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Base manifolds for fibrations of projective irreducible symplectic manifolds

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A connected complex manifold M of dimension $2n$ equipped with a holomorphic symplectic from $\omega \in H^0(M, \Omega^2_M)$ is called a *holomorphic symplectic manifold*. A subvariety V of M is said to be *Lagrangian* if V has dimension n and the restriction of ω on the smooth part of V is identically zero. A simply connected projective algebraic manifold M is called a *projective irreducible symplectic manifold* if M has a symplectic form ω such that $H^0(M, \Omega^2_M) = \mathbf{C}\omega$. It is remarkable that fibrations of projective irreducible symplectic manifolds are of very special form, as described in the following theorem due to D. Matsushita.

Theorem 1 *Let M be a projective irreducible symplectic manifold of dimension $2n$. For a projective manifold X and a surjective holomorphic map $f : M \to X$ with connected fibers of positive dimension, the following holds.*

(1) X is a Fano manifold of dimension n with Picard number 1.

(2) A general fiber of f is biholomorphic to an abelian variety.

(3) The underlying subvariety of every fiber of f is Lagrangian.

(4) All even Betti numbers of X are equal to 1 and all odd Betti numbers of X are equal to 0.

(1), (2) and (3) in Theorem 1 were proved in [Ma1] and [Ma2]. These results led to the question whether the base manifold X is the complex projective space (cf. [Hu, 21.4]). The result of [Ma3] verifies Theorem 1 (4), i.e., that the Betti numbers of X are indeed equal to those of $\mathbf{P}_n$.

Our goal is to give an affirmative answer to the question as follows.

Theorem 2 *In the setting of Theorem 1, X is biholomorphic to $\mathbf{P}_n$.*

There are two geometric ingredients in the proof of Theorem 2: the theory of varieties of minimal rational tangents and the theory of Lagrangian fibrations. On the one hand, the theory of varieties of minimal rational tangents describes a certain geometric structure arising from minimal rational curves at general points of a Fano manifold X with $b_2(X) = 1$ (cf. [HwMo1], [HwMo2]). This geometric structure has differential geometric properties reflecting special features of the deformation theory of minimal rational curves. On the other hand, the theory of Lagrangian fibrations, or equivalently, the theory of completely integrable Hamiltonian systems, provides an affine structure at general points of the base manifold X via the classical action variables (cf. [GuSt, Section 44]). Our strategy to prove Theorem 2 is to exploit the interplay of these two geometric structures on the base manifold X. Under the assumption that X is different from $\mathbf{P}_n$, the condition $b_2(X) = 1$ forces the geometric structure arising from the variety of minimal rational tangents to be 'non-flat', while the affine structure arising from the action variables is naturally 'flat'. These two structures interact via the monodromy of the Lagrangian fibration, leading to a contradiction. To be precise, two separate arguments are needed depending on whether the dimension p of the variety of minimal rational tangents is positive or zero. The easier case of $p > 0$ is handled by a topological argument using $b_4(X) = 1$, using the result of [Hw]. The more difficult case of $p = 0$ needs a deeper argument, depending on the local differential geometry of the variety of minimal rational tangents.

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