X is said to be a $\mathbf{Q}$*-factorial terminalization* of X if Y has only $\mathbf{Q}$-factorial terminal singularities and f is a birational projective morphism such that $K_Y = f^*K_X$. A $\mathbf{Q}$-factorial terminalization is a crepant resolution exactly when Y is smooth. Recently, Birkar-Cascini-Hacon-McKernan [B-C-H-M] have established the existence of minimal models of complex algebraic varieties of general type. As a corollary of this, we know that X always has a $\mathbf{Q}$-factorial terminalization. In particular, $\tilde{O}$ should have a $\mathbf{Q}$-factorial terminalization. The author would like to pose the following conjecture.

Conjecture. *Let O be a nilpotent orbit of a complex simple Lie algebra $\mathfrak{g}$. Let $\tilde{O}$ be the normalization of $\bar{O}$. Then one of the following holds:*

(1) *$\tilde{O}$ has $\mathbf{Q}$-factorial terminal singularities.*

(2) *There are a parabolic subalgebra $\mathfrak{q}$ of $\mathfrak{g}$ with Levi decomposition $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{n}$ and a nilpotent orbit O' of $\mathfrak{l}$ such that (a): $O = \mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{l}}(O')$ and (b): the normalization of $G \times^Q (n(\mathfrak{q}) + \bar{O}')$ is a $\mathbf{Q}$-factorial terminalization of $\tilde{O}$ via the generalized Springer map.*

Moreover, if $\tilde{O}$ does not have $\mathbf{Q}$-factorial terminal singularities, then every $\mathbf{Q}$-factorial terminalization of $\tilde{O}$ is of the form (2). Two $\mathbf{Q}$-factorial terminalizations are connected by Mukai flops (cf. [Na], p.91).

The main result of this report is that Conjecture is true when $\mathfrak{g}$ is classical. Recently, Fu checked Conjecture for $\mathfrak{g}$ exceptional by a case-by-case method using the computer program GAP 4 (arxiv: 0809.5109, version 2). Combining this with our result, Conjecture holds true in full generality. However, a conceptual proof without the classification of nilpotent orbits, is still missing. This is a summary of [Na -1]. For details on proofs, see the original paper [Na -1].

§1. Preliminaries

(1.1) *Nilpotent orbits and resolutions*: Let G be a complex simple algebraic group and let $\mathfrak{g}$ be its Lie algebra. G has the adjoint action on $\mathfrak{g}$. The orbit $\mathcal{O}_x$ of a nilpotent element $x \in \mathfrak{g}$ for this action is called a nilpotent orbit. By the Jacobson-Morozov theorem, one can find a semi-simple element $h \in \mathfrak{g}$, and a nilpotent element $y \in \mathfrak{g}$ in such a way that $[h, x] = 2x$, $[h, y] = -2y$ and $[x, y] = h$. For $i \in \mathbf{Z}$, let

$$\mathfrak{g}_i := \{z \in \mathfrak{g} \ [h, z] = iz\}.$$

Then one can write

$$\mathfrak{g} = \oplus_{i \in \mathbf{Z}} \mathfrak{g}_i.$$

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ with $h \in \mathfrak{h}$. Let Φ be the corresponding root system and let Δ be a base of simple roots such that h is Δ-dominant, i.e. $\alpha(h) \geq 0$ for all $\alpha \in \Delta$. In this situation,

$$\alpha(h) \in \{0, 1, 2\}.$$

The weighted Dynkin diagram of $\mathcal{O}_x$ is the Dynkin diagram of $\mathfrak{g}$ where each vertex α is labeled with $\alpha(h)$. A nilpotent orbit $\mathcal{O}_x$ is completely determined by its weighted Dynkin diagram. A Jacobson-Morozov parabolic subalgebra for x is the parabolic subalgebra $\mathfrak{p}$ defined by

$$\mathfrak{p} := \oplus_{i \geq 0} \mathfrak{g}_i.$$

Let P be the parabolic subgroup of G determined by $\mathfrak{p}$. We put

$$\mathfrak{n}_2 := \oplus_{i \geq 2} \mathfrak{g}_i.$$

Then $\mathfrak{n}_2$ is an ideal of $\mathfrak{p}$; hence, P has the adjoint action on $\mathfrak{n}_2$. Let us consider the vector bundle $G \times^P \mathfrak{n}_2$ over G/P and the map

$$\mu : G \times^P \mathfrak{n}_2 \to \mathfrak{g}$$

defined by $\mu([g, z]) := Ad_g(z)$. Then the image of μ coincides with the closure $\bar{\mathcal{O}}_x$ of $\mathcal{O}_x$ and μ gives a resolution of $\bar{\mathcal{O}}_x$(cf. [K-P], Proposition 7.4). We call μ the *Jacobson-Morozov resolution* of $\bar{\mathcal{O}}_x$. The orbit $\mathcal{O}_x$ has a natural closed non-degenerate 2-form ω (cf. [C-G], Prop. 1.1.5., [C-M], 1.3). By μ, ω is regarded as a 2-form on a Zariski open subset of $G \times^P \mathfrak{n}_2$. By [Pa], [Hi], it extends to a 2-form on $G \times^P \mathfrak{n}_2$. In other words, $\bar{\mathcal{O}}_x$ has symplectic singularity. Let $\tilde{\mathcal{O}}_x$ be the normalization of $\bar{\mathcal{O}}_x$. In many cases, one can check the $\mathbf{Q}$-factoriality of $\tilde{\mathcal{O}}_x$ by applying the following lemma to the Jacobson-Morozov resolution:

Lemma (1.1.1). *Let $\pi : Y \to X$ be a projective resolution of an affine variety X with rational singularities. Let ρ be the relative Picard number for π. If* $\mathrm{Exc}(\pi)$ *contains ρ different prime divisors, then X is $\mathbf{Q}$-factorial.*

(1.2) *Induced orbits*

(1.2.1). Let G and $\mathfrak{g}$ be the same as in (1.1). Let Q be a parabolic subgroup of G and let $\mathfrak{q}$ be its Lie algebra with Levi decomposition $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{n}$. Here n is the nil-radical of $\mathfrak{q}$ and $\mathfrak{l}$ is a Levi-part of $\mathfrak{q}$. Fix a nilpotent orbit $\mathcal{O}'$ in $\mathfrak{l}$. Then there is a unique nilpotent orbit $\mathcal{O}$ in $\mathfrak{g}$ meeting $n + \mathcal{O}'$ in an open dense subset ([L-S]). Such an orbit $\mathcal{O}$ is called the nilpotent orbit induced from $\mathcal{O}'$ and we write

$$\mathcal{O} = \mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{g}}(\mathcal{O}').$$

Note that when $\mathcal{O}' = 0$, $\mathcal{O}$ is the Richardson orbit for Q. Since the adjoint action of Q on $\mathfrak{q}$ stabilizes $n + \bar{\mathcal{O}}'$, one can consider the variety $G \times^Q (n + \bar{\mathcal{O}}')$. There is a map

$$\nu : G \times^Q (n + \bar{\mathcal{O}}') \to \bar{\mathcal{O}}$$

defined by $\nu([g, z]) := Ad_g(z)$. Since $\mathrm{Codim}_{\mathfrak{l}}(\mathcal{O}') = \mathrm{Codim}_{\mathfrak{g}}(\mathcal{O})$ (cf. [C-M], Prop. 7.1.4), ν is a generically finite dominating map. Moreover, ν is factorized as

$$G \times^Q (n + \bar{\mathcal{O}}') \to G/Q \times \bar{\mathcal{O}} \to \bar{\mathcal{O}}$$

where the first map is a closed embedding and the second map is the 2-nd projection; this implies that ν is a projective map. In the remainder, we call ν the generalized Springer map for $(Q, \mathcal{O}')$.

(1.2.2). Assume that Q is contained in another parabolic subgroup $\bar{Q}$ of G. Let $\bar{L}$ be the Levi part of $\bar{Q}$ which contains the Levi part L of Q. Let $\bar{\mathfrak{q}} = \bar{\mathfrak{l}} \oplus \bar{\mathfrak{n}}$ be the Levi decomposition. Note that $\bar{L} \cap Q$ is a parabolic subgroup of $\bar{L}$ and $\mathfrak{l}(\bar{L} \cap Q) = \mathfrak{l}$. Let $\mathcal{O}_1 \subset \bar{\mathfrak{l}}$ be the nilpotent orbit induced from $(\bar{L} \cap Q, \mathcal{O}')$. Then there is a natural map

$$\pi : G \times^Q (n + \bar{\mathcal{O}}') \to G \times^{\bar{Q}} (\bar{n} + \bar{\mathcal{O}}_1)$$

which factorizes ν as $\bar{\nu} \circ \pi = \nu$. Here $\bar{\nu}$ is the generalized Springer map for $(\bar{Q}, \mathcal{O}_1)$.

(1.2.3). Assume that there are a parabolic subgroup Q_L of L and a nilpotent orbit $\mathcal{O}_2$ in the Levi subalgebra $\mathfrak{l}(Q_L)$ such that $\mathcal{O}'$ is the nilpotent orbit induced from $(Q_L, \mathcal{O}_2)$. Then there is a parabolic subgroup Q' of G

such that $Q' \subset Q$, $\mathfrak{l}(Q') = \mathfrak{l}(Q_L)$ and $\mathcal{O}$ is the nilpotent orbit induced from $(Q', \mathcal{O}_2)$. The generalized Springer map ν' for $(Q', \mathcal{O}_2)$ is factorized as

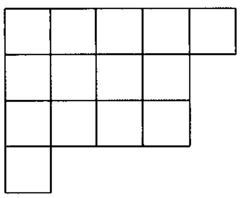
$$G \times^{Q'} (\mathfrak{n}' + \bar{\mathcal{O}}_2) \to G \times^{Q} (\mathfrak{n} + \bar{\mathcal{O}}') \to \bar{\mathcal{O}}.$$

Lemma (1.2.4). *Let*

$$\nu : G \times^{Q} (n + \bar{\mathcal{O}}') \to \bar{\mathcal{O}}$$

be a generalized Springer map defined in (1.2.1). Then the normalization of $G \times^{Q} (n + \bar{\mathcal{O}}')$ *is a symplectic variety.*

(1.3) *Nilpotent orbits in classical Lie algebras*: When $\mathfrak{g}$ is a classical Lie algebra, $\mathfrak{g}$ is naturally a Lie subalgebra of $\mathrm{End}(V)$ for a complex vector space V. Then we can attach a partition $\mathbf{d}$ of $n := \dim V$ to each orbit as the Jordan type of an element contained in the orbit. Here a partition $\mathbf{d} := [d_1, d_2, ..., d_k]$ of n is a set of positive integers with $\Sigma d_i = n$ and $d_1 \geq d_2 \geq ... \geq d_k$. Another way of writing $\mathbf{d}$ is $[d_1^{s_1}, ..., d_k^{s_k}]$ with $d_1 > d_2 ... > d_k > 0$. Here $d_i^{s_i}$ is an s_i times d_i's: $d_i, d_i, ..., d_i$. The partition $\mathbf{d}$ corresponds to a Young diagram. For example, $[5, 4^2, 1]$ corresponds to



When an integer e appears in the partition $\mathbf{d}$, we say that e is a *member* of $\mathbf{d}$. We call $\mathbf{d}$ *very even* when $\mathbf{d}$ consists with only even members, each having even multiplicity.

Let us denote by ϵ the number 1 or -1. Then a partition $\mathbf{d}$ is ϵ-admissible if all even (resp. odd) members of $\mathbf{d}$ have even multiplicities when $\epsilon = 1$ (resp. $\epsilon = -1$). The following result can be found, for example, in [C-M, §5].

Proposition (1.3.1) *Let* $\mathcal{N}o(\mathfrak{g})$ *be the set of nilpotent orbits of* $\mathfrak{g}$.

(1)(A_{n-1}): *When* $\mathfrak{g} = \mathfrak{sl}(n)$, *there is a bijection between* $\mathcal{N}o(\mathfrak{g})$ *and the set of partitions* $\mathbf{d}$ *of* n.

(2)(B_n): *When* $\mathfrak{g} = \mathfrak{so}(2n+1)$, *there is a bijection between* $\mathcal{N}o(\mathfrak{g})$ *and the set of* ϵ*-admissible partitions* $\mathbf{d}$ *of* $2n+1$ *with* $\epsilon = 1$.

(3)(C_n): *When* $\mathfrak{g} = \mathfrak{sp}(2n)$, *there is a bijection between* $\mathcal{N}o(\mathfrak{g})$ *and the set of* ϵ*-admissible partitions* $\mathbf{d}$ *of* $2n$ *with* $\epsilon = -1$.

(4)(D_n): *When* $\mathfrak{g} = \mathfrak{so}(2n)$, *there is a surjection* f *from* $\mathcal{N}o(\mathfrak{g})$ *to the set of* ϵ*-admissible partitions* $\mathbf{d}$ *of* $2n$ *with* $\epsilon = 1$. *For a partition* $\mathbf{d}$ *which is not very even,* $f^{-1}(\mathbf{d})$ *consists of exactly one orbit, but, for very even* $\mathbf{d}$, $f^{-1}(\mathbf{d})$ *consists of exactly two different orbits.*

Take an ϵ-admissible partition $\mathbf{d}$ of a positive integer m. If $\epsilon = 1$, we put $\mathfrak{g} = so(m)$ and if $\epsilon = -1$, we put $\mathfrak{g} = sp(m)$. We denote by $\mathcal{O}_{\mathbf{d}}$ a nilpotent orbit in $\mathfrak{g}$ with Jordan type $\mathbf{d}$. Note that, except when $\epsilon = 1$ and $\mathbf{d}$ is very even, $\mathcal{O}_{\mathbf{d}}$ is uniquely determined. When $\epsilon = 1$ and $\mathbf{d}$ is very even, there are two possibilities for $\mathcal{O}_{\mathbf{d}}$. If necessary, we distinguish the two orbits by the labelling: $\mathcal{O}^I_{\mathbf{d}}$ and $\mathcal{O}^{II}_{\mathbf{d}}$. Let us fix a classical Lie algebra $\mathfrak{g}$ and study the relationship among nilpotent orbits in $\mathfrak{g}$. When $\mathfrak{g}$ is of type B or D (resp. C), we only consider the ϵ-admissible partitions with $\epsilon = 1$ (resp. $\epsilon = -1$). We introduce a partial order in the set of the partitions of (the same number): for two partitions $\mathbf{d}$ and $\mathbf{f}$, $\mathbf{d} \geq \mathbf{f}$ if $\Sigma_{i\leq k} d_i \geq \Sigma_{i\leq k} f_i$ for all $k \geq 1$. On the other hand, for two nilpotent orbits $\mathcal{O}$ and $\mathcal{O}'$ in $\mathfrak{g}$, we write $\mathcal{O} \geq \mathcal{O}'$ if $\mathcal{O}' \subset \bar{\mathcal{O}}$. Then, $\mathcal{O}_{\mathbf{d}} \geq \mathcal{O}_{\mathbf{f}}$ if and only if $\mathbf{d} \geq \mathbf{f}$. When $\mathbf{d}$ and $\mathbf{f}$ are ϵ-admissible partitions with $\mathbf{f} \geq \mathbf{g}$, we call this pair an ϵ*-degeneration* or simply a *degeneration.*

Now let us consider the case $\mathfrak{g}$ is of type B, C or D.

Assume that an ϵ- degeneration $\mathbf{d} \geq \mathbf{f}$ is *minimal* in the sense that there is no ϵ-admissible partition $\mathbf{d}'$ (except $\mathbf{d}$ and $\mathbf{f}$) such that $\mathbf{d} \geq \mathbf{d}' \geq \mathbf{f}$. Kraft and Procesi [K-P] have studied the *normal slice* $N_{\mathbf{d},\mathbf{f}}$ of $\mathcal{O}_{\mathbf{f}} \subset \bar{\mathcal{O}}_{\mathbf{d}}$ in such cases. If, for two integers r and s, the first r rows and the first s columns of $\mathbf{d}$ and $\mathbf{f}$ coincide and the partition $(d_1, ..., d_r)$ is ϵ-admissible, then one can erase these rows and columns from $\mathbf{d}$ and $\mathbf{f}$ respectively to get new partitions $\mathbf{d}'$ and $\mathbf{f}'$ with $\mathbf{d}' \geq \mathbf{f}'$. If we put $\epsilon' := (-1)^s\epsilon$, then $\mathbf{d}'$ and $\mathbf{f}'$ are both ϵ'-admissible. The pair $(\mathbf{d}', \mathbf{f}')$ is also minimal. Repeating such process, one can reach a degeneration $\mathbf{d}_{irr} \geq \mathbf{f}_{irr}$ which is *irreducible* in the sense that there are no rows and columns to be erased. By [K-P], Theorem 2, $N_{\mathbf{d},\mathbf{f}}$ is analytically isomorphic to $N_{\mathbf{d}_{irr},\mathbf{f}_{irr}}$ around the origin. According to [K-P], a minimal and irreducible degeneration $\mathbf{d} \geq \mathbf{f}$ is one of the following:

a: $\mathfrak{g} = sp(2)$, $\mathbf{d} = (2)$, and $\mathbf{f} = (1^2)$.

b: $\mathfrak{g} = sp(2n)$ $(n > 1)$, $\mathbf{d} = (2n)$, and $\mathbf{f} = (2n-2, 2)$.

c: $\mathfrak{g} = so(2n+1)$ $(n > 0)$, $\mathbf{d} = (2n+1)$, and $\mathbf{f} = (2n-1, 1^2)$.

d: $\mathfrak{g} = sp(4n+2)$ $(n > 0)$, $\mathbf{d} = (2n+1, 2n+1)$, and $\mathbf{f} = (2n, 2n, 2)$.

e: $\mathfrak{g} = so(4n)$ $(n > 0)$, $\mathbf{d} = (2n, 2n)$, and $\mathbf{f} = (2n-1, 2n-1, 1^2)$.
f: $\mathfrak{g} = so(2n+1)$ $(n > 1)$, $\mathbf{d} = (2^2, 1^{2n-3})$, and $\mathbf{f} = (1^{2n+1})$.
g: $\mathfrak{g} = sp(2n)$ $(n > 1)$, $\mathbf{d} = (2, 1^{2n-2})$, and $\mathbf{f} = (1^{2n})$.
h: $\mathfrak{g} = so(2n)$ $(n > 2)$, $\mathbf{d} = (2^2, 1^{2n-4})$, and $\mathbf{f} = (1^{2n})$.
In the first 4 cases (a,b,c,d,e), $\mathcal{O}_{\mathbf{f}}$ have codimension 2 in $\bar{\mathcal{O}}_{\mathbf{d}}$. In the last 3 cases (f,g,h), $\mathcal{O}_{\mathbf{f}}$ have codimension ≥ 4 in $\bar{\mathcal{O}}_{\mathbf{d}}$.

Proposition (1.3.2) *Let $\mathcal{O}$ be a nilpotent orbit in a classical Lie algebra $\mathfrak{g}$ of type B, C or D with Jordan type* $\mathbf{d} := [(d_1)^{s_1}, ..., (d_k)^{s_k}]$ $(d_1 > d_2 > ... > d_k)$*. Let Σ be the singular locus of $\bar{\mathcal{O}}$. Then* $\mathrm{Codim}_{\bar{\mathcal{O}}}(\Sigma) \geq 4$ *if and only if the partition* $\mathbf{d}$ *has full members, that is, any integer j with $1 \leq j \leq d_1$ is a member of* $\mathbf{d}$. *Otherwise,* $\mathrm{Codim}_{\bar{\mathcal{O}}}(\Sigma) = 2$.

(1.4.1) *Jacobson-Morozov resolutions in the case of classical Lie algebras*(cf. [CM], 5.3): Let V be a complex vector space of dimension m with a non-degenerate symmetric (or skew-symmetric) form $< , >$. In the symmetric case, we take a basis $\{e_i\}_{1\leq i\leq m}$ of V in such a way that $< e_j, e_k >= 1$ if $j + k = m + 1$ and otherwise $< e_j, e_k >= 0$. In the skew-symmetric case, we take a basis $\{e_i\}_{1\leq i\leq m}$ of V in such a way that $< e_j, e_k >= 1$ if $j < k$ and $j + k = m + 1$, and $< e_j, e_k >= 0$ if $j + k \neq m + 1$. When $(V, < , >)$ is a symmetric vector space, $\mathfrak{g} := so(V)$ is the Lie algebra of type $B_{(m-1)/2}$ (resp. $D_{m/2}$) if m is odd (resp. even). When $(V, < , >)$ is a skew-symmetric vector space, $\mathfrak{g} := sp(V)$ is the Lie algebra of type $C_{m/2}$. In the remainder of this paragraph, $\mathfrak{g}$ is one of these Lie algebra contained in $\mathrm{End}(V)$. Let $\mathfrak{h} \subset \mathfrak{g}$ be the Cartan subalgebra consisting of all diagonal matrices, and let Δ be the standard base of simple roots. Let $x \in \mathfrak{g}$ be a nilpotent element. As in (1.1), one can choose h, $y \in \mathfrak{g}$ in such a way that $\{x, y, h\}$ is a $sl(2)$-triple. If necessary, by replacing x by its conjugate element, one may assume that $h \in \mathfrak{h}$ and h is Δ-dominant. Assume that x has Jordan type $\mathbf{d} = [d_1, ..., d_k]$. The diagonal matrix h is described as follows. Let us consider the sequence of integers of length m:

$d_1 - 1, d_1 - 3, ..., -d_1 + 3, -d_1 + 1, d_2 - 1, d_2 - 3, ..., -d_2 + 3, -d_2 + 1, ..., d_k - 1, d_k - 3, ..., -d_k + 3, -d_k + 1$.

Rearrange this sequence in the non-increasing order and get a new sequence $p_1^{t_1}, ..., p_l^{t_l}$ with $p_1 > p_2 ... > p_l$ and $\Sigma t_i = m$. Then

$$h = \mathrm{diag}(p_1^{t_1}, ..., p_l^{t_l}).$$

Here $p_i^{t_i}$ means the t_i times of p_i's: $p_i, p_i, ..., p_i$. It is then easy to describe explicitly the Jacobson-Morozov parabolic subalgebra $\mathfrak{p}$ of x and its ideal $\mathfrak{n}_2$ (cf. (1.1)). The Jacobson-Morozov parabolic subgroup P is the stabilizer group of certain isotropic flag $\{F_i\}_{1\leq i\leq r}$ of V. Here, an isotropic flag of V (of length r) is a increasing filtration $0 \subset F_1 \subset F_2 \subset ... \subset F_r \subset V$ such that $F_{r+1-i} = F_i^{\perp}$ for all i. The flag type of P is $(t_1, ..., t_l)$. The nilradical $\mathfrak{n} := \oplus_{i>0}\mathfrak{g}_i$ of $\mathfrak{p}$ consists of the elements z of $\mathfrak{g}$ such that $z(F_i) \subset F_{i-1}$ for all i. On the other hand, it depends on the weighted Dynkin diagram for x how $\mathfrak{n}_2$ takes its place in $\mathfrak{n}$.

Lemma (1.4.2) *Assume that* $\mathbf{d}$ *has full members. For each minimal* ϵ*-degeneration* $\mathbf{d} \geq \mathbf{f}$, *the fiber* $\mu^{-1}(\mathcal{O}_{\mathbf{f}})$ *has codimension* 1 *in* $G \times^P \mathfrak{n}_2$.

Corollary (1.4.3) *Assume that* $\mathbf{d}$ *is an* ϵ*-admissible partition and it has full members. Let* $\tilde{\mathcal{O}}_{\mathbf{d}}$ *be the normalization of* $\bar{\mathcal{O}}_{\mathbf{d}}$. *Then,* $\tilde{\mathcal{O}}_{\mathbf{d}}$ *has only* $\mathbf{Q}$*-factorial termainal singularities except when* $\mathfrak{g} = so(4n+2)$, $n \geq 1$ *and* $\mathbf{d} = [2^{2n}, 1^2]$.

Proof. Let k be the maximal member of $\mathbf{d}$. Then there are $k-1$ minimal degenerations $\mathbf{d} \geq \mathbf{f}$. By Lemma (1.4.2), $\mathrm{Exc}(\mu)$ contains at least $k-1$ irreducible divisors. When $\epsilon = 1$ (i.e, $\mathfrak{g} = so(V)$) and there is a minimal degeneration $\mathbf{d} \geq \mathbf{f}$ with $\mathbf{f}$ very even, there are two nilpotent orbits with Jordan type $\mathbf{f}$. Thus, in this case, $\mathrm{Exc}(\mu)$ contains at least k irreducible divisors. On the other hand, for the Jacobson-Morozov parabolic subgroup P, $b_2(G/P) = k-1$ when $\mathfrak{g} = sp(V)$, or $\mathfrak{g} = so(V)$ with $\dim V$ odd. When $\mathfrak{g} = so(V)$ and $\dim V$ is even, we must be careful; if the flag type of P is of the form $(p_1, ..., p_{k-1}; 2; p_{k-1}, ..., p_1)$, $b_2(G/P) = k$. This happens when $\dim V = 4n+2$ and $\mathbf{d} = [2^{2n}, 1^2]$ or when $\dim V = 8m+4n+4$ and $\mathbf{d} = [4^{2m}, 3, 2^{2n}, 1]$. In the latter case, $\mathbf{d}$ has a minimal degeneration $\mathbf{d} \geq \mathbf{f}$ with $\mathbf{f} = [4^{2m}, 2^{2n+2}]$, which is very even. Note that $b_2(G/P)$ coincides with the relative Picard number ρ of the Jacobson-Morozov resolution. By these observations, we know that μ has at least ρ exceptional divisors except when $\mathfrak{g} = so(4n+2)$, $n \geq 1$ and $\mathbf{d} = [2^{2n}, 1^2]$. Therefore, $\tilde{\mathcal{O}}_{\mathbf{d}}$ are $\mathbf{Q}$-factorial in these cases. By (1.3.2) they have terminal singularities. When $\mathfrak{g} = so(4n+2)$, $n \geq 1$ and $\mathbf{d} = [2^{2n}, 1^2]$, $\mathcal{O}_{\mathbf{d}}$ is a Richardson orbit and the Springer map gives a small resolution of $\bar{\mathcal{O}}_{\mathbf{d}}$. Therefore, $\tilde{\mathcal{O}}_{\mathbf{d}}$ has non-$\mathbf{Q}$-factorial terminal singularities.

(1.5) *Induced orbits in classical Lie algebras*: Let $\mathbf{d} = [d_1^{s_1}, ..., d_k^{s_k}]$ be an ϵ-admissible partition of m. According as $\epsilon = 1$ or $\epsilon = -1$, we put $G = SO(m)$ or $G = Sp(m)$ respectively. Assume that $\mathbf{d}$ does not have full members. In other words, for some p, $d_p \geq d_{p+1} + 2$ or $d_k \geq 2$. We put $r = \Sigma_{1 \leq j \leq p} s_j$. Then $\mathcal{O}_{\mathbf{d}}$ is an induced orbit (cf. [C-M], 7.3). More explicitly, there are a parabolic subgroup Q of G with (isotropic) flag type $(r, m-2r, r)$ with Levi decomposition $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{n}$, and a nilpotent orbit $\mathcal{O}'$ of $\mathfrak{l}$ such that $\mathcal{O}_{\mathbf{d}} = \mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{g}}(\mathcal{O}')$. Here, $\mathfrak{l}$ has a direct sum decomposition $\mathfrak{l} = gl(r) \oplus \mathfrak{g}'$, where $\mathfrak{g}'$ is a simple Lie algebra of type $B_{(m-2r-1)/2}$ (resp. $D_{(m-2r)/2}$, resp. $C_{(m-2r)/2}$) when $\epsilon = 1$ and m is odd (resp. $\epsilon = 1$ and m is even, resp. $\epsilon = -1$). Moreover, $\mathcal{O}'$ is a nilpotent orbit of $\mathfrak{g}'$ with Jordan type $[(d_1-2)^{s_1}, ..., (d_p-2)^{s_p}, d_{p+1}^{s_{p+1}}, ..., d_k^{s_k}]$. Let us consider the generalized Springer map

$$\nu : G \times^Q (n(\mathfrak{q}) + \bar{\mathcal{O}}') \to \bar{\mathcal{O}}_{\mathbf{d}}$$

(cf. (1.2)).

Lemma (1.5.1). *The map ν is birational. In other words,* $\deg(\nu) = 1$.

§2. Main Results

(2.1) Let X be a complex algebraic variety with rational Gorenstein singularities. A partial resolution $f : Y \to X$ of X is said to be a **Q**-*factorial terminalization* of X if Y has only **Q**-factorial terminal singularities and f is a birational projective morphism such that $K_Y = f^*K_X$. In particular, when Y is smooth, f is called a crepant resolution. In general, X has no crepant resolution; however, by [B-C-H-M], X always has a **Q**-factorial terminalization. But, in our case, the **Q**-factorial terminalization can be constructed very explicitly without using the general theory in [B-C-H-M].

Proposition (2.1.1). *Let $\mathcal{O}$ be a nilpotent orbit of a classical simple Lie algebra $\mathfrak{g}$. Let $\tilde{\mathcal{O}}$ be the normalization of $\bar{\mathcal{O}}$. Then one of the following holds:*

(1) $\tilde{\mathcal{O}}$ has **Q**-*factorial terminal singularities.*

(2) There are a parabolic subalgebra $\mathfrak{q}$ of $\mathfrak{g}$ with Levi decomposition $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{n}$ and a nilpotent orbit $\mathcal{O}'$ of $\mathfrak{l}$ such that (a): $\mathcal{O} = \mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{g}}(\mathcal{O}')$ and (b): the normalization of $G \times^Q (n(\mathfrak{q}) + \bar{\mathcal{O}}')$ is a **Q**-*factorial terminalization of $\tilde{\mathcal{O}}$ via the generalized Springer map.*

Proof. When $\mathfrak{g}$ is of type A, every $\tilde{\mathcal{O}}$ has a Springer resolution; hence (2) always holds. Let us consider the case $\mathfrak{g}$ is of B, C or D. Assume that (1) does not hold. Then, by (1.4.3), the Jordan type $\mathbf{d}$ of $\mathcal{O}$ does not have full members except when $\mathfrak{g} = so(4n+2)$, $n \geq 1$ and $\mathbf{d} = [2^{2n}, 1^2]$. In the exceptional case, $\mathcal{O}$ is a Richardson orbit and the Springer map gives a crepant resolution of $\tilde{\mathcal{O}}$; hence (2) holds. Now assume that $\mathbf{d}$ does not have full members. Then, by (1.5), $\mathcal{O}$ is an induced nilpotent orbit and there is a generalized Springer map

$$\nu : G \times^Q (n(\mathfrak{q}) + \bar{\mathcal{O}}') \to \bar{\mathcal{O}}.$$

This map is birational by (1.5.1). Let us consider the orbit $\mathcal{O}'$ instead of $\mathcal{O}$. If (1) holds for $\mathcal{O}'$, then ν induces a $\mathbf{Q}$-factorial terminalization of $\tilde{\mathcal{O}}$. If (1) does not hold for $\mathcal{O}'$, then $\mathcal{O}'$ is an induced orbit. By (1.2.3), one can replace Q with a smaller parabolic subgroup Q' in such a way that $\mathcal{O}$ is induced from $(Q', \mathcal{O}_2)$ for some nilpotent orbit $\mathcal{O}_2 \subset \mathfrak{l}(Q')$. The generalized Springer map ν' for $(Q', \mathcal{O}_2)$ is factorized as

$$G \times^{Q'} (\mathfrak{n}' + \bar{\mathcal{O}}_2) \to G \times^Q (\mathfrak{n} + \bar{\mathcal{O}}') \to \bar{\mathcal{O}}.$$

The second map is birational as explained above. The first map is locally obtained by a base change of the generalized Springer map

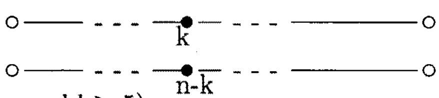
$$L(Q) \times^{L(Q) \cap Q'} (\mathfrak{n}(L(Q) \cap Q') + \bar{\mathcal{O}}_2) \to \bar{\mathcal{O}}'.$$

This map is birational by (1.5.1). Therefore, the first map is also birational, and ν' is birational. This induction step terminates and (2) finally holds.

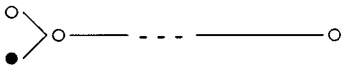
(2.2) We shall next show that *every* $\mathbf{Q}$-factorial terminalization of $\tilde{\mathcal{O}}$ is of the form in Proposition (2.1.1) except when $\tilde{\mathcal{O}}$ itself has $\mathbf{Q}$-factorial terminal singularities. In order to do that, we need the following Proposition.

Proposition (2.2.1). *Let $\mathcal{O}$ be a nilpotent orbit of a classical simple Lie algebra $\mathfrak{g}$. Assume that a $\mathbf{Q}$-factorial terminalization of $\tilde{\mathcal{O}}$ is given by the normalization of $G \times^Q (n(\mathfrak{q}) + \bar{\mathcal{O}}'))$ for some $(Q, \mathcal{O}')$ as in (2.1.1). Assume that Q is a maximal parabolic subgroup of G (i.e. $b_2(G/Q) = 1$), and this $\mathbf{Q}$-factorial terminalization is small. Then Q is a parabolic subgroup corresponding to one of the following marked Dynkin diagrams and $\mathcal{O}' = 0$:*

A_{n-1} $(k < n/2)$



D_n $(n : \text{odd} \geq 5)$



The following is the main theorem:

Theorem (2.2.2). *Let $\mathcal{O}$ be a nilpotent orbit of a classical simple Lie algebra $\mathfrak{g}$. Then $\tilde{\mathcal{O}}$ always has a $\mathbf{Q}$-factorial terminalization. If $\tilde{\mathcal{O}}$ itself does not have $\mathbf{Q}$-factorial terminal singularities, then every $\mathbf{Q}$-factorial terminalization is given by the normalization of $G \times^Q (n(\mathfrak{q}) + \bar{\mathcal{O}}'))$ in (2.1.1). Moreover, any two such $\mathbf{Q}$-factorial terminalizations are connected by a sequence of Mukai flops of type A or D defined in [Na], pp. 91, 92.*

Proof. The first statement is nothing but (2.1.1). The proof of the second statement is quite similar to that of [Na], Theorem 6.1. Assume that $\tilde{\mathcal{O}}$ does not have $\mathbf{Q}$-factorial terminal singularities. Then, by (2.1.1), one can find a generalized Springer (birational) map

$$\nu : G \times^Q (n(\mathfrak{q}) + \bar{\mathcal{O}}') \to \bar{\mathcal{O}}.$$

Let X_Q be the normalization of $G \times^Q (n(\mathfrak{q}) + \bar{\mathcal{O}}')$. Then ν induces a $\mathbf{Q}$-factorial terminalization $f : X_Q \to \tilde{\mathcal{O}}$. The relative nef cone $\overline{\mathrm{Amp}}(f)$ is a rational, simplicial, polyhedral cone of dimension $b_2(G/Q)$ (cf. (1.2.2) and [Na], Lemma 6.3). Each codimension one face F of $\overline{\mathrm{Amp}}(f)$ corresponds to a birational contraction map $\phi_F : X_Q \to Y_Q$. The construction of ϕ_F is as follows. The parabolic subgroup Q corresponds to a marked Dynkin diagram D. In this diagram, there are exactly $b_2(G/Q)$ marked vertexes. Choose a marked vertex v from D. The choice of v determines a codimension one face F of $\overline{\mathrm{Amp}}(f)$. Let D_v be the maximal, connected, single marked Dynkin subdiagram of D which contains v. Let $\bar{D}$ be the marked Dynkin diagram obtained from D by erasing the marking of v. Let $\bar{Q}$ be the parabolic subgroup of G corresponding to $\bar{D}$. Then, as in (1.2.2), we have a map

$$\pi : G \times^Q (\mathfrak{n} + \bar{\mathcal{O}}') \to G \times^{\bar{Q}} (\bar{\mathfrak{n}} + \bar{\mathcal{O}}_1).$$

Let Y_Q be the normalization of $G \times^{\bar{Q}} (\bar{\mathfrak{n}} + \bar{\mathcal{O}}_1)$. Then π induces a birational map $X_Q \to Y_Q$. This is the map ϕ_F. Note that π is locally obtained by a base change of the generalized Springer map

$$L(\bar{Q}) \times^{L(\bar{Q}) \cap Q} (\mathfrak{n}(L(\bar{Q}) \cap Q) + \bar{\mathcal{O}}') \to \bar{\mathcal{O}}_1.$$

Let $Z(\mathfrak{l}(\mathfrak{q}))$ (resp. $Z(\mathfrak{l}(\bar{\mathfrak{q}}))$) be the center of $\mathfrak{l}(\mathfrak{q})$ (resp. $\mathfrak{l}(\bar{\mathfrak{q}})$). By the definition of $\bar{Q}$, the simple factors of $\mathfrak{l}(\bar{\mathfrak{q}})/Z(\mathfrak{l}(\bar{\mathfrak{q}}))$ are common to those of $\mathfrak{l}(\mathfrak{q})/Z(\mathfrak{l}(\mathfrak{q}))$

except one factor, say $\mathfrak{m}$. Put $\mathcal{O}'' := \mathcal{O}' \cap \mathfrak{m}$. By (2.2.1), π (or ϕ_F) is a small birational map if and only if $\mathcal{O}'' = 0$ and D_v is one of the single Dynkin diagrams listed in (2.2.1). In this case, one can make a new marked Dynkin diagramD' from D by replacing D_v by its dual D_v^* (cf. [Na], Definition 1). Let Q' be the parabolic subgroup of G corresponding to D'. We may assume that Q and Q' are both contained in $\bar{Q}$. The Levi part of Q' is conjugate to that of Q; hence there is a nilpotent orbit in $\mathfrak{l}(\mathfrak{q}')$ corresponding to $\mathcal{O}'$. We denote this orbit by the same $\mathcal{O}'$. Then $\mathcal{O}$ is induced from $(Q', \mathcal{O}')$. As above, let $X_{Q'}$ be the normalization of $G \times^Q (n(\mathfrak{q}') + \bar{\mathcal{O}}')$. Then we have a birational map $\phi'_F : X_{Q'} \to Y_Q$. The diagram

$$X_Q \to Y_Q \leftarrow X_{Q'}$$

is a flop. Assume that $g : X \to \tilde{\mathcal{O}}$ is a $\mathbf{Q}$-factorial terminalization. Then, the natural birational map $X - - \to X_Q$ is an isomorphism in codimension one. Let L be a g-ample line bundle on X and let $L_0 \in \mathrm{Pic}(X_Q)$ be its proper transform of L by this birational map. If L_0 is f-nef, then $X = X_Q$ and $f = g$. Assume that L_0 is not f-nef. Then one can find a codimension one face F of $\overline{\mathrm{Amp}}(f)$ which is negative with respect to L_0. Since L_0 is f-movable, the birational map $\phi_F : X_Q \to Y_Q$ is small. Then, as seen above, there is a new (small) birational map $\phi'_F : X_{Q'} \to Y_Q$. Let $f' : X_{Q'} \to \tilde{\mathcal{O}}$ be the composition of ϕ'_F with the map $Y_Q \to \tilde{\mathcal{O}}$. Then f' is a $\mathbf{Q}$-factorial terminanization of $\tilde{\mathcal{O}}$. Replace f by this f' and repeat the same procedure; but this procedure ends in finite times (cf. [Na], Proof of Theorem 6.1 on pp. 104, 105). More explicitly, there is a finite sequence of $\mathbf{Q}$-factorial terminalizations of $\tilde{\mathcal{O}}$:

$$X_0(:= X_Q) - - \to X_1(:= X_{Q'}) - - \to .... - - \to X_k(= X_{Q_k})$$

such that $L_k \in \mathrm{Pic}(X_k)$ is f_k-nef. This means that $X = X_{Q_k}$.

Example (2.3). We put $G = SP(12)$. Let $\mathcal{O}$ be the nilpotent orbit in $sp(12)$ with Jordan type $[6, 3^2]$. Let $Q_1 \subset G$ be a parabolic subgroup with flag type $(3, 6, 3)$. The Levi part $\mathfrak{l}_1$ of $\mathfrak{q}_1$ has a direct sum decomposition

$$\mathfrak{l}_1 = \mathfrak{g}l(3) \oplus sp(6).$$

Let $\mathcal{O}'$ be the nilpotent orbit in $sp(6)$ with Jordan type $[4, 1^2]$. Then $\mathcal{O} = \mathrm{Ind}_{\mathfrak{l}_1}^{sp(12)}(\mathcal{O}')$. Next consider the parabolic subgroup $Q_2 \subset SP(6)$ with flag type $(1, 4, 1)$. The Levi part $\mathfrak{l}_2$ of $\mathfrak{q}_2$ has a direct sum decomposition

$$\mathfrak{l}_2 = \mathfrak{g}l(1) \oplus sp(4).$$

Let $\mathcal{O}''$ be the nilpotent orbit in $sp(4)$ with Jordan type $[2, 1^2]$. Then $\mathcal{O}' = \mathrm{Ind}_{\mathfrak{l}_2}^{sp(6)}(\mathcal{O}'')$. One can take a parabolic subgroup Q of $SP(12)$ with flag type $(3, 1, 4, 1, 3)$ in such a way that the Levi part $\mathfrak{l}$ of $\mathfrak{q}$ contains the nilpotent orbit $\mathcal{O}''$. Then $\mathcal{O}$ is the nilpotent orbit induced from $\mathcal{O}''$. We shall illustrate the induction step above by

$$([2, 1^2], sp(4)) \to ([4, 1^2], sp(6)) \to ([6, 3^2], sp(12)).$$

Since $\tilde{O}''$ has only $\mathbf{Q}$-factorial terminal singularities, the $\mathbf{Q}$-factorial terminalization of $\tilde{\mathcal{O}}$ is given by the generalized Springer map

$$\nu : G \times^Q (n(\mathfrak{q}) + \bar{\mathcal{O}}'') \to \bar{\mathcal{O}}.$$

The induction step is not unique; we have another induction step

$$([2, 1^2], sp(4)) \to ([4, 3^2], sp(10)) \to ([6, 3^2], sp(12)).$$

By these inductions, we get another generalized Springer map

$$\nu' : G \times^{Q'} (n(\mathfrak{q}') + \bar{\mathcal{O}}'') \to \bar{\mathcal{O}},$$

where Q' is a parabolic subgroup of G with flag type $(1, 3, 4, 3, 1)$. This gives another $\mathbf{Q}$-factorial terminalization of $\tilde{\mathcal{O}}$. The two $\mathbf{Q}$-factorial terminalizations of $\tilde{\mathcal{O}}$ are connected by a Mukai flop of type A_3.

References

[BCHM] Birkar, C., Cascini, P., Hacon, C., McKernan, J.: Existence of minimal models for varieties of general type, math.AG/0610203

[Be] Beauville, A. : Symplectic singularities, Invent. Math. **139** (2000), 541-549

[C-G] Chriss, M. , Ginzburg, V. : Representation theory and complex geometry, Progress in Math. , Birkhauser, 1997

[C-M] Collingwood, D. , McGovern, W. : Nilpotent orbits in semi-simple Lie algebras, van Nostrand Reinhold, Math. Series, 1993

[Fu] Fu, B. : Symplectic resolutions for nilpotent orbits, Invent. Math. **151**. (2003), 167-186

[Fu 2] Fu, B.: On $\mathbf{Q}$-factorial terminalizations of nilpotent orbits, math.arxiv: 0809.5109

[Hi] Hinich, V. : On the singularities of nilpotent orbits, Israel J. Math. **73** (1991), 297-308

[Ka] Kawamata, Y.: Crepant blowing-up of 3-dimensional canonical singularities and its application to degenerations of algebraic surfaces, Ann. Math. **127** (1988), 93-163

[K-P] Kraft, H., Procesi, C.: On the geometry of conjugacy classes in classical groups, Comment. Math. Helv. **57**. (1982), 539-602

[L-S] Lusztig, G., Spaltenstein, N.: Induced unipotent classes, J. London Math. Soc. **19**. (1979), 41-52

[Na -1] Namikawa, Y.: Induced nilpotent orbits and birational geometry, math.arxiv: 0809.2320

[Na] Namikawa, Y. : Birational geometry of symplectic resolutions of nilpotent orbits, Advances Studies in Pure Mathematics **45**, (2006), Moduli Spaces and Arithmetic Geometry (Kyoto, 2004), pp. 75-116

[Na 2] Namikawa, Y.: Flops and Poisson deformations of symplectic varieties, Publ. RIMS **44**. No 2. (2008), 259-314

[Na 3] Namikawa, Y.: Extensions of 2-forms and symplectic varieties, J. Reine Angew. Math. **539** (2001), 123-147

[Na 4] Namikawa, Y.: Birational geometry and deformations of nilpotent orbits, Duke Math. J. **143** (2008), 375-405

[Pa] Panyushev, D. I. : Rationality of singularities and the Gorenstein property of nilpotent orbits, Funct. Anal. Appl. **25** (1991), 225-226

[Ri] Richardson, R. : Conjugacy classes in parabolic subgroups of semisimple algebraic groups, Bull. London Math. Soc. **6** (1974), 21-24