代数幾何学シンポジウム記録
2008年度 pp.102-111

Pandharipande-Thomas theory and wall-crossings in derived categories

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Abstract

In [18], Pandharipande and Thomas introduced the notion of stable pairs on Calabi-Yau 3-folds and constructed the counting invariant of them. Conjecturally such invariant is equivalent to Donaldson-Thomas invariants and Gromov-Witten invariants via generating functions. In this article, we give a transformation formula of generating series of invariants counting stable pairs under flops. We use wall-crossing formula in the derived category.

1 Curve counting on Calabi-Yau 3-folds

Let X be a smooth projective Calabi-Yau 3-fold over $\mathbb{C}$, i.e. there is a nowhere vanishing holomorphic 3-form on X. We are interested in the curve counting theories on X. There are three such theories, called Gromov-Witten (GW) theory, Donaldson-Thomas (DT) theory, and Pandharipande-Thomas (PT) theory. Conjecturally these theories are equivalent in terms of generating functions. Let us recall these theories.

For $g \geq 0$ and $\beta \in H_2(X, \mathbb{Z})$, the *GW-invariant* $N_{g,\beta}$ is defined by the integration of the virtual class,

$$N_{g,\beta} = \int_{[\overline{M}_g(X,\beta)]^{\mathrm{vir}}} 1 \in \mathbb{Q},$$

where $\overline{M}_g(X, \beta)$ is the moduli stack of stable maps $f \colon C \to X$ with $g(C) = g$ and $f_*[C] = \beta$. We consider the following generating series,

$$\mathrm{GW}(X) = \exp\left(\sum_{g,\beta\neq 0} N_{g,\beta}\lambda^{2g-2}v^{\beta}\right).$$

For $n \in \mathbb{Z}$ and $\beta \in H_2(X, \mathbb{Z})$, let $I_n(X, \beta)$ be the Hilbert scheme of 1-dimensional subschemes $Z \subset X$ satisfying

$$[Z] = \beta, \quad \chi(\mathcal{O}_Z) = n.$$

The obstruction theory on $I_n(X, \beta)$ is obtained by viewing it as a moduli space of ideal sheaves, (cf. [21],) and the *DT-invariant* $I_{n,\beta}$ is defined by

$$I_{n,\beta} = \int_{[I_n(X,\beta)]^{\mathrm{vir}}} 1 \in \mathbb{Z}.$$

The generating function of the *reduced DT-theory* is

$$\mathrm{DT}(X) = \sum_{n,\beta} I_{n,\beta} q^n v^\beta / \sum_n I_{n,0} q^n.$$

The theory of stable pairs and their counting invariants are introduced and studied by Pandharipande and Thomas [18] to give a geometric interpretation of the reduced DT-theory. By definition, a *stable pair* is data (F, s),

$$s \colon \mathcal{O}_X \longrightarrow F,$$

where F is a pure one dimensional sheaf on X, and s is a morphism with a zero dimensional cokernel. For $\beta \in H_2(X, \mathbb{Z})$ and $n \in \mathbb{Z}$, the moduli space of stable pairs (F, s) with

$$[F] = \beta, \quad \chi(F) = n,$$

is constructed in [18], denoted by $P_n(X, \beta)$. The obstruction theory on $P_n(X, \beta)$ is obtained by viewing stable pairs (F, s) as two term complexes,

$$\cdots \longrightarrow 0 \longrightarrow \mathcal{O}_X \xrightarrow{s} F \longrightarrow 0 \longrightarrow \cdots. \tag{1}$$

Here the degree of $\mathcal{O}_X$ is -1 and the degree of F is 0. The *PT-invariant* $P_{n,\beta}$ is defined by

$$P_{n,\beta} = \int_{[P_n(X,\beta)]^{\mathrm{vir}}} 1 \in \mathbb{Z}.$$

The corresponding generating function is

$$\mathrm{PT}(X) = \sum_{n,\beta} P_{n,\beta} q^v v^\beta.$$

The series $\mathrm{GW}(X)$, $\mathrm{DT}(X)$ and $\mathrm{PT}(X)$ are conjecturally equal after suitable variable change. In order to state this, we need the following conjecture, called *rationality conjecture.*

Conjecture 1.1. [16, Conjecture 2], [18, Conjecture 3.2] *For a fixed β, the generating series*

$$\mathrm{DT}_\beta(X) = \sum_{n \in \mathbb{Z}} I_{n,\beta} q^n / \sum_{n \in \mathbb{Z}} I_{n,0} q^n, \quad \mathrm{PT}_\beta(X) = \sum_{n \in \mathbb{Z}} P_{n,\beta} q^n,$$

are Laurent expansions of rational functions of q, invariant under $q \leftrightarrow 1/q$.

The above conjecture is solved for $\mathrm{DT}_\beta(X)$ when X is a toric local Calabi-Yau 3-fold [16], and for $\mathrm{PT}_\beta(X)$ when β is an irreducible curve class [19]. Now we can state the conjectural GW-DT-PT-correspondences.

Conjecture 1.2. [16, Conjecture 3], [18, Conjecture 3.3] *After the variable change $q = -e^{i\lambda}$, we have*

$$\mathrm{GW}(X) = \mathrm{DT}(X) = \mathrm{PT}(X).$$

The variable change $q = -e^{i\lambda}$ is well-defined by Conjecture 1.1.

Note that ideal sheaves $I \subset \mathcal{O}_X$ are objects in $D^b(X)$, where $D^b(X)$ is the bounded derived category of coherent sheaves on X. We can also interpret stable pairs (F, s) as objects in $D^b(X)$ by viewing them as two term complexes (1). As discussed in [18, Secction 3], the equality $\mathrm{DT}(X) = \mathrm{PT}(X)$ should be interpreted as a wall-crossing formula for counting invariants in the category $D^b(X)$. Our purpose is to interpret PT-invariant as counting "stable" objects in the derived category with respect to some stability condition on $D^b(X)$, and study $\mathrm{PT}(X)$ via wall-crossing phenomena in the derived category.

2 Motivations

Before stating our result, we give a rough sketch of our motivation. Let $\mathcal{D}$ be a triangulated category, e.g. bounded derived category of coherent sheaves $D^b(X)$ on an algebraic variety X. Its objects consist of bounded complexes of coherent sheaves,

$$\cdots \to \mathcal{F}^i \to \mathcal{F}^{i+1} \to \cdots \to \mathcal{F}^j \to 0 \to \cdots,$$

where $\mathcal{F}^i \in \mathrm{Coh}(X)$. Historically such a class of categories was introduced to formulate the generalization of several duality theories, such as Poincaré duality, Serre duality. (cf. [2], [6].) On the other hand, the notion of triangulated categories draw much attention recently from the viewpoint of string theory. In terms of string theory, an object in the derived category of coherent sheaves is considered to represent a D-brane of type B, and a conjectural symmetry (Homological mirror symmetry) between the category of A-branes (Fukaya category) and B-branes (derived category) is proposed by Kontsevich [13].

In 2002, an important notion of stability conditions on triangulated categories was introduced by Bridgeland [4]. For a triangulated category $\mathcal{D}$, he associates a space $\mathrm{Stab}(\mathcal{D})$, which has a structure of complex manifold. So we have the following correspondence,

$$\text{triangulated category} \longrightarrow \text{complex manifold}$$

There are several motivations to introduce the complex manifold $\mathrm{Stab}(\mathcal{D})$.

- Classically there is a notion of stability condition on vector bundles on curves. (cf. [17].) We want to generalize this notion to objects in derived categories. For each $\sigma \in \mathrm{Stab}(\mathcal{D})$, there is the associated notion of σ-semistable objects in $\mathcal{D}$. So each point $\sigma \in \mathrm{Stab}(\mathcal{D})$ gives a generalization of the classical notion of stability condition. In terms of string theory, σ-semistable objects are considered to be the D-branes of BPS-state.

- The space $\mathrm{Stab}(\mathcal{D})$ is considered to describe the (extended) stringy Kähler moduli space, which should be isomorphic to the moduli space of complex structures on the mirror side. Thus it is an interesting problem to compare the space $\mathrm{Stab}(\mathcal{D})$ with the moduli space of the complex structures under mirror symmetry.

Since the theory of stability conditions on triangulated categories has been proposed recently, the theory is not so developed yet. One of the big issues is the existence problem of stability conditions, especially on the derived category of coherent sheaves on projective Calabi-Yau 3-folds. We will address this problem later.

Conjecturally the objects (1) are stable with respect to a certain stability condition on $D^b(X)$. Note that an object E given in (1) satisfies the following condition,

$$\mathrm{ch}(E) = (-1, 0, \beta, n) \in H^0 \oplus H^2 \oplus H^4 \oplus H^6, \quad \det E = \mathcal{O}_X, \tag{2}$$

Under the above background, we suggest the following story.

- For a projective Calabi-Yau 3-fold X, let $\mathcal{D} = D^b(X)$. We expect that there are stability conditions $\sigma, \tau \in \mathrm{Stab}(\mathcal{D})$ such that ideal sheaves $I_C[1]$ and objects (1) become stable with respect to σ, τ respectively.

- We expect that for any $\sigma \in \mathrm{Stab}(\mathcal{D})$, there is the algebraic moduli stack of σ-semistable objects $E \in \mathcal{D}$ with fixed phase and satisfy (2). We denote that moduli stack $\mathcal{M}^{(-1,0,\beta,n)}(\sigma)$. For a particular choice of σ, the stack $\mathcal{M}^{(-1,0,\beta,n)}(\sigma)$ should be the gerb over $I_n(X, \beta)$ or $P_n(X, \beta)$.

- We expect that there is the generalized Donaldson-Thomas invariant,

$$DT_{n,\beta}(\sigma) \in \mathbb{Q},$$

counting σ-semistable objects $E \in \mathcal{D}$ satisfying (2). $DT_{n,\beta}(\sigma)$ should be defined as the integration of the "logarithm" of the moduli stack $\mathcal{M}^{(-1,0,\beta,n)}(\sigma)$ in the Hall algebra of $\mathcal{D}$, after multiplying Behrend's weight function [1]. This procedure (expect multiplication of weight function) follows from Joyce's sequent works [9], [10], [11], [11], [12]. It should be possible to use the motivic milnor fiber idea of Kontsevich-Soibelman [14] to involve weight function into Joyce's invariants. A particular choice of σ yields $DT_{n,\beta}(\sigma) = I_{n,\beta}$ or $P_{n,\beta}$. However $DT_{n,\beta}(\sigma)$ give new invariants by deforming σ.

- We want to know how $DT_{n,\beta}(\sigma)$ varies under change of σ. In principle, there is a wall and chamber structure on $\mathrm{Stab}(\mathcal{D})$ so that $DT_{n,\beta}(\sigma)$ does not change if σ deforms in a chamber. However if σ crosses a wall, then the invariant $DT_{n,\beta}(\sigma)$ jumps and its difference should be expressed in terms of the structure of the Ringel-Hall Lie algebra associated to $\mathcal{D}$. Thus we should have the wall-crossing formula of the invariants $DT_{n,\beta}(\sigma)$.

- Applying the wall-crossing formula of $DT_{n,\beta}(\sigma)$, we expect that several properties or equalities of the generating functions of sheaf counting are realized. For instance, DT-PT correspondence [18], DT-NCDT correspondence [20], flop formula of DT-invariants [7], and the rationality conjecture of the generating functions of DT or PT-invariants should be explained by wall-crossing formula. (cf. [22].)

At this moment, there are several technical difficulties to realize the above story. One of them is to find stability conditions, which will be discussed in the next section.

3 Stability conditions

First let us give the definition of stability conditions introduced in [4].

Definition 3.1. A stability condition on a triangulated category $\mathcal{D}$ consists of data $\sigma = (Z, \mathcal{A})$, where $\mathcal{A} \subset \mathcal{D}$ is the heart of a bounded t-structure on $\mathcal{D}$, and Z is a group homomorphism,

$$Z \colon K(\mathcal{D}) \longrightarrow \mathbb{C},$$

which satisfies the following axiom.

- For a non-zero object $0 \neq E \in \mathcal{A}$, we have

  $$Z(E) \in \mathbb{H} := \{r \exp(i\pi\phi) \mid 0 < \phi \leq 1, r > 0\}.$$

  Especially one can choose the argument $\arg Z(E) \in (0, \pi]$ uniquely. An object $E \in \mathcal{A}$ is said to be Z-(semi)stable if for any non-zero object $F \subset E$, one has

  $$\arg Z(F) \leq (<) \arg Z(E).$$

- There is a Harder-Narasimhan property, i.e. any $E \in \mathcal{A}$ admits a filtration

  $$0 = E_0 \subset E_1 \subset \cdots \subset E_n = E,$$

  such that each $F_i = E_i / E_{i-1}$ is Z-semistable with $\arg Z(F_i) > \arg Z(F_{i+1})$.

Here we give some examples.

Example 3.2. (i) Let $\mathcal{D} = D^b(C)$ for a smooth projective curve C, and $Z \colon K(C) \to \mathbb{C}$ be

$$Z(E) = -\deg(E) + \mathrm{rk}(E) \cdot i.$$

Then the pair $(Z, \mathrm{Coh}(C))$ determines a stability condition on $\mathcal{D}$. In this case, an object $E \in \mathrm{Coh}(C)$ is Z-semistable if and only if it is a semistable sheaf in the sense of [17].

(ii) Let A be a finite dimensional k-algebra with k a field, and $\mathcal{D} = D^b(\mathcal{A})$ where $\mathcal{A} = \mathrm{mod}\, A$ is the abelian category of finitely generated right A-modules. Then there is a finite number of simple objects $S_1, \cdots, S_N \in \mathcal{A}$ which generates $\mathcal{A}$. One can choose $Z \colon K(\mathcal{A}) \to \mathbb{C}$ such that $Z(S_i) \in \mathbb{H}$ for all $1 \leq i \leq N$. Then the pair $(Z, \mathcal{A})$ determines a stability condition on $\mathcal{D}$.

So far, the spaces $\mathrm{Stab}(\mathcal{D})$ for several $\mathcal{D}$ have been studied in detail. For instance, see [5], [15], [8], [23]. On the other hand, the following problem has been a big issue in studying stability conditions.

Problem 3.3. *Given a triangulated category $\mathcal{D}$, do we have an example of a stability condition on $\mathcal{D}$, i.e.* $\mathrm{Stab}(\mathcal{D}) \neq \emptyset$*?*

The above problem is non-trivial especially for the case $\mathcal{D} = D^b(X)$, where X is a smooth projective variety with $\dim X \geq 2$. In this case, one can show that there is no stability condition $(Z, \mathcal{A})$ with $\mathcal{A} = \mathrm{Coh}(X)$. As an analogue of Example 3.2 (i), one might try to construct Z to be the group homomorphism

$$Z(E) = -c_1(E) \cdot \omega + \mathrm{rk}(E) \cdot i,$$

for a fixed ample divisor ω. However the pair $(Z, \mathrm{Coh}(X))$ does not give a stability condition since $Z([\mathcal{O}_x]) = 0$ for a closed point $x \in X$. When $\dim X = 2$, the examples of stability conditions are constructed by tilting the abelian category $\mathrm{Coh}(X)$, (cf. [5].) However we do not know any example of stability conditions when $\dim X \geq 3$, except the case that there is a derived equivalence $D^b(X) \cong D^b(A)$ for a finite dimensional algebra A. (e.g. $X = \mathbb{P}^3$.)

From the viewpoint of mirror symmetry, the most important case is when X is a projective Calabi-Yau 3-fold. In this case, there are some ideas coming from string theory. Let $A(X)_{\mathbb{C}}$ be the complexified ample cone,

$$A(X)_{\mathbb{C}} := \{B + i\omega \in H^2(X, \mathbb{C}) \mid \omega \text{ is ample } \}.$$

Let $Z_{(B,\omega)} \colon K(X) \to \mathbb{C}$ be

$$Z_{(B,\omega)}(E) = -\int e^{-(B+i\omega)} \mathrm{ch}(E) \sqrt{\mathrm{td}_X}.$$

We can state the following conjecture.

Conjecture 3.4. *For $\omega \gg 0$, there should exist the heart of a bounded t-structure $\mathcal{A}_{(B,\omega)} \subset D^b(X)$ such that the pair $\sigma_{(B,\omega)} = (Z_{(B,\omega)}, \mathcal{A}_{(B,\omega)})$ is a stability condition on $D^b(X)$.*

The above conjecture holds true if $\dim X \leq 2$.

4 Stability conditions on D0-D2-D6 bound states

Let $\mathrm{Coh}_{\leq 1}(X)$ be

$$\mathrm{Coh}_{\leq 1}(X) := \{E \in \mathrm{Coh}(X) \mid \dim \mathrm{Supp}(E) \leq 1\}.$$

Instead of working with $D^b(X)$, we study stability conditions on $\mathcal{D}_X$,

$$\mathcal{D}_X = \langle \mathcal{O}_X, \mathrm{Coh}_{\leq 1}(X) \rangle_{\mathrm{tr}} \subset D^b(X).$$

Here for a set of objects $S \subset D^b(X)$, we denote by $\langle S \rangle_{\mathrm{tr}}$ the smallest triangulated subcategory of $D^b(X)$, which contains S. We also denote by $\langle S \rangle_{\mathrm{ex}}$ the smallest extension closed subcategory of $D^b(X)$, which contains S. We have the following lemma, whose proof will be appear in [24].

Lemma 4.1. *There is a bounded t-structure on $\mathcal{D}_X$, whose heart $\mathcal{A}_X$ satisfies*

$$\mathcal{A}_X = \langle \mathcal{O}_X[1], \mathrm{Coh}_{\leq 1}(X) \rangle.$$

Let $A(X)_{\mathbb{C}}$ be the complexified ample cone,

$$A(X)_{\mathbb{C}} := \{B + i\omega \in H^2(X, \mathbb{C}) \mid \omega \text{ is ample. }\}.$$

For a following data,

$$s \in \mathbb{R}_{<0}, \quad t \in A(X)_{\mathbb{C}}, \quad u \in \mathbb{H},$$

we define a map

$$Z_{(s,t,u)} \colon K(\mathcal{D}_X) \longrightarrow \mathbb{C},$$

as

$$Z_{(s,t,u)}(E) = s\,\mathrm{ch}_3(E) + t\,\mathrm{ch}_2(E) - u\,\mathrm{ch}_0(E).$$

We have the following, which will be appear in [24].

Lemma 4.2. *The pair $(Z_{(s,t,u)}, \mathcal{A}_X)$ determines points in* $\mathrm{Stab}(\mathcal{D}_X)$. *In particular* $\mathrm{Stab}(\mathcal{D}_X) \neq \emptyset$.

We have the following embedding,

$$\mathcal{U}_X := \mathbb{R}_{<0} \times A(X)_{\mathbb{C}} \times \mathbb{H} \subset \mathrm{Stab}(\mathcal{D}_X).$$

The following result will be proved in [24].

Theorem 4.3. *(i) For $\sigma \in \mathcal{U}_X$, there is the algebraic moduli stack of finite type*

$$\mathcal{M}^{(-1,0,\beta,n)}(\sigma),$$

which parameterizes σ-semistable objects $E \in \mathcal{A}_X$ with

$$\mathrm{ch}(E) = (-1, 0, \beta, n), \quad \det(E) = \mathcal{O}_X.$$

(ii) Suppose that $u \in \mathbb{R}_{<0}$ for $\sigma = (s, t, u)$. For $u \ll 0$, we have

$$\mathcal{M}^{(-1,0,\beta,n)}(\sigma) = [P_n(X, \beta)/\mathbb{G}_m],$$

where $\mathbb{G}_m$ acts on $P_n(X, \beta)$ trivially.

5 Flop formula

Applying Theorem 4.3 and wall-crossing formula developed by Joyce [12], Kontsevich-Soibelman [14], we can study how generating series of invariants counting stable pairs transform under flops. Instead of working with $\mathrm{PT}(X)$, let us consider the generating series,

$$\hat{\mathrm{PT}}(X) = \sum_{n,\beta} \chi(P_n(X, \beta)) q^n v^\beta.$$

The series $\hat{\mathrm{PT}}(X)$ is closely related to $\mathrm{PT}(X)$ in the following sense.

- Suppose that $P_n(X, \beta)$ is smooth and connected. Then we have

$$P_{n,\beta} = (-1)^{\dim P_n(X,\beta)} \chi(P_n(X, \beta)).$$

- In general, there is a constructible function $\nu\colon P_n(X,\beta) \to \mathbb{Z}$, constructed by Behrend [1], such that
$$P_{n,\beta} = \sum_{n\in\mathbb{Z}} n\chi(\nu^{-1}(n)).$$

Let us consider a diagram of flop of Calabi-Yau 3-folds,

$$\begin{tikzcd} X^+ \arrow[rr, dashed, "\phi"] \arrow[dr, "f^+"'] & & X \arrow[dl, "f"] \\ & Y. & \end{tikzcd}$$

In this situation, Bridgeland [3] showed the equivalence of derived categories,

$$\Phi\colon D^b(X^+) \xrightarrow{\sim} D^b(X).$$

It is easy to see that Φ restricts to the equivalence,

$$\Phi\colon \mathcal{D}_{X^+} \xrightarrow{\sim} \mathcal{D}_X,$$

hence we have the isomorphism,

$$\Phi_*\colon \mathrm{Stab}(\mathcal{D}_{X^+}) \xrightarrow{\cong} \mathrm{Stab}(\mathcal{D}_X).$$

We have the following. (cf. [24].)

Lemma 5.1. *We have*
$$\Phi_*\overline{\mathcal{U}}_{X^+} \cap \overline{\mathcal{U}}_X \neq \emptyset.$$

The above lemma implies that we can relate stability conditions relevant to stable pairs on X to those on X^+. Let $\hat{\mathrm{PT}}(X/Y)$ be the subseries

$$\hat{\mathrm{PT}}(X/Y) = \sum_{n, f_*\beta=0} \chi(P_n(X,\beta))q^n v^\beta.$$

Applying wall-crossing formula by Joyce [12], from a point in $\mathcal{U}_X$ to $\Phi_*\mathcal{U}_{X^+}$, we obtain the following. (cf. [24].)

Theorem 5.2. *Under the above situation, we have*

$$\frac{\hat{\mathrm{PT}}(X)}{\hat{\mathrm{PT}}(X/Y)} = \phi_*\frac{\hat{\mathrm{PT}}(X^+)}{\hat{\mathrm{PT}}(X^+/Y)},$$
$$\hat{\mathrm{PT}}(X/Y) = i\circ\phi_*\hat{\mathrm{PT}}(X^+/Y).$$

Here the variable change is $\phi_*(\beta,n) = (\phi_*\beta, n)$ *and* $i(\beta,n) = (-\beta, n)$.

We can apply Joyce's wall-crossing formula of counting invariants. Unfortunately we are unable to involve Behrend's constructible function into Joyce's work, so our application is restricted to Euler number of version of the relevant moduli spaces at this moment.

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