Holographic Entanglement Entropy in the dS/CFT Correspondence and Entanglement Entropy in the $Sp(N)$ Model

Yoshiki Sato[1]

Department of Physics, Kyoto University
Kyoto 606-8502, Japan

Ph. D. Thesis

[1]E-mail: yoshiki@gauge.scphys.kyoto-u.ac.jp

Abstract

The AdS/CFT correspondence has been getting understood and we are now in a position to explore gravitational theory by using conformal field theory by recent progress on the holographic entanglement entropy. Nevertheless, the dS/CFT correspondence, which has a possibility to describe our Universe holographically, is still unclear.

We investigate the entanglement entropy in the dS/CFT correspondence. In Einstein gravity on de Sitter spacetime we propose the holographic entanglement entropy as the analytic continuation of the extremal surface in Euclidean anti-de Sitter spacetime. Even though dual conformal field theories for Einstein gravity on de Sitter spacetime are not known yet, we analyze the free $Sp(N)$ model, which is holographically dual to Vasiliev's higher-spin gauge theory on de Sitter spacetime, as a toy model. In this model we confirm the behaviour similar to our holographic result from Einstein gravity.

This Ph. D. thesis is based on the papers [1, 2].

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Publication List

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During my master and Ph. D. course, I published the other papers and the review article [3–8] concerned with the holographic description of the Schwinger effect. This Ph. D. thesis is not based on those paper [3–8].

1. Y. Sato and K. Yoshida, "Holographic description of the Schwinger effect in electric and magnetic fields," JHEP **1304** (2013) 111 [arXiv:1303.0112 [hep-th]].

2. Y. Sato and K. Yoshida, "Potential Analysis in Holographic Schwinger Effect," JHEP **1308** (2013) 002 [arXiv:1304.7917 [hep-th]].

3. Y. Sato and K. Yoshida, "Holographic Schwinger effect in confining phase," JHEP **1309** (2013) 134 [arXiv:1306.5512 [hep-th]].

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5. D. Kawai, Y. Sato and K. Yoshida, "Schwinger pair production rate in confining theories via holography," Phys. Rev. D **89** (2014) 10, 101901 [arXiv:1312.4341 [hep-th]].

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Contents

Chapter 1

Introduction

It is known that black holes (BH) have thermodynamic properties [9–11]. A BH entropy is proportional to the horizon area,

$$S_{\mathrm{BH}} = \frac{\text{Area of horizon}}{4G_{\mathrm{N}}}\,, \tag{1.0.1}$$

where G_{N} is Newton's constant. Entropy represents degree of freedom of the system and is typically proportional to a volume of the system. Nevertheless, the BH entropy shows the area law not the volume law. This fact suggests that gravitational theories can be described by lower dimensional field theories. This is called a holographic principle [12, 13].

The AdS/CFT correspondence is a concrete realization of the holographic principle [14–16]. It provides a remarkable connection between gravitational theories in anti-de Sitter spacetime (AdS) and nongravitational conformal field theories (CFT). A validity of the AdS/CFT correspondence has been checked only in the region that the gravitational theories can be approximated by classical gravitational theories. Despite of this limitation, the holographic principle, especially the AdS/CFT correspondence, provides us to analyze quantum gravitational theories by using nongravitational theories.

A useful quantity to analyze gravitational theories in the context of the AdS/CFT correspondence, is the holographic entanglement entropy proposed by Ryu and

Takayanagi [17, 18][1]. The holographic entanglement entropy is literally a holographic dual of the entanglement entropy, which is quantum information, and is related to geometrical quantities which are extremal surfaces in AdS spacetime. The holographic entanglement entropy contains information on gravitational theories [20, 21]. For instance, linearised Einstein's equation on AdS spacetime can be constructed from the holographic entanglement entropy [22, 23]. Furthermore, by using continuous multi-entanglement renormalization ansatz (cMERA), the radial component of AdS spacetime is constructed as an information metric [24].

It is natural to apply the holographic principle to our Universe. However, since it is known that our Universe is approximately the de Sitter spacetime (dS), not AdS spacetime, we cannot use the AdS/CFT correspondence to analyze our Universe. Then, we need the dS/CFT correspondence which is a duality between gravitational theories on dS spacetime and some nongravitational conformal field theories.

The dS/CFT correspondence has been proposed in [25–27]. These papers [25–27] have given some evidence that the dual field theory lives in past or future infinity of dS spacetime. Despite of these evidence, concrete examples of the dS/CFT correspondence did not exist, and there was no remarkable progress on the dS/CFT correspondence for a decade. Recently, Anninos, Hartman and Strominger have proposed a concrete example of the dS/CFT correspondence based on Giombi-Klebanov-Polyakov-Yin duality [28,29] (the duality between Vasiliev's four-dimensional higher-spin gauge theory on Euclidean AdS (EAdS) spacetime and the three-dimensional $O(N)$ vector model). The authors showed that EAdS spacetime and the $O(N)$ vector model are related to dS spacetime and the $Sp(N)$ vector model via an analytic continuation, respectively [30] (see also [31] for a review). It follows that Vasiliev's higher-spin gauge theory on dS spacetime is the holographic dual of the Euclidean $Sp(N)$ vector model which lives in $\mathcal{I}^+$ in dS spacetime. We are now in a position to analyze the dS/CFT correspondence using the concrete example.

In analogy with the AdS/CFT correspondence, we should find the holographic entanglement entropy formula for the dS/CFT correspondence towards constructing gravitational theories on dS spacetime by dual field theories. Although we have

[1] The covariant generalisation was proposed by Hubeny, Rangamani and Takayanagi [19].

the concrete example of the dS/CFT correspondence, Vasiliev's higher-spin gauge theory contains infinite massless particles and is not suitable for a description of our Universe. Furthermore, it is difficult to analyze Vasiliev's higher-spin gauge theory since an equation of motion is only known but a satisfactory action is not known yet.

In this Ph. D. thesis, we summarize our papers [1,2] which discuss the holographic entanglement entropy in the dS/CFT correspondence and the entanglement entropy in the $Sp(N)$ model. In [1], we investigate the connection between bulk geometry and the holographic entanglement entropy in Einstein gravity on dS spacetime not the Vasiliev's higher-spin gauge theory on dS spacetime. We also compare the proposed holographic entanglement entropy with the entanglement entropy in the free $Sp(N)$ model, which is the holographic dual of Vasiliev's higher-spin gauge theory on dS spacetime. Furthermore, we investigate the detail of the entanglement entropy in the $Sp(N)$ model in [2].

Outline

The organization of this Ph. D. thesis is as follows.

Chapter 2 and 3 are devoted to preliminaries of chapter 4 and chapter 5. In chapter 2 we review the AdS/CFT correspondence and the dS/CFT correspondence. In section 2.1, we will give a definition of AdS spacetime, a heuristic derivation of the AdS/CFT correspondence, and a useful relation in the AdS/CFT correspondence, GKPW relation. Then, we move to the dS/CFT correspondence in section 2.2. Firstly, we define dS geometry. Next, we explain difficulties to construct the dS/CFT correspondence and summarize fundamental works of the dS/CFT correspondence. In chapter 3 we will explain a notion of the entanglement entropy in quantum field theory. Also, the holographic dual of the entanglement entropy will be introduced. We will give a proof of the Ryu-Takayanagi formula and explain that the holographic entanglement entropy is regarded as a generalised quantity of the BH entropy.

Chapter 4 and 5 are a summary of the paper [1] and the work in progress [2]. In chapter 4 we give a proposal for the holographic entanglement entropy formula for Einstein gravity on dS spacetime based on the paper [1]. We find extremal surfaces in Poincaré dS coordinate by using a double Wick rotation from EAdS in Poincaré coordinates. We also comment on extremal surfaces in more general set of asymptotically dS spacetime. In chapter 5 we calculate the entanglement entropy in the free Euclidean $Sp(N)$ model. We compare the entanglement entropy in the $Sp(N)$ model with the proposed holographic entanglement entropy in chapter 4 and confirm that our proposal is sensible qualitatively. We also study the entanglement entropy in the $Sp(N)$ model in more detail. Chapter 6 is devoted to a conclusion and discussion.

Notation

We summarize our notations, here.

Dimension

Gravitational theories are defined on $(d+1)$-dimensional spacetime, while dual field theories are defined on d-dimensional spacetime.

Index

We use indices $M, N, \cdots$ for $(d+1)$-dimensional spacetime and indices $\mu, \nu, \cdots$ for d-dimensional spacetime.

Chapter 2

Holography

In this chapter, we give an overview of the AdS/CFT correspondence, firstly. Then, we will review the proposed dS/CFT correspondence and some related works.

2.1 AdS/CFT correspondence

2.1.1 Anti de Sitter spacetime

Before explanations of the AdS/CFT correspondence, we summarize AdS geometry.

$(d+1)$-dimensional anti-de Sitter spacetime (AdS) is defined as a hypersurface in Minkowski spacetime $(X_0, \cdots, X_{d+1})$ satisfying the relation

$$-X_0^2 + X_1^2 + \cdots + X_d^2 - X_{d+1}^2 = \ell_{\mathrm{AdS}}^2\,, \tag{2.1.1}$$

where ℓ_{AdS} is an AdS radius. When we take a time-direction Euclidean, the geometry becomes a hyperbolic space $\mathbb{H}_{d+1}$. The hyperbolic space is also called Euclidean AdS spacetime (EAdS). The metric of AdS spacetime is introduced by

$$\mathrm{d}s^2 = -\,\mathrm{d}X_0^2 + \mathrm{d}X_1^2 + \cdots + \mathrm{d}X_d^2 - \mathrm{d}X_{d+1}^2\,. \tag{2.1.2}$$

Naively, it seems that the metric (2.1.2) contains two time-directions. Nevertheless we will see that the metric (2.1.2) contains only one time-direction in the following discussion.

A global patch and a Poincaré patch are well-known coordinates of AdS spacetime. The global patch is obtained by the following parametrization;

$$\begin{aligned} X_0 &= \ell_{\text{AdS}} \cosh\rho \cos t\,, \qquad X_{d+1} = \ell_{\text{AdS}} \cosh\rho \sin t \\ X_i &= \ell_{\text{AdS}}\, \omega_i \sinh\rho \qquad (i = 1, \cdots, d)\,, \end{aligned} \tag{2.1.3}$$

where the range of ρ is $0 \le \rho < \infty$ and ω_i are polar coordinates defined as

$$\begin{aligned} \omega_1 = \cos\theta_1\,, \qquad \omega_2 = \sin\theta_1 \cos\theta_2\,, \qquad \cdots \\ \omega_{d-1} = \sin\theta_1 \cdots \sin\theta_{d-2} \cos\theta_{d-1}\,, \qquad \omega_d = \sin\theta_1 \cdots \sin\theta_{d-2} \sin\theta_{d-1}\,, \end{aligned} \tag{2.1.4}$$

respectively. The ranges of θ_i are $0 \le \theta_1 < \pi$ and $0 \le \theta_i < 2\pi$ for $i = 2, \cdots, d$. Then, the metric of the global coordinate is

$$\mathrm{d}s^2 = \ell_{\text{AdS}}^2 \left(-\cosh^2\rho\, \mathrm{d}\tau^2 + \mathrm{d}\rho^2 + \sinh^2\rho\, \mathrm{d}\Omega^2 \right)\,, \tag{2.1.5}$$

where $\mathrm{d}\Omega^2 = \mathrm{d}\theta_1^2 + \sin^2\theta_1 \mathrm{d}\theta_2^2 + \cdots + \sin^2\theta_1 \cdots \sin^2\theta_{d-2} \mathrm{d}\theta_{d-1}^2$. Here, we assume that the range of t is $-\infty < t < \infty$ although it is originally periodic with a period 2π. ρ is a radial direction of AdS spacetime, and the conformal boundary sits at $\rho \to \infty$. Dual CFTs live in the conformal boundary $\rho \to \infty$ if they exist.

The Poincaré patch is obtained by the following parametrization of X;

$$\begin{aligned} X_0 &= \frac{z}{2}\left(1 + \frac{\ell_{\text{AdS}}^2 + \boldsymbol{x}^2 - x_0^2}{z^2}\right)\,, \qquad X_d = \frac{z}{2}\left(1 - \frac{\ell_{\text{AdS}}^2 - \boldsymbol{x}^2 + x_0^2}{z^2}\right) \\ X_{d+1} &= \ell_{\text{AdS}} \frac{x_0}{z}\,, \qquad X_i = \ell_{\text{AdS}} \frac{x_i}{z} \qquad (i = 1, \cdots, d-1)\,, \end{aligned} \tag{2.1.6}$$

where z is a radial direction and its range is $0 \le z < \infty$. The horizon sits on $z \to \infty$ and the boundary is at $z = 0$. Then, the metric of the Poincaré coordinate is

$$\mathrm{d}s^2 = \ell_{\text{AdS}}^2 \frac{\mathrm{d}z^2 - \mathrm{d}x_0^2 + \sum_{i=1}^{d-1} \mathrm{d}x_i^2}{z^2}\,. \tag{2.1.7}$$

From (2.1.3) and (2.1.6), the Poincaré coordinates can be written as

$$\begin{aligned} z &= \ell_{\mathrm{AdS}} \frac{\cos\rho}{\cos t - \omega_d \sin\rho}\,, \quad x_0 = \ell_{\mathrm{AdS}} \frac{\sin t}{\cos t - \omega_d \sin\rho}\,, \\ x_i &= \ell_{\mathrm{AdS}} \frac{\omega_i \sin\rho}{\cos t - \omega_d \sin\rho} \qquad (i = 1, \cdots, d-1)\,. \end{aligned} \tag{2.1.8}$$

by using the global coordinate. From this parametrization one can see that the Poincaré patch covers a part of the global patch.

AdS geometry is a solution of the Einstein equation,

$$R_{MN} - \frac{1}{2} R g_{MN} + \Lambda g_{MN} = 0 \tag{2.1.9}$$

with a negative cosmological constant

$$\Lambda = -\frac{d(d-1)}{2\ell_{\mathrm{AdS}}^2}\,. \tag{2.1.10}$$

Here, R_{MN}, R and g_{MN} are the Ricci tensor, the Ricci scalar and the metric, respectively.

2.1.2 Derivation of the AdS/CFT correspondence from string theory

The AdS/CFT correspondence is a duality between conformal field theory and string theory on AdS spacetime. The most well-known example of the AdS/CFT correspondence is a duality between the $\mathcal{N} = 4$ $SU(N)$ super Yang-Mills theory and type IIB string theory on $\mathrm{AdS}_5 \times S^5$. In this subsection, we give a heuristic derivation of the AdS/CFT correspondence, especially the duality between the type IIB string theory on $\mathrm{AdS}_5 \times S^5$ and the $\mathcal{N} = 4$ super Yang-Mills theory, from D-brane set-ups in type IIB string theory.

Let us consider type IIB string theory and a low energy effective theory of a stack of N coincident D3-branes in flat spacetime. Energy of the D3-branes makes the geometry curved. The curved geometry is given by the black 3-brane solution;

$$\mathrm{d}s^2 = H(r)^{-1/2}(-\mathrm{d}x_0^2 + \mathrm{d}\boldsymbol{x}^2) + H(r)^{1/2}(\mathrm{d}r^2 + r^2\mathrm{d}\Omega_5^2) \tag{2.1.11}$$

with

$$H(r) = 1 + \frac{Q\ell_\mathrm{s}^2}{r^4}\,, \qquad Q = 4\pi g_\mathrm{s} N\,. \tag{2.1.12}$$

Here ℓ_s is a string scale and g_s is a coupling constant in string theory. Q is the total Ramond-Ramond charges which D3-branes have and is proportional to the number of D-branes N.

Let us introduce a typical length scale, ℓ_AdS, which represents a warp of the geometry,

$$\ell_\mathrm{AdS} := (4\pi g_\mathrm{s} N)^{1/4}\ell_\mathrm{s}\,. \tag{2.1.13}$$

Since the typical length scale becomes the AdS radius in an appropriate limit as we will see later, we use ℓ_AdS for the typical length. There are two different descriptions of the stack of N coincident D3-branes by the relationship between ℓ_s and ℓ_AdS. The first description is a probe approximation for $\ell_\mathrm{AdS} \ll \ell_\mathrm{s}$. In this limit, the warp of the geometry by the stack of the N D3-branes can be ignored compared with a thickness of the stack of the N D3-branes (it is typically order ℓ_s). Then, the stack of the N D3-branes can be described as a flat object in ten-dimensional spacetime. By taking 't Hooft limit,

$$N \to \infty\,, \quad g_\mathrm{s} \to 0 \quad \text{while} \quad \lambda = 4\pi g_\mathrm{s} N \quad \text{fixed}, \tag{2.1.14}$$

interactions with closed strings can be ignored because ten-dimensional Newton's constant becomes zero. In low energy limit $\ell_\mathrm{s} \to 0$, massless modes of open strings attaching D3-branes give the $\mathcal{N} = 4$ $SU(N)$ super Yang-Mills theory. The gauge coupling constant g_YM is related by the string coupling constant by the relation $g_\mathrm{YM}^2 = 4\pi g_\mathrm{s}$. On the other hand, massless modes of closed strings give ten-dimensional type IIB supergravity (SUGRA) on flat spacetime. Summarizing the above, we have obtained the low energy effective action,

$$S_{\lambda\ll 1} = \text{SYM on the D3-brane} + \text{SUGRA on flat spacetime}\,. \tag{2.1.15}$$

We can also describe the stack of the coincident D3-branes as a gravitational solution. When the Plank scale is enough smaller than the typical length scale, gravitational quantum corrections can be ignored. This condition can be written as

$$N \gg 1\,. \tag{2.1.16}$$

Furthermore, since massive modes have mass proportional to $1/\ell_{\mathrm{s}}$, corrections coming from these massive modes can be ignored if the typical length scale is enough larger than the string scale. That is, it is justified that the stack of the D3-branes is described as the gravitational solution. This condition is $\lambda \gg 1$ and is the opposite limit of the previous description.

In the region where $\ell_{\mathrm{AdS}} < r$, the warp factor $H(r)$ is approximately $H(r) = 1$, and the black 3-brane solution becomes flat spacetime. In this region, an effective theory is type IIB supergravity on flat spacetime. On the other hand, in the region where $0 \leq r < \ell_{\mathrm{AdS}}$, $H(r) = \ell_{\mathrm{AdS}}^4/r^4$ and the metric becomes

$$\mathrm{d}s^2 = \frac{r^2}{\ell_{\mathrm{AdS}}^2}(-\mathrm{d}x_0^2 + \mathrm{d}\boldsymbol{x}^2) + \frac{\ell_{\mathrm{AdS}}^2}{r^2}\mathrm{d}r^2 + \ell_{\mathrm{AdS}}^2\mathrm{d}\Omega_5^2 \tag{2.1.17}$$

The first two terms are AdS_5, and the third term is S^5 with a radius ℓ_{AdS}. Then, an effective theory is type IIB supergravity on $\mathrm{AdS}_5 \times S^5$. Summarizing the above, we have obtained the low energy effective action,

$$S_{\lambda\gg 1} = \text{SUGRA on } \mathrm{AdS}_5 \times S^5 + \text{SUGRA on flat spacetime}\,. \tag{2.1.18}$$

We have obtained two descriptions for the stack of the coincident N D3-branes. Dependent on the parameter λ, they are described as the effective actions (2.1.15) and (2.1.18). The AdS/CFT correspondence states that these two descriptions are same for all parameter region independent on the parameter λ. By subtracting "type IIB supergravity on flat spacetime," it means that the four-dimensional $\mathcal{N}=4$ $SU(N)$ super Yang-Mills theory is equivalent to type IIB string theory on $\mathrm{AdS}_5 \times S^5$ for arbitrary λ. The AdS/CFT correspondence is a merely conjecture, and there is no rigorous proof. Nevertheless, if we believe that the AdS/CFT description is valid for all parameter regions N and λ, we can regard that gauge theory defines "quantum" gravitational theory via the AdS/CFT correspondence. Therefore, the AdS/CFT correspondence is very fascinating.

From now on we take 't Hooft limit and $\lambda \to \infty$ limit unless otherwise noted.

GKPY relation

A useful relation in the AdS/CFT correspondence is the Gubser-Klebanov-Polyakov-Witten (GKPW) relation [15, 16]. It gives us the fundamental principle which represents how physical quantities in both gravity side and gauge side are related. It states that partition functions of both gravity side and gauge side are equivalent,

$$Z_{\mathrm{G}}[\phi_i] = \left\langle \exp\left(- \int \mathrm{d}^d x \, \mathcal{O}_i J_i(x) \right) \right\rangle_{\mathrm{CFT}} . \tag{2.1.19}$$

(2.1.19) is the Euclidean version of the GKPW relation, and we will see its detail from now on.

The left hand side in (2.1.19), $Z_{\mathrm{G}}[\phi_i]$, is the partition function of gravitational theory. The subscript G in Z_{G} means Gravity. ϕ_i represent all fields in the gravitational theory and are proportional to the external fields J_i at the conformal boundary. $Z_{\mathrm{G}}[\phi_i]$ is given by the path integral about ϕ_i with the boundary condition $\phi_i \propto J_i$ at the AdS boundary. In the case that the gravitational theory can be approximated by the classical gravity, the partition function can be evaluated as

$$Z_{\mathrm{G}}[\phi_i] = \mathrm{e}^{-S_{\mathrm{G}}[\phi_i]} , \tag{2.1.20}$$

where $S_{\mathrm{G}}[\phi_i]$ means the on-shell action with the boundary condition $\phi_i \propto J_i$ at the AdS boundary.

The right hand side in (2.1.19) is the expectation value in CFT. $\mathcal{O}_i$ are operators in CFT, and J_i are external fields coupled to $\mathcal{O}_i$. For instance, an energy-momentum tensor $T_{\mu\nu}$ in CFT couples to the metric $g_{\mu\nu}$.

By using the GKPW relation, we can evaluate physical quantities at strong coupling such as correlation functions. We can also evaluate thermodynamical quantities such as entropy. In subsection 3.3, we will see that the Ryu-Takayanagi conjecture is derived via the GKPW relation (2.1.19).

2.2 dS/CFT correspondence

The dS/CFT correspondence are proposed dualities between gravitational theories on dS spacetime and Euclidean conformal field theory on the future infinity $\mathcal{I}^+$. After we introduce dS geometry in the next subsection, problems to construct the dS/CFT correspondence are introduced in subsection 2.2.2. And, we discuss fundamental works in the rest of subsection 2.2.2 and the higher-spin dS/CFT correspondence in subsection 2.2.3.

2.2.1 de Sitter spacetime

$(d+1)$-dimensional de Sitter spacetime (dS) is defined as a hepersurface satisfying the following relation,

$$-X_0^2 + X_1^2 + \cdots + X_{d+1}^2 = \ell_{\mathrm{dS}}^2 \,, \tag{2.2.1}$$

where ℓ_{dS} is a dS radius. dS satisfies the Einstein equation with a positive cosmological constant $\Lambda = d(d-1)/2\ell_{\mathrm{dS}}^2$. A dS metric is introduced by

$$\mathrm{d}s^2 = -\,\mathrm{d}X_0^2 + \mathrm{d}X_1^2 + \cdots + \mathrm{d}X_{d+1}^2 \,. \tag{2.2.2}$$

Next, we introduce a global patch and a Poincaré patch of dS spacetime. Assume that coordinates X are

$$X_0 = \ell_{\mathrm{dS}} \sinh\tau \,, \qquad X_i = \ell_{\mathrm{dS}}\, \omega_i \cosh\tau \qquad (i = 1, \cdots, d+1) \,, \tag{2.2.3}$$

where $-\infty < \tau < \infty$ and ω_i are polar coordinates introduced in (2.1.4). The metric becomes

$$\mathrm{d}s^2 = \ell_{\mathrm{dS}}^2(-\mathrm{d}\tau^2 + \cosh^2\tau\,\mathrm{d}\Omega^2)\,. \tag{2.2.4}$$

This patch is called a global coordinate and covers a whole of dS spacetime. One can see that the dS spacetime has no spatial boundaries in opposite to AdS spacetime or flat spacetime. Nevertheless, dS spacetime has two boundaries, a future boundary $\mathcal{I}^+$ and a past boundary $\mathcal{I}^-$.

Next we move to the Poincaré patch. By substituting

$$\begin{aligned} X_0 &= -\ell_{\mathrm{dS}}\left(\sinh T + \frac{\boldsymbol{x}^2}{2}\mathrm{e}^T\right)\,, \\ X_i &= \ell_{\mathrm{dS}}\,x_i\,\mathrm{e}^T \qquad (i = 1,\cdots,d)\,, \\ X_{d+1} &= \ell_{\mathrm{dS}}\left(\cosh T - \frac{\boldsymbol{x}^2}{2}\mathrm{e}^T\right)\,, \end{aligned} \tag{2.2.5}$$

to (2.2.2), we obtain the Poincaré patch of dS spacetime. The metric becomes

$$\mathrm{d}s^2 = \ell_{\mathrm{dS}}^2(-\,\mathrm{d}T^2 + \mathrm{e}^{2T}\mathrm{d}\boldsymbol{x}^2)\,. \tag{2.2.6}$$

Note that the Poincaré patch covers only a half of dS spacetime. The future infinity corresponds to $T = \infty$. By introducing a conformal time,

$$\eta = \mathrm{e}^T\,, \tag{2.2.7}$$

the metric of the Poincaré patch of dS spacetime can be written as

$$\mathrm{d}s^2 = \ell_{\mathrm{dS}}^2\frac{-\mathrm{d}\eta^2 + \sum_{i=1}^d \mathrm{d}x_i^2}{\eta^2}\,. \tag{2.2.8}$$

By performing a double Wick rotation,

$$\eta \to iz\,, \quad \ell_{\mathrm{dS}} \to i\ell_{\mathrm{AdS}}\,, \tag{2.2.9}$$

the metric (2.2.8) becomes the metric in the Poincaré EAdS spacetime,

$$\mathrm{d}s^2 = \ell_{\mathrm{AdS}}^2\frac{\mathrm{d}z^2 + \sum_{i=0}^{d-1}\mathrm{d}x_i^2}{z^2}\,. \tag{2.2.10}$$

2.2.2 dS/CFT correspondence

While the AdS/CFT correspondence has been well-studied, the dS/CFT correspondence has been unclear even at the classical level. There are many problems to construct the dS/CFT correspondence. Main problems are follows;

1. For holography to work, it seems that spatial boundaries are needed like AdS spacetime. Since dS spacetime is topologically $\mathbb{R} \times S^d$ and has no spatial boundary, the holography might not work in dS spacetime, naively.

2. In analogy with the heuristic derivation of the AdS/CFT correspondence reviewed in subsection 2.1.2, we would like to derive the dS/CFT correspondence form string theory. Nevertheless, it is impossible to construct dS spacetime as solutions of supergravity [32]. Furthermore, α' corrections are not useful to resolve this problem [33]. That is, dS spacetime might not be constructed in string theory.

3. dS spacetime can be obtained by the analytical continuation of AdS spacetime. Then, it is expected that the dS/CFT correspondence might be obtained by the analytical continuation of the AdS/CFT correspondence. Nevertheless, we encounter imaginary fluxes or conformal weight, and ghosts fields in the gravity, when we perform the analytical continuation of AdS solutions in string theory. The analytical continuation is typically failed.

4. Analytical continuations of the AdS/CFT correspondence give non-unitary CFT. This fact seems to contradict with unitarity of dS spacetime since CFTs holographic dual to dS spacetime should have been unitary if the dS/CFT correspondence has held.

The problem 1 is resolved by assuming that dual field theories live on the past or future infinities. In fact, conformal field theories are assumed to live in $\mathcal{I}^-$ or $\mathcal{I}^+$ in the proposed dS/CFT correspondence. The problem 2 suggests that we should consider string theory without supersymmetry. There is no remarkable progress on this point in the context of the dS/CFT correspondence. In terms of the problem 3, a remarkable progress occurred. A concrete example was found. Although the analytical continuation of the AdS/CFT correspondence obtained from D-brane setups is not appropriate, we can obtain examples of the dS/CFT correspondence by the analytical continuation of higher-spin gauge theory. We will see a detail in subsection 2.2.3. The problem 4 may be resolved by pseudo unitarity which field theory holographic dual to dS has in general. Note however that there is a paper [34] which states that the dS/CFT correspondence does not exist.

From now on, we will review some fundamental works in the dS/CFT correspondence in the rest of subsection 2.2.2 and shows the concrete example of the dS/CFT correspondence in subsection 2.2.3.

Asymptotic symmetries

Consider asymptotic symmetries of dS spacetime [26] (see [35] for a review), which are defined as the group of allowed symmetries divided by the group of trivial symmetries. The allowed symmetries are symmetries which satisfy the boundary condition we impose, and the trivial symmetries are symmetries which have vanishing generators under constraints.

For simplicity, we consider a dS_3 case. By introducing a complex coordinate $z = x_1 + ix_2$ and its complex conjugate $\bar{z} = x - iy$, the Poincaré metric can be written as

$$\mathrm{d}s^2 = \ell_{\mathrm{dS}}^2(-\mathrm{d}\tau^2 + \mathrm{e}^{2\tau}\,\mathrm{d}z\mathrm{d}\bar{z})\,. \tag{2.2.11}$$

We impose a boundary condition at $\mathcal{I}^+$ as

$$g_{z\bar{z}} = \frac{\ell_{\mathrm{dS}}^2}{2}\mathrm{e}^{2\tau} + \mathcal{O}(1)\,, \quad g_{\tau\tau} = -\ell_{\mathrm{dS}}^2 + \mathcal{O}(\mathrm{e}^{-2\tau})\,, \quad g_{zz} = g_{\tau z} = \mathcal{O}(1)\,. \tag{2.2.12}$$

This boundary condition (2.2.12) is that in [35]. It is different from that in [26] and an analytical continuations of the AdS boundary condition in [36]. We choose the boundary condition (2.2.12) such that a perturbed Brown-York tensor should be finite. Here the Brown-York tensor for dS_3 is given by

$$T_{\mu\nu} = \frac{1}{4G_{\mathrm{N}}}\left(K_{\mu\nu} - \left(K + \frac{1}{\ell_{\mathrm{dS}}}\right)\gamma_{\mu\nu}\right)\,, \tag{2.2.13}$$

where $\gamma_{\mu\nu}$ is the induced metric on the future boundary $\mathcal{I}^+$ and $K_{\mu\nu}$ is an extrinsic curvature. See Appendix A for detail. The Brown-York tensor is zero for the planar metric. For the perturbed metric $g_{\mu\nu} + h_{\mu\nu}$, the Brown-York tensor becomes

$$\begin{aligned} T_{zz} &= \frac{1}{4G_{\mathrm{N}}}\left(h_{zz} - \partial_z h_{\tau z} + \frac{1}{2}\partial_\tau h_{zz}\right) , \\ T_{z\bar{z}} &= \frac{1}{4G_{\mathrm{N}}}\left(h_{zz} - \partial_z h_{\tau z} + \frac{1}{2}\partial_\tau h_{zz}\right) . \end{aligned} \tag{2.2.14}$$

The most general diffeomorphism which preserves the boundary condition can be written as

$$\zeta = U\partial_z + \frac{1}{2}\mathrm{e}^{-2\tau}U''\partial_{\bar{z}} - \frac{1}{2}U'\partial_\tau + \mathcal{O}(\mathrm{e}^{-2\tau}) + \text{complex conjugate}, \tag{2.2.15}$$

where $U = U(z)$ is a holomorphic function about z and the prime denote z-derivative. In the following, we omit the anti-holomorphic part, for simplicity. The reason that the most general diffeomorphism is (2.2.15) is that the Lie derivative of the metric $\delta_\zeta g_{\mu\nu} = -\mathcal{L}_\zeta g_{\mu\nu} = -(\nabla_\mu\zeta_\nu + \nabla_\nu\zeta_\mu) = -(g_{\mu\rho}\partial_\nu\zeta^\rho + g_{\nu\rho}\partial_\mu\zeta^\rho + \zeta^\rho\partial_\rho g_{\mu\nu})$[1], becomes

$$\delta_\zeta g_{zz} = -\frac{\ell_{\mathrm{dS}}^2}{2}U''' , \quad \delta_\zeta g_{\tau\tau} = \delta_\zeta g_{z\bar{z}} = \delta_\zeta g_{\bar{z}\bar{z}} = 0 , \tag{2.2.16}$$

and the change of the metric preserves the boundary condition (2.2.12).

A special case is

$$U = \alpha + \beta z + \gamma z^2 \tag{2.2.17}$$

with complex constant parameters α, β, γ. In this case, U''' vanishes. It means that the metric is invariant under the diffeomorphism. Consider a commutation relation between Lie derivatives with the vector fields ζ_1 and ζ_2, which have parameters U_1 and U_2, respectively. The commutation relation becomes

$$[\mathcal{L}_{\zeta_1}, \mathcal{L}_{\zeta_2}] = \mathcal{L}_{[\zeta_1,\zeta_2]} = \mathcal{L}_{\zeta_3} \tag{2.2.18}$$

where ζ_3 is a vector field with parameter $U_3 = U_1U_2' - U_1'U_2$. The vector fields with

[1] The Lie derivative of a rank two tensor $T_{\mu\nu}$ is given by $\mathcal{L}_\zeta T_{\mu\nu} = T_{\mu\rho}\partial_\nu\zeta^\rho + T_{\rho\nu}\partial_\mu\zeta^\rho + \zeta^\rho\partial_\rho T_{\mu\nu}$.

parameters $U=1, U=2z$, and $U=z^2$ satisfy commutation relations,

$$\begin{gathered}[\zeta_{U=1},\zeta_{U=1}]=[\zeta_{U=2z},\zeta_{U=2z}]=[\zeta_{U=z^2},\zeta_{U=z^2}]=0\,,\\ [\zeta_{U=1},\zeta_{U=2z}]=2\zeta_{U=1}\,,\quad [\zeta_{U=2z},\zeta_{U=z^2}]=2\zeta_{U=z^2}\,,\quad [\zeta_{U=1},\zeta_{U=z^2}]=\zeta_{U=2z}\,.\end{gathered} \tag{2.2.19}$$

This is an $SL(2,\mathbb{C})$ algebra. In conclusion the diffeomorphism with the parameter $U=\alpha+\beta z+\gamma z^2$ generates the $SL(2,\mathbb{C})$ isometry.

Next, we will find the central charge. The diffeomorphism acts on the perturbative Brown-York tensor as

$$\delta_\zeta T_{zz}=-U\partial T_{zz}-2U'T_{zz}-\frac{\ell_{\rm dS}}{8G_{\rm N}}U'''\,. \tag{2.2.20}$$

Since the stress energy tensor T transforms as

$$\delta_\zeta T=-\frac{c}{12}U'''-2U'T-U\partial T\,, \tag{2.2.21}$$

in general CFTs with the central charge c, we notice that the central charge is

$$c=\frac{3\ell_{\rm dS}}{2G_{\rm N}}\,. \tag{2.2.22}$$

State/operator relation

The dS/CFT correspondence proposes that the wave function of a universe which is asymptotically dS spacetime is evaluated by a partition function of Euclidean CFT,

$$\Psi[g_{ij}]=Z_{\rm CFT}[g_{ij}]\,. \tag{2.2.23}$$

The left hand side is the wave function of a universe with a boundary metric g_{ij}, and the right hand side is the partition function on the manifold with the metric g_{ij}. We develop the state/operator relation from the discussion about propagator below.

Consider propagators in dS_3 [26], firstly. It is expected that propagators from the future infinity to the future infinity in dS spacetime become those of the dual

CFT[2]. We will see this from now on.

The Klein-Gordon equation of a scalar field with mass m is given by

$$m^2\ell_{\rm dS}^2\phi = \ell_{\rm dS}^2\nabla^2\phi = -\partial_\tau^2\phi - 2\partial_\tau\phi + 4\mathrm{e}^{-2\tau}\partial_z\partial_{\bar{z}}\phi\,. \tag{2.2.24}$$

Since the last term can be negligible near the future infinity, solutions of the wave equation behave as

$$\phi(\tau, z, \bar{z}) \sim \mathrm{e}^{-h_\pm\tau}\phi_\pm(z,\bar{z})\,, \quad \tau\to\infty \tag{2.2.25}$$

where $h_\pm$ are defined as

$$h_\pm = 1 \pm \sqrt{1 - m^2\ell_{\rm dS}^2}\,. \tag{2.2.26}$$

We only consider the case where $0 < m^2\ell_{\rm dS}^2 < 1$[3]. In this case, $h_\pm$ are real and satisfy the relation $h_- < 1 < h_+$. We impose the boundary condition on $\mathcal{I}^+$,

$$\lim_{\tau\to\infty}\phi(\tau, z, \bar{z}) = \mathrm{e}^{-h_-\tau}\phi_-(z,\bar{z})\,. \tag{2.2.27}$$

In analogy with the AdS/CFT correspondence, the dS/CFT proposes that ϕ_- is holographic dual to an operator $\mathcal{O}_\phi$ with a conformal dimension h_+ in the dual CFT. The two-point function of $\mathcal{O}_\phi$ is proportional to the quadratic coefficient of ϕ_- in the on-shell action. If we perform the same calculation in the AdS/CFT correspondence case, we can obtain the two-point function of $\mathcal{O}_\phi$,

$$\langle\mathcal{O}_\phi(z,\bar{z})\mathcal{O}_\phi(v,\bar{v})\rangle = \frac{\text{const.}}{|z-v|^{2h_+}}\,. \tag{2.2.28}$$

In conclusion, we reproduce the two-point function of $\mathcal{O}_\phi$ of h_+ from the gravity side. This is another evidence that the dual CFT lives on the future infinity. There are similar works [37–39].

Next, we consider the two-point function, more precisely following by [27]. We want to focus on the result and skip the detail discussion here. We consider four-

[2] In [26], propagators from the past infinity from the past infinity are considered. There is no change in the discussion.

[3] In the case $m^2\ell_{\rm dS}^2 > 1$, $h_\pm$ become complex. This suggests that the dual CFT is non-unitary.

dimensional Pincaré dS spacetime and assume that all fields start in the Bunch-Davies vacuum. To begin with, calculate the wave function as a function of a massless scalar field at a reference time $\eta_{\rm c}$. We assume that the scaler field is small.

By substituting the classical solution in momentum space,

$$\phi = \phi^0_{\boldsymbol{k}} \frac{(1-ik\eta)\mathrm{e}^{ik\eta}}{(1-ik\eta_{\rm c})\mathrm{e}^{ik\eta_{\rm c}}} \tag{2.2.29}$$

to the action, the quadratic term of the action is computed as

$$\begin{aligned} iS &= i\int \frac{\mathrm{d}k^3}{(2\pi)^3}\frac{1}{2}\frac{\ell_{\rm dS}^2}{\eta_{\rm c}^2}\phi^0_{-\boldsymbol{k}}\partial_\eta\phi^0_{\boldsymbol{k}}|_{\eta=\eta_{\rm c}} = i\int \frac{\mathrm{d}k^3}{(2\pi)^3}\frac{1}{2}\frac{\ell_{\rm dS}^2 k^2}{\eta_{\rm c}^2(1-ik\eta_{\rm c})}\phi^0_{-\boldsymbol{k}}\phi^0_{\boldsymbol{k}} \\ &\sim \int \frac{\mathrm{d}k^3}{(2\pi)^3}\frac{1}{2}\ell_{\rm dS}^2\left(i\frac{k^2}{\eta_{\rm c}} - k^3 + \cdots\right)\phi^0_{-\boldsymbol{k}}\phi^0_{\boldsymbol{k}}\,, \end{aligned} \tag{2.2.30}$$

where we ignore oscillation terms. We obtain the two-point function in momentum space,

$$\langle \mathcal{O}(\boldsymbol{k})\mathcal{O}(\boldsymbol{k}')\rangle := \left.\frac{\delta^2 Z}{\delta\phi^0_{\boldsymbol{k}}\delta\phi^0_{\boldsymbol{k}'}}\right|_{\phi^0=0} \sim -(2\pi)^3\delta^3(\boldsymbol{k}+\boldsymbol{k}')\ell_{\rm dS}^2 k^3\,. \tag{2.2.31}$$

Compare this result with a corresponding EAdS computation. In EAdS computation, we obtain the similar result,

$$\langle \mathcal{O}(\boldsymbol{k})\mathcal{O}(\boldsymbol{k}')\rangle_{\rm EAdS} := \left.\frac{\delta^2 Z}{\delta\phi^0_{\boldsymbol{k}}\delta\phi^0_{\boldsymbol{k}'}}\right|_{\phi^0=0} \sim (2\pi)^3\delta^3(\boldsymbol{k}+\boldsymbol{k}')\ell_{\rm AdS}^2 k^3\,. \tag{2.2.32}$$

This differs by a sign from the dS computation in four-dimensional case. We can explain this fact by the analytical continuation from EAdS spacetime to dS spacetime. Since the degree of the dS radius $\ell_{\rm dS}$ is different from two in other dimension, the extra i appears in dS computation. The dS computation and the EAdS computation are not related each other by the analytical continuation. In conclusion, the $\mathrm{dS}_4/\mathrm{CFT}_3$ correspondence, or physical observables in the $\mathrm{dS}_4/\mathrm{CFT}_3$ correspondence at least, would be obtained by the analytical continuation. But the dS/CFT correspondence in other dimension may not be obtained by the analytical continuation.

Other related works

There are many works related to the dS/CFT correspondence. We summarize these works;

i. T-duality in a time direction

Hull and his collaborators considered T-duality in a time direction [40, 41]. The time-like T-duality turns type IIA and IIB string theories to type IIB* and IIA* string theories, respectively, where type II* theories are new theories obtained by time-like T-duality. D-branes in type II string theories are translated to branes at which open strings are confined in type II* string theories. Such branes are called E-branes. Dp-brane and Ep-brane are connected by time-like T-duality.

As explained in the previous section, by considering N coincident D3-branes it has been conjectured that the type IIB string theory on $\mathrm{AdS}_5 \times S^5$ has a dual description by the $\mathcal{N} = 4$ super Yang-Mills theory with gauge group $SU(N)$. Similarly, Hull has proposed that type IIB* string theory on $\mathrm{dS}_5 \times H^5$ is dual to the Euclidean[4] $\mathcal{N} = 4$ super Yang-Mills theory with gauge group $SU(N)$ by considering the stuck of N E4-branes. Since the theories in both sides include ghost fields which have the wrong sign of the kinetic terms, this duality is pathological.

ii. Inflation in the dS/CFT correspondence

It is known that our universe is approximately dS geometry near the past infinity and future infinity. The paper [42] discusses the inflation in the dS/CFT correspondence.

The dS geometry[5],

$$\mathrm{d}s^2 = \ell_{\mathrm{dS}}^2(-\mathrm{d}\tau^2 + \mathrm{e}^{2H\tau}\mathrm{d}\boldsymbol{x}^2)\,, \tag{2.2.33}$$

is invariant under the following transformations,

$$\tau \to \tau + \lambda\,, \quad x_i \to \mathrm{e}^{-\lambda H} x_i\,. \tag{2.2.34}$$

[4]In Hull's paper [40], the word "Euclidean" means the time-like reduction. On the other hand, the words "Euclideanised" or "Wick-rotated" means the Wick-rotated theory.

[5]Since we consider geometries which approach dS spacetime near the past and future infinities later, we introduce the Hubble parameter H.

This transformation generates time evolution in the bulk theory and scale transformations in the dual field theory. Then, the future infinity in the bulk gravity corresponds to the UV of the dual conformal field theory, while the past infinity corresponds to the IR.

When our universe can be approximated by the Robertson-Walker metric,

$$\mathrm{d}s^2 = \ell_{\mathrm{dS}}^2(-\mathrm{d}\tau^2 + R(\tau)^2\mathrm{d}\boldsymbol{x}^2)\,, \tag{2.2.35}$$

the dual field theory has no conformal symmetry, and the renormalisation flow from UV to IR in the dual field theory corresponds to the inverse of time evolution in the gravity. Furthermore, the Hubble parameter $H(\tau) = \dot{R}/R$ is related by the central charge as

$$c = \frac{3}{2HG_{\mathrm{N}}}\,. \tag{2.2.36}$$

This suggests that the Hubble parameter decreases monotonically in time evolution.

Inflation and cosmological observables

In [43, 44], the authors proposed a holographic description of inflation with single scalar field in four-dimensional universe and a relation between cosmological observables and correlation functions in a dual three-dimensional quantum field theory. It is shown that the holographic description gives correct predictions of the standard inflation in the region where gravity is weak.

2.2.3 Concrete example of the dS/CFT correspondence

Although the dS/CFT was proposed in 2001 by Witten [25] and Strominger [26] independently, concrete examples had not been known yet until 2011. In 2011, Anninos, Hartman and Strominger have conjectured that four-dimensional Vasiliev's higher-spin theory [45] on dS spacetime is holographic dual to the three-dimensional symplectic fermion model [46, 47]. This is the dS/CFT correspondence version of GKPY duality [28, 29], which is a duality between Vasiliev's higher-spin theory on AdS_4 and three-dimensional $O(N)$ vector model. As noted in the previous subsection, the analytical continuation from AdS spacetime to dS spacetime produces something pathological in the bulk. Nevertheless AdS spacetime and dS spacetime in Vasiliev's higher-spin gauge theory are simply related by the reverse of the cosmological constant, $\Lambda \to -\Lambda$, while Newton's constant $G_{\rm N}$ fixed. Since N is proportional to $1/\Lambda G_{\rm N}$, $N \sim 1/\Lambda G_{\rm N}$, in the GKPY duality, the transformation in the bulk becomes $N \to -N$ in CFT. The global symmetry is changed from $O(N)$ to $Sp(N)$. Furthermore, if we reconcile propagators in the bulk and the boundary, we should take the filed in the $Sp(N)$ model be anti-commuting. We can summarize the above discussion as follows.

$$
\begin{array}{ccc}
\text{Vasiliev's HS gauge theory on AdS} & \overset{\text{GKPY duality}}{\longleftrightarrow} & O(N) \text{ vector model} \\
\updownarrow \Lambda \to \Lambda, G_{\rm N} \text{ fixed} & & \updownarrow N \to -N \\
\text{Vasiliev's HS gauge theory on dS} & & Sp(N) \text{ vector model}
\end{array}
$$

The higher-spin dS/CFT correspondence is supported by the GKPY duality. If GKPY duality holds, the higher-spin dS/CFT correspondence holds automatically.

Before concluding this chapter, we comment on some works after the realization of the higher-spin dS correspondence. The state/operator correspondence in higher-spin dS/CFT correspondence was considered in [48]. The higher-spin dS/CFT correspondence has been extended to non-minimal Vasiliev's higher spin gauge theory [49]. The three-dimensional extension of higher spin dS/CFT correspondence has been discussed in [50–52]. The bulk theory is $SL(n, \mathbb{C})$ Chern-Simons theory.

Chapter 3

Entanglement Entropy and its Holographic Dual

3.1 Entanglement entropy

3.1.1 Definition

In this subsection, we define an entanglement entropy and show some examples in quantum mechanics and quantum field theory.

Let us define the entanglement entropy. We consider a quantum system defined on the Hilbert space $\mathcal{H}_{\rm tot}$. Firstly, we divide the total Hilbert space $\mathcal{H}_{\rm tot}$ into $\mathcal{H}_A$ and $\mathcal{H}_B$. It means that the total Hilbert space is decomposed as

$$\mathcal{H}_{\rm tot} = \mathcal{H}_A \times \mathcal{H}_B \,. \tag{3.1.1}$$

In quantum field theory, we take a time-slice of spacetime and define the Hilbert space on it. The division of the total Hilbert space into two Hilbert space corresponds to a division of the time-slice into a region A and a region B. See Fig. 3.1.

When the quantum system is described by a wave function $|\Psi\rangle$, a total density matrix is given by

$$\rho_{\rm tot} = |\Psi\rangle\langle\Psi| \,. \tag{3.1.2}$$

By tracing out the degree of freedom of the region B, a reduced density matrix ρ_A

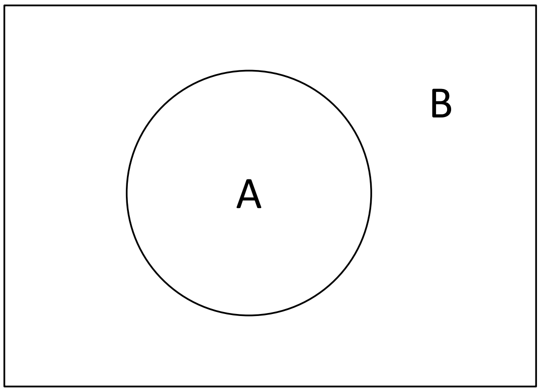

Figure 3.1: Division of the time-slice into the region A and the region B. It is not necessary for the region A to be connected.

is defined as

$$\rho_A := \mathrm{tr}_B\, \rho_{\mathrm{tot}}\,. \tag{3.1.3}$$

The entanglement entropy is defined as a von Neumann entropy,

$$S_A := -\mathrm{tr}_A\, \rho_A \log \rho_A\,. \tag{3.1.4}$$

To clarify aspects of the entanglement entropy, let us consider a spin-2 systems as an example in quantum mechanics. Suppose that the wave function is

$$|\Psi\rangle = \cos\theta |0\rangle_A |1\rangle_B + \sin\theta |1\rangle_A |0\rangle_B\,. \tag{3.1.5}$$

The reduced density matrix becomes

$$\rho_A = \mathrm{tr}_{\mathcal{H}_B} |\Psi\rangle\langle\Psi| = \cos^2\theta |0\rangle_A \langle 0|_A + \sin^2\theta |1\rangle_A \langle 1|_A\,. \tag{3.1.6}$$

Since eigenvalues of ρ_A are $\cos^2\theta$ and $\sin^2\theta$, the entanglement entropy becomes

$$S_A = -\cos^2\theta \log\cos^2\theta - \sin^2\theta \log\sin^2\theta\,. \tag{3.1.7}$$

For $\theta = 0$ and $\theta = \pi/2$, the states are $|\Psi\rangle = |0\rangle_A|1\rangle_B$ and $|\Psi\rangle = |1\rangle_A|0\rangle_B$, respectively, and the entanglement entropy vanishes as $S_A = 0$. This is because the state is a pure state and the density matrix can be written by a direct product as $\rho_{\rm tot} = \rho_A \times \rho_B$. For $\theta = \pm\pi/4$, the states are $|\Psi\rangle = (|0\rangle_A|1\rangle_B \pm |1\rangle_A|0\rangle_B)/\sqrt{2}$, and the entanglement entropy is $S_A = \log 2$. The 2 in the logarithm represents that degree of freedom of the two-spin system is two. Entanglement entropies measures the strength of the correlations between the subsystems A and B.

3.1.2 Replica method

Next, we would like to discuss the entanglement entropy in quantum field theory. In quantum field theory it is difficult to calculate the von Neumann entropy (3.1.4) directly because quantum field theory has infinite degrees of freedom defined on each points of spacetime and the von Neumann entropy includes the logarithm of the reduced density matrix. Then, we use analytical expressions of (3.1.4),

$$S_A = -\lim_{n\to 1}\frac{\partial}{\partial n}{\rm tr}_A\rho_A^n = -\lim_{n\to 1}\frac{\partial}{\partial n}\log{\rm tr}_A\rho_A^n\,, \tag{3.1.8}$$

instead of calculating (3.1.4). We calculate ${\rm tr}_A\rho_A^n$ to evaluate the entanglement entropy. We make a comment on the analytical continuation about n. n is assumed to be integer and ${\rm tr}_A\rho_A^n$ is calculated for integer n. Nevertheless, when we calculate the entanglement entropy, we assume that n is a real number. Although the analytical continuation about n does not always give a correct result, counterexamples about the entanglement entropy are not known yet. We don't discuss the validity of the analytical continuation about n anymore.

From now on, we explain a replica method [53]. Let us calculate the entanglement entropy on the ground state $|\Psi\rangle$[1]. For simplicity we consider Euclidean scalar field theory. When the system is invariant under a time-translation, we can take $x_0 = 0$ without loss of generality. The wave function for the ground state $\langle\phi|\Psi\rangle$ is given by

$$\Psi[\phi(\boldsymbol{x}), x_0 = 0] = \frac{1}{\sqrt{Z}}\int\prod_{-\infty<x_0<0}\prod_{\boldsymbol{x}}\mathcal{D}\phi\,{\rm e}^{-S[\phi]}\delta[\phi(0,\boldsymbol{x}) - \phi(\boldsymbol{x})] \tag{3.1.9}$$

in path-integral representation. Here, $S[\phi]$ is an action, and Z is a partition function. We insert $1/\sqrt{Z}$ as a normalisation of the wave function. A complex conjugate of

[1]We should write $|0\rangle$ instead of $|\Psi\rangle$ because we consider the vacuum state.

the wave function is defined as

$$\Psi^*[\phi(\boldsymbol{x}), x_0 = 0] = \frac{1}{\sqrt{Z}} \int \prod_{0<x_0<\infty} \prod_{\boldsymbol{x}} \mathcal{D}\phi \, \mathrm{e}^{-S[\phi]} \delta[\phi(0, \boldsymbol{x}) - \phi(\boldsymbol{x})] \,. \tag{3.1.10}$$

One can see that the wave function satisfies the normalisation condition

$$\int \mathcal{D}\phi \, \Psi^*[\phi(\boldsymbol{x})] \Psi[\phi(\boldsymbol{x})] = 1 \,. \tag{3.1.11}$$

The reduced density matrix becomes

$$[\rho_A]_{\phi_-\phi_+} = \frac{1}{Z} \int \mathcal{D}\phi \, \mathrm{e}^{-S[\phi]} \prod_{\boldsymbol{x} \in A} \delta[\phi(-0, \boldsymbol{x}) - \phi_-(\boldsymbol{x})] \delta[\phi(+0, \boldsymbol{x}) - \phi_+(\boldsymbol{x})] \tag{3.1.12}$$

where ϕ_+ and ϕ_- are boundary conditions. The reduced density matrix $[\rho_A]_{\phi_-\phi_+}$ is represented pictorially in Fig. 3.2. By using this expression of the reduced density

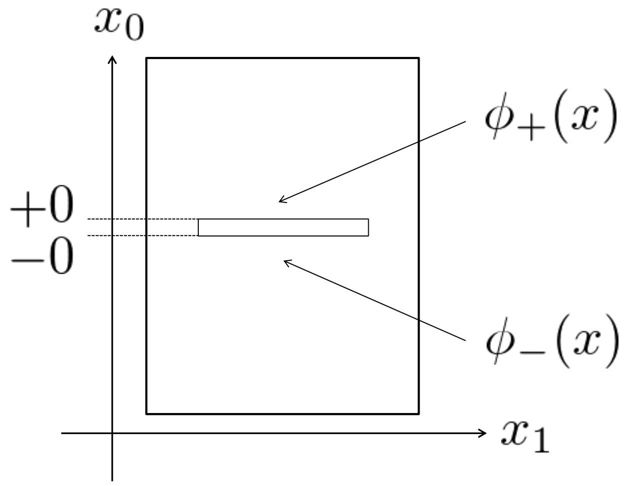

Figure 3.2: Path-integral representation of the reduced density matrix. The other directions $x_2, \cdots, x_{d-1}$ are omitted.

matrix, we obtain

$$\mathrm{tr}_A \, \rho_A^n = \frac{1}{Z^n} \int \prod_{x \in \Sigma_n} \mathcal{D}\phi \, \mathrm{e}^{-S[\phi]} \tag{3.1.13}$$

where Σ_n is an n-sheeted Riemann surface constructed as follows. Firstly, we prepare n sheets of the original spacetime Σ. We number each spacetime Σ and each field

ϕ on each Σ, and refer to them as $\Sigma^{(1)}, \cdots, \Sigma^{(n)}$ and $\phi^{(1)}, \cdots, \phi^{(n)}$, called replica fields, respectively. Σ_n is the spacetime obtained by gluing each Σ with the boundary condition[2]

$$\phi_+^{(1)} = \phi_-^{(2)}\,, \quad \phi_+^{(2)} = \phi_-^{(3)}\,, \quad \cdots \quad, \phi_+^{(n)} = \phi_-^{(1)}\,. \tag{3.1.14}$$

See Fig. 3.3. From this expression, we notice that $\mathrm{tr}_A\,\rho_A^n$ is the same as a partition function on Σ_n (up to a normalisation).

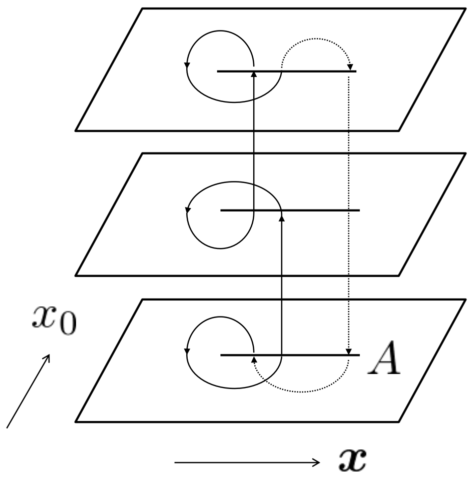

Figure 3.3: n-sheeted Riemann surface Σ_n. For simplicity, the figure is $n = 3$ case.

Let us consider a free scalar field theory and calculate the entanglement entropy for a half-plane for an example. We take the region A as $x_1 > 0$. In this case, we take n a fraction like $1/M$ where M is integer, to make calculations very easy[3]. According to the replica trick, we need to evaluate the partition function on $\mathbb{R}^2/\mathbb{Z}_M \times \mathbb{R}^{d-2}$. In the following we show a detail calculation of the entanglement entropy.

An Euclidean Lagrangian is

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi(x))^2 + \frac{1}{2}m^2\phi(x)^2\,. \tag{3.1.15}$$

[2] For fermions, we need to take anti-periodic boundary condition instead of the periodic boundary condition.

[3] If we don't take n a fraction, we need to diagonalize a set of replica fields $(\phi^{(1)}, \cdots, \phi^{(n)})$, and find eigenvalues of a non-diagonal matrix. See [54], for detail.

By using a Fourier expansion of $\phi(x)$,

$$\phi(x) = \frac{1}{V_d} \sum_n \mathrm{e}^{-ik_n \cdot x} \phi(k_n) \,, \tag{3.1.16}$$

the action can be written as

$$S = \frac{1}{V_d} \sum_{k_n^0 > 0} (k_n^2 + m^2)[(\mathrm{Re}\, \phi(k_n))^2 + (\mathrm{Im}\, \phi(k_n))^2] \,. \tag{3.1.17}$$

A measure of path integral is written as

$$\mathcal{D}\phi(x) = \prod_{k_n^0 > 0} \mathrm{d}\, \mathrm{Re}\, \phi(k_n)\, \mathrm{d}\, \mathrm{Im}\, \phi(k_n) \tag{3.1.18}$$

in a Fourier space. In continuous limit of momentum, the summation of n is replaced by an integral of momentum,

$$\frac{1}{V_d} \sum_n \to \int \frac{\mathrm{d}^d k}{(2\pi)^d} \,. \tag{3.1.19}$$

By using the above expressions, the partition function on $\mathbb{R}^d$ becomes

$$Z_{\mathbb{R}^d} = \int \prod_{k_n^0 > 0} \mathrm{d}\, \mathrm{Re}\, \phi(k_n) \mathrm{d}\, \mathrm{Im}\, \phi(k_n) \exp\left(-S\right) = \det \left(\frac{\pi V_d}{k_n^2 + m^2} \right)^{1/2} \,. \tag{3.1.20}$$

Then, the logarithm of the partition function can be calculated as

$$\begin{aligned}
\log Z_{\mathbb{R}^d} &= \log \det \left(\frac{\pi V_d}{k_n^2 + m^2} \right)^{1/2} = \log \prod_{k_n} \left(\frac{\pi V_d}{k_n^2 + m^2} \right)^{1/2} \\
&= \mathrm{Tr} \log \left(\frac{\pi V_d}{k_n^2 + m^2} \right)^{1/2} = \sum_{k_n} \log \left(\frac{\pi V_d}{k_n^2 + m^2} \right)^{1/2} \\
&= -\frac{1}{2} \sum_{k_n} \log \left(k_n^2 + m^2 \right) \quad \text{(ignore a factor coming from } \pi V_{d+1}) \\
&= -\frac{V_d}{2} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \log \left(k^2 + m^2 \right) \\
&= V_d \int_{\varepsilon^2}^{\infty} \frac{\mathrm{d}s}{2s} \int \frac{\mathrm{d}^d k}{(2\pi)^d}\, \mathrm{e}^{-s(k^2 + m^2)}
\end{aligned}$$

$$= V_d \int_{\varepsilon^2}^{\infty} \frac{\mathrm{d}s}{2s} (4\pi s)^{-d/2} \mathrm{e}^{-sm^2} \tag{3.1.21}$$

where we use the Schwinger's parametrization,

$$\log \alpha = -\int_0^{\infty} \frac{\mathrm{d}t}{t} \mathrm{e}^{-\alpha t} \, . \tag{3.1.22}$$

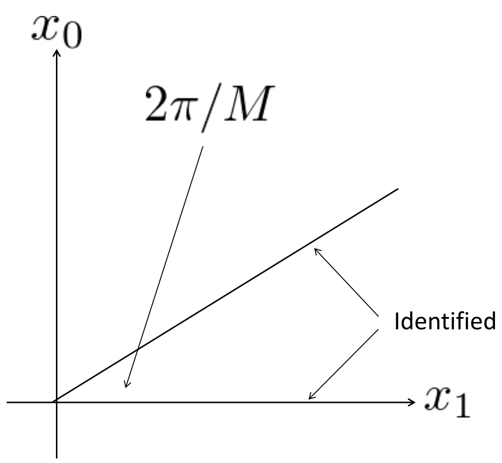

Figure 3.4: $\mathbb{R}^2/\mathbb{Z}_M$. These two lines are identified.

Next, we calculate the partition function on $\mathbb{R}^2/\mathbb{Z}_M \times \mathbb{R}^{d-2}$. $\mathbb{R}^2/\mathbb{Z}_M$ is a space such that $\mathbb{R}^2$ spanned by (x_0, x_1) is identified under the rotated transformation g with the angle $2\pi/M$ as

$$g : (x_0, x_1) \to \left(\cos \frac{2\pi}{M} x_0 - \sin \frac{2\pi}{M} x_1, \sin \frac{2\pi}{M} x_0 + \cos \frac{2\pi}{M} x_1 \right) \, . \tag{3.1.23}$$

See Fig. 3.4. Also, under the rotating transformation g , the field is identified as

$$\phi(r, \theta + 2\pi/M) = \phi(r, \theta) \tag{3.1.24}$$

where we introduce polar coordinates (r, θ). The logarithm of the partition function is given by

$$\log Z_{\mathbb{R}^2/\mathbb{Z}_M \times \mathbb{R}^{d-2}} = -\sum_{k,\ell} \log(k^2 + \ell^2 + m^2) \tag{3.1.25}$$

where k and ℓ are conjugate momentums of spacetimes spanned by $\mathbb{R}^{d-2}$ and $\mathbb{R}^2/\mathbb{Z}_M$, respectively. The continuous limit of the summation of k gives

$$\sum_k \to V_{d-2} \int \frac{\mathrm{d}^{d-2}k}{(2\pi)^{d-2}} \tag{3.1.26}$$

as before. In order to perform the summation of ℓ, we insert a projection operator $\sum_{j=0}^{M-1} g^j/M$ into the summation of ℓ. That is, the summation is written as

$$\sum_\ell \log(k^2+\ell^2+m^2) = \int \frac{\mathrm{d}^2\ell}{(2\pi)^2} \langle \ell | \log(k^2+\ell^2+m^2) \sum_{j=0}^{M-1} \frac{g^j}{M} |\ell\rangle \,. \tag{3.1.27}$$

g operates on the ket vector $|\ell\rangle = |\ell_0,\ell_1\rangle$, which is normalised as $\langle \ell|\ell'\rangle = (2\pi)^2\delta^2(\ell-\ell')/V_2$ and $\langle\ell|\ell\rangle = 1$, as

$$g|\ell_0,\ell_1\rangle = \left|\cos\frac{2\pi}{M}\ell_0 - \sin\frac{2\pi}{M}\ell_1, \sin\frac{2\pi}{M}\ell_0 + \cos\frac{2\pi}{M}\ell_1\right\rangle \,. \tag{3.1.28}$$

By introducing the Schwinger's parametrization, the partition function becomes

$$\log Z_{\mathbb{R}^2/\mathbb{Z}_M\times\mathbb{R}^{d-2}} = V_{d-2}V_2 \int_{\varepsilon^2}^\infty \frac{\mathrm{d}s}{2s} \int \frac{\mathrm{d}^{d-2}k}{(2\pi)^{d-2}} \mathrm{e}^{-sk^2-sm^2} \int \frac{\mathrm{d}^2\ell}{(2\pi)^2} \sum_{j=0}^{M-1} \langle\ell|\mathrm{e}^{-s\ell^2}\frac{g^j}{M}|\ell\rangle \tag{3.1.29}$$

For $j=0$, it is just the partition function on $\mathbb{R}^d$ divided by M. For $j\neq 0$, we obtain

$$V_2 \int \frac{\mathrm{d}^2\ell}{(2\pi)^2} \langle\ell|\mathrm{e}^{-s\ell^2}\frac{g^j}{M}|\ell\rangle = \frac{1}{M}\int \mathrm{d}^2\ell\, \mathrm{e}^{-s\ell^2}\delta^2(\ell - g^j\cdot\ell) = \frac{1}{4M\sin^2\frac{\pi j}{M}} \,. \tag{3.1.30}$$

By using the identity,

$$\sum_{j=1}^{M-1} \frac{1}{\sin^2\frac{\pi j}{M}} = \frac{M^2-1}{3} \,, \tag{3.1.31}$$

we obtain the partition function,

$$\begin{aligned} \log Z_{\mathbb{R}^2/\mathbb{Z}_M\times\mathbb{R}^{d-2}} &= V_{d-2}\frac{M^2-1}{12M} \int_{\varepsilon^2}^\infty \frac{\mathrm{d}s}{2s} \int \frac{\mathrm{d}^{d-2}k}{(2\pi)^{d-2}} \mathrm{e}^{-sk^2-sm^2} + \frac{1}{M}\log Z_{\mathbb{R}^d} \\ &= V_{d-2}\frac{M^2-1}{12M} \int_{\varepsilon^2}^\infty \frac{\mathrm{d}s}{2s} \frac{\mathrm{e}^{-sm^2}}{(4\pi s)^{\frac{d-2}{2}}} + \frac{1}{M}\log Z_{\mathbb{R}^d} \,. \end{aligned} \tag{3.1.32}$$

The entanglement entropy becomes

$$\begin{aligned} S_A &= -\lim_{M\to 1} \frac{\partial}{\partial(1/M)} \left(\log Z_{\mathbb{R}^2/Z_M \times \mathbb{R}^{d-2}} - \frac{1}{M} \log Z_{\mathbb{R}^d} \right) \\ &= \frac{V_{d-2}}{6(d-2)(4\pi)^{\frac{d-2}{2}}} \cdot \frac{1}{\varepsilon^{d-2}} + \mathcal{O}(\varepsilon^{-(d-4)})\,. \end{aligned} \tag{3.1.33}$$

This result shows that the entanglement entropy satisfies the area law,

$$S_A = \gamma \frac{\text{Area of } \partial A}{\varepsilon^{d-2}} + \mathcal{O}(\varepsilon^{-(d-4)})\,, \tag{3.1.34}$$

where γ is a numerical constant which depends on the detail of the theory and ∂A a circumference of A. Finally we comment on physical quantity in the entanglement entropy. It is known that the entanglement entropy behaves as

$$S_A = p_1 \frac{\ell^{d-2}}{\varepsilon^{d-2}} + p_3 \frac{\ell^{d-4}}{\varepsilon^{d-4}} + \cdots + \begin{cases} p_{d-2}\frac{\ell}{\varepsilon} + p_d & d : \text{odd} \\ p_{d-3}\frac{\ell^2}{\varepsilon^2} + q \log\left(\frac{\ell}{\varepsilon}\right) & d : \text{even} \end{cases} \tag{3.1.35}$$

where p and q are constants and depend on the shape of the entanglement surface. ℓ is a typical size of the region A. The most divergent term which shows the area law of the entanglement entropy is not a physical quantity because it depends on the cut-off parameter ε. That is, if we change the cut-off parameter, the coefficient is changed. For the same reason, other divergent terms are not physical quantities, also. On the other hand, the constant term and the logarithm term are physical. In two dimensional case, the coefficient of the logarithm term is a central charge which represents degree of freedom.

3.2 Holographic entanglement entropy

We discuss the entanglement entropy in the context of the holography. A holographic dual of the entanglement entropy is given by the Ryu-Takayanagi formula [17, 18]. The Ryu-Takayanagi formula states that the entanglement entropy for the region A is given by

$$S_A = \frac{\text{Area of } \gamma_A}{4G_{\mathrm{N}}}\,, \tag{3.2.1}$$

where $G_{\rm N}$ is $(d+1)$-dimensional Newton's constant. Here γ_A is an extremal surface in AdS spacetime and its boundary is the same as that of the region A. We explain γ_A in more detail. Firstly, we take a time-slice in AdS spacetime and the region A on the time-slice at the AdS boundary. γ_A is $(d-1)$-dimensional spacelike surface and its boundary is the same as the one of the region A. Furthermore, A and γ_A are assumed to be homologous. The topologies of A and γ_A are same. γ_A is the minimal surface in such surfaces. See Fig. 3.5. In Lorentzian signature, the 'minimal' means minimal at the time-slice. When AdS backgrounds depend on time, we need to use the covariant generalisation of the Ryu-Takayanagi formula [19]. In this case, γ_A is extremal not minimal. Hereafter we use the word "extremal" to represent the surface in the Ryu-Takayanagi formula or the holographic entanglement entropy formula.

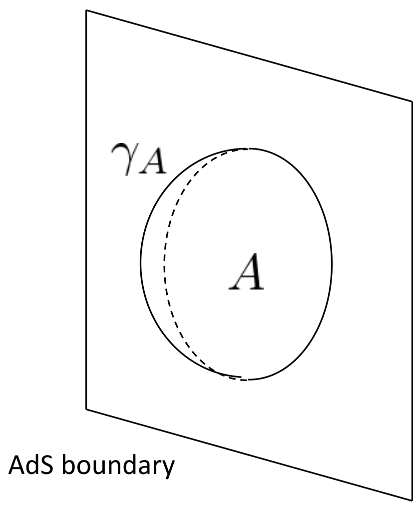

Figure 3.5: Ryu-Takayanagi formula. In this figure, the time-direction is not drawn. The extremal surface γ_A dips in the interior of AdS spacetime.

Let us explain two simple examples; a finite interval in the $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence and a half plane in the $\mathrm{AdS}_{d+1}/\mathrm{CFT}_d$ correspondence in the Poincaré patch. First example is the holographic entanglement entropy for the finite interval. In Euclidean signature, the Poincaré metric of AdS_3 is

$$\mathrm{d}s^2 = \ell_{\rm AdS}^2 \frac{\mathrm{d}z^2 + \mathrm{d}t^2 + \mathrm{d}x^2}{z^2}\,. \tag{3.2.2}$$

The boundary of the region A is given by $(t, x) = (0, -R)$ and $(t, x) = (0, R)$. The extremal surface is given by

$$z = \sqrt{R^2 - x^2}\,. \tag{3.2.3}$$

See Fig. 3.6. The induced metric on the extremal surface is

$$\mathrm{d}s^2 = \frac{\ell_{\mathrm{AdS}}^2 R^2 \mathrm{d}z^2}{4z^2(R^2 - z^2)}\,. \tag{3.2.4}$$

Then, by using the Ryu-Takayanagi formula the entanglement entropy becomes

$$\begin{aligned} S_A &= \frac{1}{4G_{\mathrm{N}}} \int \mathrm{d}s = \frac{\ell_{\mathrm{AdS}} R}{4G_{\mathrm{N}}} \int_{\varepsilon}^{R} \frac{\mathrm{d}z}{z\sqrt{R^2 - z^2}} \\ &= \frac{\ell_{\mathrm{AdS}}}{2G_{\mathrm{N}}} \log\left(\frac{R}{\varepsilon}\right)\,. \end{aligned} \tag{3.2.5}$$

Here ε is a cut-off parameter and corresponds to a UV cut-off in the dual CFT. By using the well-known result about the central charge in the $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence,

$$c = \frac{3\ell_{\mathrm{AdS}}}{2G_{\mathrm{N}}}\,, \tag{3.2.6}$$

we obtain the logarithm behaviour of the entanglement entropy

$$S_A = \frac{c}{6} \log\left(\frac{R}{\varepsilon}\right)\,. \tag{3.2.7}$$

We skip the discussion about the entanglement entropy in two dimensional CFTs. We can derive (3.2.7) by using the replica trick.

Next example is the holographic entanglement entropy of the half plane. See Fig. 3.7. From the symmetry, the extremal surface is given by

$$0 \leq z < \infty\,. \tag{3.2.8}$$

According to the Ryu-Takayanagi formula, the holographic entanglement entropy of the half plane is given by

$$S_A = \frac{V_{d-2}}{4G_{\mathrm{N}}} \int_{\varepsilon}^{\infty} \mathrm{d}z \left(\frac{\ell_{\mathrm{AdS}}}{z}\right)^{d-1} = \frac{V_{d-2}\,\ell_{\mathrm{AdS}}^{d-1}}{4G_{\mathrm{N}}(d-2)} \cdot \frac{1}{\varepsilon^{d-2}} \tag{3.2.9}$$

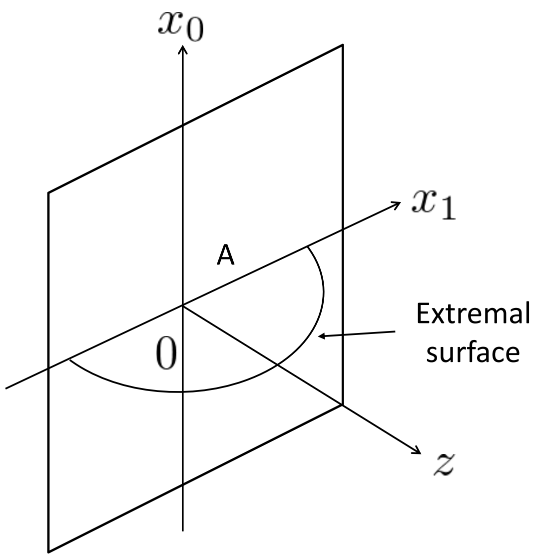

Figure 3.6: Set-ups of a finite interval and extremal surface in AdS$_3$.

where V_{d-2} is an infinite volume spanned by $x_2, \cdots, x_d$, and ε corresponds to a UV cutoff in field theory. The entanglement entropy satisfies the area law as expected. The leading behaviour is the same as the result of the replica method (3.1.33).

3.3 Proof of the Ryu-Takayanagi formula

In this section, we give a proof of the Ryu-Takayanagi formula following by Lewkowycz and Maldacena [55]. After that, we will see that the Ryu-Takayanagi formula is a generalised formula of the BH entropy formula.

As explained in subsection 3.1.2, the entanglement entropy can be calculated by the replica method. By considering the spacetime Σ_n, which is the n sheets of the original spacetime Σ, and by using the partition functions on Σ_n and Σ, the entanglement entropy is expressed as

$$S_A = -\lim_{n\to 1}\frac{\partial}{\partial n}(\log Z_{\Sigma_n} - n\log Z_\Sigma)\,. \tag{3.3.1}$$

When we pile the original spacetimes, n is assumed to be integer. Nevertheless, when we calculate the entanglement entropy using the replica method, n is assumed to be non-integer.

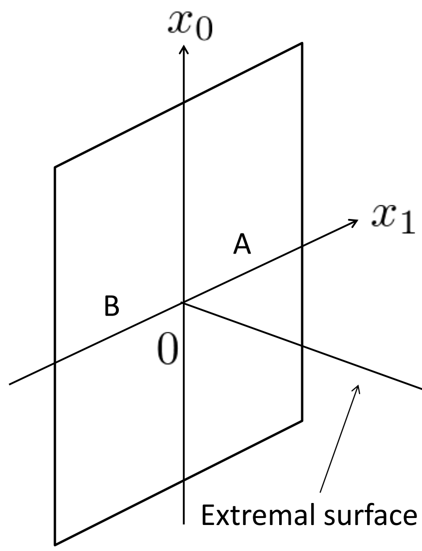

Figure 3.7: Set-ups of a half plane and extremal surface in AdS spacetime. We omit other directions.

We need to evaluate the partition function on Σ_n in order to calculate the entanglement entropy. According to the GKPW relation, the partition function on Σ_n is mapped to that of gravitational theory,

$$Z_{\Sigma_n} = Z_{\mathrm{G}}(M_n)\,. \tag{3.3.2}$$

Here, $(d+1)$-dimensional spacetime M_n approaches to the d-dimensional spacetime Σ_n at the AdS boundary and it is a solution of the equation of motion of gravitational theory such as the Einstein's equation. M_n must be a smooth geometry because it is a solution of the equation of motion although Σ_n has the deficit angle.

Consider the case where n is integer. To begin with, let us introduce the polar coordinate τ, $0 \leq \tau < 2\pi n$. When τ is shifted by 2π, the present spacetime moves to the next spacetime. Then, Σ_n is invariant under $\tau \to \tau + 2\pi$, and has a $\mathbb{Z}_n$ symmetry, which is called a replica symmetry. In the following discussion, we assume that the replica symmetry is not broken. Σ_n is the d-dimensional spacetime spanned by τ and other coordinates (We don't write coordinates except τ explicitly.). M_n is the $(d+1)$-dimensional manifold, and has a new spatial coordinate. It corresponds to a radial coordinate and we will refer to it as r. Although there is an ambiguity to choose r, we select r such that τ direction shrinks smoothly at $r = 0$. In this case, the metric behaves

$$\mathrm{d}s^2 \simeq \mathrm{d}r^2 + \frac{r^2}{n^2}\mathrm{d}\tau^2 + \cdots \tag{3.3.3}$$

near $r = 0$. $\cdots$ represents other directions. Since the period of τ is $2\pi n$, there is no conical singularity at $r = 0$. Also, we impose the replica symmetry on fields as

$$\phi(\tau + 2\pi) = \phi(\tau)\,, \tag{3.3.4}$$

where ϕ represents all fields included in gravity such as metric or scalar fields.

Note that the logarithm of $Z_{\mathrm{G}}(M_n)$ can be written as

$$\log Z_{\mathrm{G}}(M_n) = n \log Z_{\mathrm{G}}(\hat{M}_n) \tag{3.3.5}$$

where $\hat{M}_n$ is division into n parts of the spacetime M_n. Since $\hat{M}_n$ has the period 2π, a conical singularity exists at $r = 0$. Nevertheless, we ignore a contribution coming from the conical singularity when we calculate the partition function on $\hat{M}_n$. When we calculate the entanglement entropy, we need to perform the analytical continuation about n. In that case, we define $Z_{\mathrm{G}}(M_n)$ for general n by using the relation (3.3.5).

We introduce a $(d+1)$-dimensional manifold $\hat{N}_n$, and decompose the calculation of the entanglement entropy as follows;

$$\begin{aligned} \log Z_{\Sigma_n} - n \log Z_{\Sigma} &= n \log Z_{\mathrm{G}}(\hat{M}_n) - n \log Z_{\mathrm{G}}(M) \\ &= -n\left(S_{\mathrm{G}}(\hat{M}_n) - S_{\mathrm{G}}(\hat{N}_n)\right) - n\left(S_{\mathrm{G}}(\hat{N}_n) - S_{\mathrm{G}}(M)\right)\,. \end{aligned} \tag{3.3.6}$$

Here $\hat{N}_n$ is a smooth spacetime corresponding to M except a vicinity of $r = 0$ and $\hat{M}_n$ at $r < \varepsilon$ for an arbitrary small quantity ε. $\hat{N}_n$ does not satisfy the equation of motion, and there are many candidates such a spacetime. We choose one of them.

To calculate the entanglement entropy, we need to find $\mathcal{O}(n-1)$ contributions in (3.3.6) in the $n \to 1$ limit. The first parentheses in (3.3.6) gives no contribution to the entanglement entropy because $\hat{M}_n$ is the solution of the equation of motion and a deviation $\mathcal{O}(n-1)$ of the action does not appear,

$$\lim_{n\to 1}\left(S_{\mathrm{G}}(\hat{M}_n)-S_{\mathrm{G}}(\hat{N}_n)\right)=0\,. \tag{3.3.7}$$

In the second parentheses in (3.3.6), the $\mathcal{O}(n-1)$ contribution is coming from the region $r<\epsilon$. $n\hat{N}_n$ has no conical singularity but nM has a conical singularity with a deficit angle $2\pi(1-n)$. Then, a difference between Ricci scalars becomes

$$R_{n\hat{N}}-R_{nM}=4\pi(1-n)\cdot\delta^2(r)\,. \tag{3.3.8}$$

The proof is given in Appendix B.

Summarizing the above discussion, we obtain

$$\log Z_{\Sigma_n}-n\log Z_{\Sigma}=-n\left(S_{\mathrm{G}}(\hat{N}_n)-S_{\mathrm{G}}(M)\right)=\frac{1-n}{4G_{\mathrm{N}}}\int_{\gamma_A}\sqrt{g}\,. \tag{3.3.9}$$

where γ_A is $(d-1)$-dimensional spacelike surface at $r=0$ and g the determinant of the induced metric on $r=0$. According to the replica trick, we finally obtain the entanglement entropy

$$S_A=\frac{1}{4G_{\mathrm{N}}}\int_{\gamma_A}\sqrt{g} \tag{3.3.10}$$

Furthermore, it is shown that γ_A is an extremal surface from the equation of motion. This is a proof of the Ryu-Takayanagi formula.

Finally, we comment on the relation between the BH entropy and the holographic entanglement entropy. Consider a Euclidean gravity and its black hole solution. In this case, the boundary condition for the metric or other fields, is invariant a $U(1)$ symmetry along the τ direction. It is known that the BH entropy,

$$S_{\mathrm{BH}}=-\lim_{n\to 1}\frac{\partial}{\partial n}\left(\log Z_{\mathrm{G}}(M_n)-n\log Z_{\mathrm{G}}(M_1)\right), \tag{3.3.11}$$

is equal to the area of the co-dimension two surface which is invariant the $U(1)$ symmetry. This co-dimension two surface is a horizon of the black hole.

In the holographic entanglement entropy formula, the $U(1)$ symmetry is replaced by the replica symmetry $\mathbb{Z}_n$, and the boundary is changed from the horizon to the asymptotically AdS boundary. In this sense, the Ryu-Takayanagi formula can be regarded as a generalised quantity of the BH entropy.

Chapter 4

Holographic Entanglement Entropy in Einstein Gravity on dS

4.1 Proposal

As noted in the Introduction, it is known that the BH entropy is given by an event horizon area divided by four times Newton's constant,

$$S_{\mathrm{BH}} = \frac{\text{Horizon Area}}{4G_{\mathrm{N}}} \,. \tag{4.1.1}$$

The BH entropy formula holds not only in asymptotically flat spacetime or AdS spacetime but also in asymptotically dS spacetime. As reviewed in the previous chapter, the holographic entanglement entropy can be regarded as a generalised quantity of the BH entropy [55], and it is given by an area of an extremal surface divided by four times Newton's constant,

$$S_{\mathrm{HEE}} = \frac{\text{Extremal Surface Area}}{4G_{\mathrm{N}}} \,. \tag{4.1.2}$$

Then it is natural to expect that a formula similar to the Ryu-Takayanagi formula holds even in dS spacetime as the BH entropy formula.

Thus we need to find "extremal surfaces" which extend to the future infinity $\mathcal{I}^+$, where the dual CFTs are assumed to live, in Einstein gravity on dS spacetime in order to construct the holographic entanglement entropy formula in the dS/CFT correspondence. The notion of the extremal surface is however obscure. If the surfaces were space-like, their area would be smaller and smaller as the surfaces approached null. If the surfaces were time-like, their area would be imaginary, and the surfaces would not be closed in general. We discuss this issue based on analytic continuation because the analytic continuation enables us to obtain surfaces in dS spacetime, which satisfy the equation of motion obtained from the variation of the area functional. Our proposal is that the extremal surfaces in dS spacetime are given by the analytic continuation of extremal surfaces in EAdS spacetime. This proposal allows for extremal surfaces which extend in complex-valued coordinate spacetime, and lets us find complex surfaces as extremal surfaces. In the next chapter, we will check the consistency of our proposal using a toy model.

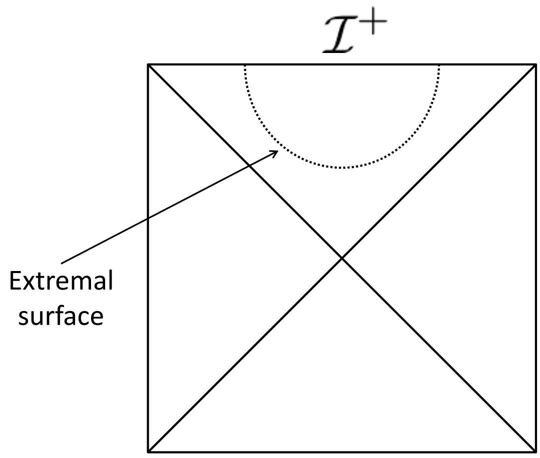

Figure 4.1: Extremal surface in dS spacetime.

Before an explanation of details of our proposal, we comment on other possibilities of extremal surfaces in dS spacetime. The extremal surfaces in our proposal are space-like. However, it would be possible to regard time-like surfaces as the extremal surfaces. The time-like surfaces might not be appropriate for the holographic entanglement entropy formula because these surfaces are not closed in general. See

Fig. 4.2. To see this, let us consider the global patch of dS spacetime,

$$\mathrm{d}s^2 = \ell_{\mathrm{dS}}^2(-\mathrm{d}\tau^2 + \cosh^2\tau\, \mathrm{d}\Omega^2)\,, \tag{4.1.3}$$

and take the subregion A as $0 \le \theta_1 < \pi/2$, $0 \le \theta_i < 2\pi$ $(i = 2, \cdots, d-2)$, for example. The θ_{d-1} direction is assumed to be the time direction in the dual field theory[1]. In this case, the extremal surface would be given by $\tau = \tau(\theta_1)$ or $\theta_1 = \theta_1(\tau)$. The induced metric on the extremal surface is

$$\begin{aligned} \mathrm{d}s^2 = \ell_{\mathrm{dS}}^2\left(-1 + \cosh^2\tau\, \theta_1'^2\right)\mathrm{d}\tau^2 + \ell_{\mathrm{dS}}^2 \sin^2\theta_1 \mathrm{d}\theta_2^2 \\ + \cdots + \ell_{\mathrm{dS}}^2 \sin^2\theta_1 \cdots \sin^2\theta_{d-3}\mathrm{d}\theta_{d-2}^2\,. \end{aligned} \tag{4.1.4}$$

The area functional which determines the extremal surfaces become

$$\int \mathrm{d}\tau \int \mathrm{d}\theta_2 \cdots \int \mathrm{d}\theta_{d-2}\, \ell_{\mathrm{dS}}^{d-2} \sin^{d-3}\theta_1 \sin^{d-4}\theta_2 \cdots \sin\theta_{d-3}\sqrt{1 - \cosh^2\tau\, \theta_1'^2}\,. \tag{4.1.5}$$

The extremal surfaces obtained from the above area functional are trivial,

$$\theta_1 = \mathrm{const.} \tag{4.1.6}$$

Then the extremal surfaces can't be closed. Thus, time-like surfaces are not adequate for the holographic entanglement entropy formula. One can also check this fact using the Poincaré patch.

The time-like surfaces would be closed if we consider that we attach a half sphere to a half of dS spacetime so that it represents the Hartle-Hawking state. See Fig. 4.3. In this case, the extremal surfaces in the half of dS spacetime are the same as that of the above discussion, and the extremal surfaces in the half sphere are given by analytical continuation of that in dS spacetime. Since the extremal surfaces in dS spacetime are time-like, those give imaginary areas. On the other hand, the extremal surfaces in the sphere are space-like and give real areas. Then the holographic entanglement entropy would become a sum of a pure real part coming from the sphere and a pure imaginary part coming from dS spacetime. We will

[1] Although the dual field theory is Euclidean field theory, we assume that the theory has the time direction to introduce the entanglement entropy.

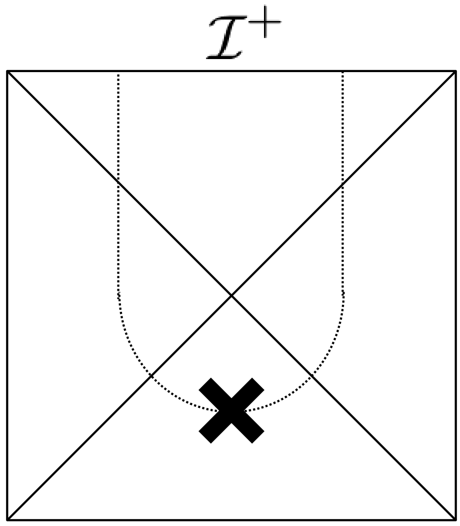

Figure 4.2: Time-like surface in dS spacetime. The mark X means that the surface cannot be closed.

not discuss this possibility in this Ph. D. thesis anymore since this result largely disagrees with our result in chapter 5.

Let us switch the subject to our proposal. As explained in the above, it is difficult to find meaningful extremal surfaces whose boundaries sit on $\mathcal{I}^+$ in dS spacetime. As dS spacetime is obtained by analytical continuation of AdS spacetime, it is expected that extremal surfaces in dS spacetime can be obtained by the analytical continuation. The conformal boundary of AdS spacetime is mapped to the future infinity of dS spacetime by the analytical continuation. Then surfaces whose boundaries sit on the conformal boundary of AdS spacetime are mapped to surfaces whose boundaries sit on the future infinity of dS spacetime. Furthermore, if surfaces in AdS spacetime satisfy the equation of motion obtained from the area functional, surfaces in dS spacetime also satisfy the equation of motion obtained from the area functional. Our proposal is that

the extremal surfaces in dS spacetime are given by the analytic continuation of the extremal surfaces in EAdS spacetime.

As we will see later, the extremal surfaces in dS spacetime extend in complex-valued coordinates in general. We obtain or define extremal surfaces by analytical

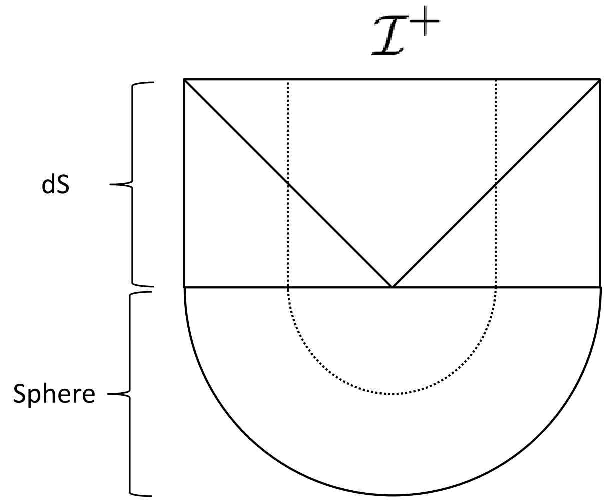

Figure 4.3: Hartle-Hawking vacuum and a candidate of an extremal surface.

continuation in our proposal, but one can also obtain the same extremal surfaces by using the equation of motion obtained from the area functional if one allow surfaces to extend in complex-valued coordinates. The idea of complex surfaces has also appeared in the AdS case [56]. Complex surface in the dS case is explored in [57, 58].

Let us consider the Poincaré coordinate of dS spacetime, as an example. We take the subregion A a half plane $x_1 > 0$, and regard x_d as a time-direction. In the AdS case, the extremal surface is $0 \leq z < \infty$ at $x_1 = 0$ (3.2.8). After the double Wick rotation, the extremal surface in dS spacetime is given by

$$0 \leq \eta < i\infty \,. \tag{4.1.7}$$

The extremal surface in dS spacetime is not real-valued but *complex*-valued. Performing the double Wick rotation (2.2.9) while Newton's constant G_{N} and the UV cutoff parameter ε are held fixed, the holographic entanglement entropy (3.2.9) in the AdS/CFT correspondence is transformed as

$$S_A = (-i)^{d-1} \frac{V_{d-2}\ell_{\mathrm{dS}}^{d-1}}{4G_{\mathrm{N}}(d-2)} \cdot \frac{1}{\varepsilon^{d-2}} \,. \tag{4.1.8}$$

In conclusion, the holographic entanglement entropy in the dS/CFT correspondence is given by that of the AdS/CFT correspondence, where ℓ_{AdS} is replaced by ℓ_{dS}, times $(-i)^{d-1}$. It can become positive, negative, or imaginary for the dimension.

4.2 Extremal surfaces in asymptotically dS spacetime

In the previous section, we proposed the holographic entanglement entropy formula in the dS/CFT correspondence via the double Wick rotation. And we found the extremal surfaces in Poincaré dS spacetime and obtained the holographic entanglement entropy, as an example. In this section, we comment on extremal surfaces in a more general set of asymptotically dS spacetime.

To define extremal surfaces in asymptotically dS spacetime, we need to find a double Wick rotation between the asymptotically dS spacetime and the corresponding asymptotically EAdS spacetime. One Wick rotation is

$$\ell_{\mathrm{dS}} \to i\ell_{\mathrm{AdS}} \tag{4.2.1}$$

to make the cosmological constant positive. A second analytic continuation is concerned with the time coordinate in asymptotically dS spacetime and the radial direction in corresponding asymptotically EAdS spacetime. Since the second analytical continuation depends on coordinate systems, we cannot typically say the second analytical continuation concretely like (4.2.1).

Our proposal is that the holographic entanglement entropy in the dS/CFT correspondence is defined as

$$S_{\mathrm{HEE}} := \frac{\mathrm{Area}_{\mathrm{dS}}}{4G_{\mathrm{N}}} \,. \tag{4.2.2}$$

Here $\mathrm{Area}_{\mathrm{dS}}$ is the area of the "extremal surfaces" in asymptotically dS spacetime defined as follows. Firstly, we find extremal surfaces in the corresponding asymptotically EAdS spacetime. Next, performing the double Wick rotation of the extremal surfaces in asymptotically EAdS spacetime, we define "extremal surfaces" in asymptotically dS spacetime. As in the previous section, the extremal surfaces in asymptotically dS spacetime are *complex*-valued in general although the extremal surfaces in asymptotically EAdS spacetime are real-valued. Then, Area_{dS} is the area of the extremal surfaces in asymptotically dS spacetime. The holographic entanglement entropy (4.2.2) is uniquely defined by using the extremal surfaces in asymptotically EAdS spacetime.

It is intriguing to apply our proposal for the asymptotically dS spacetime case to a Schwarzschild dS black hole. By analogy from the AdS/CFT correspondence case, it is expected that the holographic entanglement entropy includes the contribution of the black hole. We could obtain an interpretation of the black hole entropy in dS spacetime.

Chapter 5

Entanglement Entropy in the $Sp(N)$ Model

Since the CFT dual to Einstein gravity on dS spacetime is not known yet, we analyze the $Sp(N)$ model as a toy model. Since the $Sp(N)$ model is the holographic dual of Vasiliev's higher-spin gauge theory on dS spacetime, we can quantitatively compare such results only with Vasiliev's higher-spin gauge theory, not with Einstein gravity. However, it is natural to expect that their basic qualitative behaviours do not change between these two theories. Then, we explore the $Sp(N)$ model [46, 47] in this chapter.

5.1 $Sp(N)$ model

The interacting $Sp(N)$ model on a Lorentzian spacetime with the metric $g_{\mu\nu}$ is defined by the action

$$S = -\frac{1}{2}\int \mathrm{d}^d x\,\sqrt{-g}\,\left(\Omega_{ab}g^{\mu\nu}\partial_\mu\chi^a\partial_\nu\chi^b + m^2\Omega_{ab}\chi^a\chi^b + \lambda(\Omega_{ab}\chi^a\chi^b)^2\right)\,, \qquad (5.1.1)$$

where χ^a ($a = 1, \cdots, N$) are anticommuting scalars, N an even integer, m mass, λ coupling constant and

$$\Omega_{ab} = \begin{pmatrix} 0 & 1_{N/2\times N/2} \\ -1_{N/2\times N/2} & 0 \end{pmatrix}\,, \qquad (5.1.2)$$

the anti-symmetric matrix [46]. The metric sign is mostly plus. Here we ignore whether the theory can be renormalised or not. When $d = 4$, the theory is renormalisable and violates spin-static theorem.

By introducing

$$\eta^a = \chi^a + i\chi^{a+\frac{N}{2}}\,, \quad \bar{\eta}^a = -i\chi^a - \chi^{a+\frac{N}{2}} \qquad \left(a = 1, \cdots, \frac{N}{2}\right), \tag{5.1.3}$$

the action is rewritten as

$$S = -\int \mathrm{d}^d x\, \sqrt{-g}\, \left(g^{\mu\nu}\partial_\mu\bar{\eta}\,\partial_\nu\eta + m^2\bar{\eta}\eta + \lambda(\bar{\eta}\eta)^2\right)\,. \tag{5.1.4}$$

When we write the action in terms of η and $\bar{\eta}$, the action seems to be that of complex scalars. We will use both actions.

Let us quantize the $Sp(N)$ model on flat spacetime. Conjugate momentum fields are introduced as

$$\pi_\eta := \frac{\partial\mathcal{L}}{\partial(\partial_0\eta)} = -\partial_0\bar{\eta}\,, \qquad \pi_{\bar{\eta}} := \frac{\partial\mathcal{L}}{\partial(\partial_0\bar{\eta})} = \partial_0\eta\,, \tag{5.1.5}$$

respectively. Here $\mathcal{L}$ is a Lagrangian, and derivatives in terms of η and $\bar{\eta}$ mean left-derivatives. For simplicity, we use π and $\bar{\pi}$ instead of π_η and $\pi_{\bar{\eta}}$, respectively. Then, a Hamiltonian is given by

$$\begin{aligned}\mathcal{H} :&= \partial_0\eta\cdot\pi + \partial_0\bar{\eta}\cdot\bar{\pi} - \mathcal{L}\\ &= \bar{\pi}\pi + \partial_i\bar{\eta}\partial_i\eta + m^2\bar{\eta}\eta + \lambda(\bar{\eta}\eta)^2\,.\end{aligned} \tag{5.1.6}$$

Anti-commutation relations at the same time are given by

$$\{\eta^a(t,\boldsymbol{x}), \pi^b(t,\boldsymbol{x}')\} = \{\bar{\eta}^a(t,\boldsymbol{x}), \bar{\pi}^b(t,\boldsymbol{x}')\} = i\delta^{ab}\delta^{d-1}(\boldsymbol{x}-\boldsymbol{x}')\,. \tag{5.1.7}$$

From now on, we focus on the free $Sp(N)$ model, for simplicity. Fourier transformations of the fields are

$$\eta(x) = \int \frac{\mathrm{d}^{d-1}k}{(2\pi)^{d-1}\sqrt{2\omega_k}}(b^\dagger_{\boldsymbol{k},-}\mathrm{e}^{ik\cdot x} + b_{\boldsymbol{k},+}\mathrm{e}^{-ik\cdot x})\,, \tag{5.1.8}$$

$$\bar{\eta}(x) = \int \frac{\mathrm{d}^{d-1}k}{(2\pi)^{d-1}\sqrt{2\omega_k}}(-b_{\boldsymbol{k},-}\mathrm{e}^{ik\cdot x} + b^\dagger_{\boldsymbol{k},+}\mathrm{e}^{-ik\cdot x})\,, \tag{5.1.9}$$

where $k \cdot x = -\omega_k t + \boldsymbol{k}\cdot\boldsymbol{x}$. By inserting Fourier transformations (5.1.8) and (5.1.9) to the anti-commutation relations (5.1.7), the anti-commutation relations in terms of $b_{\boldsymbol{k},\pm}$ become

$$\{b_{\boldsymbol{k},+}, b^\dagger_{\boldsymbol{k}',+}\} = \{b_{\boldsymbol{k},-}, b^\dagger_{\boldsymbol{k}',-}\} = \delta^{d-1}(\boldsymbol{k}-\boldsymbol{k}')\,. \tag{5.1.10}$$

Other anti-commutation relations vanish. The Hamiltonian becomes

$$\begin{aligned} H &= \int \mathrm{d}^{d-1}\boldsymbol{k}\,\omega_k(b^\dagger_{\boldsymbol{k},+}b_{\boldsymbol{k},+} - b_{\boldsymbol{k},-}b^\dagger_{\boldsymbol{k},-}) \\ &= \int \mathrm{d}^{d-1}\boldsymbol{k}\,\omega_k(b^\dagger_{\boldsymbol{k},+}b_{\boldsymbol{k},+} + b^\dagger_{\boldsymbol{k},-}b_{\boldsymbol{k},-}) + \text{const.} \end{aligned} \tag{5.1.11}$$

in terms of the creation and annihilation operators $b_{\boldsymbol{k},\pm}$ and $b^\dagger_{\boldsymbol{k},\pm}$. The vacuum state $|0\rangle$ is defined such that it is annihilated by the operators $b_{\boldsymbol{k},\pm}$,

$$b_{\boldsymbol{k},\pm}|0\rangle = 0\,. \tag{5.1.12}$$

Let's discuss a pseudo-unitarity of the $Sp(N)$ model. The $Sp(N)$ model is not unitary in the sense that the inner product $\langle\psi|\phi\rangle$ is not preserved under time evolution because the Hamiltonian is not Hermite,

$$H^\dagger \neq H\,. \tag{5.1.13}$$

Nevertheless, $Sp(N)$ model has a special property and we can define a new inner product preserved under time evolution. Firstly, we introduce a unitary operator C which satisfies

$$C^\dagger C = 1\,, \qquad C^\dagger = C\,. \tag{5.1.14}$$

The vacuum $|0\rangle$ is assumed to be invariant, $C|0\rangle = |0\rangle$. We define that the unitary operator C operates on the fields as[1]

$$C\bar{\eta}C = -\eta \,. \tag{5.1.15}$$

In terms real fields χ^a and $\chi^{a+N/2}$, they transforms as

$$C\chi^a C = \chi^{a+N/2}\,, \qquad C\chi^{a+N/2}C = \chi^a \,. \tag{5.1.16}$$

The unitary operator C operates on the creation and annihilation operators as

$$Cb_\pm C = \mp b_\pm^\dagger \,. \tag{5.1.17}$$

Although the Hamiltonian is not Hermite, the Hamiltonian satisfies the relation,

$$H^\dagger = CHC \,. \tag{5.1.18}$$

We refer to it as *pseudo unitarity*. We introduce a new inner product $\langle\psi|\phi\rangle_C$ as

$$\langle\psi|\phi\rangle_C := \langle\psi|C|\phi\rangle \,. \tag{5.1.19}$$

We refer to the new norm as C-norm. This inner product is preserved under time evolution,

$$\langle\psi(t)|\phi(t)\rangle_C := \langle\psi|\mathrm{e}^{iH^\dagger t}C\mathrm{e}^{-iHt}|\phi\rangle = \langle\psi|\phi\rangle_C \,, \tag{5.1.20}$$

as opposed to the usual inner product $\langle\psi|\phi\rangle$.

5.2 Entanglement entropy

In this section, we introduce the entanglement entropy in the $Sp(N)$ model [1, 2]. Naively, the density matrix is given by $\rho_{\text{tot}} = |\Psi\rangle\langle\Psi|$ and the entanglement entropy is defined as a von-Neumann entropy $S_A = -\mathrm{tr}_A\,\rho_A \log\rho_A$ with a reduced density

[1] In [46], the unitary operator is defined to operate on η and $\bar{\eta}$ as $C\bar{\eta}C = \eta^\dagger$. But, if we define so, we would obtain $C\chi^a C = (\chi^{a+N/2})^\dagger$ and $C\chi^{a+N/2}C = -(\chi^a)^\dagger$. This is a contradiction. We should define operations by C as $C\bar{\eta}C = \pm\eta$. There is an ambiguity to choose a sign and we adopt $C\bar{\eta}C = -\eta$ in this Ph. D. thesis not to include negative signs in (5.1.16).

matrix $\rho_A = \mathrm{tr}_A \rho_{\mathrm{tot}}$. However, it seems that this definition is inadequate because of $H^\dagger \neq H$ as we will see below. To compute $[\rho_{\mathrm{tot}}]_{\chi_- \chi_+}$, we write $\langle \chi_- | \Psi \rangle$ and $\langle \Psi | \chi_+ \rangle$ in path-integral. Doing the same calculation, we obtain the wave function at $x_0 = 0$,

$$\langle \chi_- | \Psi \rangle = \int \prod_{-\infty < x_0 < 0} \mathcal{D}\chi \, \mathrm{e}^{-S[\chi]} \delta(\chi(0, \boldsymbol{x}) - \chi_-(\boldsymbol{x})) \,, \tag{5.2.1}$$

in Euclidean signature. Unlike $\langle \chi_- | \Psi \rangle$, $\langle \Psi | \chi_+ \rangle$ is different from that of standard scalar field theory. A complex conjugate of the wave function at t becomes

$$\langle \Psi(t) | = \langle \Psi(t_0) | \mathrm{e}^{iH^\dagger (t - t_0)} \,, \tag{5.2.2}$$

in Lorentzian signature. Since the Hamiltonian is not Hermite, we cannot express the vacuum state using the path integral. The complex conjugate of the wave function can not be expressed in path-integral representation.

Instead of $\rho_{\mathrm{tot}} = |\Psi\rangle\langle\Psi|$, we use a new density matrix,

$$\rho^C_{\mathrm{tot}} := |\Psi\rangle\langle\Psi| C \,. \tag{5.2.3}$$

As long as there is no confusion, we use ρ_{tot} instead of ρ^C_{tot}. Next task is to evaluate $\langle \Psi | C | \chi_+ \rangle$ to calculate a reduced density matrix. It is evaluated as

$$\langle \Psi(t_0) | C | \chi_+ \rangle \sim \int \prod_{t_0 < x_0 < \infty} \mathcal{D}\chi \, \mathrm{e}^{iS[\chi]} \delta(\chi(0, \boldsymbol{x}) - \chi_+(\boldsymbol{x})) \tag{5.2.4}$$

in Lorentzian signature. Here, we assume that the state is a vacuum state. When the state is an exciting state, we need to some operators in path integral. In Euclidean signature, the wave function and its complex conjugate with the pseudo unitary operator C are

$$\langle \chi_- | \Psi \rangle = \frac{1}{\sqrt{Z'}} \int \prod_{-\infty < x_0 < 0} \mathcal{D}\chi \, \mathrm{e}^{-S[\chi]} \delta(\chi(0, \boldsymbol{x}) - \chi_-(\boldsymbol{x})) \,, \tag{5.2.5}$$

$$\langle \Psi | C | \chi_+ \rangle = \frac{1}{\sqrt{Z'}} \int \prod_{0 < x_0 < \infty} \mathcal{D}\chi \, \mathrm{e}^{-S[\chi]} \delta(\chi(0, \boldsymbol{x}) - \chi_+(\boldsymbol{x})) \,, \tag{5.2.6}$$

where Z' is a normalisation factor such that $\mathrm{tr}_A \rho_A = 1$. As we will see later, Z' is just a partition function Z when we impose the periodic boundary condition on χ. On the other hand, Z' is different from the partition function when we impose the anti-periodic boundary condition on χ like fermion. By using (5.2.5) and (5.2.6), the path-integral representation of the reduced density matrix is given by,

$$[\rho_A]_{\chi_-\chi_+} = \frac{1}{Z'}\int \mathcal{D}\chi\, \mathrm{e}^{-S[\chi]} \prod_{\boldsymbol{x}\in A} \delta(\chi(-0,\boldsymbol{x}) - \chi_-(\boldsymbol{x}))\delta(\chi(+0,\boldsymbol{x}) - \chi_+(\boldsymbol{x}))\,. \quad (5.2.7)$$

Then, the entanglement entropy can be defined as

$$S_A = -\mathrm{tr}_A\, \rho_A \log \rho_A\,. \quad (5.2.8)$$

In the following, we calculate the entanglement entropy for a half plane.

Let us consider the periodic boundary condition case, firstly. The entanglement entropy for the half plane on $\mathbb{R}^d$ can be calculated from partition functions on $\mathbb{R}^2/\mathbb{Z}_M \times \mathbb{R}^{d-2}$ and $\mathbb{R}^d$ as explained in section 3.1.2,

$$S_A^{\mathrm{P}} = -\lim_{M\to 1} \frac{\partial}{\partial(1/M)}\left(\log Z_{\mathbb{R}^2/\mathbb{Z}_M\times\mathbb{R}^{d-2}} - \frac{1}{M}\log Z_{\mathbb{R}^d}\right)\,, \quad (5.2.9)$$

where the index P means periodic. The logarithm of the partition function is evaluated as

$$\log Z_{\mathbb{R}^d} = \log \int \mathcal{D}\chi\, \mathrm{e}^{-S} = NV_d \log \int \frac{\mathrm{d}^d k}{(2\pi)^d} \log k^2\,, \quad (5.2.10)$$

where V_d is the volume of $\mathbb{R}^d$. This result is minus that of standard scalar field theories. It comes from the statics of the fields. Since $\log Z_{\mathbb{R}^2/\mathbb{Z}_M\times\mathbb{R}^{d-2}}$ is also minus that of the standard scalar field theories, the entanglement entropy is given by

$$S_A^{\mathrm{P}} = -\frac{NV_{d-2}}{6(d-2)(4\pi)^{\frac{d-2}{2}}} \cdot \frac{1}{\varepsilon^{d-2}}\,, \quad (5.2.11)$$

where V_{d-2} is a $(d-2)$-dimensional infinite volume and ε a UV cutoff. The entanglement entropy (5.2.11) is minus that of standard scalar field theories. We can also obtain similar results for arbitrary subsystems on $\mathbb{R}^d$ or other curved spacetimes.

Next, we consider the anti-periodic case. In this case, the fields χ^a transforms as

$$\chi^a(r,\theta+2\pi) = -\chi^a(r,\theta)\,. \tag{5.2.12}$$

The logarithm of the partition function on $\mathbb{R}^d$ is evaluated as

$$\log Z_{\mathbb{R}^d} = \log \int \mathcal{D}\chi\, \mathrm{e}^{-S} = -NV_d \log \int \frac{\mathrm{d}^d k}{(2\pi)^d} \log k^2\,. \tag{5.2.13}$$

Since we impose the anti-periodic boundary condition (5.2.12) instead of the periodic boundary condition, the negative sign appears in (5.2.13). We also obtain a similar behaviour of the partition function on $\mathbb{R}^2/\mathbb{Z}_M \times \mathbb{R}^{d-2}$. Then, the entanglement entropy becomes

$$S_A^{\mathrm{AP}} = \frac{NV_{d-2}}{6(d-2)(4\pi)^{\frac{d-2}{2}}} \cdot \frac{1}{\varepsilon^{d-2}}\,, \tag{5.2.14}$$

where the index AP means anti-periodic. We can also generalize it to arbitrary entanglement surfaces on arbitrary curved spacetimes.

5.3 Comparison with A Toy CFT Model

Let us compare the result in the previous section with our proposal discussed in chapter 4. According to our proposal, we obtain the holographic entanglement entropy (4.1.8),

$$S_A = (-i)^{d-1} \frac{V_{d-2}\ell_{\mathrm{dS}}^{d-1}}{4G_{\mathrm{N}}(d-2)} \cdot \frac{1}{\varepsilon^{d-2}}\,. \tag{5.3.1}$$

Note that the holographic entanglement entropy has a coefficient $(-i)^{d-1}$ in general. In [1], we adopt the entanglement entropy with the periodic boundary condition on the grounds that the fundamental fields of the $Sp(N)$ model are scalar. We have obtained the entanglement entropy,

$$S_A^{\mathrm{P}} = -\frac{NV_{d-2}}{6(d-2)(4\pi)^{\frac{d-2}{2}}} \cdot \frac{1}{\varepsilon^{d-2}}\,. \tag{5.3.2}$$

Note that S_A^{P} is minus that of standard scalar field theories. The result (5.2.11) holds for any dimensions. Note however that it is known that the duality between

Vasiliev's higher-spin gauge theory and the $Sp(N)$ model holds only when $d = 3$.

Summarizing the above, the holographic entanglement entropy behaves as $S_A \propto (-i)^{d-1}$ in dS_{d+1} (4.1.8), and the entanglement entropy is given by minus that of standard field theories (5.2.11). This result may suggest that the $\mathrm{dS}_{d+1}/\mathrm{CFT}_d$ correspondence makes senses only when $d \in 4\mathbb{Z} - 1$. Note that the most interesting case, the $\mathrm{dS}_4/\mathrm{CFT}_3$ correspondence, is included, while the simplest case, the $\mathrm{dS}_3/\mathrm{CFT}_2$ correspondence, is excluded. This is consistent with results in subsection 5.2 in [27], which we reviewed in "surface/state operator" in subsection 2.2.2.

Our proposal has been checked only in the simple case, where the subsystem is the half plane. We need to check our proposal in more nontrivial setups with circular or some other shaped entanglement surfaces for example. However, it is expected that our proposal holds in any entanglement surfaces because the factor $(-i)^{d-1}$ appears in Einstein gravity via the double Wick rotation (4.1.8), and the minus sign appears in the $Sp(N)$ model (5.2.11). It is interesting to apply our proposal to Schwarzschild dS spacetime. Naively, it is expected that the holographic entanglement entropy is a sum of that of pure dS spacetime and the Schwarzschild BH entropy.

Finally, we comment on the negativity of the entanglement entropy (5.2.11) in the free $Sp(N)$ model. In standard field theories, the entanglement entropy is positive definite. In contrast, our result (5.2.11) is negative definite. The negativity comes from the fact that the scalars of the $Sp(N)$ model are anticommuting, and implies that the inner products of the Hilbert space are not positive definite. This negativity might be a key ingredient of the dS/CFT correspondence.

Chapter 6

Conclusion and Discussion

In this Ph. D. thesis, we reviewed the (A)dS/CFT correspondence in chapter 2 and the (holographic) entanglement entropy in chapter 3 as preliminaries of chapter 4 and chapter 5.

We have discussed the entanglement entropy in the proposed dS/CFT correspondence. We have proposed the holographic entanglement formula for Einstein gravity on dS spacetime and more general set of asymptotically dS spacetime via the double Wick rotation. In our proposal we found extremal surfaces which extend in complex-valued coordinate spacetime. The holographic entanglement entropy behaves as $S_A \propto (-i)^{d-1}$ in dS_{d+1} spacetime (4.1.8).

In chapter 5, we have summarized the $Sp(N)$ model, which is holographically dual to Vasiliev's higher-spin gauge theory on dS spacetime. We have calculated the entanglement entropy in the free $Sp(N)$ model and compared it with our proposal to check that our proposal works. We have found that the entanglement entropy is given by minus that of standard scalar field theories (5.2.11) when we imposed the periodic boundary condition on the fields[1].

These results obtained in chapter 4 and chapter 5 may suggest that the $\mathrm{dS}_{d+1}/\mathrm{CFT}_d$ correspondence makes senses only when $d \in 4\mathbb{Z} - 1$. Note that the most interesting case, the $\mathrm{dS}_4/\mathrm{CFT}_3$ correspondence, is included, while the simplest case, the

[1]As discussed in section 5.2, if we impose anti-periodic boundary condition on the fields, the entanglement entropy becomes positive. In this Ph. D. thesis, we use the one of the periodic boundary condition.

dS_3/CFT_2 correspondence, is excluded.

Our proposal has been checked only in the simple case, where the subsystem is the half plane. We need to check our proposal in more nontrivial setups with circular or some other shaped entanglement surfaces, for example. However, it is expected that our proposal holds in any entanglement surfaces because the factor $(-i)^{d-1}$ appears in Einstein gravity via the double Wick rotation (4.1.8), and the minus sign appears in the $Sp(N)$ model (5.2.11) in general.

Future perspective

We would like to conclude this Ph. D. thesis to give a future perspective on the holographic entanglement entropy in the dS/CFT correspondence.

To compare the entanglement entropy in the $Sp(N)$ model with a holographic result more precisely, we need to find a holographic entanglement entropy formula in Vasiliev's higher spin gauge theory on (A)dS spacetime. It is however difficult to find the formula even in the AdS case because a satisfactory action of Vasiliev's higher spin gauge theory is not known yet. If we found the action of Vasiliev's higher spin gauge theory, we would construct the holographic entanglement entropy formula in the AdS/CFT correspondence based on the Lewkowycz-Maldacena's procedure explained in section 3.3. By the double Wick rotation, we would obtain the holographic entanglement entropy formula in the dS/CFT correspondence. The holographic entanglement entropy formula for three-dimensional higher spin gauge theory described by the $SL(n,\mathbb{R})$ Chern-Simons gauge theory is proposed in [59,60]. We may use this formula. Note however that this holographic entanglement entropy doesn't satisfy a strong subadditivity.

Other way is to construct a holographically dual to Einstein gravity on dS spacetime to check our proposal more precisely. However, as noted in section 2.2, it is very difficult to do it.

What we should do is to reveal the pseudo-unitarity of the $Sp(N)$ model. Although this property would be important in the dS/CFT correspondence, this property isn't examined in detail yet. We will report the property of the pseudo-unitarity of the $Sp(N)$ model by using the entanglement entropy for some excited states [2].

The pseudo unitarity is crucial for the excited states because operators are not usually invariant under the pseudo operator C.

Appendix A

Brown-York stress tensor

In this appendix, we summarize the Brown-York tensor. The Brown-York tensor is defined as

$$T^{\mu\nu} := -\frac{4\pi}{\sqrt{\gamma}}\frac{\delta S}{\delta\gamma_{\mu\nu}}\,, \tag{A.0.1}$$

where $\gamma_{\mu\nu}$ is the induced metric on the boundary.

From now on, we concentrate on three-dimensional gravity with a positive cosmological constant. The most general action with a matter action S_{matter} is given by

$$S = \frac{1}{16\pi G_{\text{N}}}\int_{\mathcal{M}} \mathrm{d}^3x\sqrt{-g}\left(R - \frac{2}{\ell_{\text{dS}}^2}\right) + \frac{1}{8\pi G_{\text{N}}}\int_{\partial\mathcal{M}}\sqrt{\gamma}K + \frac{1}{8\pi G_{\text{N}}\ell_{\text{dS}}}\int_{\partial\mathcal{M}}\sqrt{\gamma} + S_{\text{matter}}\,. \tag{A.0.2}$$

Here $K_{\mu\nu}$ is an extrinsic curvature defined as

$$K_{\mu\nu} = -\nabla_{(\mu}n_{\nu)} = -\frac{1}{2}\mathcal{L}_n\gamma_{\mu\nu} \tag{A.0.3}$$

where n_μ is the outgoing unit vector. K is the trace of the extrinsic curvature. Inserting the three-dimensional gravity action to the definition of the Brown-York tensor, we obtain

$$T_{\mu\nu} = \frac{1}{4G_{\text{N}}}\left(K_{\mu\nu} - \left(K + \frac{1}{\ell_{\text{dS}}}\right)\gamma_{\mu\nu}\right)\,. \tag{A.0.4}$$

Appendix B

Ricci scalar with a conical singularity

Selecting two-dimensional surface transverse to the entangle surface in the n-sheeted Riemann surface Σ_n, the metric behaves as

$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2 \mathrm{d}\theta^2 \,, \tag{B.0.1}$$

while the polar coordinate has a period $2\pi n$,

$$\theta \sim \theta + 2\pi n \,. \tag{B.0.2}$$

That is, Σ_n has a deficit angle $2\pi(1-n)$ at $r=0$. In this case, it is known that the curvature diverges delta functionally as

$$R = 4\pi(1-n) \cdot \delta^2(r) \,. \tag{B.0.3}$$

A proof of (B.0.3) is following. We replace the metric with the deficit angle $2\pi(1-n)$ at $r=0$ by a limit of a smooth metric

$$\mathrm{d}s^2 = \frac{r^2 + n^2\xi^2}{r^2+\xi^2}\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 \,, \qquad \theta \sim \theta + 2\pi n \,. \tag{B.0.4}$$

For $r \ll \xi$, the metric approaches to $\mathbb{R}^2$ and the singularity point vanishes. On the other hand, for $r \gg \xi$, the metric (B.0.4) approaches to (B.0.1). It is expected that the metric (B.0.4) approaches to (B.0.1) under the limit $\xi \to 0$. By using the metric (B.0.4), the integrand of the curvature gives the value independent on ξ,

$$\int \mathrm{d}^2 r \sqrt{g} R = 4\pi(1-n)\,. \tag{B.0.5}$$

Then, we obtain the relation (B.0.3).

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