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ON DEFORMATIONS OF LAGRANGIAN FIBRATIONS

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ABSTRACT. Let X be an irreducible symplectic manifold and $\mathrm{Def}(X)$ the Kuranishi family. Assume that X admits a Lagrangian fibration. We prove that there exists a smooth hypersurface H of $\mathrm{Def}(X)$ such that the restriction family $\mathscr{X} \times_{\mathrm{Def}(X)} H$ admits a family of Lagrangian fibrations over H.

1. INTRODUCTION

A compact Kähler manifold X is said to be *symplectic* if X carries a holomorphic symplectic form. Moreover X is said to be *irreducible symplectic* if X satisfies the following two properties:

(1) $\dim H^0(X.\Omega^2_X) = 1$ and;
(2) $\pi_1(X) = \{1\}$.

A surjective morphism between Kähler spaces is said to be *fibration* if it is surjective and has only connected fibres. A fibration from a symplectic manifold is said to be *Lagrangian* if a general fibre is a Lagrangian submanifold. The plainest example of an irreducible symplectic is a $K3$ surface. An elliptic fibration from a $K3$ surface gives an example of a Lagrangian fibration. It is expected that a $K3$ surface and an irreducible symplectic manifold share many geometric properties. Let S be a $K3$ surface and $g : S \to \mathbb{P}^1$ an elliptic fibration. Kodaira proves that there exists a smooth hypersurface H_S in the Kuranishi space $\mathrm{Def}(S)$ of S which has the following three properties:

(1) The hypersurface H_S passes the reference point.
(2) For the Kuranishi family $\mathscr{S}$ of S, the base change $\mathscr{S} \times_{\mathrm{Def}(S)} H_S$ admits a surjective morphsim over $\mathbb{P}^1_{H_S}$. Moreover they satisfy the following diagram:

$$\begin{tikzcd} \mathscr{S} \times_{\mathrm{Def}(S)} H_S \arrow[r] \arrow[d] & \mathbb{P}^1_{H_S} \arrow[dl] \\ H_S. & \end{tikzcd}$$

(3) The original fibration g coincides with the restriction of the above diagram over the reference point. The restriction of the diagram over a every point of H_S gives an elliptic fibration.

The following is the main theorem, which induces a higher dimensional analog of the above statement.

THEOREM 1.1. *Let X be an irreducible holomorphic symplectic manifold and $\mathscr{X} \to \mathrm{Def}(X)$ the Kuranishi family of X. Assume that X admits a Lagrangian fibration $f : X \to B$ over a projective variety B. Let L be a line bundle which is a pull back of an ample line bundle on B. Then we have a smooth hypersurface H of $\mathrm{Def}(X)$ and a line bundle $\mathscr{L}$ on $\mathscr{X} \times_{\mathrm{Def}(X)} H$ which satisfies the following two properties:*

(1) *The hypersurface H passes the reference point.*
(2) *The restriction of $\mathscr{L}$ to X is isomorphic to L.*
(3) *For the projection $\pi : \mathscr{X} \times_{\mathrm{Def}(X)} H \to H$, $R^i\pi_*\mathscr{L}$ is locally free for every i.*

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COROLLARY **1.2.** *Let $f: X \longrightarrow B$ be as in Theorem 1.1. We also let L be a pull back of a very ample line buncle of S. The symbols π, $\mathscr{X}$, H and $\mathscr{L}$ denote same objects as in Theorem 1.1. Then there exists a morphism $f_H : \mathscr{X} \times_{\mathrm{Def}(X)} H \longrightarrow \mathbb{P}(\pi_* \mathscr{L})$. Together with π, they form the following diagram:*

$$\begin{tikzcd} \mathscr{X} \times_{\mathrm{Def}(X)} H \arrow[r, "f_H"] \arrow[d, "\pi"'] & \mathbb{P}(\pi_* \mathscr{L}) \arrow[dl] \\ H, & \end{tikzcd}$$

The orginal fibration f coincides with the restriction of the above diagram over the reference point. The restriction of the diagram over a every point of H gives a Lagrangian fibration.

REMARK **1.3.** *If X be an irreducible symplectic manifold. Assume that X admits a surjective morphism $f: X \longrightarrow S$ such that f has connected fibres and $0 < \dim S < \dim X$. If X and S are projective or X and S are smooth and Kähler then f is Lagrangian over a projective base S by* [8], [9] *and* [5, Proposition 24.8].

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2. PROOF OF THEOREM

PROPOSITION **2.1.** *Let $f: X \longrightarrow B$ and L be as in Theorem 1.1. We denote by A a general fiber of f. Then there exists a smooth hypersurface H of $\mathrm{Def}(X)$ such that*

1. *The base change $\mathscr{X} \times_{\mathrm{Def}(X)} H$ carries the line bundle $\mathscr{L}$ on $\mathscr{X} \times_{\mathrm{Def}(X)} H$ whose restiction to the fibre over the reference point is isomorphic to L.*
2. *The relative Douady space $D(\mathscr{X}/\mathrm{Def}(X))$ of the morphism $\mathscr{X} \longrightarrow \mathrm{Def}(X)$ is smooth at A.*
3. *Let $D(\mathscr{X}/\mathrm{Def}(X))_A$ be the irreducible component of $D(\mathscr{X}/\mathrm{Def}(X))$ which contains A. The image of the induced morphism $D(\mathscr{X}/\mathrm{Def}(X))_A \longrightarrow \mathrm{Def}(X)$ coincides with H.*

Proof of Proposition 2.1. (1) By [6, (1.14)], there exists a universal deformation $(\mathscr{X}, \mathscr{L})$ of the pair (X, L). The parameter space of the universal family forms a smooth hypersurface H of $\mathrm{Def}(X)$. The hypersurface H and the line bundle $\mathscr{L}$ satisfy the assertion (1) of Proposition 2.1.
(2) Let $D(\mathscr{X}/\mathrm{Def}(X))$ be the relative Douady space of the morphism $\mathscr{X} \longrightarrow \mathrm{Def}(X)$. Since A is smooth and Lagrangian, $D(\mathscr{X}/\mathrm{Def}(X))$ is smooth at A by [11, Theorem 0.1].
(3) We need the following Lemma.

LEMMA **2.2.** *Let X, L and A be as in Propositon 2.1. For an element z of $H^1(X, \Omega^1_X)$, the restriction $z|_A = 0$ in $H^1(A, \Omega^1_A)$ if $q_X(z, L) = 0$, where q_X is the Beauville-Bogomolov-Fujiki form on X.*

Proof. Let σ be a Kähler class of X. It is enough to prove that

$$z\sigma^{n-1}L^n = z^2\sigma^{n-2}L^n = 0,$$

where $2n = \dim X$. By [3, Theorem 4.7], we have the following equation;

$$c_X q_X(z + s\sigma + tL)^n = (z + s\sigma + tL)^{2n}, \tag{1}$$

where s and t are indeterminacy and c_X is a constant only depending on X. By the assumption,

$$c_X q_X(z + s\sigma + tL)^n = c_X(q_X(z) + s^2 q_X(\sigma) + 2s q_X(z, \sigma) + 2st q_X(\sigma, L))^n.$$

If we compare the $s^{n-1}t^n$ and $s^{n-2}t^n$ terms of the both hand sides of the above equation (1), we obtain the assertions. $\square$

We go back to the proof of the assertion (3) of Proposition 2.1. Let $j : H^2(X,\mathbb{C}) \to H^2(A,\mathbb{C})$ be the natural induced morphism by the inclusion $A \to X$. We denote by L_X the intersection of $H^2(X,\mathbb{Q})$ and the orthogonal space of $\mathrm{Ker}(j)$ with respect to the Beauville-Bogomolov-Fujiki form. Since A is Lagrangian, the image of the natural projection $D(\mathscr{X}/\mathrm{Def}(X))_A \to \mathrm{Def}(X)$ is a smooth proper subanalytic space H_A of $\mathrm{Def}(X)$ by [11, 0.1 Theorem]. Moreover, the family over H_A perserves the subspace L_X of $\mathrm{NS}(X)\otimes\mathbb{Q}$ by [11, 0.2 Corollary]. The tangent space of H_A is $\mathrm{Ker}(j) = \mathrm{Ker}\{H^1(X,\Omega^1_X) \to H^1(A,\Omega^1_A)\}$ by [11, 0.1 Theorem]. Let $L^{\perp}$ be the orthogonal space of L in $H^1(X,\Omega^1_X)$ with the Beauville-Bogomolov-Fujiki form. We note that $L^{\perp}$ is tha tangent space of H at the reference point. By Lemma 2.2, $L^{\perp}$ is contained in $\mathrm{Ker}(j)$. This implies that $\mathrm{Ker}(j) = L^{\perp}$. Moreover L_X is spaned by L, because the Beauville-Bogomolov-Fujiki form is nondegenerate. Since H is the universal family of the pair (X,L), we obtain that $H_A \subset H$. Comparing the dimension of the tangent spaces, we have $H_A = H$. $\square$

PROPOSITION **2.3.** *Let* $\mathscr{X}$, $\mathrm{Def}(X)$, $\mathscr{L}$ *and* H *be as in Proposition 2.1. We also let* Δ *be a unit disk in* H *which has the following two properties:*

(1) Δ *passes the reference point of* $\mathrm{Def}(X)$.

(2) *For a very general point* t *of* Δ, *the Picard number of the fibre* $\mathscr{X}_t$ *of* π *over* t *is one.*

The symbols $\mathscr{X}_\Delta$, π_Δ *and* $\mathscr{L}_\Delta$ *denote the base change* $\mathscr{X}\times_H \Delta$, *the induced morphism* $\mathscr{X}_\Delta \to \Delta$ *and the restriction* $\mathscr{L}$ *to* $\mathscr{X}_\Delta$, *respectively. Then*

$$R^i(\pi_\Delta)_*\mathscr{L}_\Delta,$$

are locally free for all i *at the reference point.*

Proof. For a point u of Δ, $\mathscr{X}_u$ and $\mathscr{L}_u$ denote the fibre of π_Δ over u and the restriction of $\mathscr{L}_\Delta$ to $\mathscr{X}_u$, respectively. We consider whether $\mathscr{L}_\Delta$ has the following property:

(2) For every $u \neq o$, $\mathscr{L}_u$ is semi-ample

If $\mathscr{L}_u$ has the above property, the assertion of Proposition 2.3 follows from [10, Corollary 3.14]. To prove it, we need the following two lemmata.

LEMMA **2.4.** *For a very general point* u *of* Δ, $\mathscr{L}_u$ *is semi-ample.*

Proof. We start with proving the following claim.

CLAIM **2.5.** *There exists a dominant meromorphic map* $\Phi : \mathscr{X}_u \dashrightarrow B_u$ *such that a general fibre of* Φ *is compact,* B_u *is a Kähler manifold and* $\dim B_u > 0$.

Proof. We use the notation as in Proposition 2.1. By Propositon 2.1 (2), there exists a smooth open neighborhood V of A in $D(\mathscr{X}/\mathrm{Def}(X))$. Let $D(\mathscr{X}_u)$ be the irreducible component of $D(\mathscr{X}/\mathrm{Def}(X))\times_H \{u\}$ which intersects V. We note that $D(\mathscr{X}_u)$ is an irreucible component of the Douady space of $\mathscr{X}_u$. We take a resolution $D(\mathscr{X}_u)^{\sim} \to D(\mathscr{X}_u)$ and denote by $U(\mathscr{X}_u)^{-}$ the normalization of $U(\mathscr{X}_u)\times_{D(\mathscr{X}_u)} D(\mathscr{X}_u)^{\sim}$, where $U(\mathscr{X}_u)$ is the universal family over $D(\mathscr{X}_u)$. We also denote by by p and q the natural projections $U(\mathscr{X}_u)^{\sim} \to \mathscr{X}_u$ and $U(\mathscr{X}_u)^{\sim} \to D(\mathscr{X}_u)^{\sim}$. The relations of these objects are summerized in the following diagram:

$$\begin{tikzcd}
\mathscr{X}_u & U(\mathscr{X}_u) \arrow[l] \arrow[d] & U(\mathscr{X}_u)^{-} \arrow[l] \arrow[d, "q"] \arrow[ll, bend right, "p"'] \\
 & D(\mathscr{X}_u) & D(\mathscr{X}_u)^{\sim} \arrow[l]
\end{tikzcd}$$

Let a be a point of $\mathscr{X}_u$. We define the subvarieties $G_i(a)$ of $\mathscr{X}_u$ by

$$G_0 := a$$
$$G_{i+1} := p(q^{-1}(q(p^{-1}(G_i(a)))))$$

We also define

$$G_\infty(a) := \bigcup_{i=0}^{\infty} G_i(a).$$

Let $B(\mathscr{X}_u)$ be the Barlet space of $\mathscr{X}_u$. By [1, Théorème A.3], $G_\infty(a)$ is compact for a general point a of $\mathscr{X}_u$ and there exists a meromorphic map $\Phi : \mathscr{X}_u \dashrightarrow B(\mathscr{X}_u)$ whose general fibre is $G_\infty(a)$. By [3, (5.2) Theorem], $B(\mathscr{X}_u)$ is of class $\mathscr{C}$. Hence there exists an embedded resolution $B(\mathscr{X}_u)^\sim \to B(\mathscr{X}_u)$ of the image of Φ whose proper transformation is smooth and Kähler. We denote by B_u the proper transformation. The composition map

$$\mathscr{X}_u \dashrightarrow \mathrm{Im}(\Phi) \dashrightarrow B_u$$

gives the desired meromorphic map if the image of Φ is not a point. Hence we show that Φ is not a trivial. Let a be a general point of $\mathscr{X}_u$. Then $G_1(a)$ is a complex torus. Moreover $G_1(a)$ is a Lagrangian submanifold of $\mathscr{X}_u$. Thus $D(\mathscr{X}_u)$ is smooth at $G_1(a)$ and its dimension is half of those of $\mathscr{X}_u$. The normal bundle of $G_1(a)$ is the direct sum of the trivial bundles. Therefore p is locally isomorphic in a neighborhood of $p^{-1}(G_1(a))$ and p is generically finite. If p is bimeromorphic, then $G_\infty(a) = G_1(a)$ and we are done. If p is not bimeromorphic, we consider the branch locus of the Stein factorization of p. Since $\mathscr{X}_u$ is smooth, the branch locus defines an effective divisor E of $\mathscr{X}_u$. We will prove that $G_\infty(a) \cap E = \emptyset$ if a is general. Since the Picard number of $\mathscr{X}_u$ is one, $\mathscr{L}_u$ and E should be numerically propotional. The pull back $p^*\mathscr{L}_u$ is numerically trivial on fibres of q. Hence the restiction of $\mathscr{L}_u$ to $G_1(a)$ is a numerically trivial bundle if a is general. Therefore $q(p^{-1}(E)) \neq D(\mathscr{X}_u)^\sim$. This implies that $E \cap G_1(a) = \emptyset$ for a general point a of $\mathscr{X}_u$. Since E is effective, E is nef. By [10, Lemma 2.15], there exists an effective $\mathbb{Q}$-divisor E' on $D(\mathscr{X}_u)^\sim$ such that

$$p^*E = q^*E'.$$

Hence $G_\infty(a) \cap E = \emptyset$ if $G_1(a) \cap E = \emptyset$. □

We go back to the proof of Lemma. By blowing ups and flattening, we have the following diagram:

$$\begin{tikzcd}
\mathscr{X}_u \arrow[d, dotted] & \mathscr{Y}_u \arrow[l] \arrow[d] & \mathscr{Z}_u \arrow[l] \arrow[d] & \mathscr{W}_u \arrow[l] \arrow[d, "r"] \arrow[lll, bend right, "\nu"'] \\
B_u \arrow[r, equal] & B_u & B_u^\sim \arrow[l] \arrow[r, equal] & B_u^\sim,
\end{tikzcd}$$

where

(1) $\mathscr{Y}_u \to \mathscr{X}_u$ is a resolution of indeteminancy of Φ.
(2) $\mathscr{Z} \to \mathscr{Y}_u$ and $B_u^\sim \to B_u$ are bimeromorphic.
(3) $B_u^\sim$ is smooth and Kähler.
(4) $\mathscr{Z}_u \to B_u^\sim$ is flat.
(5) $\mathscr{W}_u \to \mathscr{Z}_u$ is the normalization.

We denote by ν and r the induced morphisms $\mathscr{W}_u \to \mathscr{X}_u$ and $\mathscr{W}_u \to B_u^\sim$, respectively. The proof consists of three steps.

Step 1. We prove that $B_u^\sim$ is projective. Since $B_u^\sim$ is Kähler, it is enough to prove that $\dim H^0(B_u^\sim, \Omega^2) = 0$. We derive a contradiction assuming that $\dim H^0(B_u^\sim, \Omega^2) > 0$. Under this assumption, there exists a holomorphic 2-form ω on $B_u^\sim$. The pull back $r^*\omega$ defines a degenerate holomorphic 2-form on

$\mathscr{W}_u$. On the other hand, $H^0(\mathscr{W}_u,\Omega^2) \cong H^0(\mathscr{X}_u,\Omega^2)$ because ν is birational and $\mathscr{X}_u$ is smooth. Hence $\dim H^0(\mathscr{W}_u,\Omega^2) = 1$ and it should be generated by a generically nondegenerate holomorphic 2-form. That is a contradiction.

Step 2. We prove that $\mathscr{L}_u$ is nef. This is [2, 3.4 Theorem]. For the convinience of readers, we copy their arguments. By [7, Proposition 3.2] it is enoght to prove that $\mathscr{L}_u.C \geq 0$ for every effective curve of $\mathscr{X}_u$. Since the Beauville-Bogomolov-Fujiki form $q_{\mathscr{X}_u}$ is non-degenerate and defined over $H^2(\mathscr{X}_u,\mathbb{Q})$, there exists an isomorphic

$$\iota : H^{1,1}(\mathscr{X}_u,\mathbb{C})_{\mathbb{R}} \to H^{2n-1,2n-1}(\mathscr{X}_u,\mathbb{C})_{\mathbb{R}}$$

such that

$$q_{\mathscr{X}_u}(\mathscr{L}_u,\iota^{-1}([C])) = \mathscr{L}_u.C.$$

If $q(\mathscr{L}_u,z) \neq 0$ for an element z of $H^{1,1}(\mathscr{X}_u,\mathbb{Q})$, then there exists a rational number d such that $q(\mathscr{L}_u + dz) > 0$. By [6, Corollary 3.9], this implies that $\mathscr{X}_u$ is projective. That is a contradiction. Thus $\mathscr{L}_u.C = 0$ for every curve C.

Step 3. Let M be a very ample divisor on $B_u^{\sim}$. We prove that there exists a rational number c such that

$$\mathscr{L}_u \sim_{\mathbb{Q}} c\nu_* r^* M.$$

It is enough to prove that

$$q_{\mathscr{X}_u}(\nu_* r^* M) = q_{\mathscr{X}_u}(\mathscr{L}_u) = q_{\mathscr{X}_u}(\nu_* r^* M, \mathscr{L}_u) = 0.$$

Since $\mathscr{X}_u$ is non projective, $q_{\mathscr{X}_u}(\nu_* r^* M) \leq 0$ and $q_{\mathscr{X}_u}(\mathscr{L}_u) \leq 0$ by [6, Corollary 3.8]. On the other hand, $q_{\mathscr{X}_u}(\mathscr{L}_u) \geq 0$ because $\mathscr{L}_u$ is nef. The linear system $|r^*M|$ contains members M_1 and M_2 such that $M_1 \cap M_2$ has a codimension two. By the definition

$$q_{\mathscr{X}_u}(\nu_* r^* M) = \int (\nu_* r^* M)^2 \sigma^{n-1} \bar{\sigma}^{n-1},$$

where σ is a symplectic form on $\mathscr{X}_u$. Thus $q_{\mathscr{X}_u}(\nu_* r^* M) \geq 0$. Therefore $q_{\mathscr{X}_u}(\nu_* r^* M) = q(\mathscr{L}_u) = 0$. Since $\nu_* r^* M$ is effective and $\mathscr{L}_u$ is nef, $q_{\mathscr{X}_u}(\nu_* r^* M, \mathscr{L}_u) \geq 0$. Again by [6, Corollary 3.8], $q_{\mathscr{X}_u}(\nu_* r^* M + \mathscr{L}_u) \leq 0$. Thus $q_{\mathscr{X}_u}(\nu_* r^* M, \mathscr{L}_u) = 0$ and we are done.

Step 4. We prove that $\mathscr{L}_u$ is semi-ample. By [10, Remark 2.11.1] and [10, Theorem 5.5], it is enough to prove that there exists a nef and big divisor M' on $B_u^{\sim}$ such that

$$\nu^* \mathscr{L}_u \sim_{\mathbb{Q}} r^* M'$$

By Step 2 and Step 3, $\nu^* \mathscr{L}_u \sim_{\mathbb{Q}} r^* M + \sum e_i E_i$ where E_i are ν-exceptional divisors and e_i are positive rational numbers. By Step 2, $\sum e_i E_i$ is nef for every irreducible component of every fibre of r. By [10, Lemma 2.15], there exists a $\mathbb{Q}$-effective divisor M_0 such that

$$\sum e_i E_i = r^* M_0.$$

If we put $M' := M + M_0$, we are done.

□

LEMMA 2.6. *If $\mathscr{L}_u$ is semi-ample for very general point u of Δ, then $\mathscr{L}_u$ is semi-ample for every $u \neq o$.*

Proof. Let $\Delta(k)$ be an open set of Δ which has the following two properties:

(1) $\pi_* \mathscr{L}_{\Delta}^{\otimes k}$ is locally free.
(2) $\mathscr{L}_{\Delta}^{\otimes k} \otimes k(u) \cong H^0(\mathscr{X}_u, \mathscr{L}_{\Delta}|_{\mathscr{X}_u})$.

We also define

$$\Delta^{\circ} := \{u \in \Delta; \rho(\mathscr{X}_u) = 1.\},$$

where $\mathscr{X}_u$ is the fibre over u and $\rho(\mathscr{X}_u)$ stands for the Picard number of $\mathscr{X}_u$. We fix a compact set K of Δ which contains the reference point. Then $K \setminus (K \cap \Delta(k))$ consists of finite points. Thus

$$\bigcup_{k=1}^{\infty} (K \setminus K \cap \Delta(k))$$

consists of countable infinite points. Hence

$$\left(\bigcap_{k=1}^{\infty} \Delta(k) \right) \cap K \cap \Delta^{\circ} \neq \emptyset.$$

Thus there exists a point t_0 of Δ and an integer k such that

$$\pi_{\Delta}^{*}(\pi_{\Delta})_{*}\mathscr{L}_{\Delta} \to \mathscr{L}_{\Delta}$$

is surjective on $\mathscr{X}_{t_0}$. This implies that the support Z of the cokernel sheaf of $\pi_{\Delta}^{*}(\pi_{\Delta})_{*}\mathscr{L}_{\Delta} \to \mathscr{L}_{\Delta}$ is a proper closed subset of $\mathscr{X}_{\Delta}$. Hence $\mathscr{L}_u$ is semi-ample if $u \in \Delta \setminus \pi(Z)$. □

We complete the proof of Proposition 2.3. □

Proof of Theorem 1.1. By Proposition 2.1, there exists a smooth hypersurface H of $\mathrm{Def}(X)$ and a line bundle $\mathscr{L}$ which have the properties of (1) and (2) of Theorem 1.1. Assume that $R^i\pi_*\mathscr{L}$ is not locally free. We define the function $\varphi(t)$ as

$$\varphi(t) := \dim H^i(\mathscr{X}_t, \mathscr{L}_t)$$

where $\mathscr{X}_t$ is the fibre of π over t and $\mathscr{L}_t$ is the restriction of $\mathscr{L}$ to $\mathscr{X}_t$. Then

$$\varphi(o) > \varphi(t),$$

where o is the reference point and t is a general point of H. The Picard number of a fibre $\mathscr{X}_t$ over a very general point of H is one. Hence there exists a unit disk Δ such that $o \in \Delta$ and the Picard number of a very general fibre of the induced morphsim $\mathscr{X} \times_H \Delta \to \Delta$ is one. By Proposition 2.3, $R^i(\pi_\Delta)_*\mathscr{L}_\Delta$ is locally free for every i. This implies that $\varphi(o) = \varphi(t)$. That is a contradiction. □

REFERENCES

[1] F. Campana, *Remarques sur le revêtement uviversel des variétés kähleriennes compactes*, Bull S.M.F., **122**, 255–284, (1994).

[2] F. Campana, K, Oguiso and T. Peternell *Non-algebraic hyperKähler manifolds* , math.AG/0804.1682.

[3] A. Fujiki, *On the de Rham cohomology group of a compact Kähler symplectic manifold*, Algebraic geometry, Sendai, 1985, Adv. Stud. Pure Math., **10**, 105–165, North-Holland, Amsterdam, (1987).

[4] ———, *On the Douady space of a compact complex space in the category $\mathscr{C}$. II*, Publ. Res. Inst. Math. Sci., **20**, (1984), 461–489,

[5] M. Gross, D. Huybrechts and D. Joyce, *Calabi-Yau manifolds and related geometries*, Lectures from the Summer School held in Nordfjordeid, June 2001, Springer-Verlag, Berlin, (2003), viii+239.

[6] D. Huybrechts, *Compact hyperkähler manifolds: basic results*, Invent. Math., **135**, (1999), 63–113.

[7] ———, *The Kähler cone of a compact hyperkähler manifold*, Math. Ann. **326**, (2003), 499–513.

[8] D. Matsushita, *On fibre space structures of a projective irreducibe symplectic manifold*, Topology, **38** (1999), 79-83.

[9] ———, *Addendum to: On fibre space structures of a projective irreducible symplectic manifold*, Topology, **40** (2001), 431–432.

[10] N. Nakayama, *The lower semicontinuity of the plurigenera of complex varieties*, in Algebraic geometry, Sendai, 1985, Adv. Stud. Pure Math., **10**, 551–590.

[11] C. Voisin, *Sur la stabilité des sous-variétés lagrangiennes des variétés symplectiques holomorphes*, in Complex projective geometry (Trieste, 1989/Bergen, 1989), London Math. Soc. Lecture Note Ser., **179**, (1992), 294–303.

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