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FINITE SUBGROUPS OF THE PLANAR CREMONA GROUPO

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ABSTRACT. In this talk we review some new results about classification of conjugacy classes of finite subgroups of the planar Cremona group.

1. INTRODUCTION

Let $\mathrm{Cr}_n(k)$ denote the Cremona group of birational automorphisms of the projective space $\mathbb{P}_k^n$ over a field k. From algebraic point of view

$$\mathrm{Cr}_n(k) = \mathrm{Aut}_k(k(t_1, \ldots, t_n)).$$

When $n = 1$, the group of $\mathrm{Cr}_1(k)$ is isomorphic to the linear algebraic group $\mathrm{PGL}_2(k)$. The description of its finite subgroups is well known. There is one conjugacy class of each group, and the groups are isomorphic to either a cyclic, or dihedral, or the group of symmetries of a platonic solid. We will be concerned with the case $n = 2$.

There are three different aspects of the theory depending on the field k.

(i) $k = \mathbb{C}$, the field of complex numbers;
(ii) k is any perfect field and groups are of order prime to the characteristic of k;
(iii) k is algebraically closed of characteristic dividing the order of the group.

In any case the classification of finite subgroups uses the following simple idea. For each finite subgroup $G \subset \mathrm{Cr}_2(k)$ one can find a smooth rational projective algebraic surface X such that G acts biregularly on X inducing the same action on the field of rational functions. Two subgroups are conjugate in $\mathrm{Cr}_2(k)$ if and only if the corresponding surfaces are birationally G-equivariantly isomorphic. Among all surfaces X which "regularize" the subgroup G one can choose minimal one in the sense that it does not allow a non-trivial birational G-equivariant morphism $X \to Y$. It follows from Mori's theory of minimal models that a minimal G-surface X belongs to one of the following classes of surfaces

(i) X has a structure of a conic bundle $f : X \to \mathbb{P}^1_k$ with $m \geq 0$ singular fibres,
(ii) X is a Del Pezzo surface of degree d with $\mathrm{Pic}(X)^G \cong \mathbb{Z}$.

So the problem in each case is reduced to the classification of finite subgroups of the automorphism groups of minimal G-surfaces as above and the classification of birational G-equivariant isomorphisms between minimal surfaces (which are decomposed in elementary links classified by V. Iskovskikh [5]).

2. The case of complex numbers

This is the most classical case, the theory originates from the work of Bertini who classified the conjugacy classes of involutions in $\mathrm{Cr}_2(\mathbb{C})$. We refer to the history and references to the modern work to our paper [4].

Theorem 1. *Each involution in* $\mathrm{Cr}_2(\mathbb{C})$ *is conjugate to either*

(i) *de Jonquières involution;*
(ii) *a Geiser involution;*
(iii) *a Bertini involution.*

Recall the definitions. A de Jonquières involution is defined (in its algebraic form) by the transformation $(x, y) \mapsto (x, \frac{F_{2g+1}(x)}{y})$, where $F(x)$ is a polynomial of degree $2g + 1$ without multiple roots.

A Geiser involution is defined geometrically as the deck transformation of the rational map of degree 2 given by the linear system of plane cubic curves through 7 general points in the plane. A Bertini involution is defined similarly by the linear system of plane curves of degree 6 with double points at 8 general points in the plane.

Already in this special case one sees the dramatic difference of conjugacy classes of finite subgroups of the Cremona group and a linear algebraic group. Namely, the set of conjugacy classes is infinite, and in fact can be parametrized by points of algebraic varieties.

One starts the classification from considering subgroups of a conic bundle. They belong to the class of de Jonquières transformations, i.e. transformations which can be algebraically given in the form $(x, y) \mapsto (x, \frac{a(x)y+b(x)}{c(x)y+d(x)}$, where $a(x, b(x), c(x), d(x)$ are rational functions in x. In geometric forms a de Jonquières transformation can be define as a birational transformation of the plane leaving invariant a pencil of lines.

Let $f : X \to \mathbb{P}^1$ be a conic bundle and $p_1, \ldots, p_m$ be the set of points over which the fibres X_{p_i} are reducible conics. Any finite subgroup G of $\mathrm{Aut}(X)$ fits into an exact sequence of groups

$$1 \to H \to G \to \bar{G} \to 1,$$

where $\bar{G} \subset \mathrm{Aut}(\mathbb{P}^1)$ is the image of G in its action on the base of the fibration. The group G also acts on the group $\mathrm{Pic}(X) \otimes \mathbb{Q}$ of X which is generated by K_X and components of fibres, taken one from each. A conic bundle G-surface X is called *exceptional* if the latter action is not trivial. One can describe such conic bundles explicitly. They are isomorphic to a hypersurface in $\mathbb{P}(1,1,g,g)$

$$F_{2g+2}(t_0,t_1) + t_2t_3 = 0,$$

where $F_{2g+2}(t_0,t_1)$ is a binary form of degree $2g$. The group of automorphisms of such a surface is easy to describe. It is isomorphic to the extension $N.P$, where P is the subgroup of $\mathrm{PGL}_2(\mathbb{C})$ leaving the binary form $F_{2g}(t_0,t_1)$ invariant, and $N = \mathbb{C} : 2$ is the group of matrices with determinant ± 1 leaving the binary form t_1t_2 invariant. This allows one to describe all finite subgroups of automorphism of X.

Assume that X is not an exceptional conic bundle. Then the group H leave each fibre F_{x_i} invariant and embeds injectively in the group $2^m := (\mathbb{Z}/2\mathbb{Z})^m$ via switching the components. Since H is a subgroup of the general fibre of $f : X \longrightarrow \mathbb{P}^1$, it is isomorphic to a subgroup of $\mathrm{PGL}_2(K)$, where K is the field of rational functions on the base. It is known that no finite subgroup of this group is isomorphic to a group 2^s with $s > 2$. This shows that $H \cong 2^s$ with $s = 1$ or 2.

The previous argument shows that G is either isomorphic to an extension $2.P$ or $2^2.P$, where P is a finite subgroup of $\mathrm{PGL}_2(\mathbb{C})$. In the first case, the fixed locus of the non-trivial element in 2 is a hyperelliptic curve of genus g ramified over the set $\Sigma = \{p_1,\ldots,p_m\}, m = 2g+2$, (or a rational, or elliptic curve if $g < 2$)), and P is its group of automorphisms. In the second case, the fixed locus of each non-trivial involution $\tau_i \in 2^2$ is a hyperelliptic curve of some genus p_i such that ramifies over a subset Σ_i of Σ of cardinality n_i such that Σ is partitioned into three subsets A, B, C with $\Sigma_1 = A+B, \Sigma_2 = B+C, \Sigma_3 = A+C$. An example of such surface X is the surface in $\mathbb{P}^1 \times \mathbb{P}^2$ given by the equation

$$a_0(t_0,t_1)z_0^2 + a_1(t_0,t_1)z_0^2 + a_2(t_0,t_1)z_0^2 = 0,$$

where a_0, a_1, a_2 are binary forms of some degree m.

Let us now pass to the case when G is realized on a Del Pezzo surface of degree d. Recall that d takes values between 1 and 9, any surface of degree $d < 8$ is isomorphic to the blow-up of $9-d$ distinct points in an "unnodal position" (e.g. no three points are collinear). When $n = 8$, X is isomorphic to either $\mathbb{P}^1 \times \mathbb{P}^1$ or to the blow-up of one point which is not minimal, and can be omitted from consideration. When $n = 8$, the surface X is isomorphic to the projective plane. The classification of the conjugacy classes of finite subgroups of $\mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$

and $\mathrm{Aut}(\mathbb{P}^2) \cong \mathrm{PGL}_3(\mathbb{C})$ is one-hundred years old. Assume now that $d < 8$. If $d = 7$, the surface X is not G-minimal since the proper inverse transform of the line joining the two points is a G-invariant (-1)-curve. For $d < 7$ one uses the representation of G in the group of orthogonal transformations of the Picard lattice generated by reflections in vectors $v \in \mathrm{Pic}(X)$ such that $v^2 = -2$. This is the notorious Weyl group of a root system of type E_{9-d}, where, by definition, E_3 is equal to type $A_2 \times A_1$, E_4 is equal to A_4, and E_5 is equal to D_5. The representation is always faithful except in the case $d = 6$, where the kernel consists of the group of projective transformations fixing the three points. Dealing with the case by case, it is possible to classify first the conjugacy classes of cyclic subgroups. Then one finds the corresponding surfaces and studies possible additional symmetries. In this way one achieves a complete classification of minimal Del Pezzo G-surfaces. The tables are given in [4].

The final step in the classification is to find out when two minimal Del Pezzo G-surfaces are birationally isomorphic. This is achieved by Iskovskikh's classification of elementary links birationally relating two Del Pezzo surfaces. For example, minimal Del Pezzo G-surfaces with $d \leq 3$ are rigid, in the sense that cannot be birationally isomorphic to other surfaces.

3. The case when k is a perfect field and $(\#G, \mathrm{char}(k)) = 1)$

Here the work has only began since, essentially, only the case of cyclic groups has been studied. Recall that in the case k is algebraically closed any cyclic group of order prime to characteristic can be realized by a subgroup of projective transformations. It is not anymore true if k is not algebraically closed. The relevant useful information about the field k is given by the following two numbers. From now on ℓ is a prime number different from $\mathrm{char}(k)$ and ζ_ℓ be the generator of the group of elements of order ℓ in $\bar{k}$.

Set

$$t_\ell = [k(\zeta_\ell) : k], \quad m_\ell = \sup\{d \geq 1 : \zeta_{\ell^d} \in k(\zeta_\ell)\}.$$

For any group A let $\ell^{\nu_l(A)}$ be the order of its Sylow ℓ-subgroup. The classification of finite subgroups of $\mathrm{PGL}_{n+1}(k) = \mathrm{Aut}(\mathbb{P}^n_k)$ is based on the following result [6].

Theorem 2. *Let A be a finite subgroup of $\mathrm{PGL}_{n+1}(k)$. For any $\ell > 2$,*

$$\nu_\ell(A) \leq \sum_{2 \leq s \leq n+1, t_\ell | s} (m_\ell + \nu_\ell(s)).$$

It follows that $\mathrm{PGL}_{n+1}(k)$ does not contain elements of prime order ℓ if $t_\ell \geq n+2$. For example, $\mathrm{Aut}(\mathbb{P}^2_{\mathbb{Q}})$ does not contain elements of order $\ell > 5$.

The first example of an automorphism of order 7 in $\mathrm{Cr}_{\mathbb{Q}}(2)$ was given by J.-P. Serre. It is based on the following idea.

First he uses the fact from [9] that an algebraic 2-dimensional torus over an arbitrary field k is a rational variety (as always over k). Then he proves the following.

Theorem 3. *Let T be an algebraic k-torus and A be a finite subgroup of $T(k)$. Then*

$$\nu_\ell(A) \leq m_\ell \left[\frac{\dim T}{\phi(t_\ell)}\right],$$

where ϕ is the Euler function. Assume $m_\ell < \infty$ (e.g. k is finitely generated over its prime subfield). For any $n \geq 1$ there exists an n-dimensional k-torus T and a finite subgroup A of $T(k)$ such that $\nu_\ell(A) = m_\ell\left[\frac{\dim T}{\phi(t_\ell)}\right]$.

Corollary 4. *A 2-dimensional k-torus T with $T(k)$ containing an element of prime order $\ell > 2$ exists if and only if t_ℓ takes values in the set $\{1, 2, 3, 4, 6\}$.*

We can realize a 2-dimensional k-torus T as an open subset of a Del Pezzo surface of degree 6 that has a structure of a toric k-surface.

The main result of our paper [3] is the following

Theorem 5. *Let k be a perfect field of characteristic $p \geq 0$. Then $\mathrm{Cr}_2(k)$ contains an element of prime order $\ell > 5$ not equal to p if and only if there exists a 2-dimensional algebraic k-torus T such that $T(k)$ contains an element of order ℓ.*

We will also prove the following uniqueness result.

Theorem 6. *Assume that k is of characteristic 0 and does not contain a primitive ℓ-th root of unity. Then $\mathrm{Cr}_2(k)$ does not contain elements of prime order $\ell > 7$ and all elements of order 7 in $\mathrm{Cr}_2(k)$ are conjugate to an automorphism of a Del Pezzo surface of degree 6.*

Using the description of minimal G-surfaces given in the previous section and the fact that a Del Pezzo surface of degree $d \geq 3$ embeds in $\mathbb{P}^d_k$ or admits a canonical double cover of the plane or of quadratic cone for $d = 2$ or $d = 1$, Serre uses his theorem 2 to estimate an order of any finite subgroup of $\mathrm{Cr}_2(k)$ of order prime to $\mathrm{char}(k)$ (see [7].

Theorem 7. *Let* $M(k,\ell) = 2m+3$ *if* $\ell = 2$,

$$M(k,\ell) = \begin{cases} 2m & \text{if } t_\ell = 1,2, \ell > 3, \\ m & \text{if } t_\ell = 3,4, \ell > 3, \\ 0 & \text{if } t_\ell = 5, \ell > 6 \\ 4 & \text{if } t_\ell = 1,2, \ell > 3 \text{or } \ell = 3, t_\ell = m_\ell = 1, \\ 2m+1 & \text{otherwise.} \end{cases}$$

For any finite subgroup A *of* $\mathrm{Cr}_2(k)$

$$\nu_\ell \leq M(k,\ell).$$

Moreover, $M(k,\ell)$ *is the upper bound of the* $\nu_\ell(A)$.

4. Wild subgroups

Here we discuss finite subgroups of $\mathrm{Cr}_2(k)$ of order divisible by $p = \mathrm{char}(k)$. Again the work is still in progress. We study only cyclic groups of order p^s. We will also describe conjugacy classes of elements of order p^2 over algebraically closed field of characteristic $p > 0$.

Using the Jordan form it is easy to prove the following

Lemma 8. *For any element of order* p^s *in* $\mathrm{Aut}(\mathbb{P}^r_k)$ *we have* $s < 1 + \log_p(r+1)$.

For example, when $r = 1$, no elements of order $p^s, s \geq 2$, exist in $\mathrm{Aut}(\mathbb{P}^1_k)$. This easily implies that

Theorem 9. *Let* $f : X \to \mathbb{P}^1_k$ *be a conic bundle and* σ *be an automorphism of* X *of order* p^s *preserving the conic bundle. Then* $s \leq 2$.

A closer look at elements of order p^2 shows that a minimal automorphism of order p^2 of a conic bundle $X \to \mathbb{P}^1_k$ exist only when $p = 2$.

Next we consider the case of Del Pezzo surfaces. For example, if $d = 9$, $X = \mathbb{P}^2_k$, by Lemma 8 we get $s \leq 2$. All elements of order p^2 are conjugate in $\mathrm{Aut}(\mathbb{P}^2_k)$.

If $d = 8$, then $X \cong \mathbb{P}^1_k \times \mathbb{P}^1_k$ because the ruled surface $\mathbf{F}_1$ is not σ-minimal. We know that $\mathrm{Aut}(\mathbf{F}_0)$ contains a subgroup of index 2 isomorphic to $\mathrm{Aut}(\mathbb{P}^1_k) \times \mathrm{Aut}(\mathbb{P}^1_k)$. Applying Lemma 8 we obtain $s = 1$ if $p \neq 2$, and $s \leq 2$ otherwise. The automorphism of X given in affine coordinates by $(x,y) \mapsto (y+1, x)$ is of order 4.

If $d = 7$, as we explained in section 1, the surface is not σ-minimal.

Assume $d = 6$. Then $\mathrm{Aut}(X)$ is isomorphic to the semi-direct product $T \rtimes G$, where $T \cong k^{*2}$ is a 2-dimensional torus and G is a dihedral group $D_{12} \cong (\mathbb{Z}/2\mathbb{Z}) \times S_3$. Since T does not contain elements of order

p and D_{12} does not contain elements of order $p^s, s > 1$, we obtain that the only possibility is $s = 1$ and $p = 2, 3$.

Assume $d = 5$. We know that $\mathrm{Aut}(X)$ acts faithfully on the Picard group of X of a Del Pezzo surface of degree ≤ 5. Via this action it becomes isomorphic to a subgroup of the Weyl group $W(A_4) \cong S_5$. Thus $s = 1$ unless $p = 2$ and $s = 2$. The group $W(A_4)$ acts on $K_X^{\perp} \cong \mathbb{Z}^4$ via its standard irreducible representation on $\{(a_1, \ldots, a_5) \in \mathbb{Z}^5 : a_1 + \ldots + a_5 = 0\}$. A cyclic permutation of order 4 has a fixed vector. This shows that X is not σ-minimal.

Starting from the cases $d \leq 4$, the arguments become a little more involved. The most difficult case is the case $d = 1$ and $p = 2$. We will give the details.

The linear system $|-2K_X|$ defines a degree 2 map $f : X \to Q$, where Q is a quadratic cone in $\mathbb{P}_k^3$. Again, since $-K_X$ is ample, f is a finite map, and arguing as in the previous case we see that the map is separable. The Galois group of the cover is generated by the Bertini involution. For any divisor D we have

$$D + \gamma^*(D) \sim 2(D \cdot K_X)K_X. \tag{1}$$

This shows that β^* acts as the minus identity on the lattice $K_X^{\perp}$. The lattice $K_X^{\perp}$ is isomorphic to the root lattice of type E_8. The involution β^* generates the center of the Weyl group $W(E_8)$.

The automorphism group $\mathrm{Aut}(X)$ is a subgroup of $W(E_8)$. Possible orders $p^s, s > 1$, of minimal automorphisms are 4 and 8 (see [4]).

So we assume $p = 2$. The linear system $|-K_X|$ has one base point p_0. Blowing it up we obtain a fibration $\pi : X' \to \mathbb{P}_k^1$ whose general fibre is an irreducible curve of arithmetic genus 1. Since $-K_X$ is ample, all fibres are irreducible, and this implies that a general fibre is an elliptic curve (see [1]). Let S_0 be the exceptional curve of the blow-up. It is a section of the elliptic fibration. We take it as the zero in the Mordell-Weil group of sections of π. The map $f : X \to Q$ extends to a degree 2 separable finite map $f' : X' \to \mathbf{F}_2$, where $\mathbf{F}_2$ is the minimal ruled surface with the exceptional section E satisfying $E^2 = -2$. Its branch curve is equal to the union of E and a curve B from the divisor class $3f + e$, where f is the class of a fibre and $e = [E]$. We have $f'^*(E) = 2S_0$. The elliptic fibration on X' is the pre-image of the ruling of $\mathbf{F}_2$. We know that $\tau = \sigma^2$ acts identically on the base of the elliptic fibration. Since it also leaves invariant the section S_0, it defines an automorphism of the generic fibre considered as an abelian curve with zero section defined by S_0. If $\tau^2 = 1$, then τ is the negation automorphism, hence defines the Bertini transformation of the projective plane. Its image in the Weyl group $W(E_8)$ generates the center. The group of automorphisms of an abelian curve in characteristic 2 is of order 2 if the absolute invariant of the curve is not equal to 0 or of order 24 otherwise. In the latter case it is isomorphic to $Q_8 \rtimes \mathbb{Z}/3$, where Q_8 is the quaternion group with the center generated by the negation automorphism (see [8], Appendix A). Thus $\tau^4 = 1$ and the Weierstrass model of the generic fibre is

$$y^2 + a_3 y + x^3 + a_4 x + a_6 = 0.$$

In global terms the Weierstrass model of the elliptic fibration $\pi : X' \to \mathbb{P}^1_k$ is a surface in $\mathbb{P}(1,1,2,3)$ given by the equation

$$y^2 + a_3(u,v)y + x^3 + a_4(u,v)x + a_6(u,v),$$

where a_i are binary forms of degree i. It is obtained by blowing down the section S_0 to the point $(u,v,x,y) = (0,0,1,1)$ and is isomorphic to our Del Pezzo surface X. The image of the branch curve B is given by the equation $a_3(u,v) = 0$, i.e. B is equal to the pre-image of an effective divisor of degree 3 on the base plus the section S_0. Since a general point of B is a 2-torsion point of a general fibre, we see that all nonsingular fibres of the elliptic fibration are supersingular elliptic curves (i.e. have no non-trivial 2-torsion points). An automorphism of order 4 of X is defined by

$$(u,v,x,y) \mapsto (u,v,x+s(u,v)^2, y+s(u,v)x+t(u,v)),$$

where s is binary forms of degree 1 and t is a binary form of degree 3 satisfying

$$a_3 = s^3,\ t^2 + a_3 t + s^6 + a_4 s^2 = 0. \tag{2}$$

In particular, it shows that a_3 must be a cube, so we can change the coordinates (u,v) to assume that $s = u, a_3 = u^3$. The second equality in (2) tells that t is divisible by u, so we can write it as $t = uq$ for some binary form q of degree 2 satisfying $q^2 + u^2 q + u^4 + a_4 = 0$. Let α be a root of the equation $x^2 + x + 1 = 0$ and $b = q + \alpha u^2$. Then b satisfies $a_4 = b^2 + u^2 b$ and $t = ub + \alpha u^3$. Conversely, any surface in $\mathbb{P}(1,1,2,3)$ with equation

$$y^2 + u^3 y + x^3 + (b(u,v)^2 + u^2 b(u,v))x + a_6(u,v) = 0 \tag{3}$$

where b is a quadratic form in (u,v) and the coefficient at uv^5 in a_6 is not zero (this is equivalent to that the surface is nonsingular) is a Del Pezzo surface of degree 1 admitting an automorphism of order 4

$$\tau : (u,v,x,y) \mapsto (u,v,x+u^2, y+ux+ub+\alpha u^3).$$

Note that $\tau^2 : (u,v,x,y) \mapsto (u,v,x,y+u^3)$ coincides with the Bertini transformation.

Theorem 10. *Let X be a Del Pezzo surface* (3). *Then it does not admit an automorphism of order 8.*

Proof. Assume $\tau = \sigma^2$. Since σ leaves invariant $|-K_X|$, it fixes its unique base point, and lifts to an automorphism of the elliptic surface X' preserving the zero section S_0. Since the general fibre of the elliptic fibration $f : X' \to \mathbb{P}^1_k$ has no automorphism of order 8, the transformation σ acts nontrivially on the base of the fibration. Note that the fibration has only one singular fibre F_0 over $(u, v) = (0, 1)$. It is a cuspidal cubic. The transformation σ leaves this fibre invariant and hence acts on $\mathbb{P}^1_k$ by $(u, v) \mapsto (u, u + cv)$ for some $c \in k$. Since the restriction of σ to F_0 has at least two distinct fixed points: the cusp and the origin $F_0 \cap S_0$, it acts identically on F_0 and freely on its complement $X' \setminus F_0$.

Recall that X' is obtained by blowing up 9 points $p_1, \ldots, p_9$ in $\mathbb{P}^2_k$, the base points of a pencil of cubic curves. We may assume that X is the blow-up of the first 8 points, and the exceptional curve over p_9 is the zero section S_0. Let S be the exceptional curve over any other point. We know that $\beta = \sigma^4$ is the Bertini involution of X. Applying formula (1), we obtain that $S \cdot \beta(S) = 3$. Identifying $\beta(S)$ and S with their pre-images in X', we see that $\beta(S) + S = S_0$ in the Mordell-Weil group of sections of $\pi : X' \to \mathbb{P}^1_k$. Thus S and $\beta(S)$ meet at 2-torsion points of fibres. However, all nonsingular fibres of our fibration are supersingular elliptic curves, hence S and $\beta(S)$ can meet only at the singular fibre F_0. Let $Q \in F_0$ be the intersection point. The sections S and $\beta(S)$ are tangent to each other at Q with multiplicity 3. Now consider the orbit of the pair $(S, \beta(S))$ under the cyclic group $\langle\sigma\rangle$. It consists of 4 pairs

$$(S, \sigma^4(S)),\ (\sigma(S), \sigma^5(S)),\ (\sigma^2(S), \sigma^6(S)),\ (\sigma^3(S), \sigma^7(S)).$$

Let $D_i = \sigma^i(S) + \sigma^{i+4}(S)$, $i = 1, 2, 3, 4$. We have $D_1 + \ldots + D_4 \sim -8K_{X},$, hence for $i \neq j$ we have $D_i \cdot D_j = (64 - 16)/12 = 4$. Let $Y \to X$ be the blow-up of Q. Since Q is a double point of each D_i, the proper transform $\bar{D}_i$ of each D_i in Y has self-intersection 0 and consists of two smooth rational curves intersecting at one point with multiplicity 2. Moreover, we have $\bar{D}_i \cdot \bar{D}_j = 0$. Applying (1), we get $D_i \in |-2K_X|$. Since Q is a double point of D_i, we obtain $\bar{D}_i \in |-2K_Y|$. The linear system $|-2K_Y|$ defines a fibration $Y \to \mathbb{P}^1_k$ with a curve of arithmetic genus 1 as a general fibre (an elliptic or a quasi-elliptic fibration) and four singular fibres $\bar{D}_i$ of Kodaira's type III. The automorphism σ acts on the base of the fibration and the four special fibres form one orbit. But the action of σ on $\mathbb{P}^1_k$ is of order 2 and this gives us a contradiction. □

Remark 1. A computational proof of Theorem 7 was given by J.-P. Serre.

To summarize one can prove the following result (see [2]).

Theorem 11. *An element of order p^2 not conjugate to a projective transformation exists only if $p = 2$. Assume that k is algebraically closed. An element of order 4 is either conjugate to a projective transformation, or conjugate to an element realized by a minimal automorphism of a conic bundle, or a Del Pezzo surface of degree 1.*

For the completeness sake let us add that elements of order p not conjugate to a projective transformations occur for any p. They can be realized as automorphisms of conic bundles, and if $p = 2, 3, 5$ as automorphisms of Del Pezzo surfaces.

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