Construction of a new model generating three-dimensional random volumes: Towards a formulation of membrane theory

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In the thesis, towards a formulation of membrane theory, we construct a new class of models generating three-dimensional discrete random volumes. The world-volume theory of a membrane actually can be regarded as three-dimensional quantum gravity with matter fields corresponding to the target space coordinates. We thus hope that we can study the properties of membranes in a random volume approach.

For a class of string theories, the perspective of worldsheet theory as two-dimensional quantum gravity has achieved a success. In particular, a discrete approach based on matrix models has played a major role in the achievement. In the discrete approach, the path-integral of metric is replaced by the summation over random triangulations of worldsheets, which are generated by matrix models as the Feynman diagrams. Furthermore, one can analytically determine the free energy of a class of matrix models and can understand the critical behaviors. This fact enables us to take the continuum limit of models. The double scaling limit especially makes it possible to sum over all topologies at least perturbatively. Moreover, it is believed that they also include the nonperturbative information. Such analytical treatments of matrix models and the relation to the Liouville theory are summarized in section 2.1, section 3.1 and appendix A.

We expect that a discrete approach to membrane theory is also possible as in matrix models to string theory. We first review continuum approaches to membrane theory and the conjecture of M-theory in section 2.2. In section 3.2, we review tensor models as natural generalizations of matrix models to higher dimensional gravity. The three-dimensional models certainly generate random tetrahedral decompositions of three-dimensional objects as the Feynman diagrams. However, since the objects are not always manifolds, the relation to discretized gravity is not apparent. Moreover, we do not know how to take the continuum limits of general tensor models. The difficulties in the analytical treatment of tensor models are partly removed in colored tensor models, where a small class of tetrahedral decompositions are summed. In colored models, a $1/N$ expansion can be made, and it gives the degree expansion that is a generalization of the genus expansion in matrix models. The leading contributions in the large N limit come from part of diagrams representing three-dimensional sphere. In the so called invariant models, the leading free energy can be evaluated analytically by using the Schwinger-Dyson equation. The double scaling limit also can be taken. Nevertheless, the relation to three-dimensional gravity is still not clear.

In chapter 4, we propose a new class of models which generate three-dimensional random volumes. We call the models triangle-hinge models, because each Feynman diagram can be interpreted as an object consisting of triangles and multiple hinges, where multiple hinges are objects connecting edges of triangles. Unlike tensor models, the dynamical variables of triangle-hinge models are two real symmetric $N \times N$ matrices. The forms of interaction terms are characterized by semisimple associative algebras $\mathcal{A}$. With specific choices of interaction terms, it is shown that the weight of each Feynman diagram is given by a product of the index functions $\zeta(v)$ of vertices v in the diagram. It is shown that $\zeta(v)$ depends only on the topology of the neighborhood around v.

In chapter 5, we investigate the models whose defining algebras $\mathcal{A}$ are matrix rings. In fact, although there are also diagrams which do not represent tetrahedral decompositions of three-dimensional manifolds, it is shown that, by choosing specific interaction terms and taking an appropriate large N limit, the set of diagrams can be restricted such that only (and all of the) orientable tetrahedral decompositions survive. Furthermore, one can single out tetrahedral decompositions of three-dimensional manifolds by introducing a parameter to count the number of vertices of diagrams. For example, one can realize such a parameter by considering the direct sum of algebras $\mathcal{A}$. If we carry out the restriction, we can relate parameters in the models to those in discretized Euclidean three-dimensional pure gravity. Thus, triangle-hinge models can give a discretized formulation of three-dimensional pure gravity. We further demonstrate that in the models for matrix rings there is a duality which interchanges the roles of triangles and hinges. We also investigate the case where the defining algebras $\mathcal{A}$ are group rings.

We can introduce local matter degrees of freedom to triangle-hinge models, by assigning colors to simplices in tetrahedral decompositions. In chapter 6, we give general prescriptions to assign colors to each simplices of arbitrary dimensions (that is tetrahedra, triangles, edges and vertices). The coloring is realized by setting the definitive algebras $\mathcal{A}$ to the form $\mathcal{A}_{\mathrm{grav}} \otimes \mathcal{A}_{\mathrm{mat}}$ and modifying the interaction terms appropriately. Since the coupling constants between colors are local ones, the obtained matter fields have local interactions. For example, one can introduce matter fields corresponding to the target space coordinates of a membrane. One can also construct various spin systems, such as the Ising model and the q-state Potts models, on random volumes. We also show in section 6.2 that the Feynman rules of colored tensor models are reproduced in triangle-hinge models by introducing specific local matter degrees of freedom to tetrahedra, triangles and edges.