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TOWARD RESOLUTION OF SINGULARITIES FOR ARBITRARY CHARACTERISTICS

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1. Introduction

We work over an algebraically closed field $k = \bar{k}$.

Our theme is resolution of singularities for a variety X/k. As is well known, resolution of singularities exist in the following cases:

- $\operatorname{char} k = 0, \forall d = \dim X$
  - '64 Hironaka,
  - '89 Villamayor (simplification)
  - '91 Bierstone-Milman (simplification)
- $\operatorname{char} k = p > 0, \forall d \leq 3$
  - '66 Abhyankar ($\forall p \geq 3! = 6$),
  - '07 Cossart-Piltant ($d = 3,\ p = 2, 3, 5$)

and it is *open* problem for the case of $\operatorname{char} k = p > 0$ and $\forall d \geq 4$.

Our goal is to give constructive proof for the existence of resolution of singularities for the case of $\forall p = \operatorname{char} k$ and $\forall d = \dim X$.

We introduced the **Idealistic Filtration Program (IFP)** for this goal. Here we present the introduction of the idea of IFP.

This is joint work with Kenji Matsuki.

2. Review for $\operatorname{char} k = 0$ approach

We review briefly the known algorithm for resolution in characteristic 0, after Villamayor and Bierstone-Milman. We only deal with the case with *no exceptional divisors*, for simplicity.

One of major approaches for resolution of singularities is to give *embedded resolution*. Namely, for pair $(X \subset M)$, where M is a nonsingular variety and X is a closed subset of M, we construct

$$f : M^{\sim} \to M : \text{ sequence of blowups along nonsingular centers}$$

such that

(1) f is isomorphic over $M \setminus X$
(2) $f^{-1}(X)$ is a simple normal crossing divisor.

It is well known that the existence of resolution of singularities is deduced from the existence of "canonical" embedded resolution.

In the known approach for resolution in characteristic 0, they construct embedded resolution along the following strategy:

- For each closed point $P \in M$, attach the invariant inv_P.

- Blowup along the maximal locus of invariant, and see the decrease of invariant after blowup.

The invariant inv_P is of the form

$$\mathrm{inv}_P = (\mu_0, \mu_1, \ldots, \mu_l, \infty) \quad (\mu_i \in \mathbb{Q}),$$

and μ_i's are defined inductively as follows:

(0) Put $R_0 = \mathcal{O}_{M,P}$ and $b_0 = 1$. Let $I_0 \subset R_0$ be the defining ideal of X at $P \in M$. We regard the triplet $\boxed{(I_0 \subset R_0, b_0)}$ as the *initial data*. We denote the multiplicity at P by ord_P and define $\boxed{\mu_0 = \mathrm{ord}_P(I_0)/b_0}$.

(1) We denote the set of all derivations $\partial\colon R \to R$ over k as $\mathrm{Der}(R)$. For an ideal $J \subset R$, we define an ideal $D(J) \subset R$ as

$$DJ = J + (\partial g \mid g \in J,\ \partial \in \mathrm{Der}(R)).$$

By definition of $\mathrm{ord}_P(I_0)$, we can find $f_0 \in D^{\mathrm{ord}_P(I_0)-1}(I_0)$ with $\mathrm{ord}_P(f_0) = 1$. Fix one such $f_0 \in R$ and put

$$R_1 = R_0/f_0R_0,\ b_1 = \mathrm{ord}_P(I_0)!,\ I_1 = \sum_{0 \le i < \mathrm{ord}_P(I_0)} \left(D^{\mathrm{ord}_P(I_0)-i}(I_0)\right)^{b_1/i}.$$

Now we obtain new data $\boxed{(I_1 \subset R_1, b_1)}$. We define $\boxed{\mu_1 = \mathrm{ord}_P(I_1)/b_1}$.

(2) Do the same routine as in (1). Namely, take $f_1 \in D^{\mathrm{ord}_P(I_1)-1}(I_1)$ with $\mathrm{ord}_P(f_1) = 1$, and put

$$R_2 = R_1/f_1R_1,\ b_2 = \mathrm{ord}_P(I_1)!,\ I_2 = \sum_{0 \le i < \mathrm{ord}_P(I_1)} \left(D^{\mathrm{ord}_P(I_1)-i}(I_1)\right)^{b_2/i}.$$

We have data $\boxed{(I_2 \subset R_2, b_2)}$ and define $\boxed{\mu_2 = \mathrm{ord}_P(I_2)/b_2}$.

(3) Repeat this procedure until $I_{l+1} = (0)$.

Remark.

(1) The procedure presented above *does* depend on the choice of f_i's. Nevertheless, μ_i's *do not* depend on this ambiguity, and thus invariant is well-defined.

(2) $H = V(f_0)$ is called a *hypersurface of maximal contact* (abbreviated simply as "maximal contact" in the rest of this article) of X at P. A maximal contact is a kind of nonsingular local hypersurface, but we do not give here its precise definition. One of the feature of a maximal contact is the following:

$$\{Q \in M \mid \mathrm{ord}_Q(I_0) \ge \mathrm{ord}_P(I_0)\} \subset H \quad \text{near } P.$$

The data $(I_1 \subset R_1, b_1)$ is regarded as the information of the left hand side in the above equation. In fact, we have

$$\{Q \in M \mid \mathrm{ord}_Q(I_0) \ge \mathrm{ord}_P(I_0)\} = \{Q \in H \mid \mathrm{ord}_Q(I_1) \ge b_1\},$$

which yields the scheme for "induction on dimension" in characteristic 0 case.

In the case of positive characteristic, it is known that a maximal contact *does not* exist in general. This is the main hurdle when we try to apply the known algorithm to the case of positive characteristic.

In IFP, we generalize the object. In the known algorithm, main object is the pair (I, b) with an ideal I and rational number b. We generalize this notion and define idealistic filtration $\mathbb{I}$, introduced in the next section. By analyzing this idealistic filtration algebraically, we can find the substitute of maximal contact, called LGS.

	known case	IFP case
object	pair (I, b)	idealistic filtration $\mathbb{I}$
local hypersurface	maximal contact	LGS of $\mathbb{I}$

3. Idealistic filtration

We introduce the idealistic filtration, the main object of IFP.

Let R be a k-algebra. For a subset $\mathbb{J} \subset R \times \mathbb{R}$ of $R \times \mathbb{R}$, we denote the level a set of $\mathbb{J}$ as $\mathbb{J}_a$. Namely,

$$\mathbb{J}_a = \{f \in R \mid (f, a) \in \mathbb{J}\}.$$

Definition. A subset $\mathbb{I} \subset R \times \mathbb{R}$ is called an *idealistic filtration* on R if $\mathbb{I}$ satisfies the following conditions:

(1) $\mathbb{I}_0 = R$.
(2) $\mathbb{I}_a$ is an ideal of R.
(3) $\mathbb{I}_a \mathbb{I}_b \subset \mathbb{I}_{a+b}$.
(4) $\mathbb{I}_a \supset \mathbb{I}_b \quad (a \leq b)$.

Remark. We interpret "$f \in \mathbb{I}_a$" as the information "$\mathrm{ord}_P(f) \geq a$".

Definition. For a subset $\mathbb{J} \subset R \times \mathbb{R}$, We define the support of $\mathbb{J}$ as

$$\mathrm{Supp}(\mathbb{J}) = \left\{ P \in \mathrm{maxSpec}(R) \mid \inf_{a>0} \frac{\mathrm{ord}_P(\mathbb{J}_a)}{a} \geq 1 \right\}$$

Definition. For a subset $\mathbb{J} \subset R \times \mathbb{R}$, the minimum idealistic filtration containing $\mathbb{J}$ is called the idealistic filtration *generated by* $\mathbb{J}$ and denoted as $\mathrm{G}(\mathbb{J})$.

We introduce the saturation of idealistic filtration to obtain much information for resolution problem.

Definition. Let $\mathbb{I}$ be an idealistic filtration on R. We denote the set of differential operators of degree $\leq t$ on R over k as $\mathrm{Diff}_{\leq t}(R/k)$.

We say $\mathbb{I}$ is $\mathfrak{D}$*-saturated* if the following condition holds:

$$\partial(\mathbb{I}_a) \subset \mathbb{I}_{a-t} \quad \text{for any } t \in \mathbb{Z}_{\geq 0},\ \partial \in \mathrm{Diff}_{\leq t}(R/k) \text{ and } a \in \mathbb{R}.$$

The minimum $\mathfrak{D}$-saturated idealistic filtration containing $\mathbb{I}$ is called the $\mathfrak{D}$*-saturation* of $\mathbb{I}$, and denoted as $\mathfrak{D}(\mathbb{I})$.

Remark. "A differential operator ∂ of degree $\leq t$ on R over k" is a k-linear map $\partial\colon R \to R$ characterized by "generalized Leibnitz rule"

$$\sum_{T \subset S} (-1)^{\#T} F_{S\setminus T}\, \partial(F_T) = 0,$$

where $S = \{0, 1, \ldots, t\}$, $F_I = \prod_{i \in I} f_i$ and f_i's are arbitrary elements in R. One can find in EGA IV §16 more detailed account.

From now on, replacing $\mathbb{I}$ by $\mathfrak{D}(\mathbb{I})$ if necessary, we always deal with only $\mathfrak{D}$-saturated idealistic filtrations. In the next section, we analyze $\mathfrak{D}$-saturated idealistic filtration and give the definition of LGS.

4. Leading generator system

We introduce the notion of LGS, substitute of maximal contact in any characteristics.

We restrict our attention to the local situation. Thus, in the rest of this article, we assume the following conditions unless specified:

- $R = (R, \mathfrak{m}) = O_{M,P}$ is a local ring at $P \in M$, where M is a nonsingular variety and $P \in M$ is a closed point.
- $\mathbb{I}$ is a $\mathfrak{D}$-saturated idealistic filtration on R.
- We assume $\mu(\mathbb{I}) \geq 1$, where $\mu(\mathbb{I}) = \inf_{a \in \mathbb{R}_{>0}} \dfrac{\mathrm{ord}_P(\mathbb{I}_a)}{a}$. In other words, we assume $P \in \mathrm{Supp}(\mathbb{I})$.

First we introduce the leading algebra $\mathrm{L}(\mathbb{I})$.

Definition. For $n \in \mathbb{Z}_{\geq 0}$, let $\pi_n : R \to R/\mathfrak{m}^{n+1}$ be the natural projection. Since $\mu(\mathrm{I}) \geq 1$, we have $\pi_n(\mathbb{I}_n) \subset \mathfrak{m}^n$. We define the *leading algebra* $\mathrm{L}(\mathbb{I})$ of $\mathbb{I}$ as

$$\mathrm{L}(\mathbb{I}) = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} \pi_n(\mathbb{I}_n) \subset \mathrm{Gr}(R) = \bigoplus \mathfrak{m}^n/\mathfrak{m}^{n+1}.$$

In our setting, it is clear that the graded ring $\mathrm{Gr}(\mathbb{I})$ is isomorphic to a polynomial ring over k, i.e. $\mathrm{Gr}(R) \cong k[Y]$. Therefore $\mathrm{L}(\mathbb{I})$ corresponds to a graded k-subalgebra $L \subset k[Y]$. Since $\mathbb{I}$ is $\mathfrak{D}$-saturated, we also see that L is stable under differential operations. That is, we see

$$\partial_{Y^I} L \subset L \quad (\forall I\text{: multi-index}),$$

where ∂_{Y^I} is defined by the formula $\partial_{Y^I} Y^J = \binom{J}{I} Y^{J-I}$.

Now we face the following question:

> Let $L \subset k[Y]$ be a graded k-subalgebra which is stable under differential operations. What can we say on such L?

In fact, in characteristic 0, L is generated by homogeneous part $S_1 \subset S$ of degree 1. Moreover, if $P \in X \subset M$ and $\mathbb{I} = \mathfrak{D}(\mathrm{G}(I_X \times \{\mathrm{ord}_P(I_X)\}))$, non-zero elements of S_1 correspond to maximal contact of X. It is important nature of maximal contact if one try to define invariant, since it is necessary to eliminate $\mathrm{ord}_P(I_X)$ part to detect higher order $\mu_1, \mu_2, \ldots$. Therefore, the substitute of maximal contact should correspond to generators of $\mathrm{L}(\mathbb{I})$ as k-algebra.

In positive characteristic case, we have the following result:

Proposition (Hironaka-Oda). *Assume* $\mathrm{char}\, k = p > 0$. *Let* $L \subset k[Y]$ *be a graded* k-*subalgebra stable under differential operations. Then, there exist integers* $N, e_1, \ldots, e_N \in \mathbb{Z}_{\geq 0}$ *and homogeneous elements* $f_1, \ldots, f_N \in k[Y]_1$ *of degree* 1 *such that* $f_1, \ldots, f_N$ *are* k-*linearly independent and* $\{f_1^{p^{e_1}}, \ldots, f_N^{p^{e_N}}\}$ *generates* L *as a* k-*algebra.*

Remark. As stated above, generators of L live in several degree p-th power parts. In characteristic 0 case, we should regard $p = \infty$. Then, all elements of degree p^i with $i > 0$ disappear, and generators only appear at degree $p^0 = 1$ part.

By virtue of the above proposition, we define LGS.

Definition. The representative $\mathbb{H} \subset \mathbb{I}$ of generators of $\mathrm{L}(\mathbb{I})$, in the shape of above proposition, is called a *leading generator system (LGS)* of $\mathbb{I}$. By definition,

$$\mathbb{H} = \{(h_i, p^{e_i}) \mid 1 \leq i \leq N\} \subset \mathbb{I}, \quad h_i = f_i^{p^{e_i}} + (\text{higher order part})$$

Remark.
(1) Note that LGS is not unique.
(2) If $e_i > 0$, h_i defines a singular local hypersurface. This is big difference to the original maximal contact.

5. Paired invariants

We introduce invariants σ and $\mu^{\sim}$, which are necessary to translate the known algorithm to IFP.

5.1. **Framework.** In the known algorithm for characteristic 0 case, we repeat the procedure of "restrict data to maximal contact H (= low dimensional ambient space)" and "estimating order on H".

object	(I_0, b_0)	(I_1, b_1)	$\cdots$	(I_t, b_t)	$(0, b_{t+1})$
ambient space	M	$\supset H_1$	$\cdots$	$\supset H_t$	$\supset H_{t+1}$
(higher) order	μ_0	μ_1	$\cdots$	μ_t	∞

Initial data is a pair $(I_0, b_0) = (I_X, 1)$ on M, and invariant is

$$\text{inv}_P = (\mu_0, \mu_1, \ldots, \mu_t, \infty).$$

In IFP, we cannot restrict data to low dimensional space since LGS may give singular local hypersurfaces. Therefore, we continue to stay the *same* (original) ambient space. Instead of restriction, we

- *enlarge* I and *enlarge* its LGS, and
- estimate order *modulo LGS*.

object	$\mathbb{I}_0$	$\subset \mathbb{I}_1$	$\cdots$	$\subset \mathbb{I}_t$	$\subset \mathbb{I}_{t+1}$
ambient space	M	M	$\cdots$	M	M
(higher) order	$(\sigma_0, \mu_0^{\sim})$	$\cdots$	$\cdots$	$(\sigma_t, \mu_t^{\sim})$	(σ_{t+1}, ∞)

Initial data is an idealistic filtration $\mathbb{I}_0 = G(I_X \times \{1\})$ on M, and invariant is

$$\text{inv}_P = ((\sigma_0, \mu_0^{\sim}), \ldots, (\sigma_t, \mu_t^{\sim}), (\sigma_{t+1}, \infty)).$$

By this translation, we need 2 new invariants σ and $\mu^{\sim}$ introduced below:

5.2. **Definitions.** Settings and notations are same to the ones in §4. We denote an LGS of $\mathbb{I}$ as

$$\mathbb{H} = \{(h_i, p^{e_i}) \mid 1 \le i \le N\}.$$

Definition ($\sigma \leftrightarrow$ "dimension of ambient space".).
$\sigma(\mathbb{I})$ is defined as an infinite sequence

$$\sigma(\mathbb{I}) = (\sigma_0, \sigma_1, \ldots) \in \mathbb{Z}_{\ge 0}^{\infty},$$

where each σ_i is defined by the formula $\sigma_e = \dim R - \#\{i \mid e_i \le e\}$.

In characteristic 0 case, $\sigma(\mathbb{I})$ is automatically a constant sequence. Namely,

$$\sigma(\mathbb{I}) = (\sigma_0, \sigma_0, \ldots), \quad \sigma_0 = \dim R - \text{"\# of maximal contact"},$$

Definition ($\mu^{\sim}$ ↔"order on low dimensional ambient space".).
We define the order modulo $\mathbb{H}$ of an ideal $J \subset R$ as

$$\mathrm{ord}_{\mathbb{H}}(J) = \sup\left\{ n \geq 0 \mid J \subset \mathfrak{m}^n + \sum_{i=1}^{N} Rh_i \right\}.$$

$\mu^{\sim}(\mathbb{I})$ is defined by

$$\mu^{\sim}(\mathbb{I}) = \inf_{a>0} \frac{\mathrm{ord}_{\mathbb{H}}(\mathbb{I}_a)}{a}.$$

In characteristic 0 case, all $H_i = V(h_i)$ and $\bigcap_i H_i$ are nonsingular. Thus $\mathrm{ord}_{\mathbb{H}}(J)$ is the order of J, estimated on $\bigcap_i H_i$.

Proposition. *$\sigma(\mathbb{I})$ and $\mu^{\sim}(\mathbb{I})$ are independent of the choice of $\mathbb{H}$.*

6. Results

As is already repeated, LGS may define singular local hypersurface, and it causes several serious problems. We present 2 results to overcome such problems.

In the known algorithm for characteristic 0 case, the maximum locus of invariant defines the center of next blowup. It is given by the intersection of maximal contact, which is automatically nonsingular due to the nonsingularity of maximal contact. In IFP case, the maximum locus of invariant corresponds to the support of last enlarged idealistic filtration $\mathbb{I}_{l+1}$, where $\mu^{\sim}(\mathbb{I}_{l+1}) = \infty$. The following theorem guarantees the nonsingularity of maximum locus of invariant in IFP.

Theorem (Nonsingulaity principle). *Settings and notations are same to the ones in §4. Assume $\mu^{\sim}(\mathbb{I}) = \infty$. Then, the following holds:*

(1) *$\mathbb{I}$ is generated by any LGS $\mathbb{H}$, i.e.* $\mathrm{G}(\mathbb{H}) = \mathbb{I}$.
(2) *There exist a part of regular system of parameters $\{g_i \mid 1 \leq i \leq N\} \subset R$ and non-negative integers $\{e_i \mid 1 \leq i \leq N\} \subset \mathbb{Z}_{\geq 0}$ such that $\{(g_i^{p^{e_i}}, p^{e_i}) \mid 1 \leq i \leq N\}$ is an LGS of $\mathbb{I}$.*
Especially, $\mathrm{Supp}(\mathbb{I})$ *is a germ of nonsingular variety at $P \in M$.*

We explain only the last statement. By (1), we have $\mathrm{Supp}(\mathbb{I}) = \mathrm{Supp}(\mathbb{H})$. Choose $\mathbb{H}$ given in (2). Then, $\mathrm{Supp}(\mathbb{H}) = V(g_i \mid 1 \leq i \leq N)$, which is nonsingular at P.

On the nonsingular ambient space, multiplicity is upper semi-continuous. Therefore, so is the invariant in known characteristic 0 case. In IFP, as we use *the order modulo LGS*, we have to verify the upper semi-continuity of the invariant.

Theorem. *Let $M = \mathrm{Spec}\, R$ be a nonsingular affine variety over k, and $\mathbb{I}$ a $\mathfrak{D}$-saturated idealistic filtration on R. We denote $\mathbb{I}_P$ as the localization of $\mathbb{I}$ at $P \in M$. Then, the pair $(\sigma(\mathbb{I}_P), \mu^{\sim}(\mathbb{I}_P))$ with lexicographical order defines an upper semi-continuous function on* $\mathrm{maxSpec}\, R$.

References

[1] H. Kawanoue. Toward resolution of singularities over a field of positive characteristic. Part I. Foundation of the program: the language of the idealistic filtration. *Publ. Res. Inst. Math. Sci.*, 43(3):819–909, 2007.

[2] H. Kawanoue and K. Matsuki. Toward resolution of singularities over a field of positive characteristic (The Idealistic Filtration Program) Part II. Basic invariants associated to the idealistic filtration and their properties. (preprint, `http://arxiv.org/abs/math/0612008`)

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