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POLARIZED ENDOMORPHISMS ON NORMAL PROJECTIVE VARIETIES

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ABSTRACT. This is the summary of the paper [14]. We show that polarized endomorphisms of rationally connected threefolds with at worst terminal singularities are equivariantly built up from those on $\mathbb{Q}$-Fano threefolds, Gorenstein log del Pezzo surfaces and $\mathbb{P}^1$. Similar results are obtained for polarized endomorphisms of uniruled threefolds and fourfolds. As a consequence, we show conceptually that every smooth Fano threefold with a polarized endomorphism of degree > 1, is rational.

1. INTRODUCTION

We work over the field $\mathbb{C}$ of complex numbers. We study *polarized* endomorphisms $f : X \to X$ of varieties X, i.e., those f with $f^*H \sim qH$ for some $q > 0$ and some ample line bundle H. Every surjective endomorphism of a projective variety of Picard number one, is polarized. If $f = [F_0 : F_1 : \cdots : F_n] : \mathbb{P}^n \to \mathbb{P}^n$ is a surjective morphism and $X \subset \mathbb{P}^n$ a f-stable subvariety, then $f^*H \sim qH$ and hence $f|X : X \to X$ is polarized; here $H \subset X$ is a hyperplane and $q = \deg(F_i)$. If A is an abelian variety and $m_A : A \to A$ the multiplication map by an integer $m \neq 0$, then $m_A^*H \sim m^2H$ and hence m_A is polarized; here $H = L + (-1)^*L$ with L an ample divisor, or H is any ample divisor with $(-1)^*H \sim H$. One can also construct polarized endomorphisms on quotients of $\mathbb{P}^n$ or A. So there are many examples of polarized endomorphisms f. See [16] for the many conjectures on such f.

In [11], it is proved that a normal variety X with a non-isomorphic polarized endomorphism f either has only canonical singularities with $K_X \sim_{\mathbb{Q}} 0$ (and further is a quotient of an abelian variety when $\dim X \leq 3$), or is uniruled so that f descends to a polarized endomorphism f_Y of the non-uniruled base variety Y (so $K_Y \sim_{\mathbb{Q}} 0$) of a specially chosen maximal rationally connected fibration $X \dashrightarrow Y$. By the induction on dimension and since Y has a dense set of f_Y-periodic points $y_0, y_1, \ldots$ (cf. [2, Theorem 5.1]), the study of polarized endomorphisms is then reduced to that of rationally connected varieties Γ_{y_i} as fibres of the graph $\Gamma = \Gamma(X/Y)$ (cf. [11, Remark 4.3]).

The study of non-isomorphic endomorphisms of singular varieties (like Γ_{y_i} above) is very important from the dynamics point of view, but is very hard

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even in dimension two and especially for rational surfaces; see [9] (about 150 pages).

We consider polarized endomorphisms of rationally connected varieties (or more generally of uniruled varieties) of dimension ≥ 3. Theorem 1.1 – 1.8 below are our main results.

Theorem 1.1. *Let X be a $\mathbb{Q}$-factorial threefold having only terminal singularities and a polarized endomorphism of degree $q^3 > 1$. Suppose that X is rationally connected. Then we have :*

1. *There is an $s > 0$ such that $(f^s)^*_{|N^1(X)} = q^s \,\mathrm{id}$.*
2. *Either X is rational, or $-K_X$ is big.*
3. *There are only finitely many irreducible divisors $M_i \subset X$ with the Iitaka D-dimension $\kappa(X, M_i) = 0$.*

Theorem 1.1 (3) apparently does not hold on an abelian variety A with a subtorus of codimension one, though the multiplication map m_A is polarized as mentioned above. Neither it holds for $X = S \times \mathbb{P}^1$, where S is a rational surface with infinitely many (-1)-curves (the blowup of nine general points of $\mathbb{P}^2$ is such S as observed by Nagata).

Theorem 1.1 (1) above strengthens (in our situation) Serre's result [12] on a conjecture of Weil (in the projective case): **(Serre)** If f is a polarized endomorphism of degree $q^{\dim X} > 1$ of a smooth variety X then every eigenvalue of $f^*|N^1(X)$ has the same modulus q.

The proof of Theorem 1.2 below is conceptually done. In a recent paper [15], we have removed the polarizedness assumption in Theorem 1.2.

Theorem 1.2. *Let X be a smooth Fano threefold with a polarized endomorphism of degree > 1. Then X is rational.*

A klt $\mathbb{Q}$-Fano variety has only finitely many extremal rays. A similar phenomenon occurs in the quasi-polarized case.

Theorem 1.3. *Let X be a $\mathbb{Q}$-factorial rationally connected threefold having only Gorenstein terminal singularities and a quasi-polarized endomorphism of degree > 1. Then X has only finitely many K_X-negative extremal rays.*

We expect a possible application of Theorem 1.4 below (see Theorem 1.7 for a more detailed version) to the Dynamic Manin-Mumford conjecture for (X, f) formulated by S. -W. Zhang in [16, Conjecture 1.2.1]. This conjecture for (X, f) is essentially equivalent to that for (X_r, g_r) because f^{-1}, as seen in Theorem 1.7, preserves the maximal subset of X where the birational map $X \cdots\rightarrow X_r$ is not holomorphic.

Further, X_r is better to be dealt with because it has a fibration structure preserved by g_r. *The existence of such a fibration $\pi : X_r \to Y$ is guaranteed* when X is uniruled by the recent development in MMP.

Theorem 1.4. *Let X be a $\mathbb{Q}$-factorial n-fold, with $n \in \{3, 4\}$, having only log terminal singularities and a polarized endomorphism f of degree $q^n > 1$.*

Let $X = X_0 \dashrightarrow X_1 \cdots \dashrightarrow X_r$ be a composition of divisorial contractions and flips. Replacing f by its positive power, we have:

(1) *The dominant rational maps $g_i : X_i \dashrightarrow X_i$ $(0 \le i \le r)$ (with $g_0 = f$) induced from f, are all holomorphic.*

(2) *Let $\pi : X_r \to Y$ be an extremal contraction with $\dim Y \le 2$. Then g_r is polarized and it descends to a polarized endomorphism $h : Y \to Y$ of degree $q^{\dim Y}$ with $\pi \circ g_r = h \circ \pi$.*

The claim in the abstract about the building blocks of polarized endomorphisms, is justified by the remark below.

Remark 1.5.

(1) The Y in Theorem 1.4 is $\mathbb{Q}$-factorial and has at worst log terminal singularities.

(2) Suppose that the X in Theorem 1.4 is rationally connected. Then Y is also rationally connected. Suppose further that X has at worst terminal singularities and $(\dim X, \dim Y) = (3, 2)$. Then Y has at worst Du Val singularities by [8, Theorem 1.2.7]. So there is a composition $Y \to \hat{Y}$ of divisorial contractions and an extremal contraction $\hat{Y} \to B$ such that either $\dim B = 0$ and $\hat{Y}$ is a Du Val del Pezzo surface of Picard number 1, or $\dim B = 1$ and $\hat{Y} \to B \cong \mathbb{P}^1$ is a $\mathbb{P}^1$-fibration with all fibres irreducible. After replacing f by its power, h descends to polarized endomorphisms $\hat{h} : \hat{Y} \to \hat{Y}$, and $k : B \to B$ (of degree $q^{\dim B}$); see Theorems 1.6.

(3) By [2, Theorem 5.1], there are dense subsets $Y_0 \subset Y$ (for the Y in Theorem 1.4) and $B_0 \subset B$ (when $\dim B = 1$) such that for every $y \in Y_0$ (resp. $b \in B_0$) and for some $r(y) > 0$ (resp. $r(b) > 0$), $g^{r(y)}|W_y$ (resp. $\hat{h}^{r(b)}|\hat{Y}_b$) is a well-defined polarized endomorphism of the Fano fibre.

We remark that Noboru Nakayama has produced many examples of polarized f on abelian surfaces which are not scalar. The result below shows that this happens only on abelian surfaces and their quotients.

Theorem 1.6. *Let X be a normal projective surface. Suppose that $f : X \to X$ is an endomorphism such that $f^*P \equiv qP$ for some $q > 1$ and some big Weil $\mathbb{Q}$-divisor P. Then we have:*

(1) *f is polarized of degree q^2.*

(2) *There is an $s > 0$ such that $(f^s)^*|\mathrm{Weil}(X) = q^s\,\mathrm{id}$ unless X is Q-abelian with $\mathrm{rank}\mathrm{Weil}(X) \in \{3, 4\}$.*

More generally, we prove the two theorems below. Theorem 1.7 below includes Theorem 1.4 as a special case.

Theorem 1.7. *Let X be a $\mathbb{Q}$-factorial n-fold, with $n \in \{3, 4\}$, having only log terminal singularities and a polarized endomorphism f of degree $q^n > 1$. Let $X = X_0 \dashrightarrow X_1 \cdots \dashrightarrow X_r$ be a composition of divisorial contractions and flips. Replacing f by its positive power,* (I) *and* (II) *hold:*

(I) *The dominant rational maps $g_i : X_i \dashrightarrow X_i$ $(0 \le i \le r)$ (with $g_0 = f$) induced from f, are all holomorphic. Further, g_i^{-1} preserves* *each irreducible component of the exceptional locus of* $X_i \to X_{i+1}$ *(when it is divisorial) or of the flipping contraction* $X_i \to Z_i$ *(when* $X_i \cdots\to X_{i+1} = X_i^+$ *is a flip).*

(II) *Let* $\pi : W = X_r \to Y$ *be the contraction of a* K_W*-negative extremal ray* $\mathbb{R}_{\ge 0}[C]$*, with* $\dim Y \le n-1$*. Then* $g := g_r$ *descends to a surjective endomorphism* $h : Y \to Y$ *of degree* $q^{\dim Y}$ *such that*

$$\pi \circ g = h \circ \pi.$$

For all $0 \le i \le r$*, all eigenvalues of* $g_i^*|N^1(X_i)$ *and* $h^*|N^1(Y)$ *are of modulus* q*; there are big line bundles* H_{X_i} *and* H_Y *satisfying*

$$g_i^* H_{X_i} \sim q H_{X_i}, \quad h^* H_Y \sim q H_Y.$$

Suppose further that either $\dim Y \le 2$ *or* $\rho(Y) = 1$*. Then* H_W *and* H_Y *can be chosen to be ample and* g *and* h *are polarized.*

The contraction π below exists by the MMP for threefolds.

Theorem 1.8. *Let* X *be a* $\mathbb{Q}$*-factorial rationally connected threefold having at worst terminal singularities and a polarized endomorphism of degree* $>$ 1*. Let* $X \cdots\to W$ *be a composition of divisorial contractions and flips, and* $\pi : W \to Y$ *an extremal contraction of non-birational type. Suppose either* $\dim Y \ge 1$*, or* $\dim Y = 0$ *and* W *is smooth. Then* X *is rational.*

The difficulty 1.9. In Theorem 1.4, if $X \to X_1$ is a divisorial contraction, one can descend a polarized endomorphism f on X to an one on X_1, but the latter may not be polarized any more because the pushfoward of a nef divisor may not be nef in dimension ≥ 3 (the first difficulty). If $X \cdots\to X_1$ is a flip, then in order to descend f on X to some holomorphic f_1 on X_1, one has to show that a power of f preserves the centre of the flipping contraction (the second difficulty). The second difficulty is taken care by a key lemma where the polarizedness is essentially used.

The question below is the generalization of Theorem 1.2 and the famous conjecture: every smooth Fano n-fold of *Picard number one* with a non-isomorphic surjective endomorophism, is $\mathbb{P}^n$ (for its affirmative solution when $n = 3$, see Amerik-Rovinsky-Van de Ven [1] and Hwang-Mok [4]).

Question 1.10. *Let* X *be a smooth Fano* n*-fold with a non-isomorphic polarized endomorphism. Is* X *rational ?*

For the recent development on endomorphisms of algebraic varieties, we refer to Amerik-Rovinsky-Van de Ven [1], Fujimoto-Nakayama [3], Hwang-Mok[4], S. -W. Zhang [16], as well as [10], [13].

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