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SPIN CURVES AND SCORZA QUARTICS

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Contents

This is the joint work with Francesco Zucconi. We give some applications of 3-fold birational geometry to the study of even spin curves. Much explanation is taken from the Dolgachev-Kanev's paper [DK93] and our preprints [TZ08a] and [TZ08b].

1. Even spin curves

Let $\mathcal{H}$ be a smooth projective curve of genus g and θ a theta characteristic on $\mathcal{H}$, namely, $2\theta \sim K_{\mathcal{H}}$. A couple $(\mathcal{H}, \theta)$ is called a *spin curve* and *even* if so is $h^0(\mathcal{H}, \theta)$. Let $\mathcal{S}_g^+$ be the moduli space of even spin curves of genus g. It is known that $h^0(\mathcal{H}, \theta) = 0$ (called *ineffective theta characteristics*) for a general pair $(\mathcal{H}, \theta) \in \mathcal{S}_g^+$.

2. Scorza correspondence

The basic of our study is the following correspondence originally studied by G. Scorza.

Definition 2.0.1. Given an ineffective θ, $h^0(\mathcal{H}, \theta + x) = 1$ for every $x \in \mathcal{H}$ by the Riemann-Roch theorem, hence θ gives the correspondence $I_\theta \subset \mathcal{H} \times \mathcal{H}$ such that $(x, y) \in I_\theta$ if and only if y is in the support of the unique member of $|\theta + x|$. This is called *the Scorza correspondence.*

We denote by $I_\theta(x)$ the fiber of $I_\theta \to \mathcal{H}$ over x. In other words, $I_\theta(x)$ is the unique member of $|\theta + x|$.

We can easily verify the following properties of I_θ by the Riemann-Roch theorem, etc:

(a) $\theta = I_\theta(x) - x$ is (of course) independent of x,
(b) $h^0(\mathcal{H}, \theta + x) = 1$ for any $x \in \mathcal{H}$,
(c) I_θ is disjoint from the diagonal,
(d) I_θ is symmetric, and
(e) I_θ is a (g, g)-correspondence.

By [DK93, Lemma 7.2.1], conversely, for any reduced correspondence I' satisfying the above conditions, there exists a unique ineffective theta characteristic such that $I' = I_\theta$.

Here we mention two known applications of the Scorza correspondence:

Rationality of S_3^+. We learned the following by [DK93]. Let V be a 3-dimensional vector space and $\check{V}$ its dual. For a homogeneous form $G \in S^m\check{V}$ of degree m on V, we define the (first) polar $P_a(G)$ of G at $a \in \mathbb{P}(V)$ by $P_a(G) := \frac{1}{m}\sum a_i \frac{\partial G}{\partial x_i}$, where a_i and x_i are coordinates of a, and on V, respectively.

Let $F \in S^4\check{V}$ be a general ternary quartic form on V. Set

$$S^o(F) := \{a \in \mathbb{P}(V) \mid P_a(F) \text{ is projectively equivalent to the Fermat cubic}\}.$$

Then the closure $S(F) := \overline{S^o}(F)$ is again a smooth quartic curve, which is called the Clebsch covariant quartic of F. By taking the second polars of $S(F)$, we have the following correspondence:

$$T(F) := \{(a, b) \in S(F) \times S(F) \mid \operatorname{rank} P_{a,b}(S(F)) \leq 1\}. \tag{2.1}$$

For example, if $P_a(F) = \{x^3 + y^3 + z^3 = 0\}$, then $b = (1:0:0), (0:1:0)$ or $(0:0:1)$, thus $T(F)$ is a $(3,3)$-correspondence. In the end, $T(F)$ turns out to be the Scorza correspondence I_θ defined by a unique theta characteristic θ.

So we have the map $\mathrm{Sc}\colon \mathcal{M}_3^0 \to S_3^+$ such that $\mathrm{Sc}\colon [F=0] \mapsto [S(F), \theta]$ defined over the open set $\mathcal{M}_3^0 \subset \mathcal{M}_3$ where $S(F)$ is nonsingular. This association map was discovered by Scorza and is called the *Scorza map*. Scorza showed it is an injective birational map. Thus S_3^+ is rational since $\mathcal{M}_3$ is known to be rational by [Kat96] (see also [Boh]). The curve F corresponding to a couple $(S(F), \theta)$ is called the *Scorza quartic* of $(S(F), \theta)$. In other words, by setting $\mathcal{H} = S(F)$, F is the unique quartic such that if $(a, b) \in I_\theta (\subset \mathcal{H} \times \mathcal{H})$, then $\operatorname{rk} P_{a,b}(F) = 1$ holds.

Mukai's description of a Fano threefold. *A prime Fano threefold of genus* 12 is a smooth projective threefold A_{22} such that $-K_{A_{22}}$ is ample, the class of $-K_{A_{22}}$ generates $\operatorname{Pic} A_{22}$, and such that the genus $g(A_{22}) := \frac{(-K_{A_{22}})^3}{2} + 1 = 12$. Mukai found the description of such a Fano as a variety of power sums.

Definition 2.0.2. Let V be a $(v+1)$-dimensional vector space and let $F \in S^m \check{V}$ be a homogeneous forms of degree m on V. Set

$$\mathrm{VSP}\,(F, n)^o := \{([H_1], \dots, [H_n]) \mid H_1^m + \cdots + H_n^m = F\} \subset \mathrm{Hilb}^n \mathbb{P}(\check{V}).$$

The closure $\mathrm{VSP}\,(F, n) := \overline{\mathrm{VSP}\,(F, n)^o}$ is called the *varieties of power sums* of F.

Theorem 2.0.3 (S. Mukai). *Let $\{F_4 = 0\} \subset \mathbb{P}(V) = \mathbb{P}^2$ be a general plane quartic curve. Then*

(1) $\mathrm{VSP}(F_4, 6) \subset \mathrm{Hilb}^6 \check{\mathbb{P}}^2$ *is a general prime Fano threefold of genus* 12*; and conversely,*
(2) *every general prime Fano threefold of genus* 12 *is of this form.*

See [Muk92] and [Muk04]. Mukai observed the following:

(a) The Hilbert scheme of lines on A_{22} is isomorphic to a smooth plane quartic $\mathcal{H}_1$ and the correspondence on $\mathcal{H}_1 \times \mathcal{H}_1$ defined by intersections of lines on A_{22} gives an ineffective theta characteristic θ on $\mathcal{H}_1$. More precisely, θ is constructed so that the Scorza correspondence I_θ is equal to

$$\{([l], [m]) \in \mathcal{H}_1 \times \mathcal{H}_1 \mid l \cap m \neq \emptyset, l \neq m\}.$$

By the result of Scorza recalled above, the Scorza quartic $\{F_4 = 0\}$ is associated to the pair $(\mathcal{H}_1, \theta)$ in the same ambient plane as the canonically embedded $\mathcal{H}_1$. Theorem 2.0.3 (2) claims that X is recovered as $\mathrm{VSP}\,(F_4, 6)$. (1) follows from (2) since the number of the moduli of prime Fano threefolds of genus 12 is equal to $\dim \mathcal{M}_3 = 6$.

(b) The Hilbert scheme of conics on A_{22} is isomorphic to the plane $\mathcal{H}_2$ and $\mathcal{H}_2$ is naturally considered as the plane $\check{\mathbb{P}}^2$ dual to $\mathbb{P}^2$ since, for a conic q on A_{22}, the lines intersecting q form a hyperplane section of $\mathcal{H}_1$. Further, he showed the six points $[H_1], \dots, [H_6]$ such that $([H_1], \dots, [H_6]) \in \mathrm{VSP}^o(F_4, 6)$ correspond to six conics through one point of A_{22}.

3. Scorza quartics

Scorza succeeded in associating a unique quartic hypersurface, which is also called *the Scorza quartic*, to a spin curve of any genus g with ineffective theta. In the case $g = 3$, this association turns out to be the inverse of the Scorza map. Dolgachev and Kanev, however, pointed out Scorza overlooked three conditions on spin curves mentioned below to consturct the Scorza quartic.

Let $\mathcal{H} \subset \mathbb{P}^{g-1}$ be a canonical curve of genus g, θ an ineffective theta characteristic on it and $I_\theta \subset \mathcal{H} \times \mathcal{H}$ the Scorza correspondence. Since the linear hull $\langle I_\theta(x) - y\rangle$ for $(x, y) \in I_\theta$ is a hyperplane of $\mathbb{P}^{g-1}$, we can define a morphism $\pi_\theta \colon I_\theta \to |\omega_{\mathcal{H}}| = \check{\mathbb{P}}^{g-1}$ by $(x, y) \mapsto \langle I_\theta(x) - y\rangle$.

The following is a crucial object to construct the Scorza quartic:

Definition 3.0.4. The image $\Gamma(\theta)$ of the above morphism $\pi_\theta \colon I_\theta \to \check{\mathbb{P}}^{g-1}$ (with reduced structure) is called the *discriminant locus* of the pair $(\mathcal{H}, \theta)$.

By Definition 3.0.4, we have the following diagram:

$$\begin{tikzcd} & I_\theta \subset \mathcal{H} \times \mathcal{H} \arrow[dl, "\pi_\theta"'] \arrow[dr, "p"] & \\ \Gamma(\theta) \subset \check{\mathbb{P}}^{g-1} & & \mathcal{H} \subset \mathbb{P}^{g-1}. \end{tikzcd} \tag{3.1}$$

The three conditions mentioned above is the following, which are a kind of generality conditions:

(A1) the degree of the map $I_\theta \to \Gamma(\theta)$ is two, namely, $\langle I_\theta(x') - y'\rangle = \langle I_\theta(x) - y\rangle$ implies $(x', y') = (x, y)$ or (y, x),
(A2) $\Gamma(\theta)$ is not contained in a quadric, and
(A3) I_θ is smooth.

From now on in this section, we assume these conditions.

We can define:

$$\overline{D}_H := \pi_{\theta *} p^*(H \cap \mathcal{H})$$

as a divisor, where H is an hyperplane of $\mathbb{P}^{g-1}$.

By using (A1)–(A3), it is not difficult to see $\deg \Gamma(\theta) = g(g-1)$ and $\deg \overline{D}_H = 2g(g-1)$. Therefore we may expect that $\overline{D}_H$ is a quadric section of $\Gamma(\theta)$. Actually this is true:

Proposition 3.0.5. *$\overline{D}_H$ is cut out by a quadric in $\check{\mathbb{P}}^{g-1}$.*

Now we define the following correspondence:

$$\mathcal{D} := \{(q_1, q_2) \mid q_1 \in \overline{D}_{H_{q_2}}\} \subset \Gamma(\theta) \times \Gamma(\theta),$$

where H_q is the hyperplane of $\mathbb{P}^{g-1}$ corresponding to $q \in \check{\mathbb{P}}^{g-1}$. It is easy to see that $\mathcal{D}$ is symmetric. By Proposition 3.0.5, we see that $\mathcal{D}$ is the restriction of a symmetric $(2, 2)$ divisor $\mathcal{D}'$ of $\check{\mathbb{P}}^{g-1} \times \check{\mathbb{P}}^{g-1}$. Let $\{\check{F}_4 = 0\}$ be the quartic hypersurface obtained by restricting $\mathcal{D}'$ to the diagonal of $\check{\mathbb{P}}^{g-1} \times \check{\mathbb{P}}^{g-1}$. The Scorza quartic is the 'dual' quartic in $\mathbb{P}^{g-1}$ of $\{\check{F}_4 = 0\}$.

To explain this more precisely, we give a quick review of some generality of the theory of polarity. Set $V := H^0(\mathcal{H}, \omega_{\mathcal{H}})\check{}$. Each homogeneous form $F \in S^4\check{V}$ defines a linear map:

$$\begin{array}{rcl} \mathrm{ap}_F \colon S^2V & \to & S^2\check{V} \\ G & \mapsto & P_G(F). \end{array}$$

called the *apolarity map*, which is nothing but the linear extension of iterating polar maps. If ap_F is an isomorphism, F is called *non-degenerate* and then the inverse isomorphism is given by a $\check{F} \in S^4V$, that is $\mathrm{ap}_F{}^{-1} = \mathrm{ap}_{\check{F}}$. The form $\check{F} \in S^4V$ is called *the dual form* of F.

It turns out that the constructed $\{\check{F}_4 = 0\}$ is non-degenerate and we can take the dual $\{F_4 = 0\}$, which is the Scorza quartic.

To explain this construction of the Scorza quartic is actually the inverse of the Scorza map in genus 3 case, we remark one of the important properties of the Scorza quartic. By the theory of polarity and the definition of $\check{F}_4$, the fiber of $\mathcal{D} \to \Gamma(\theta)$ over a point $q \in \Gamma(\theta)$ is defined by the second polar $P_{H_q^2}(\check{F}_4)$. Moreover, by definition of $\Gamma(\theta)$, it is easy to derive that $P_{H_q^2}(\check{F}_4) = ab$ for some $a, b \in \mathcal{H}$ such that $(a, b) \in I_\theta$, where $a, b \in \mathbb{P}^{g-1}$ is considered as a linear form on $\check{\mathbb{P}}^{g-1}$. By definition of the dual, we have $P_{a,b}(F_4) = H_q^2$. Thus we have verified the association of the Scorza quartic is the inverse of the Scorza map in the case $g = 3$.

4. Special quartics arising from quintic del Pezzo 3-fold

Now we start explanation of our results.

Trigonal even spin curves of any genus and their Scorza quartics arise from some 3-folds as in Mukai's case.

Let B be the smooth quintic del Pezzo threefold, that is B is a smooth projective threefold such that $-K_B = 2H$, where H is the ample generator of $\operatorname{Pic} B$ and $H^3 = 5$. It is well known that the linear system $|H|$ embeds B into $\mathbb{P}^6$.

Let d be an arbitrary integer greater than or equal to 6. We consider a general smooth rational curves C of degree d on B obtained inductively from lines, more precisely, smoothings of the union of a degree $d-1$ rational curve and a line intersecting it. Let $f \colon A \to B$ be the blow-up along C and E_C the f-exceptional divisor.

We explain the relation of this with A_{22}. If we take the blow-up $A' \to A_{22}$ along a general line on it, then there is a unique flop $A' \dashrightarrow A$

and birational contraction $A \to B$, which is the blow-up of B along a smooth rational curve of degree 5. Thus the above situation is a generalization of this. Moreover, a general line is mapped to a line on B intersecting C, and a general conic is mapped to a conic on B intersecting C twice or more.

We consider the notions of lines and conics on A, which correspond to lines on B intersecting C, and conics on B intersecting C twice or more.

Definition 4.0.6. A connected and reduced curve $l \subset A$ is called a *line* on A if $-K_A \cdot l = 1$ and $E_C \cdot l = 1$.

By $-K_A = f^*(-K_B) - E_C$ and $E_C \cdot l = 1$, $f(l)$ is a line on B *intersecting* C.

Proposition 4.0.7. *The Hilbert scheme of lines on A is a smooth trigonal curve $\mathcal{H}_1$ of genus $d-2$.*

Definition 4.0.8. A connected and reduced curve $q \subset A$ is called a *conic* on A if $-K_A \cdot q = 2$ and $E_C \cdot q = 2$.

We showed that the Hilbert scheme of conics on A is an irreducible surface and the normalization morphism is injective, namely, the normalization $\mathcal{H}_2$ parameterizes conics on A in one to one way.

Moreover we have the full description of $\mathcal{H}_2$ as follows. For this, let $D_l \subset \mathcal{H}_2$ be the locus parameterizing conics on A which intersect a fixed line l on A.

Theorem 4.0.9. *$\mathcal{H}_2$ is so-called the* White surface, *namely, the surface obtained by blowing up $S^2C \simeq \mathbb{P}^2$ at $s := \binom{d-2}{2}$ points and embedded by $|D_l| = |(d-3)h - \sum_{i=1}^{s} e_i|$, where h is the pull-back of a line, e_i are the exceptional curves of $\eta\colon \mathcal{H}_2 \to \mathbb{P}^2$. Moreover, $\mathcal{H}_2$ is given by intersection of cubics.*

Here we use the notation $\check{\mathbb{P}}^{d-3}$ since the ambient projective space of $\mathcal{H}_2$ and that of the canonical embedding of $\mathcal{H}_1$ can be considered as reciprocally dual as in Mukai's case. We write the ambient of $\mathcal{H}_1$ by $\mathbb{P}^{d-3}$ and that of $\mathcal{H}_2$ by $\check{\mathbb{P}}^{d-3}$.

Set

$$\mathcal{D}_2 := \{([q_1],[q_2]) \in \mathcal{H}_2 \times \mathcal{H}_2 \mid q_1 \cap q_2 \neq \emptyset\}$$

and denote by D_q the fiber of $\mathcal{D}_2 \to \mathcal{H}_2$ over a point $[q]$. Then $D_q \sim 2D_l$ and $\mathcal{D}_2 \sim p_1^* D_q + p_2^* D_q$. $\mathcal{D}_2$ is obviously symmetric. Thus $\mathcal{D}_2$ is the restriction of a unique symmetric $(2,2)$-divisor $\mathcal{D}_2'$ on $\check{\mathbb{P}}^{d-3} \times \check{\mathbb{P}}^{d-3}$. The restriction of $\mathcal{D}_2'$ to the diagonal is a quartic hypersurface $\{\check{F}_4' = 0\}$

in $\check{\mathbb{P}}^{d-3}$. We can show that $\check{F}'_4$ is non-degenerate. Then we obtain the unique quartic hypersurface $\{F'_4 = 0\}$ in $\mathbb{P}^{d-3}$ dual to $\check{F}'_4$.

The following is a generalization of Theorem 2.0.3 (2):

Theorem 4.0.10. *Let $f\colon A \to B$ be the blow-up along C, and let $\rho\colon \widetilde{A} \to A$ be the blow-up of A along the strict transforms of $\binom{d-2}{2}$ bi-secant lines of C on B. Then there is an injection from $\widetilde{A}$ to $\mathrm{VSP}\,(F'_4, n)$, where $n := \binom{d-1}{2}$. Moreover the image of $\widetilde{A}$ is uniquely determined by $\mathcal{D}_2$ and is an irreducible component of*

$$\mathrm{VSP}\,(F'_4, n; \mathcal{H}_2) := \overline{\{([H_1], \dots, [H_n]) \mid [H_i] \in \mathcal{H}_2\}} \subset \mathrm{VSP}\,(F'_4, n).$$

To characterize 3-fold $\widetilde{A}$, we need extra deta $\mathcal{H}_2$, which is implicit in Mukai's case. See [TZ08a].

5. Existence of the Scorza quartic

Notice that the construction of F'_4 is quite similar to that of the Scorza quartic. This similarity will be clear once we define a theta characteristic on $\mathcal{H}_1$ and clarify the relation of $\mathcal{H}_1$ and $\mathcal{H}_2$.

For the curve $\mathcal{H}_1$ parameterizing lines on A, we can introduce the incidence correspondence as in Mukai's case:

$$I := \{([l], [m]) \mid l \neq m, l \cap m \neq \emptyset\} \subset \mathcal{H}_1 \times \mathcal{H}_1 \tag{5.1}$$

with reduced structure. We can prove I satisfies the conditions (a)–(e) whence there exists a unique ineffective theta characteristic such that $I = I_\theta$.

Moreover, as we mentioned above, there is a natural duality between the ambient spaces of $\mathcal{H}_1$ and $\mathcal{H}_2$. This gives us a very computable way to describe the discriminant loci $\Gamma(\theta)$ of θ.

Proposition 5.0.11. *For the pair $(\mathcal{H}_1, \theta)$, $\Gamma(\theta)$ is contained in $\mathcal{H}_2$, and the generic point of the curve $\Gamma(\theta)$ parameterizes line pairs on A. Moreover, $\Gamma(\theta) \sim 3(d-2)h - 4\sum_{i=1}^{s} e_i$ on $\mathcal{H}_2$. In particular $\Gamma(\theta)$ is not contained in a cubic section of $\mathcal{H}_2$.*

Moreover, we can consider $\{F'_4 = 0\}$ lives in the same ambient space as canonically embedded $\mathcal{H}_1$.

Proposition 5.0.12. *The special quartic $\{F'_4 = 0\} \subset \mathbb{P}^{d-3}$ of Theorem 4.0.10 coincides with the Scorza quartic of $(\mathcal{H}_1, \theta)$.*

Proof. Noting $\Gamma(\theta) \subset \mathcal{H}_2$, we can show that the restriction of the correspondence defining F'_4 to $\Gamma(\theta) \times \Gamma(\theta)$ coincides with the correspondence defining the Scorza F_4. □

The story goes further. By virtue of the above explicit computation of the discriminant, we can prove that the pair $(\mathcal{H}_1, \theta)$ satisfies the conditions (A1)–(A3). Then, by a standard deformation theoretic argument, we can then verify that the conditions (A1)–(A3) hold also for a general even spin curve, hence we answer affirmatively to the Dolgachev-Kanev Conjecture:

Theorem 5.0.13. *The Scorza quartic exists for a general even spin curve.*

See [TZ08b].

6. Moduli space of trigonal even spin curves

Let $\mathcal{M}_g^{\mathrm{tr}}$ and $\mathcal{S}_g^{+\mathrm{tr}}$ be the moduli space of trigonal curves of genus g and the moduli space of even trigonal spin curves of genus g, respectively. We would like to study $\mathcal{S}_g^{+\mathrm{tr}}$ using the geometry of (B, C). Denote by $\mathcal{H}_d^B$ the Hilbert scheme of general smooth rational curves of degree d as in Section 4. $\mathcal{H}_d^B$ is irreducible. By $\operatorname{Aut} B \simeq \mathrm{SL}(2,\mathbb{C})$, we have the natural rational maps $\pi_{\mathcal{S}}\colon \mathcal{H}_d^B/\mathrm{SL}(2,\mathbb{C}) \dashrightarrow \mathcal{S}_{d-2}^{+\mathrm{tr}}$ mapping $C_d \mapsto (\mathcal{H}_1, \theta)$, and $\pi_{\mathcal{F}}$ from $\mathcal{H}_d^B/\mathrm{SL}(2,\mathbb{C})$ to the moduli space $\mathcal{F}_d$ of $\widetilde{A}_d$ ($= \widetilde{A}$ of degree d) mapping $C_d \mapsto \widetilde{A}_d$.

$$\begin{tikzcd} & \mathcal{H}_d^B/\mathrm{SL}(2,\mathbb{C}) \arrow[dl, "\pi_{\mathcal{S}}"'] \arrow[dr, "\pi_{\mathcal{F}}"] & \\ \mathcal{S}_{d-2}^{+\mathrm{tr}} & & \mathcal{F}_d \end{tikzcd}$$

Since $\mathcal{H}_d^B$ is irreducible and $\mathcal{H}_d^B/\mathrm{SL}(2,\mathbb{C}) \dashrightarrow \mathcal{F}_d$ is dominant, we see that $\mathcal{F}_d$ is irreducible.

Proposition 6.0.14. *The map $\pi_{\mathcal{F}}$ is finite. If $d = 6$, then $\deg \pi_{\mathcal{F}} = 2$. If $d \geq 7$, then $\pi_{\mathcal{F}}$ is birational.*

Proof. $\deg \pi_{\mathcal{F}} = 2$ for $d = 6$ follows from the following diagram:

$$\begin{tikzcd} & & \widetilde{A} \arrow[dl] \arrow[dr] & & \\ & A \arrow[dl, "f"'] \arrow[rr, dashed] & & A' \arrow[dr, "f'"] & \\ B & & & & B, \end{tikzcd}$$

where $A \dashrightarrow A'$ is a flop and $A' \to B$ is also the blow-up along a smooth rational curve C' of degree 6 on B. This reflects the fact $\mathcal{H}_1$ has two different g^1_3's (birationality of $\pi_{\mathcal{F}}$ for $d \geq 7$ will reflect the fact a general trigonal curve of genus ≥ 5 has a unique g^1_3). Indeed, there is one to one correspondence between the sets lines on A and lines on A'. Thus we identify the Hilbert schemes of lines on A and A' and denote it by $\mathcal{H}_1$. $\mathcal{H}_1$ has two triple covers $\mathcal{H}_1 \to C$ and $\mathcal{H}_1 \to C'$. These are defined by two different g^1_3's of $\mathcal{H}_1$. Thus (B, C) and (B, C') are not isomorphic to each other but correspond the same $\widetilde{A}$. □

For genus three curve, the Scorza quartic is useful to prove the rationality of S^+_3. Unfortunately, this is not the case in the higher genus case for the moment since the Scorza quartics are special quartics and there is no description of the loci of them in the space of quartics. Nevertheless, it gives another way to study of S^+_g.

Proposition 6.0.15. *$\pi_{\mathcal{S}}$ factor through $\pi_{\mathcal{F}}$ as $\mathcal{H}^B_d/\mathrm{SL}(2,\mathbb{C}) \to \mathrm{Im}\,\pi_{\mathcal{S}} \to \mathcal{F}_d$. In other words, $\widetilde{A}$ is determined from $(\mathcal{H}_1, \theta)$.*

Proof. From $(\mathcal{H}_1, \theta)$, we can define $\Gamma(\theta)$ and F_4. By Theorem 4.0.9 and Proposition 5.0.11, we obtain $\mathcal{H}_2$ as the intersection of cubics containing $\Gamma(\theta)$. We can define the divisor $\mathcal{D}_2 \subset \mathcal{H}_2 \times \mathcal{H}_2$ from the dual $\check{F}_4$. By Theorem 4.0.10, $\widetilde{A}$ is obtained from F_4 and $\mathcal{H}_2$, thus from $(\mathcal{H}_1, \theta)$. □

Corollary 6.0.16. $\mathrm{Im}\,\pi_{\mathcal{S}}$ *is an irreducible component of $\mathcal{S}^{+tr}_{d-2}$ dominating $\mathcal{M}^{\mathrm{tr}}_{d-2}$. In particular a general $\mathcal{H}_1$ is a general trigonal curve of genus $d-2$. $\pi_{\mathcal{S}} \colon \mathcal{H}^B_d/\mathrm{SL}\,(2,\mathbb{C}) \to \mathrm{Im}\,\pi_{\mathcal{S}}$ is finite of degree two if $d = 6$ and birational if $d \geq 7$.*

Proof. Since $\dim \mathcal{H}^B_d = 2d$ and $\dim \mathrm{Aut}\,(B, C_d) \leq \dim \mathrm{Aut}\, B = 3$, we see that $\dim \mathcal{F}_d \geq 2d - 3$ by Proposition 6.0.14. By Proposition 6.0.15, $\dim \mathrm{Im}\,\pi_{\mathcal{S}} \geq 2d - 3$. Thus by $\dim \mathcal{S}^{+tr}_{d-2} = 2d - 3$, the first claim follows.

If $d \geq 7$, then $\pi_{\mathcal{S}}$ is birational by Proposition 6.0.14. If $d = 6$, then, as in the proof of Proposition 6.0.14, two triple covers $\mathcal{H}_1 \to C$ and $\mathcal{H}_1 \to C'$ are defined by two different g^1_3's of $\mathcal{H}_1$, thus (B, C) and (B, C') are not isomorphic to each other. But (B, C) and (B, C') define the same theta characteristic. Thus $\pi_{\mathcal{S}}$ is of degree two. □

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