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STABLE POINTS ON STACKS

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1. Approximating algebraic stacks to schemes or algebraic spaces

A coarse moduli space for an algebraic stack[1] is an algebraic space that is the closest to the algebraic stack. First let us recall the definition of a coarse moduli space:

Definition 1.1. Let $\mathcal{X}$ be an algebraic (Artin) stack over a scheme S. A coarse moduli map for $\mathcal{X}$ is a morphism

$$\pi : \mathcal{X} \to X$$

over S such that

(1) X is an algebraic space over S,
(2) π is universal among maps to algebraic spaces,
(3) for any algebraically closed S-field K, π gives rise to a bijective map from the set of the isomorphism classes of $\mathcal{X}(K)$ to the set of K-valued points $X(K)$.

Informally speaking, we have the rough slogan:

Algebraic stack = Groupoid valued sheaf + Algebraically Geometric structures,

Scheme or Algebraic space = Set values sheaf + Algebraically Geometric structures.

From this point of view, it is clear that the coarse moduli space X for an algebraic stack $\mathcal{X}$ loses the information arising from the non-trivial morphisms which belong to groupoids. For example, in general, the category of sheaves on $\mathcal{X}$ is quite different from that of X. However, in the treatment of algebraic stacks we often need the existence of a coarse moduli space. Namely, the proof sometimes relies on the existence of a coarse moduli space. The typical use can be found in the proof of Riemann-Roch theorem for Deligne-Mumford stacks due to Toën ([13]). Thus, coarse moduli spaces provide useful bridges between the geometry of stacks and schemes and algebraic spaces.

Now we will try to construct a coarse moduli space for a given stack $\mathcal{X}$. If we ignore "Algebraically Geometric structures" in the above slogan, we easily find the way: Take a connected component of groupoids. Namely, view $\mathcal{X}$ as a functor

$$\mathcal{X} : (\text{Schemes})_{\text{ét}} \longrightarrow (\text{Groupoids})$$

and define $\pi_0(\mathcal{X})(S) = \pi_0(\mathcal{X}(S))$ for any scheme S. In other words, $\pi_0(\mathcal{X})(S)$ is the set of isomorphism classes of groupoids $\mathcal{X}(S)$. A sheafification after this truncating procedure gives rise to a sheaf $\bar{\mathcal{X}}$ on the site $(\text{Schemes})_{\text{ét}}$. This sheaf $\bar{\mathcal{X}}$ is the best approximation of $\mathcal{X}$ in the category of sheaves on $(\text{Schemes})_{\text{ét}}$. The picture of this

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[1]In this note, by an algebraic stack we mean an Artin stack. We refer to [10] as the general reference.

construction is clear and easy to understand since it is nothing but a truncation. However, if one takes account into "Algebraically Geometric structures", then the problem becomes very subtle and difficult. To understand this, let us recall the known class of algebraic stacks which have their coarse moduli spaces, and examples which do not. The theorem we first recall is a well-known result due to Keel and Mori ([9]).

Theorem 1.2 (Keel-Mori). *Let $\mathcal{X}$ be an algebraic stack locally of finite type over a noetherian base scheme S. Let $I\mathcal{X} \to \mathcal{X}$ be the first (or second) projection in the diagram*

$$\begin{array}{ccc} \mathcal{X} \times_{\mathcal{X} \times_S \mathcal{X}} \mathcal{X} & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ \mathcal{X} & \longrightarrow & \mathcal{X} \times_S \mathcal{X}. \end{array}$$

Suppose that $I\mathcal{X} \to \mathcal{X}$ is a finite morphism. Then there exists a coarse moduli map

$$\pi : \mathcal{X} \to X$$

such that $\pi_* \mathcal{O}_{\mathcal{X}} = \mathcal{O}_X$, and π is proper and quasi-finite. Moreover $\mathcal{X}$ has finite diagonal, then $\mathcal{X}$ is separated.

The stack $I\mathcal{X} \to \mathcal{X}$ is called the inertia stack of $\mathcal{X}$. This stack parametrizes the automorphisms of objects in $\mathcal{X}$. Namely, for any $\alpha : T \to \mathcal{X}$, the fiber product $\mathrm{pr}_2 : I\mathcal{X} \times_{\mathcal{X}} T \to T$ represents the functor

$$\mathrm{Aut}_T(\alpha) : (T\text{-schemes}) \to (\text{groups})$$

which to any $f : T' \to T$ associates the group $\mathrm{Aut}_T(\alpha)(T') := \{\text{automorphisms of } f^*\alpha\}$. The inertia stack can be viewed as a kind of the free loop space for $\mathcal{X}$. The condition $I\mathcal{X} \to \mathcal{X}$ is a finite morphism, is equivalent to imposing that every object in $\mathcal{X}$ has a finite automorphism group scheme. In characteristic zero, algebraic stacks whose inertia are finite, are always Deligne-Mumford.

Next we consider examples of stacks which do not admit coarse moduli spaces. Examples we will keep in mind are the moduli stack of vector bundles on an algebraic variety, and more generally, the moduli stack of G-bundles on the algebraic variety, where G is an algebraic group. Another example is the moduli stack of objects of derived category of coherent cohomology on a scheme. What happen on an algebraic stack which does not admit a coarse moduli space? In order to make an observation, consider the open immersion

$$\mathbb{G}_m \hookrightarrow \mathbb{A}^1$$

of a torus into an affine line over the complex number field. It gives rise to the natural action of $\mathbb{G}_m$ on $\mathbb{A}^1$. Take the quotient stack $[\mathbb{A}^1/\mathbb{G}_m]$. It is the moduli stack of pairs $(\mathcal{L}, s)$, where $\mathcal{L}$ is an invertible sheaf $\mathbb{A}^1$, and s is a section on $\mathbb{A}^1$. Since we have two $\mathbb{G}_m$-orbits on $\mathbb{A}^1$, there are two closed points on $[\mathbb{A}^1/\mathbb{G}_m]$. On the other hand, $[\mathbb{A}^1/\mathbb{G}_m]$ is connected, thus if we assume that $[\mathbb{A}^1/\mathbb{G}_m]$ has a coarse moduli space, then it is connected and has exactly two closed points. But such a complex analytic space does not exists, and we conclude that $[\mathbb{A}^1/\mathbb{G}_m]$ does not have a coarse moduli space. Put another way, notice that the dimension of the stabilizer at the origin on $\mathbb{A}^1$ is positive whereas the other points have 0-dimensional stabilizer groups. This collapses the "Algebraic Geometric structures" (see [11, page 6]). Therefore if one hope that an algebraic stack $\mathcal{X}$ has a coarse moduli space, then objects in $\mathcal{X}$ should have the equidimensional automorphisms, that is, $I\mathcal{X} \to \mathcal{X}$ is equidimensional. We can ask the converse: if $I\mathcal{X} \to \mathcal{X}$ is equidimesional, then does $\mathcal{X}$ have a coarse moduli space? Unfortunately, this problem is quite subtle. Even in the case where $I\mathcal{X} \to \mathcal{X}$ is quasi-finite, we (at least the author) do not know whether or not $\mathcal{X}$ has a coarse moduli space. Another point we should note concerns the problem of the finite generation of invariant rings.

2. Intrinsic Stability on Algebraic stacks

In the proceeding section, we discuss coarse moduli spaces for algebraic stacks, especially the example of an algebraic stack that does not admit a coarse moduli space. The theory dealing with the last problem was essentially proposed by Mumford in the case $\mathcal{X} = [X/G]$ where X is an algebraic scheme, and G is a reductive group acting on X, that is, Geometric Invariant Theory (GIT) ([11]). Suppose that a reductive group G acts on an algebraic scheme X. Mumford defined pre-stable points on X with respect to the action of G, and proved that G-orbit space of pre-stable points has a structure of a scheme called the geometric quotient Y (see [11]). It is rephrased that the quotient stack $[X(\mathrm{Pre})/G]$ has a "coarse moduli scheme"

$$[X(\mathrm{Pre})/G] \to Y.$$

Thus, from our point of view, Mumford' GIT provides a machinery that chooses an open substack of $[X/G]$ which admits a coarse moduli (if we further take a suitable line bundle on $[X/G]$, then we have a polarized coarse moduli of the open substack of stable points). Inspired from Mumford's theory and Keel-Mori theorem, we want to propose the idea:

> Introduce intrinsically "stability" on a general Artin stack $\mathcal{X}$ so that stable points $\mathcal{X}^s$ form an open substack which admits a coarse moduli map $\mathcal{X}^s \to X$.

We first remark that we want to define "intrinsically stable points" on $\mathcal{X}$ by using local properties on $\mathcal{X}$, and thus we do not take account into the global flavour. At this point, the reader might begin to object that if we do not use the global aspects on $\mathcal{X}$ (such as linearized line bundle in GIT), the resulting coarse moduli space is not a good space, for example, often not separated. Here we would like to call the reader's attention to the observation: Keel-Mori theorem, which we want to take a position to generalize, tells us no global information of the coarse moduli space. Recall that Theorem 1.2 says that if $\mathcal{X}$ has finite diagonal, then the coarse moduli space X for $\mathcal{X}$ is separated. Nevertheless, if we assume that the existence of a (not necessarily separated) coarse moduli space X, then the proof of the separatedness of X is quite formal. Of course, the price is that Keel-Mori theorem tells us very little about how to prove that X is separated. In my opinion, one of the reasons why Keel-Mori theorem is useful, is that the finiteness of $I\mathcal{X} \to \mathcal{X}$ is a local condition on $\mathcal{X}$, and the global aspect should be treated in the next step by case-by-case approaches.

In [8], we introduced some stabilities which have relations described as follows:

$$\text{(GIT-like p-stable)} \subset \text{(p-stable)} \supset \text{(strong p-stable)}$$

In this note, we discuss and focus on GIT-like stability, which has some remarkable properties. Also, we briefly mention strong p-stability.

Definition 2.1 (GIT-like p-stable point). Let $\mathcal{X}$ be an algebraic stack locally of finite type over a perfect field k. Let p be a closed point on $\mathcal{X}$. The point p is GIT-like p-stable if there exists an effective versal deformation $\xi \in \mathcal{X}(A)$ for p (see Remark 2.2), which has the following properties:

(a) The special fiber of the automorphism group $\mathrm{Aut}_{\mathrm{Spec}\,A}(\xi) \to \mathrm{Spec}\,A$ is linearly reductive.

(b) If I denotes the ideal generated by nilpotent elements in A, then there exists a normal subgroup scheme $\mathcal{F}$ of $\mathrm{Aut}(\xi) \times_A (A/I) \to \mathrm{Spec}\,A/I$ such that the following conditions hold: (i) $\mathcal{F}$ is smooth and affine over $\mathrm{Spec}\,A/I$, and whose geometric fibers are connected, (ii) the quotient $\mathrm{Aut}(\xi) \times_A (A/I)/\mathcal{F}$ is finite over $\mathrm{Spec}\,A/I$, and (iii) for any two morphisms $\alpha, \beta : T \rightrightarrows \mathrm{Spec}\,A/I$ such that $\alpha^*\xi \cong \beta^*\xi$, we have $\alpha^*\mathcal{F} \cong \beta^*\mathcal{F}$ in $\mathrm{Aut}_T(\alpha^*\xi) \cong \mathrm{Aut}_T(\beta^*\xi)$.

Remark 2.2. (i) The letter "p" in the terms GIT-like p-stable, p-stable.. is the initial of *pointwise*.

(ii) We say that $\xi \in \mathcal{X}(A)$ is an effective versal deformation for a closed point p if
  (a) A is a complete noetherian local k-ring whose residue field is of finite type over k,
  (b) the special fiber of $\xi : \mathrm{Spec}\,A \to \mathcal{X}$ lies over p,
  (c) the corresponding morphism $\mathrm{Spec}\,A \to \mathcal{X}$ is formally smooth, i.e., it satisfies the usual lifting property (cf. [8], [2]).

(iii) Recall the definition of linearly reductivity. An algebraic group G over k is linearly reductive if the functor

$$(G\text{-vector spaces over } k) \to (k\text{-vector spaces}) \; M \mapsto M^G$$

is exact. In characteristic zero, an algebraic group is linearly reducitve if and only if it is a reductive group.

(iv) GIT-like stability depends only on the reduced algebraic stack $\mathcal{X}_{\mathrm{red}}$ associated to $\mathcal{X}$.

(v) To verify that a given group scheme $\mathcal{G}$ over a reduced scheme S is smooth (over S), it is enough to prove that $\mathcal{G} \to S$ is equidimensional, and all fibers are smooth.

(vi) The condition (iii) in (b) in Definition 2.1 is a natural compatibility condition.

(vii) Our definition fits in with Artin's representability criterion ([3]) which is desribed in terms of deformation theory. Of course, our formulation is influenced by Artin's works.

(viii) A closed point on $\mathcal{X}$ is said to be a strong p-stable point if there exists an effective versal deformation $\xi \in \mathcal{X}(A)$ such that there exists a flat normal subgroup scheme $\mathcal{F} \subset \mathrm{Aut}(\xi)$ such that $\mathrm{Aut}(\xi)/\mathcal{F} \to \mathrm{Spec}\,A$ is a finite morphism, and the compatibility condition as in (iii) in Definition 2.1 holds.

To give a feeling for GIT-like p-stability defined above, we will consider the following example. The relationship with GIT will be discussed in the next section. Let G be a connected reductive group over $\mathbb{C}$ and C a connected smooth projective curve over $\mathbb{C}$. Let $\mathcal{M}$ be the moduli algebraic stack of Higgs G-bundles on C. The automorphism of every Higgs G-bundle $(E, \phi \in \Gamma(C, \mathfrak{G}_E \times_{\mathbb{C}} \Omega_X))$ contains the center of G. The center Cent(G) is a reductive group and for any family $(\tilde{E},\tilde{\phi})$ of Higgs G-bundle over $C\times_{\mathbb{C}}T$, $\mathrm{Cent}(G)\times_{\mathbb{C}}T$ is a normal subgroup in $\mathrm{Aut}_T((\tilde{E},\tilde{\phi}))$. A Higgs bundle (E,ϕ) is GIT-like p-stable (in other words, the corresponding point on $\mathcal{M}$ is GIT-like p-stable) if and only if an effective versal deformation $(\mathcal{E},\Phi)\in\mathcal{M}(A)$ for (E,ϕ) has a finite automorphism group scheme modulo $\mathrm{Cent}(G)\times_{\mathbb{C}}\operatorname{Spec}A$.

Now we are ready to state the existence theorem of coarse moduli spaces for GIT-like p-stable points ([8]).

Theorem 2.3. *Let $\mathcal{X}$ be an algebraic stack locally of finite type over a perfect field. Then the open substack $\mathcal{X}^{gs}$ of GIT-like p-stable points has a coarse moduli map*

$$\pi:\mathcal{X}^{gs}\longrightarrow X.$$

Moreover π is universally closed morphism and of finite type.

The construction takes three steps:

- *First Step.* Let $\mathcal{X}_0$ be the reduced stack associated to $\mathcal{X}$. Applying the algebraization, we may assume that the inertia stack $I\mathcal{X}_0\to\mathcal{X}_0$ contains a smooth and affine subgroup stack $\mathcal{F}\subset I\mathcal{X}_0$, whose geometric fibers are connected. Namely, $\mathcal{X}=\mathcal{X}^{gs}$. Then the rigidification technique removes the automorphisms in $\mathcal{F}$, and we obtain the "rigidified" stack $\mathcal{X}_0^{\mathrm{rig}}$.
- *Second Step.* By our assumption, $\mathcal{X}_0^{\mathrm{rig}}$ has a finite inertia stack $I\mathcal{X}_0^{\mathrm{rig}}\to\mathcal{X}_0^{\mathrm{rig}}$. Then by Keel-Mori theorem, there exists a coarse moduli space X_0 for $\mathcal{X}_0^{\mathrm{rig}}$. The composite $\mathcal{X}_0\to\mathcal{X}_0^{\mathrm{rig}}\to X_0$ is also a coarse moduli map for $\mathcal{X}_0$. (The first and second steps are rather formal parts in our strategy.)
- *Third Step.* Now we want to construct a coarse moduli space X for $\mathcal{X}$ by deforming X_0 as follows:

$$\begin{array}{ccc}\mathcal{X}_0 & \longrightarrow & \mathcal{X}\\ \downarrow & & \vdots\\ X_0 & \cdots\cdots\rightarrow & X\end{array}$$

At this point, there are some points we should note. Even if an algebraic stack $\mathcal{Y}$ and the associated reduced stack $\mathcal{Y}_0$ have coarse moduli spaces Y and Y_0, the natural morphism $Y_0\to Y$ is not necessarily a deformation. To make things simple, assume that $\mathcal{Y}=[\operatorname{Spec}A/G]$, $\mathcal{Y}_0=[\operatorname{Spec}(A/I)/G]$, $Y_0=\operatorname{Spec}(A/I)^G$ and $Y=\operatorname{Spec}A^G$, where G is an algebraic group and I is a nilpotent ideal of A. If G is linearly reductive, then $A^G\to(A/I)^G$ is surjective, thus $Y_0\to Y$ is a deformation as expected. However, if G is unipotent, then it happens that $A^G\to(A/I)^G$ is not surjective. Note $(A/I)^G=\Gamma(\mathcal{Y}_0,\mathcal{O}_{\mathcal{Y}_0})$ and $A^G=\Gamma(\mathcal{Y},\mathcal{O}_{\mathcal{Y}})$. Thus, we need to verify that after étale localization on X_0, $\Gamma(\mathcal{X},\mathcal{O}_{\mathcal{X}})\to\Gamma(\mathcal{X}_0,\mathcal{O}_{\mathcal{X}_0})$ is surjective. It is accomplished by constructing the "étale local quotient structure" of $\mathcal{X}$. In this part, we essentially use the linearly redutivity of automorphism groups. Finally, developing the deformation theory of coarse moduli spaces, we construct the desired deformation $X_0\hookrightarrow X$.

3. Comparing with Geometric Invariant Theory

We assume that the base field k is algebraically closed of characteristic zero. In this section, we discuss the relationship between our GIT-like p-stability and Geometric

Invariant Theory due to Mumford ([11]). Let X be an algebraic scheme over k. Let G be a reductive group scheme over k. Let $\sigma : G \times_k X \to X$ be an action on X. Let $X(\mathrm{Pre}) \subset X$ be the open subset of X, consisting of pre-stable points in the sense of [11, Definition 1.7]. The relation is described by

Theorem 3.1. *Let $[X(\mathrm{Pre})/G]$ be the open substack of $[X/G]$. Let*

$$[X/G]^{gs}$$

be the open substack consisting of GIT-like p-stable points on $[X/G]$. Let $\mathcal{S}$ be the maximal open substack of $[X/G]$, admitting a coarse moduli space that is a scheme. (The open substack $\mathcal{S} \subset \mathcal{X}$ is characterized by the following universality: If $\mathcal{U} \subset \mathcal{X}$ has a coarse moduli space which is a scheme, then $\mathcal{U} \subset \mathcal{S}$.) Then we have

$$[X(\mathrm{Pre})/G] = [X/G]^{gs} \cap \mathcal{S}.$$

From this evidence, we can say that GIT-like p-stability is an intrinsic generalization of the local part of Mumford's GIT. (Pre-stability in GIT is a local part of GIT.)

Let us briefly explain how one can view pre-stable points in the sense of GIT as GIT-like p-stable points. Let $x \in X$ be a closed pre-stable point. By the definition, there exists a G-invariant affine neighborhood U of x, such that the action of G on U is closed. That is to say, every orbit is a closed set in U. Notice that G acts also on the reduced scheme U_{red} associated to U (because the base field is perfect). Clearly, the action of G on U_{red} is closed. Let

$$\mathsf{Stab} \to U_{\mathrm{red}}$$

be the stabilizer group scheme defined to be the top horizontal arrow in the cartesian diagram

$$\begin{array}{ccc} (G \times U_{\mathrm{red}}) \times_{U_{\mathrm{red}} \times U_{\mathrm{red}}} U_{\mathrm{red}} & \longrightarrow & U_{\mathrm{red}} \\ \downarrow & & \downarrow \text{diagonal} \\ G \times U_{\mathrm{red}} & \xrightarrow{(\sigma, \mathrm{pr}_2)} & U_{\mathrm{red}} \times U_{\mathrm{red}} \end{array}$$

where $\sigma : G \times U_{\mathrm{red}} \to U_{\mathrm{red}}$ is the action. The group scheme $\mathsf{Stab} \to U_{\mathrm{red}}$ is a (non-flat) equidimensional group scheme over U_{red}. According to Matsushima's theorem, we see that each fiber of $\mathsf{Stab} \to U_{\mathrm{red}}$ is a reductive algebraic group. Let $\mathcal{F}$ be the identity component of Stab. Then by SGA3 ([4]), $\mathcal{F}$ is smooth and affine over U_{red}, whose geometric fibers are connected. Moreover it can be shown that $\mathsf{Stab}/\mathcal{F}$ is a finite scheme over U_{red}. Since the completion of the local ring $\mathcal{O}_{X,x}$ gives rise to a versal deformation of the corresponding point on $[X/G]$, thus we see that the filtration $\mathcal{F} \subset \mathsf{Stab}$ over U_{red} yields the structure of a GIT-like p-stable point.

Remark 3.2. In Mumford's GIT, it is essential to have the quotient of a scheme by a reductive group. However, an algebraic Artin stack is not necessarily of the quotient form $[X/G]$, where X is a scheme (or more generally algebraic space), and G is a group scheme. In practice, it is quite hard to prove that a given algebraic stack is a quotient stack even if it has (cf. [7]). Moreover, it is hopeless to control the quotient structure. (In a sense, a quotient form should be viewed as a good coordinate.) On the other hand, our stability is defined in the intirisic way, thus it seems to be flexible and convenient, especially in the case where stacks have modular interpretations.

4. Hidden Properness of algebraic stacks: An application

In the final section, we will discuss the finiteness of coherent cohomology. In particular, we will propose "hidden properness" of algebraic (Artin) satcks. First we would like to remind the definition of proper morphisms between algebraic stacks.

Definition 4.1 ([10]). Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. The morphism $f : \mathcal{X} \to \mathcal{Y}$ is said to be proper if the following conditions hold:

(i) f is universally closed map,
(ii) f is of finite type,
(iii) f is separated, i.e., the diagonal $\mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is proper.

We have a finiteness of coherent cohomology for algebraic stacks:

Theorem 4.2 (Laumon, Moret-Bailly, Faltings, Olsson, Gabber). *Let $f : \mathcal{X} \to \mathcal{Y}$ be a proper morphism of locally noetherian algebraic stacks over locally noetherian base scheme. Let $\mathcal{E}$ be a coherent sheaf on $\mathcal{X}$. Then for any $i \geq 0$, the sheaf $Rf^i_*\mathcal{E}$ is coherent on $\mathcal{Y}$.*

The finiteness theorem of coherent cohomology for proper algebraic stacks has been proved by Laumon and Moret-Bailly under some restrictive hypotheses (cf. [10, (15.6)]). Later, Faltings proved the finiteness theorem for general proper morphisms via a surprising method of rigid geometry (cf. [5]). Recently, Olsson-Gabber proved Chow's lemma for algebraic stacks and reproved the finiteness theorem (cf. [12]).

Now we would like to reader's attention to:

> The separatedness for algebraic (Artin) stacks is a quite strong assumption.

To understand it, let $f : \mathcal{X} \to \mathcal{Y}$ be a separated morphism. For simplicity, suppose that $\mathcal{Y}$ is a noetherian affine scheme $\operatorname{Spec} A$. (The proof of the finiteness can be reduced to the case $\mathcal{Y} = \operatorname{Spec} A$.) Let $\alpha, \beta : \operatorname{Spec} K \to \mathcal{X}$ be morphisms where K is an algebraically closed field. Then the fiber product of

$$\begin{array}{ccc} & & \operatorname{Spec} K \\ & & \downarrow{\scriptstyle (\alpha,\beta)} \\ \mathcal{X} & \longrightarrow & \mathcal{X} \times_A \mathcal{X} \end{array}$$

is the algebraic space $\operatorname{Isom}(\alpha, \beta)$, which represents the functor

$$(K\text{-schemes}) \longrightarrow (\text{sets})$$

sending $h : T \to \operatorname{Spec} K$ to the set $\operatorname{Hom}_{\mathcal{X}(T)}(h^*\alpha, h^*\beta)$. The algebraic space $\operatorname{Isom}(\alpha, \beta)$ is empty or isomorphic to the proper algebraic group $\operatorname{Aut}(\alpha)_{/K}$. (Note that $\operatorname{Isom}(\alpha, \beta) \to \operatorname{Spec} K$ is proper.) The identity component of the reduced (smooth) algebraic group associated to $\operatorname{Aut}(\alpha)_{/K}$ is a (possibly 0-dimensional) abelian variety. Thus, if $\mathcal{X}$ has an object whose automorphism is a positive-dimensional affine group scheme, then $\mathcal{X}$ is not separated, in particular, not proper. This causes one of main drawbacks of algebraic stacks. Also, this observation tells us that if $\mathcal{X}$ is separated over A, then $\mathcal{X}$ practically has finite diagonal. Consequently, in such a situation, if $\mathcal{X}$ is proper over A, then it has a proper coarse moduli space (by Keel-Mori theorem).

We are now in the position to state our finiteness.

Theorem 4.3. *Let $\mathcal{X}$ be an algebraic stack of finite type over a field k. Suppose that all closed points are GIT-like p-stable, and a coarse moduli space for $\mathcal{X}$ is proper k. Let $\mathcal{E}$ be a coherent sheaf on $\mathcal{X}$. Then for any $i \geq 0$, the cohomology $H^i(\mathcal{X}, \mathcal{E})$ is finite dimensional. Moreover, (of course) the relative version of this statement holds.*

Clearly, our finiteness does not contain Theorem 4 because $\mathcal{X}$ in Theorem 4.3 is supposed to have linearly reductive automorphisms (and we work only over a field). But, nevertheless, we would like to stress that our finiteness is applicable to a certain class of non-proper algebraic stacks (in particular, our assumption is fairly weak in characteristic zero, and it can be applied to algebraic stacks having positive dimensional affine automorphisms groups). We should think that such algebraic stacks behave like proper, and have "hidden properness" (although here we ignore the finiteness concerning constructible sheaves). The proof is different from Falting's one and Olsson-Gabber's one. Our proof is done by showing that the coarse moduli map $\mathcal{X} \to X$ has the "hidden properness". Given an algebraic stack of finite type and all closed points are GIT-like p-stable, we have a version of valuative criterion for the properness of a coarse moduli space for $\mathcal{X}$ ([8]). Using it, we can state our finiteness without making reference to the coarse moduli space.

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