M2-branes in M-theory and exact large N expansion

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Abstract

The purpose of this thesis is to present our recent studies on the exact large N expansion of the partition function of the $\mathcal{N}=4$ $U(N)$ circular quiver superconformal Chern-Simons theories. These theories are known to describe the N stack of the M2-branes in the M-theory on various orbifold. With the help of the formal relation between the partition function and a quantum statistical system of N-particle ideal Fermi gas discovered by Marino and Putrov, we achieved to determine the all order perturbative corrections in $1/N$ to the partition function in large N expansion.

We also analyzed the non-perturbative effects in $1/N$. In the context of the AdS/CFT correspondence, these non-perturbative effects in $1/N$ can be interpreted quantitatively as the effects of fundamental M2-branes winding non-trivial three-cycles in the dual eleven dimensional background geometry, and thus called the instantons. We determined the explicit expression of several instanton coefficients and found the following interesting property. If the quantum number of two different kinds of instantons satisfy particular rational relation, they exhibit the same exponential suppression in $1/N$. We discovered that the individual instanton coefficient is always singular at the coincidence, while the divergences are always precisely cancelled in the net coefficients.

Restricting ourselves onto one particular theory among the $\mathcal{N}=4$ theories which describes the N stack of the M2-branes probing the orbifold $(\mathbb{C}^2/\mathbb{Z}_2\times\mathbb{C}^2/\mathbb{Z}_2)/\mathbb{Z}_k$, we also achieved to determine the coefficients of all kinds of instantons and at arbitrary quantum numbers. We finally found that the instanton coefficients are completely reproduced by the free energy of the refined topological string theory.

Contents

1 Introduction

Quantum formulation of the gravity is a long-standing problem in the theoretical physics. One of the strong candidates is the superstring theory, where a point like degree of freedom is replaced with the fundamental string extending in $(1+1)$ dimension. In string theory the problems on the ultraviolet divergence, appearing in the field theory approach toward the quantum gravity, are avoided.

The type IIA string theory can be formulated by first giving a background geometry, a classical solution in the ten dimensional type IIA supergravity, and describing the embedding of a string worldsheet into the bulk spacetime by a two dimensional supersymmetric sigma model. The interactions among the strings can be incorporated as the non-trivial topology of the entire worldsheet. This formalism is perturbative in the coupling constant g_s of string interaction. However, it was suggested [1] that we can regard g_s as the size of the eleventh extra dimension. Various other types of the string theory with critical dimensions can be also understood as the different compactifications of the eleven dimension. In this sense, the quantum version of the eleven dimensional supergravity, called the *M-theory*, will provide the non-perturbative and unified formulation for the perturbative string theories.

To understand the M-theory, it will be helpful to focus on the solitonic objects in the eleven dimensional supergravity. There are two kinds of half BPS solitons which are extending in $(1+2)$ dimension and $(1+5)$ dimension respectively and called the M2-brane and the M5-brane. Various extending objects in string theory, the fundamental string and the Dp-branes, can be interpreted as the compactification of these M-branes with or without winding in the eleventh direction. In the string theory, a stack of D-branes can be described by some field theory, which we shall call the worldvolume theory of the branes. The fields living on the branes can be interpreted as the fundamental strings ending on the branes, and the Lagrangian of the worldvolume theory will be obtained by studying the interactions of these open strings. Interestingly, although the string theory is formulated in the perturbation, these field theory themselves can be analyzed for finite coupling constant and thus expected to exist non-perturbatively. Hence it is natural to expect that the M-branes can also be described by some field theories, which will play important roles in understanding the M-theory.

Different from the case of the D-branes, so far we do not know the origin of the fundamental degrees of freedom living on the M-branes. Thus the worldvolume theory of the M-branes have been mysterious for long time. However, the symmetries preserved by the M-brane solitons in the eleven dimentional supergravity and the relation to the D-branes under the compactification provide many hints for the candidates of such field theory. Recently the worldvolume theory on a stack of N M2-branes was finally proposed as a particular class of the $U(N)$ superconformal Chern-Simons matter theory [2, 3, 4, 5].

There have been various consistency checks for the proposals. The most striking one among them would be that the partition function of the theory behaves in the large N limit as $\log Z(N) \sim N^{3/2}$. This reproduces the behavior of the gravitational entropy of the M2-branes estimated in [6]. The correspondence is stated more concretely as the following $\mathrm{AdS}_4/\mathrm{CFT}_3$ correspondence [7]. Consider a stack of N coincident M2-branes. Then the correspondence says that, the free energy of the worldvolume theory for this setup should coincide, in the limit $N \to \infty$, with the action of the classical supergravity evaluated on the AdS_4 background which is the near horizon geometry of this setup. It was shown that the coefficient in front of $N^{3/2}$ is also precisely consistent with the calculation in the gravity side, in accordance with this correspondence.

Having the concrete worldvolume theory on the M2-branes in our hand, how can we utilize these theory to understand the M-theory? One idea is the extension of the $\mathrm{AdS}_4/\mathrm{CFT}_3$ correspondence. In the original correspondence the limit $N \to \infty$ is required to suppress the curvature of geometry so that the classical approximation for gravity will be justified. Conversely, it is natural to expect that the $1/N$ corrections in the worldvolume theory correspond to the quantum corrections to the classical supergravity. Indeed this naive idea have been supported by various computations in the case of the D3-branes.

In our recent works we studied the $1/N$ corrections to the free energy $\log Z(N)$ systematically in various theories of the M2-branes. In a large class of theories which describe the M2-branes on $(\mathrm{C}^2/\mathbb{Z}_q \times \mathrm{C}^2/\mathbb{Z}_p)/\mathbb{Z}_k$, which we shall call the $(q,p)_k$ models, we achieved to determine the all order perturbative corrections in $1/N$ to the free energy [8]. We also found interesting structure in the non-perturbative effects [9] in $1/N$ which will be explained later in this thesis. Moreover, for a special case $q = p = 2$ among these theories we also discovered an interesting correspondence [10] between the non-perturbative effects and the Gopakumar-Vafa formula in the topological string theory [11]. This implies we can generate the whole non-perturbative expansion for arbitrarily high order. Hopefully these structures and the quantitative results in the large N expansion will help us to unveil the M-theory in future.

The thesis consists of two parts other than the appendices. In part I we briefly review the three dimensional theories which describe the stack of N M2-branes, and the exact computation of their partition function in the strict large N limit. Note that the review is not fully self-contained. Various details we have omitted will be found in the references therein. In part II, after reviewing the systematic method of analysis called the Fermi gas formalism [12], we introduce our recent works on the exact large N expansion of the partition function. In contrast to part I, we describe the results in full detail and also append several technical aspects which were omitted in the original papers.

Note that the exact computation have been rapidly developed in recent years, before and after the publication of our works. Especially the data for the small k expansion in section 7 and for finite k in section 8 and 9 were remarkably improved in the independent work [13]. In section 9 we try to explain the guesswork in the determination relying only on the original data, though one will obtain the same conclusions more smoothly if he/she uses these additional data.

Part I

Gauge theory on M2-branes

In this part we have reviewed the field theoretical description of the M2-branes, which we shall call the worldvolume theory of the M2-branes. In the eleven dimensional supergravity, a stack of M2-branes is described as a half BPS soliton with $(1+2)$ dimensional extension. For single M2-brane the worldvolume theory is given by a simple Nambu-Goto type action, which coincide with the low energy effective action of the massless modes around the M2-brane soliton solution in the supergravity.

For a stack of multiple M2-branes, on the other hand, the worldvolume theory must contain the interacting modes among the M2-branes which are not contained in the naive set of these massless modes. Such theory have been completely obscure for long time due to the lack of the picture of the fundamental degrees of freedom on the M2-branes, in contrast to the case of the D-branes where the fundamental degree of freedom in the worldvolume theory can be understood as the open strings ending on the D-branes.

Moreover, a computation of the gravitational entropy or the $\mathrm{AdS}_4/\mathrm{CFT}_3$ correspondence for classical gravity suggests a strange scaling of the degree of freedom on N M2-branes: $\log Z(N) \sim N^{3/2}$ [6]. This is again in contrast to the N^2 scaling in the case of the D3-branes in type IIB string theory which allows an intuitive way of understanding by the combinatorics for the two endpoints of a fundamental string.

Another obstacle was associated to the gauge fields on the branes. Since the worldvolume theory of the M2-branes is the strong coupling limit of that on the D2-branes, the field content on the M2-branes should also contain the gauge fields. On the other hand, from the study of the massless modes it follows that there are eight scalars and eight fermions, which correspond to the Nambu-Goldstone modes of the translations and the supersymmetries broken by the M2-brane soliton respectively. Hence the additional gauge fields seem to contradict with the remaining (3d $\mathcal{N}=8$) supersymmetry preserved by the M2-branes.

In [14] a solution to this dilemma was proposed that there is a Chern-Simons gauge field on the M2-branes, which do not have on-shell degree of freedom. Based on this idea the method to realize high supersymmetry in Chern-Simons matter theory have been developed, and finally a $\mathcal{N}=8$ superconformal Chern-Simons matter theory was proposed as the worldvolume theory of two M2-branes [15, 16, 17, 18]. Though these proposal are successful only for two M2-branes, the worldvolume theories for the stack of $N \geq 3$ M2-branes have also been constructed as the $U(N)$ superconformal Chern-Simons matter theories.

Below we first construct such theories from the quiver diagrams, and also provide two evidences which support that the theory indeed describe the M2-branes in section 2. In section 3 we show that the theory have eight dimensional moduli space which can be interpreted as the translation

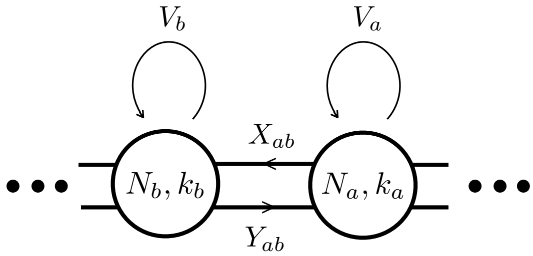

Figure 1: A part of a quiver diagram. Each vertex in the quiver diagram is assigned with a vector multiplet V_a, while each edge between the vertex a and the vertex b is assigned with a pair of chiral multiplet X_{ab} and Y_{ab}.

moduli of the M2-branes in the eleven dimensional spacetime. In section 4 we evaluate the partition function of these theory in the limit $N \to \infty$ and reproduce the $N^{3/2}$ scaling predicted from the gravity side.

2 $U(N)$ circular quiver superconformal Chern-Simons theory

In this thesis, we shall call a diagram which consist of the set of the vertices and the edges connecting the vertices like figure 1 as a quiver diagram.[1] Given a quiver diagram with a pair of numbers (N_a, k_a) on each a-th vertex, we can construct a 3d $\mathcal{N}=2$ supersymmetric Chern-Simons matter theory by assigning a $U(N_a)$ $\mathcal{N}=2$ Chern-Simons vector multiplet

$$V_a = (A_{a,\mu}, \sigma_a, \chi_a, D_a) \tag{2.1}$$

with the Chern-Simons level k_a on the a-th vertex and a pair of $\mathcal{N}=2$ chiral (matter) multiplets

$$\mathcal{X}_{ab} = (X_{ab}, \varphi_{ab}, F_{ab}), \quad \mathcal{Y}_{ab} = (Y_{ab}, \psi_{ab}, G_{ab}) \tag{2.2}$$

on each edge connecting a-th vertex and b-th vertex. Here the chiral multiplets are in the bi-fundamental representation in the gauge group $U(N_a)$ and $U(N_b)$ on the endpoints of the edge, $(\bar{N}_a, N_b)$ and $(N_a, \bar{N}_b)$ respectively.

We especially focus on the circular quivers with M vertices and the uniform rank of the gauge groups $N_1 = N_2 = \cdots = N_M = N$. In this case it is convenient to call the matter multiplets $\mathcal{X}_{a,a+1}$ and $\mathcal{Y}_{a,a+1}$ as $\mathcal{X}_a$ and $\mathcal{Y}_a$ respectively, as in figure 2. The action of these theories consist of three terms

$$S = S_{\text{CS}} + S_{\text{mat}} + S_{\text{pot}}, \tag{2.3}$$

[1]Usually a "quiver diagram" stands for a diagram which consists of the set of vertices and the arrows connecting the vertices [19]. In this thesis, however, we always consider the case where any pair of vertices is either connected by oppositely oriented two arrows or not connected at all. Hence we depict such pair of arrows just by an edge.

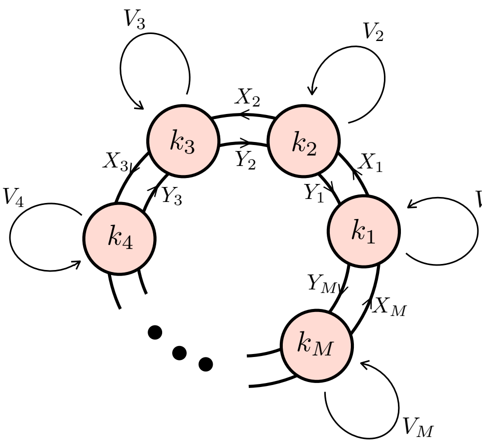

Figure 2: A $\mathcal{N}=3$ circular quiver with M nodes.

where the first term is the Chern-Simons action of the vector multiplets

$$S_{\rm CS} = \int \sum_{a=1}^{M} \frac{k_a}{4\pi} \,{\rm Tr}\Big(A_a \wedge dA_a + \frac{2}{3} A_a \wedge A_a \wedge A_a - \bar{\chi}_a \chi_a + 2 D_a \sigma_a \Big). \tag{2.4}$$

while the other two terms are the action for the matter multiplets. In terms of the superfields they can be written as

$$\begin{aligned} S_{\rm mat} &= \int d\theta^4 \sum_{a=1}^{M} {\rm Tr}(e^{-V_a} X_a^\dagger e^{V_{a+1}} X_a + e^{V_a} Y_a e^{-V_{a+1}} Y_a^\dagger), \\ S_{\rm pot} &= \int d\theta^2 W + (h.c). \end{aligned} \tag{2.5}$$

Here we choose the superpotential W as

$$W = \sum_{a=1}^{M} \frac{1}{k_a} \,{\rm Tr}(Y_a X_a - X_{a-1} Y_{a-1})^2. \tag{2.6}$$

Though we do not use the superspace formalism for the computations in this thesis, we would like to note that the Chern-Simons action (2.4) can also be written in terms of the superfields [20]. Hence the action is manifestly invariant under the 3d $\mathcal{N}=2$ supersymmetry transformation. This theory also enjoys the $\mathcal{N}=2$ superconformal symmetry, which follows from the fact that the superpotential is quartic in the matter fields [20].

Due to the requirement of the gauge invariance and the flux quantization condition, the Chern-Simons levels are restricted to integers [21]. As we shall see in the next section, the levels also have to satisfy the following condition

$$\sum_{a=1}^{M} k_a = 0 \tag{2.7}$$

if we require that the dimension of the Moduli space is eight so that the theory can describe the M2-branes in an eleven dimensional spacetime. In this case the supersymmetry enhances to $\mathcal{N} = 3$.

On the other hand it is allowed to choose a Chern-Simons level zero $k_a = 0$, where the Chern-Simons action is absent. Though the corresponding terms in the superpotential (2.6) are ill defined, we can define a well defined theory by introducing an auxiliary adjoint chiral multiplet Φ_a on each vertex with $k_a = 0$ and replace the singular superpotentials with

$$\widetilde{W}_a = \mathrm{Tr}\,\Phi_a(Y_a X_a - X_{a-1} Y_{a-1}). \tag{2.8}$$

Above we have chosen the superpotentials just by hand. Actually, if we only require the $\mathcal{N} = 2$ supersymmetry and the conformal invariance at classical level, an arbitrary quartic term in X_a, Y_a is allowed for the superpotential W. However if we consider a concrete type IIB brane setup to realize the Chern-Simons terms [22, 23], we can naturally deduce our particular choice of the superpotential (2.6), including the exceptional cases of $k_a = 0$ (2.8) [3].

3 Eight dimensional moduli space

In this section we review the study of the vacuum moduli space of the theory we have defined in the previous section [3, 24]. We show that there exists an eight dimensional branch of moduli space if and only if the Chern-Simons levels add up to zero (2.7).

3.1 Generic Chern-Simons levels

Let us write down the bosonic potential terms in the Lagrangian (2.3)

$$
\begin{aligned}
\mathcal{L}_{\rm CS} &= \sum_{a=1}^{M} \frac{k_a}{2\pi} \operatorname{Tr} D_a \sigma_a + \cdots , \\
\mathcal{L}_{\rm mat} &= \sum_{a=1}^{M} \operatorname{Tr} \Big[\frac{1}{4} (\sigma_a X_a^\dagger - X_a^\dagger \sigma_{a+1})(X_a \sigma_a - \sigma_{a+1} X_a) \\
&\qquad + \frac{1}{4} (\sigma_a Y_a - Y_a \sigma_{a+1})(Y_a^\dagger \sigma_a - \sigma_{a+1} Y_a^\dagger) \\
&\qquad + \frac{1}{2} D_a (-X_a^\dagger X_a + X_{a-1} X_{a-1}^\dagger + Y_a Y_a^\dagger - Y_{a-1}^\dagger Y_{a-1}) + F_a^\dagger F_a + G_a G_a^\dagger \Big] \\
&\quad + \sum_{a=1}^{M} \Big[2i \operatorname{Tr} \Big\{ \Big(\frac{1}{k_a} + \frac{1}{k_{a+1}} \Big) Y_a X_a Y_a - \frac{1}{k_a} X_{a-1} Y_{a-1} Y_a - \frac{1}{k_{a+1}} Y_a Y_{a+1} X_{a+1} \Big\} F_a \\
&\qquad + 2i \operatorname{Tr} G_a \Big\{ \Big(\frac{1}{k_a} + \frac{1}{k_{a+1}} \Big) X_a Y_a X_a - \frac{1}{k_a} X_a X_{a-1} Y_{a-1} - \frac{1}{k_{a+1}} Y_{a+1} X_{a+1} X_a \Big\} \\
&\qquad + (h.c) \Big] + \cdots ,
\end{aligned} \tag{3.1}
$$

where the last three lines in $\mathcal{L}_{\rm mat}$ come from the superpotential (2.6). Integrating the auxiliary fields F_a and G_a in the chiral multiplets we obtain the following F-term conditions

$$
\begin{aligned}
(k_a + k_{a+1}) Y_a X_a Y_a - k_{a+1} X_{a-1} Y_{a-1} Y_a - k_a Y_a Y_{a+1} X_{a+1} &= 0, \\
(k_a + k_{a+1}) X_a Y_a X_a - k_{a+1} X_a X_{a-1} Y_{a-1} - k_a Y_{a+1} X_{a+1} X_a &= 0.
\end{aligned} \tag{3.2}
$$

To obtain the D-term conditions we first integrate out the auxiliary fields D_a in the vector multiplet to obtain the following constraints

$$
\frac{k_a}{\pi} \sigma_a = X_a^\dagger X_a - X_{a-1} X_{a-1}^\dagger - Y_a Y_a^\dagger + Y_{a-1}^\dagger Y_{a-1}. \tag{3.3}
$$

From the first two lines in the potential terms (3.1), the D-term conditions are

$$
X_a \sigma_a - \sigma_{a+1} X_a = 0, \quad \sigma_a Y_a - Y_a \sigma_{a+1} = 0 \tag{3.4}
$$

with substitution of (3.3). The vacuum moduli space is the space of the solutions to the F-term conditions (3.2) and the D-term conditions (3.4) modulo the gauge transformations.

For simplicity we shall only display the analysis for the abelian cases N=1. We shall also concentrate ourselves on the branch where $X_a, Y_a \neq 0$. In this case, the F-term conditions and the D-term conditions simplify respectively as

$$\begin{aligned} F \to &(k_a + k_{a+1}) X_a Y_a = k_a X_{a+1} Y_{a+1} + k_{a+1} X_{a-1} Y_{a-1}, \\ D \to &(k_a + k_{a+1})(|X_a|^2 - |Y_a|^2) = k_a(|X_{a+1}|^2 - |Y_{a+1}|^2) + k_{a+1}(|X_{a-1}|^2 - |Y_{a-1}|^2). \end{aligned} \tag{3.5}$$

Let us estimate the dimension of the moduli space, assuming that the total Chern-Simons level vanishes (2.7). As commented above, the dimension of the moduli space can be written schematically as

$$\dim(\mathcal{M}_{U(1)}) = 4M - (\#\text{ of } F\text{- and } D\text{-term conditions}) - (\#\text{ of gauge d.o.f}), \tag{3.6}$$

where the first $4M$ is the total number of the matter scalar fields. To count the second ingredient, it is convenient to rearrange the F-term conditions (together with the complex conjugates) and the D-term conditions (3.5) as

$$\mathcal{K}_{ab} \begin{pmatrix} X_a & Y_a^\dagger \end{pmatrix} \sigma_\alpha \begin{pmatrix} X_b^\dagger \\ Y_b \end{pmatrix} = 0, \quad \alpha = 1, 2, 3 \tag{3.7}$$

where σ_α are the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{3.8}$$

and $\mathcal{K}$ is a matrix defined by the Chern-Simons levels as

$$\mathcal{K} = \begin{pmatrix} k_1+k_2 & k_1 & 0 & \cdots & \cdots & 0 & k_2 \\ k_3 & k_2+k_3 & k_2 & 0 & \cdots & \cdots & 0 \\ 0 & k_4 & k_3+k_4 & k_3 & 0 & \cdots & 0 \\ & & & \ddots & & & \\ 0 & \cdots & 0 & k_{M-1} & k_{M-2}+k_{M-1} & k_{M-2} & 0 \\ 0 & \cdots & \cdots & 0 & k_M & k_{M-1}+k_M & k_{M-1} \\ k_M & 0 & \cdots & \cdots & 0 & k_1 & k_M+k_1 \end{pmatrix}. \tag{3.9}$$

Under the condition (2.7), the rank of this matrix is $\text{rank}(\mathcal{K}) = M-2$, hence the F-term conditions and the D-term conditions are $3(M-2)$ real equations in total.

Next let us count the gauge transformations. Though there are M gauge groups $U(1)^M$ assigned on the vertices of the quiver, it is obvious that no fields transform under the "overall $U(1)$ gauge transformation". Moreover, due to the condition (2.7) the corresponding gauge field

$$A_{\text{over}} = \sum_{a=1}^{M} A_a \tag{3.10}$$

do not have the own Chern-Simons term coupling to itself but only couple to the other gauge fields, i.e. (up to the integration by parts)

$$\sum_{a=1}^{M} \frac{k_a}{4\pi} \int A_a dA_a = \int A_{\text{others}} dA_{\text{over}} + \cdots . \tag{3.11}$$

This implies that we can dualize the overall gauge field A_{over} into a free scalar field, which compensates another $U(1)$ gauge transformation. We conclude that there are $M-2$ gauge degrees of freedom in total.

From these arguments the dimension of the moduli space is estimated as

$$\dim(\mathcal{M}_{U(1)}) = 4M - 3(M-2) - (M-2) = 8. \tag{3.12}$$

This result is consistent with the number of transverse directions to an M2-brane in eleven dimensional spacetime $11 - (2+1) = 8$, and supports the proposal to describe the M2-branes by the superconformal quiver Chern-Simons theory. If the Chern-Simons levels do not sum up to zero, on the other hand, the total number of F- and D-term conditions increases since the rank of the coefficient matrix $\mathcal{K}$ is $\text{rank}(\mathcal{K}) = M - 1$. What is worse, there are no reduction of the gauge degree of freedom discussed above. As a result the dimension of the moduli space would be smaller than 8. Hence the condition (2.7) is indeed essential for the theory to describe the M2-branes.

3.2 Vanishing Chern-Simons levels

We need to modify the arguments if the theory contains vanishing Chern-Simons levels $k_a = 0$. For simplicity we shall only consider the special case

$$k_1 = k, \quad k_2 = k_3 = \cdots = k_q = 0, \tag{3.13}$$

$$k_{q+1} = -k, \quad k_{q+2} = k_{q+3} = \cdots k_{q+p} = 0 \tag{3.14}$$

with $k, q, p \in \mathbb{N}$. We will call this theory as $(q,p)_k$ minimal model for the reason explained in section 6, and study this theory in full detail thereafter. This model is special in the sense that the supersymmetry enhances from $\mathcal{N} = 3$ to $\mathcal{N} = 4$ [3].

Again assume the abelian gauge groups. First consider the F-term condition for the auxiliary scalars F_{Φ_a} in the adjoint chiral multiplets Φ_a. The potential terms containing these auxiliary fields are

$$S_{\text{mat}} = \sum_{a \neq 1, q+1} i F_{\Phi_a} (Y_a X_a - X_{a-1} Y_{a-1}) + (h.c.) + \cdots . \tag{3.15}$$

Since there are no Kahler term for the adjoint chiral multiplets, we obtain the δ-function constraints by integrating F_{Φ_a}

$$X_1 Y_1 = X_2 Y_2 = \cdots = X_q Y_q \equiv \mu_+, \quad X_{q+1} Y_{q+1} = X_{q+2} Y_{q+2} \cdots = X_M Y_M \equiv \mu_-. \tag{3.16}$$

Second we consider the F-term conditions for the bifundamental chiral multiplet. Since the original superpotential other than $\widetilde{W}_a$ only contains F_1, F_q, F_{q+1}, F_M and G_1, G_q, G_{q+1}, G_M, let us first consider the F-term conditions for the other auxiliary fields. From $\widetilde{W}_a$, we obtain the following condition for F_2 and G_2 respectively

$$Y_2(\phi_2 - \phi_3) = 0, \quad X_2(\phi_2 - \phi_3) = 0. \tag{3.17}$$

Here ϕ_a is the lowest scalar component of the adjoint chiral multiplet Φ_a. Assuming $X_a, Y_a \neq 0$ as in the case of non-zero Chern-Simons levels, the two conditions reduce to $\phi_2 = \phi_3$. Repeating the manipulation one by one in order, we finally obtain the following conditions

$$\phi_2 = \phi_3 = \cdots = \phi_q \equiv \phi_+, \quad \phi_{q+2} = \phi_{q+3} = \cdots = \phi_M \equiv \phi_-. \tag{3.18}$$

The F-term conditions for the other auxiliary fields F_a, G_a ($a = 1, q, q+1, M$) are the slight modifications of (3.2) due to $\widetilde{W}_a$. After applying the conditions in our hands (3.16) and (3.18), only two of the eight conditions are found to be independent:

$$\begin{aligned} F_1 &\to \frac{2}{k}(\mu_+ - \mu_-) - \phi_+ = 0, \\ F_{q+1} &\to -\frac{2}{k}(\mu_- - \mu_+) - \phi_- = 0. \end{aligned} \tag{3.19}$$

Third we shall consider the D-term conditions. From D_a with $a \neq 1, q+1$ we obtain the δ-function constraints wihch reduce to

$$|X_1|^2 - |Y_1|^2 = |X_2|^2 - |Y_2|^2 = \cdots = |X_q|^2 - |Y_q|^2 \equiv \mu_{3+}, \quad \sigma_2 = \sigma_3 = \cdots = \sigma_{q+1} \equiv \sigma_+ \tag{3.20}$$

for $a = 2, 3, \cdots q$ and

$$|X_{q+1}|^2 - |Y_{q+1}|^2 = |X_{q+2}|^2 - |Y_{q+2}|^2 = \cdots = |X_M|^2 - |Y_M|^2 \equiv \mu_{3-}, \tag{3.21}$$

$$\sigma_{q+2} = \sigma_{q+3} = \cdots = \sigma_1 \equiv \sigma_- \tag{3.22}$$

for $a = q+2, q+3, \cdots M$. On the other hand, the D-terms conditions for D_a with $a = 1, q+1$, which are again simplified with the help of these results, are

$$\mu_{3+} - \mu_{3-} = \frac{k\sigma_+}{\pi}, \quad \sigma_+ = \sigma_-. \tag{3.23}$$

In summary

- ϕ_a and σ_a do not carry independent degrees of freedom.
- We have $2q$ complex scalars X_a, Y_a ($a = 1, 2, \cdots, q$) subject to $3(q-1)$ real constraints (3.16) and (3.20), and $(q-1)$ independent gauge transformations acting on them.
- We also have $2p$ complex scalars X_a, Y_a ($a = q+1, q+2, \cdots, M$) subject to $3(p-1)$ real constraints (3.16) and (3.21), and $(p-1)$ independent gauge transformations acting on them.

Hence the real dimension of the moduli space is again 8 from (3.6)

$$\dim(\mathcal{M}_{U(1)}) = 4q - 3(q-1) - (q-1) + (4p - 3(p-1) - (p-1)) = 8. \tag{3.24}$$

The concrete expression of the moduli space of the $(q,p)_k$ model is determined in [3] as

$$(\mathbb{C}^2/\mathbb{Z}_q \times \mathbb{C}^2/\mathbb{Z}_p)/\mathbb{Z}_k. \tag{3.25}$$

4 Large N partition function

In this section we review the computation of the partition function $Z(N)$ of the $\mathcal{N}=3$ $U(N)$ circular quiver superconformal Chern-Simons theory in the limit of $N\to\infty$. In the first two subsections we will explain that the partition function of the theory realized on the round three sphere S^3 with unit radius is given by the following finite dimensional integration (which we shall call a matrix model) [25, 26, 27].

$$Z(N)=\frac{1}{(N!)^M}\prod_{a=1}^M\prod_{i=1}^N\Big[\int\frac{d\lambda_{a,i}}{2\pi}e^{\frac{ik_a}{4\pi}\lambda_{a,i}^2}\Big]\prod_{a=1}^M\frac{\prod_{i<j}2\sinh\frac{\lambda_{a,i}-\lambda_{a,j}}{2}\prod_{i<j}2\sinh\frac{\lambda_{a+1,i}-\lambda_{a+1,j}}{2}}{\prod_{i,j}2\cosh\frac{\lambda_{a,i}-\lambda_{a+1,j}}{2}}. \tag{4.1}$$

We would like to provide only the basic idea of the localization technique which is the most essential in the whole derivation and a few sketches for the other parts of the derivation, while skip almost all the detail of the derivation.[2] After that we demonstrate the evaluation of the matrix model in the large N limit for the case of the simplest quiver $\{k_a\}_{a=1}^2=\{k,-k\}$, which is called the ABJM theory [2, 4].

4.1 Localization technique

Here we shall explain the powerful computational technique in the supersymmetric gauge theories, called the localization technique [28, 29]. Consider a theory with the field content Φ and the action $S(\Phi)$, and try to compute the following expectation value of a composite $\mathcal{O}(\Phi)$

$$\langle\mathcal{O}\rangle=\int\mathcal{D}\Phi\mathcal{O}e^{-S}. \tag{4.2}$$

Then, it follows that

- Suppose there exists some symmetry Q of the theory respected by $\mathcal{O}$, $QS=Q\mathcal{O}=0$, and some Q-exact functional $QV(\Phi)$ which is positive semi definite in Φ and Q-closed $Q(QV)=0$. Then path integral partly reduces to the Gaussian integrations around the locus of $QV(\Phi)=0$

$$\langle\mathcal{O}\rangle=\sum_{\Phi_0(QV(\Phi_0)=0)}\mathcal{O}(\Phi_0)e^{-S(\Phi_0)}Z_{\text{1-loop}}\bigg(\Big[\frac{1}{2}\frac{\delta^2QV}{\delta\Phi\delta\Phi}\Big]_{\Phi_0}\bigg), \tag{4.3}$$

  where $Z_{\text{1-loop}}(\cdot)$ is the functional determinant of an operator over the fluctuations of Φ around Φ_0.

This can be shown as follows. First we introduce a deformation parameter $t>0$ to (4.2) as

$$\langle\mathcal{O}\rangle_t=\int\mathcal{D}\Phi\mathcal{O}e^{-S-tQV}. \tag{4.4}$$

[2] The readers who accept the matrix model expression of the partition function (4.1) may skip section 4.1 and 4.2.

The original expectation value is reproduced by taking $t = 0$

$$\langle \mathcal{O} \rangle = \langle \mathcal{O} \rangle_{t=0}. \tag{4.5}$$

On the other hand, differentiating (4.4) we obtain

$$\begin{aligned} \frac{d}{dt}\langle \mathcal{O} \rangle_t &= \int \mathcal{D}\Phi(-QV)\mathcal{O}e^{-S-tQV} \\ &= \int \mathcal{D}\Phi Q\Big(-V\mathcal{O}e^{-S-tQV}\Big) \\ &= 0, \end{aligned} \tag{4.6}$$

where we have used $QS = Q\mathcal{O} = Q(QV) = 0$ and have supposed the Q-invariance of the integration measure. This implies that $\langle \mathcal{O} \rangle_t$ is actually independent of the deformation parameter t. Hence we conclude that the original expectation value $\langle \mathcal{O} \rangle$ can be evaluated by $\langle \mathcal{O} \rangle_t$ in the limit $t \to \infty$. Since QV is positive semi definite, the path integral almost localizes to the configurations $QV = 0$ in this limit. The deviation from this rough estimation can be obtained by expanding the fields as

$$\Phi = \Phi_0 + \frac{1}{\sqrt{t}}\Delta\Phi, \tag{4.7}$$

with which

$$\begin{aligned} S &= S(\Phi_0) + \mathcal{O}(t^{-\frac{1}{2}}), \\ tQV &= \frac{1}{2}\Delta\Phi\left[\frac{\delta^2 QV}{\delta\Phi\delta\Phi}\right]_{\Phi_0}\Delta\Phi + \mathcal{O}(t^{-\frac{1}{2}}). \end{aligned} \tag{4.8}$$

Here we have used the fact that $QV = \delta QV/(\delta\Phi) = 0$ at $\Phi = \Phi_0$. Plugging these expressions into (4.4) and sending t to infinity we obtain (4.3).

A familiar example for such reduction would be the process of the BRST gauge fixing in a gauge theory, where Q is the BRST charge and Φ_0 are the general field configurations on some particular gauge slice. Typically the saddle point loci Φ_0 themselves keep some functional degree of freedom, as in the case of this example, and there still remain non-trivial path integrals. In some of the supersymmetric field theories with Q an appropriate choice of the supersymmetry transformation, however, the path integral localizes completely and the remaining sum over Φ_0 is discrete sum or at most finite dimensional ordinary integrals. The matrix model expression of the partition function (4.1) is obtained by applying this technique to the partition function ($\mathcal{O} = 1$) of the superconformal Chern-Simons theory on S^3.

Note that, in the above argument we have implicitly assumed that the differential with respect to t commute with the path integral, that is, that the path integral converges for arbitrary fixed t in $0 \leq t < \infty$. Roughly speaking, the action on S^3 is realized by Euclideanize the original action (2.3), replacing the derivatives with the covariant derivatives, and adding curvature coupling terms appropriately. The compactification will introduce a inflared cutoff to the theory. Moreover, due to the curvature coupling the contribution of a large constant mode will be suppressed, which regularize the divergence caused by infinite volume of the moduli space of the theory. These are the reasons for why we consider the theory on a sphere.

4.2 Partition function as matrix model – sketch of derivation

In this section we briefly explain the derivation of each factor in (4.1) for the $\mathcal{N}=3$ $U(N)$ quiver superconformal Chern-Simons theory. First, we shall quote that the localization locus which solve the conditions

$$\delta\chi_a = 0, \quad \delta\phi_{ab} = 0, \quad \delta\psi_{ab} = 0 \tag{4.9}$$

is found to be [25]

$$\begin{aligned} \sigma_a &= \sigma_{0a}, \quad (\sigma_{0a}\text{: constant matrix}) \\ D_a &= -\sigma_a, \quad \text{other fields} = 0. \end{aligned} \tag{4.10}$$

Due to the gauge invariance of the action we can choose the saddle point values σ_{0a} as diagonal matrices in exchange of the appearance of the Vandermonde determinant in the integration over the saddle point configurations (Weyl integration formula) as

$$\begin{aligned} \sigma_{0a} &= \begin{pmatrix} \sigma_{0a,1} & & & \\ & \sigma_{0a,2} & & \\ & & \ddots & \\ & & & \sigma_{0a,N} \end{pmatrix}, \\ \sum_{\text{saddle}} &\to \frac{1}{N!}\prod_{a=1}^{M}\prod_{i=1}^{N}\int d\sigma_{a,i}\prod_{a=1}^{M}\prod_{i\neq j}(\sigma_{0a,i}-\sigma_{0a,j}). \end{aligned} \tag{4.11}$$

After the substitution of the saddle point configuration into the action, only the $D\sigma$ term in S_{CS} survives and we obtain

$$e^{-S|_{\text{saddle}}} = e^{-i\int_{S^3} dx^3 \sqrt{g}\sum_{a=1}^{M}\frac{k_a}{2\pi}\operatorname{Tr} D_a\sigma_a} = e^{\pi i k\sum_{a=1}^{M}\sum_{i=1}^{N}\sigma_{0a,i}^2}, \tag{4.12}$$

where the prefactor i comes in through the Euclideanization of the action (2.4).

Next, we would like to sketch the derivation of the 1-loop determinant. Expanding the regulator potential QV as (4.8), we will obtain the kinetic terms with supersymmetric version of the covariant derivatives, i.e. the diffeomorphism/gauge covariant derivatives plus the Yukawa terms with $\sigma = \sigma_0$.[3] On the round sphere S^d we can explicitly list the eigenfunctions of the kinetic operators, labeled by d integers as well as the spin/gauge indices. Though almost all the contributions from these modes are cancelled between the bosonic fields and the fermionic fields,

[3] Notice that the 3d $\mathcal{N}=2$ supermultiplets are the dimensional reduction of the 4d $\mathcal{N}=1$ supermultiplets, where the scalar σ is originally the fourth component of the gauge field A_μ.

nevertheless there typically remains some infinite series of eigenvalues with reduced dimension. Here we shall just display the final results [25]. For the adjoint multiplet on the a-th vertex in the quiver we obtain

$$Z_{\text{1-loop}}^{\text{adj}} = \prod_{i\neq j} \frac{\prod_{\ell=1}^{\infty}(\ell + i(\sigma_{0a,i} - \sigma_{0a,j}))}{\prod_{\ell=1}^{\infty}(\ell - i(\sigma_{0a,i} - \sigma_{0a,j}))} = \prod_{i\neq j} \frac{2\sinh\pi(\sigma_{0a,i} - \sigma_{0a,j})}{\sigma_{0a,i} - \sigma_{0a,j}}, \tag{4.13}$$

while for the pair of bifundamental multiplet on the a-th edge

$$Z_{\text{1-loop}}^{\text{matter}} = \Biggl[\prod_{i,j=1}^{N}\prod_{\ell=1}^{\infty}\biggl(\frac{\ell+\frac{1}{2}+i(\sigma_{0a,i}-\sigma_{0a+1,j})}{\ell-\frac{1}{2}-i(\sigma_{0a,i}-\sigma_{0a+1,j})}\biggr)^{\ell}\Biggr]^2 = \prod_{i,j=1}^{N}\frac{1}{2\cosh\pi(\sigma_{0a,i}-\sigma_{0a+1,j})}, \tag{4.14}$$

where the index ℓ originates from the labels of the spherical Harmonics on S^3, surviving after the cancellation between the bosons and the fermions. The σ_{0a}-dependence reflects the representation of each supermultiplet under the gauge groups.

Putting above results together, with the rescaling $\sigma_{0a,i} = \lambda_{a,i}/(2\pi)$, we obtain the matrix model expression of the partition function (4.1).

4.3 Large N limit of partition function

In the last sections we have shown that the partition function of the quiver superconformal Chern-Simons theory can be evaluated by the finite dimensional ordinary integrals (4.1), without being bothered by the non-trivial path integrals. Nevertheless we have to evaluate the large number (MN) of integrals, which is still a non-trivial task. In the strict large N limit, however, we can evaluate the integrals with the help of the saddle point approximation. The result perfectly coincide with the action of the eleven dimensional supergravity on the dual background geometry, which provides a consistency check for the theory to describe N M2-branes.

Historically the matrix model was first solved for the simplest case, the ABJM theory, in the 't Hooft limit $k, N \to \infty$ with k/N fixed [30]. Later the matrix model for general $\mathcal{N} = 3$ quiver was uniformly solved in the limit $N \to \infty$ while the levels k_a kept finite [31], which is more directly interpreted as the dual of the eleven dimensional geometry and called the "M-theoretical limit" [32]. In this section we shall review the latter way of analysis, while considering the ABJM theory for simplicity.

Let us denote $\lambda_{a=1}$ and $\lambda_{a=2}$ as λ and $\widetilde{\lambda}$ respectively, and write the matrix model (4.1) as

$$Z(N) = \prod_{i=1}^{N} \int d\lambda_i d\widetilde{\lambda}_i e^{f(\lambda,\widetilde{\lambda})} \tag{4.15}$$

with

$$f(\lambda,\widetilde{\lambda}) = \frac{ik}{4\pi}\sum_{i=1}^{N}(\lambda_i^2 - \widetilde{\lambda}_i^2) + \sum_{i\neq j}\log 2\sinh\frac{\lambda_i-\lambda_j}{2} + \sum_{i\neq j}\log 2\sinh\frac{\widetilde{\lambda}_i-\widetilde{\lambda}_j}{2}$$

$$
-2\sum_{i,j=1}^{N}\log 2\cosh\frac{\lambda_i-\widetilde{\lambda}_j}{2}-2\log(2\pi N!). \tag{4.16}
$$

Then the saddle point approximation for the partition function is

$$
Z(N)\sim e^{f(\lambda,\widetilde{\lambda})}\Big|_{\text{saddle}} \tag{4.17}
$$

where "saddle" stands for the substitution of the solution of the following saddle point equations:

$$
\begin{aligned}
0=\frac{\partial f}{\partial\lambda_i}&=\frac{ik}{2\pi}\lambda_i+\sum_{j(\neq i)}\coth\frac{\lambda_i-\lambda_j}{2}-\sum_j\tanh\frac{\lambda_i-\widetilde{\lambda}_j}{2},\\
0=\frac{\partial f}{\partial\widetilde{\lambda}_i}&=-\frac{ik}{2\pi}\widetilde{\lambda}_i+\sum_{j(\neq i)}\coth\frac{\widetilde{\lambda}_i-\widetilde{\lambda}_j}{2}+\sum_j\tanh\frac{\lambda_j-\widetilde{\lambda}_i}{2}.
\end{aligned} \tag{4.18}
$$

The sum over the indices in (4.16) implies that $f(\lambda,\widetilde{\lambda})$ generically grows in some positive power of N, and hence the saddle point approximation for the partition function (4.15) becomes accurate in the limit $N\to\infty$.

We can estimate the solution of the saddle point equation by a numerical study. Consider a $2N$ pair of infinite sequences $\{\lambda_i^{(n)},\widetilde{\lambda}_i^{(n)}\}_{n=1}^{\infty}$ generated by the following recursion relation

$$
\begin{pmatrix}\lambda_i^{(n+1)}\\ \widetilde{\lambda}_i^{(n+1)}\end{pmatrix}=\begin{pmatrix}\lambda_i^{(n)}\\ \widetilde{\lambda}_i^{(n)}\end{pmatrix}+\mathcal{E}\begin{pmatrix}\frac{\partial f(\lambda_i^{(n)},\widetilde{\lambda}_i^{(n)})}{\partial\lambda_i}\\ \frac{\partial f(\lambda_i^{(n)},\widetilde{\lambda}_i^{(n)})}{\partial\widetilde{\lambda}_i}\end{pmatrix} \tag{4.19}
$$

with $\mathcal{E}$ some $2N\times 2N$ matrix. Given k, N, the initial values and the matrix $\mathcal{E}$ we can easily generate the whole sequence numerically. Now suppose the sequence converges. Then it follows from the recursion relation that the set of values of convergence $(\lambda_i,\widetilde{\lambda}_i)$ is a solution to the saddle point equation (4.18). For $k=4$ and $N=20,80$, for example, the sequence can converge for the following choice of $\mathcal{E}$

$$
\mathcal{E}=\begin{pmatrix}0.001\times i\mathbb{1} & 0\\ 0 & -0.001\times i\mathbb{1}\end{pmatrix}. \tag{4.20}
$$

The results are displayed in figure 3.

Now let us solve the saddle point equation analytically in the large N limit. Comparing the numerical results for $N=20$ and $N=80$, and for λ and $\widetilde{\lambda}$, it is reasonable to pose the following ansatz:

$$
\lambda_i=\sqrt{N}x_i+iy_i,\quad \widetilde{\lambda}_i=\sqrt{N}x_i-iy_i \tag{4.21}
$$

with x_i and y_i some real numbers of $\mathcal{O}(1)$. For these ansatz the saddle point equations for $\widetilde{\lambda}$, $\partial f/\partial\widetilde{\lambda}_i$, are simply the complex conjugate of $\partial f/\partial\lambda_i$. It is also convenient to regard the solutions as a continuous distribution by introducing the eigenvalue density $\rho(x)$ in the real axis

$$
x_i\to x,
$$

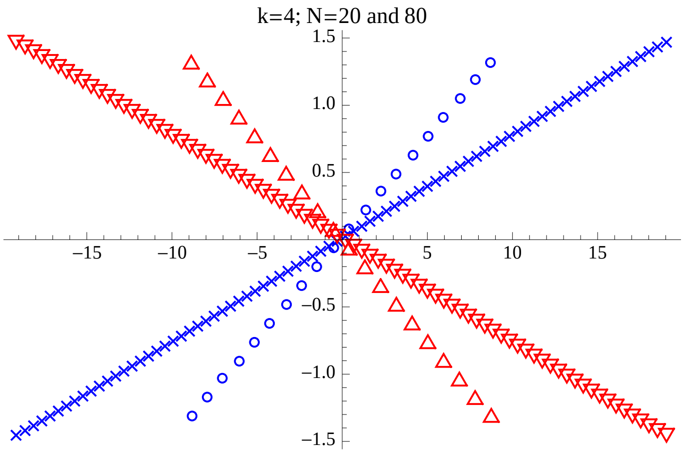

Figure 3: The numerical result of $(\{\lambda_i\}, \{\widetilde{\lambda}_i\})$ for $N = 20$ ((blue circle, red triangle)) and those for $N = 80$ ((blue cross, red inverted triangle)). We observe that $\widetilde{\lambda}_i$ are distributed on the complex conjugates of λ_i. Also notice that the real part scales as $\sqrt{N}$ (doubled as N is quadrupled) while the imaginary part is not scaled.

$$\begin{aligned} y_i &\to y(x), \\ \sum_i &\to N \int_{-L}^{L} dx \rho(x) \end{aligned} \tag{4.22}$$

where we have assumed the support of distribution to be a single segment $(-L, L)$ and have normalized $\rho(x)$ as

$$\int_{-L}^{L} dx \rho(x) = 1. \tag{4.23}$$

Expanding the saddle point equations in the large N limit under these expressions, we finally obtain the following equations from the real/imaginary part respectively

$$\begin{aligned} \frac{k}{2\pi} - 4\rho\frac{dy}{dx} - 4\frac{d\rho}{dx}y = 0, \\ \frac{kx}{2\pi} - 4\rho y = 0. \end{aligned} \tag{4.24}$$

From the numerical result we postulate $y(x)$ to be linear in x and find the following solution to the differential equation and the normalization condition (4.23).

$$y = \frac{kLx}{4\pi}, \quad \rho = \frac{1}{2L}. \tag{4.25}$$

Though the constant L remains undetermined at this stage, it can be determined by minimizing the free energy f (4.16). In the continuum notation (4.21) with (4.22), the leading part of f can be written as

$$f = N^{\frac{3}{2}} \left(-\frac{k}{\pi} \int_{-L}^{L} dx \rho y - \pi^2 \int_{-L}^{L} dx \rho^2 + 4 \int_{-L}^{L} dx y^2 \rho^2 \right). \tag{4.26}$$

Substituting the solution (4.25) we obtain

$$f = N^{\frac{3}{2}}\left(-\frac{k^2 L^3}{24\pi^2} - \frac{\pi^2}{2L}\right) \tag{4.27}$$

which is minimized at

$$L = \pi\sqrt{\frac{2}{k}}. \tag{4.28}$$

Hence we obtain the partition function in the large N limit as

$$Z(N) \sim \exp\left[-\frac{\pi\sqrt{2k}}{3}N^{\frac{3}{2}}\right]. \tag{4.29}$$

Note that the abelian moduli space of the ABJM theory is $\mathbb{C}^4/\mathbb{Z}_k$ (3.25) and this partition function can be written with the volume of the radial section of $\mathbb{C}^4/\mathbb{Z}_k$, $\mathrm{vol}(Y)$ as

$$Z(N) \sim \exp\left[-N^{\frac{3}{2}}\sqrt{\frac{2\pi^6}{27\,\mathrm{vol}(Y)}}\right] \tag{4.30}$$

and precisely consistent with the calculation in the gravity side. Though here we have only considered the ABJM theory, the computation is parallel also for the general $\mathcal{N} = 3$ quivers and we will obtain the same final expression (4.30) with Y the radial section of the moduli space of each theory [31].

5 Summary for part I

In this part we have review on the $\mathcal{N}=3$ $U(N)$ superconformal quiver Chern-Simons theories proposed as the worldvolume theory of N M2-branes. Different from the case of D-branes, so far there are no first principle derivation for the fundamental degree of freedom in the worldvolume theory on M2-branes, hence the proposal would be in some sense heuristic. Supported by a variety of circumstantial evidences, however, these theories are believed to actually describe the M2-branes.

We have introduced two of these evidences. The first one is that the theory have eight dimensional moduli space, corresponding to the location of the M2-branes in the eleven dimensional spacetime. The second one is that the large N limit of the free energy is consistent with the calculation in the gravity side: $N^{3/2}$ scaling behavior as well as its prefactor. In order the moduli space to be eight dimensional, the vanishing of the total Chern-Simons level (2.7) is crucial. Though we have assumed this condition in the computation of the partition function above, the localization and the saddle point analysis are also applicable for the non-vanishing cases and we can show that (2.7) is the necessary condition also for the leading $N^{3/2}$ scaling of the free energy [33].

Remarkably, the $\mathcal{N}=3$ $U(N)$ circular quiver superconformal Chern-Simons theories we have considered can be realized by particular D3-NS5-$(1,k)$5 brane setups in the type IIB string theory [2, 3]. Here $(1,k)$5 is the bound state of one NS5 brane and k D5-branes. These systems can be lifted up to the system of the M2-branes via appropriate duality transformations, where the D3-branes turn into the M2-branes while the five branes become the geometry called Kaluza-Klein monopole. Close to the core, the Kaluza-Klein monopole geometry is asymptotically an orbifolded plane which precisely reproduce the orbifold action on the moduli space. This strongly supports that the proposed theories are indeed the worldvolume theory of the M2-branes.

In terms of the gauge/gravity correspondence, in this part we have used the gravity side to confirm the gauge theory. Once we accept the quiver superconformal Chern-Simons theory as the worldvolume theory, however, we can provide new predictions to the gravity side from the gauge side. Hopefully the exact analysis of the superconformal Chern-Simons theory beyond the large N limit will give new predictions to the quantum effect of the gravity. In the next part we present our recent works where we compute such corrections systematically.

Part II

Beyond large N limit

In this part we review our recent works on the exact computation of the partition function of the $\mathcal{N} \geq 3$ $U(N)$ superconformal Chern-Simons theories beyond the large N limit.

In section 4 we have reviewed that the partition function of the superconformal Chern-Simons theory can be reduced to finite dimensional ordinary integrals, i.e. a matrix model, with the help of the localization technique. In the large N limit we can further evaluate the remaining integrals and obtain the result consistent with the classical supergravity.

However, it is still non-trivial how to evaluate these integrals beyond the strict limit $N \to \infty$. Generally, for small N, i.e. $N = 1, 2$, etc. it would be possible to evaluate the integrals exactly or at least numerically. In the context of the AdS/CFT correspondence, however, such results are quite difficult to translate in the language of gravity. To be successful, it will be more helpful to express the deviation from the large N limit as the large N *expansion*, where the $1/N$ corrections may be interpreted in the gravity side based on the language of classical geometry.

Though such computation is difficult due to the increasing number of integrations, there is a powerful traditional technique in the matrix model, the 't Hooft expansion. In this method we take the limit $k, N \to \infty$ while keeping the 't Hooft coupling $\lambda = N/k$ finite, and compute the partition function in perturbation with respect to λ. This method was indeed successful in the case of the ABJM theory [34] so that the authors finally achieved to determine the all order perturbative corrections to the partition function in $1/N$. However, the computation heavily relies on the non-trivial correspondence between the ABJM matrix model and the topological string theory on the local $\mathbb{P}^1 \times \mathbb{P}^1$ [35]. Also, as the dimension of the moduli space $\mathbb{C}^4/\mathbb{Z}_k$ is reduced in the limit $k \to \infty$, it correspond to the ten dimensional IIA limit rather than the eleven dimensional M-theoretical regime. Especially, the non-perturbative effects in $1/k$ would be invisible in the 't Hooft expansion.

Recently, an alternative technique was proposed in [12], which is called the Fermi gas formalism. We can compute the large N limit of the partition function and the leading coefficient for general quivers far more easily even compared with the saddle point method in section 4. As we will see below, this formalism also provide various systematic methods to compute the partition function without taking the IIA limit $k \to \infty$. Interestingly, the computation simplifies in the opposite limit $k \to 0$.[4] As a result we can obtain the all order perturbative corrections to the partition function in $1/N$ and even the non-perturbative effects in N like $\mathcal{O}(e^{-\sqrt{N}})$.

This part is organized as follows. In section 6 we first review the derivation of the Fermi gas formalism itself for general $U(N)$ $\mathcal{N} = 3$ circular quiver superconformal Chern-Simons theories.

[4] Though the Chern-Simons levels must be integers in the original field theory, in the matrix model (4.1) we can continue them to general irrational numbers.

Using the Fermi gas formalism we also provide the all order perturbative corrections to the partition function in $1/N$, which sum up to an Airy function. In section 7 we analyze the partition function in the small k expansion, and discover the non-perturbative contributions in $1/N$. These effects are non-perturbative also in $1/k$, hence invisible in the perturbative 't Hooft expansion, and can be interpreted as the D-brane instantons [36]. After establishing another method in section 8, in section 9 we determine the non-perturbative effects for finite k. We finally reveal a beautiful structure of the whole non-perturbative effects: the coincidence with the Gopakumar-Vafa formula [11] for the refined topological string theory.

6 Fermi gas formalism and all order perturbative corrections in $1/N$

In this section we first introduce the Fermi gas formalism where the partition function $Z(N)$ is re-expressed as the partition function of the quantum statistical system of an ideal Fermi gas with non-trivial one particle density matrix $\widehat{\rho}$ (6.20). After that, in section 6.1 we reveal the general structure of the large N expansion of the partition function. Surprisingly, we can show that the all order perturbative corrections to the partition function sum up to an Airy function

$$Z^{\mathrm{pert}}(N) = e^{A} C^{-\frac{1}{3}} \,\mathrm{Ai}[C^{-\frac{1}{3}}(N-B)], \tag{6.1}$$

where C, B and A are some constants depending on the detail of the theory, and the Airy function is defined by the following integration expression

$$\mathrm{Ai}(x) = \int_{-i\infty}^{i\infty} \frac{dz}{2\pi i} e^{\frac{1}{3}z^3 - xz}. \tag{6.2}$$

The large $(N-B)$ expansion of the free energy $\log Z$ is obtained from the asymptotic behavior of the Airy function as

$$\log Z = -\frac{2}{3\sqrt{C}}(N-B)^{\frac{3}{2}} - \frac{1}{4}\log(N-B) + A - \log[2\sqrt{\pi}C^{\frac{1}{4}}] + \mathcal{O}((N-B)^{-\frac{3}{2}}). \tag{6.3}$$

The leading behavior in the limit $N \to \infty$ is consistent with the results obtained from the classical supergravity on $\mathrm{AdS}_4 \times Y_7$ if $C = \mathrm{vol}(Y_7)$. We also obtain the explicit expression of C, which is indeed consistent with the requirement. In this section 6.2 we determine the explicit expression of the second coefficient B for a special class of the quivers.

Our starting point is the partition function of the $\mathcal{N}=3$ $U(N)$ circular quiver superconformal Chern-Simons theory, which is reduced to a matrix model after the application of the localization technique [25, 26, 27, 31]

$$Z(N) = \frac{1}{(N!)^M} \int D\lambda_{a,i} \prod_{a=1}^{M} \frac{\prod_{i<j} 2\sinh\frac{\lambda_{a,i}-\lambda_{a,j}}{2} \prod_{i<j} 2\sinh\frac{\lambda_{a+1,i}-\lambda_{a+1,j}}{2}}{\prod_{i,j} 2\cosh\frac{\lambda_{a,i}-\lambda_{a+1,j}}{2}}, \tag{6.4}$$

where

$$D\lambda_{a,i}=\frac{d\lambda_{a,i}}{2\pi}e^{\frac{ik_a\lambda_{a,i}^2}{4\pi}}. \tag{6.5}$$

Notice that the 1-loop determinants are decomposed into the pieces each of which is in the form of the Cauchy determinant formula

$$\frac{\prod_{i<j}(x_i-x_j)\prod_{j<j}(y_i-y_j)}{\prod_{i,j}(x_i+y_j)}=\det_{i,j}\frac{1}{x_i+y_j} \tag{6.6}$$

as

$$\frac{\prod_{i<j}2\sinh\frac{\lambda_{a,i}-\lambda_{a,j}}{2}\prod_{i<j}2\sinh\frac{\lambda_{a+1,i}-\lambda_{a+1,j}}{2}}{\prod_{i,j}2\cosh\frac{\lambda_{a,i}-\lambda_{a+1,j}}{2}}=\det_{i,j}\frac{1}{2\cosh\frac{\lambda_{a,i}-\lambda_{a+1,j}}{2}}. \tag{6.7}$$

We shall also use the following formula (see appendix A in [37]) to swap the order of the integrations and the determinant

$$\frac{1}{N!}\prod_{i=1}^{N}\int dz_i\Big[\det_{i,j}f(x_i,z_j)\Big]\Big[\det_{i,j}g(z_i,y_j)\Big]=\det_{i,j}\int dzf(x_i,z)g(z,y_j). \tag{6.8}$$

Applying these formulas repeatedly to (6.4) we finally obtain the following expression for the partition function

$$Z(N)=\frac{1}{N!}\prod_{i=1}^{N}\int\frac{dx_i}{2\pi}\det_{i,j}\rho_0(x_i,x_j)=\frac{1}{N!}\prod_{i=1}^{N}\int\frac{dx_i}{2\pi}\sum_{\sigma\in S_N}(-1)^{\sigma}\prod_{i=1}^{N}\rho_0(x_i,x_\sigma(i)) \tag{6.9}$$

with

$$\rho_0(x,y)=\prod_{a=2}^{M}\int\frac{dz_a}{2\pi}e^{\frac{ik_1}{4\pi}x^2}\frac{1}{2\cosh\frac{x-z_2}{2}}e^{\frac{ik_2}{4\pi}z_2^2}\frac{1}{2\cosh\frac{z_2-z_3}{2}}e^{\frac{ik_3}{4\pi}z_3^2}\cdots e^{\frac{ik_M}{4\pi}z_M^2}\frac{1}{2\cosh\frac{z_M-y}{2}}. \tag{6.10}$$

The expression (6.9) is the same form as the partition function of N particle ideal Fermi gas in the statistical system.

In the statistical system, it is often easier to study the grand potential $\mathcal{J}(\mu)$ defined by introducing an auxiliary parameter μ called the chemical potential as

$$e^{\mathcal{J}(\mu)}=\sum_{N=0}^{\infty}e^{N\mu}Z(N) \tag{6.11}$$

rather than the partition function itself. Indeed the grand potential can be written compactly as (see appendix A)

$$\mathcal{J}(\mu)=\mathrm{Tr}\log(1+e^{\mu}\rho_0). \tag{6.12}$$

This quantity is much easier to analyze than the partition function $Z(N)$, as we will see in the following sections.

To take advantages of the techniques of the quantum statistical mechanics, it is useful to rewrite the grand potential in the operator formalism. Since the Chern-Simons level k_a sum up to zero it is more reasonable to assign the numbers s_a on the edges of the circular quiver to define the levels as the differences

$$k_a = \frac{k(s_a - s_{a-1})}{2}. \tag{6.13}$$

In this notation the quantity ρ_0 (6.10) can be written as

$$\rho_0 = \prod_{a=2}^{M} \int \frac{dz_a}{2\pi} \left[e^{\frac{iks_1 x^2}{8\pi}} \frac{1}{2\cosh\frac{x-z_2}{2}} e^{-\frac{iks_1 z_2^2}{8\pi}} \right] \left[e^{\frac{iks_2 z_2^2}{8\pi}} \frac{1}{2\cosh\frac{z_2-z_3}{2}} e^{-\frac{iks_2 z_3^2}{8\pi}} \right] \cdots \left[e^{\frac{iks_M z_M^2}{8\pi}} \frac{1}{2\cosh\frac{z_M-y}{2}} e^{-\frac{iks_M y^2}{8\pi}} \right]. \tag{6.14}$$

Using the Fourier transformation formula

$$\frac{1}{2\cosh\frac{x}{2}} = \int \frac{dp}{2\pi} e^{\frac{ipx}{2\pi}} \frac{1}{2\cosh\frac{p}{2}}, \tag{6.15}$$

each factor in the square bracket in (6.14) can be rewritten as

$$e^{\frac{iks_a z_a^2}{8\pi}} \frac{1}{2\cosh\frac{z_a-z_{a+1}}{2}} e^{-\frac{iks_a z_{a+1}^2}{8\pi}} = k \left\langle kz_a \middle| e^{\frac{is_a \widehat{x}^2}{8\pi k}} \frac{1}{2\cosh\frac{\widehat{p}}{2}} e^{-\frac{is_a \widehat{x}^2}{8\pi k}} \middle| kz_{a+1} \right\rangle, \tag{6.16}$$

where we have introduced the canonical position/momentum operators $(\widehat{x}, \widehat{p})$ and their eigenstates $(|x\rangle, |p\rangle)$ normalized so that

$$[\widehat{x}, \widehat{p}] = i\hbar, \quad (\hbar = 2\pi k)$$
$$\langle x|x'\rangle = 2\pi\delta(x-x'), \quad \langle p|p'\rangle = 2\pi\delta(p-p'), \quad \langle x|p\rangle = \frac{1}{\sqrt{k}} e^{\frac{ipx}{2\pi k}}. \tag{6.17}$$

In the operator formalism, the $(M-1)$ integrals in ρ_0 (6.10) together with the k factored out in (6.16) are interpreted as the insertion of unity

$$1 = \int \frac{dx}{2\pi} |x\rangle\langle x|, \quad (x = kz) \tag{6.18}$$

hence the grand potential (6.12) can be rewritten as

$$\mathcal{J}(\mu) = \operatorname{Tr}\log(1 + e^{\mu}\widehat{\rho}) \tag{6.19}$$

with

$$\operatorname{Tr}(\cdot) = \int \frac{dx}{2\pi} \langle x| \cdot |x\rangle,$$
$$\widehat{\rho} = \frac{1}{2\cosh[\frac{1}{2}(\widehat{p} - \frac{s_1}{2}\widehat{x})]} \frac{1}{2\cosh[\frac{1}{2}(\widehat{p} - \frac{s_2}{2}\widehat{x})]} \cdots \frac{1}{2\cosh[\frac{1}{2}(\widehat{p} - \frac{s_M}{2}\widehat{x})]}. \tag{6.20}$$

Here we have used (6.16) and the formula

$$e^{\frac{i}{2\hbar}\widehat{x}^2} f(\widehat{p}) e^{-\frac{i}{2\hbar}\widehat{x}^2} = f(\widehat{p} + \widehat{x}). \tag{6.21}$$

Interestingly, the grand potential $\mathcal{J}(\mu)$ extremely simplifies in the limit $k \to 0$, the opposite to the IIA limit $k \to \infty$. In this limit the statistical system can be treated classically.

6.1 All order perturbative corrections in Airy function

In this section we show the Airy function structure (6.1) of the perturbative expansion of the partition function $Z(N)$ in $1/N$. Though the structure looks complicated in the partition function, this is equivalent to the following simple structure in the large μ expansion of the grand potential $J(\mu)$[5]

$$J(\mu) = \frac{C}{3}\mu^3 + B\mu + A + \mathcal{O}(e^{-\mu}). \tag{6.23}$$

Indeed the inverse transformation from the grand potential to the partition function follows from (6.11) as[6]

$$Z(N) = \int_{-i\infty}^{i\infty} \frac{d\mu}{2\pi i} e^{J(\mu)-\mu N}, \tag{6.25}$$

which immediately produce the Airy function (6.1) after the substitution of (6.23).

In this section we shall derive the large μ expansion of $J(\mu)$ (6.23) with some simple ideas of the statistical mechanics. First note that the large μ expansion (6.23) for the grand potential is equivalent to the following large E expansion for the "number of states" $n(E)$ with the energy less than E, $\widehat{H} \le E$,

$$n(E) = CE^2 + B - \frac{\pi^2 C}{3} + \mathcal{O}(e^{-E}). \tag{6.26}$$

Here $\widehat{H} = -\log\widehat{\rho}$ is the one particle Hamiltonian of the statistical system. This can be seen as follows. First, as we may often do in the statistical mechanics, we try to compute the trace in (6.12) with the energy eigenstates

$$J(\mu) = \int_0^\infty dE \frac{dn(E)}{dE} \log(1 + e^{\mu - E}). \tag{6.27}$$

Integrating by parts we obtain

$$J(\mu) = \int_0^\infty dE n(E) \frac{e^{\mu-E}}{1+e^{\mu-E}}, \tag{6.28}$$

[5] Note that $J(\mu)$ is slightly different from $\mathcal{J}(\mu)$ defined by (6.11), which is periodic in μ: $e^{\mathcal{J}(\mu+2\pi i)} = e^{\mathcal{J}(\mu)}$. The original grand potential $\mathcal{J}(\mu)$ should be understood as a periodic superposition of the large μ expansion $J(\mu)$ as [58]

$$e^{\mathcal{J}(\mu)} = \sum_{n\in\mathbb{N}} e^{J(\mu+2\pi i n)}. \tag{6.22}$$

[6] This inversion relation (6.25) follow from the following inversion relation for the original grand potential $\mathcal{J}(\mu)$

$$Z(N) = \int_{-i\pi}^{i\pi} \frac{d\mu}{2\pi i} e^{\mathcal{J}(\mu)-\mu N} \tag{6.24}$$

which follows from (6.11), and the relation between $J(\mu)$ and $\mathcal{J}(\mu)$ (6.22).

where we have assumed $n(0) = 0$. This integration convert a polynomial in E in $n(E)$ into polylogarithm functions $\mathrm{Li}(-e^{\mu})$ in the grand potential $J(\mu)$, which again produces a polynomial in μ in the large μ expansion:

$$\int_0^\infty dE E^\alpha \frac{e^{\mu-E}}{1+e^{\mu-E}} = -\Gamma(\alpha+1)\,\mathrm{Li}_{\alpha+1}(-e^{\mu}) = \frac{(2\pi i)^\alpha}{\alpha+1} B_{\alpha+1}\Big(\frac{1}{2}+\frac{\mu}{2\pi i}\Big) + \mathcal{O}(e^{-\mu}), \tag{6.29}$$

where $\alpha \in \mathbb{Z}_+$ and we have used the identity for the polylogarithm functions

$$\mathrm{Li}_n(z) + (-1)^n\,\mathrm{Li}_n\Big(\frac{1}{z}\Big) = -\frac{(2\pi i)^n}{n!} B_n\Big(\frac{1}{2}+\frac{\log(-z)}{2\pi i}\Big). \tag{6.30}$$

Here $B_n(z)$ are the Bernoulli polynomial, whose explicit expression for small n is

$$B_1(z) = z - \frac{1}{2},\quad B_2(z) = z^2 - z + \frac{1}{6},\quad B_3(z) = z^3 - \frac{3z^2}{2} + \frac{z}{2}, \cdots. \tag{6.31}$$

From these explicit expressions we can find that the large E expansion (6.26) is indeed converted into the large μ expansion (6.23). Hence we can determine the first two coefficients C and B by analyzing $n(E)$.

Now let us compute the number of states $n(E)$ in the semiclassical limit $k \to 0$. In this limit, the Hamiltonian is mere a c-numbered function on the (x,p)-phase space

$$H(x,p) = \sum_{a=1}^M \log\bigg(2\cosh\frac{p-\frac{s_a x}{2}}{2}\bigg), \tag{6.32}$$

and $n(E)$ is the phase space volume inside the Fermi surface $F = \{(x,p)\in\mathbb{R}^2 | H = E\}$ as

$$n(E) = \int \frac{dxdp}{2\pi\hbar}\theta(H-E). \tag{6.33}$$

The Fermi surface F is plotted in figure 4. From the graph we can read off that F approaches a polygon. Indeed, assuming $|p - s_a x/2| \gg 1$ for all a, each term in the Hamiltonian (6.32) approaches a linear function and the Fermi surface can be approximated with a polygon

$$F_{\mathrm{pol}} = \bigg\{(x,p)\in\mathbb{R}^2 \bigg| \sum_{a=1}^M \Big|p - \frac{s_a x}{2}\Big| = 2E\bigg\}. \tag{6.34}$$

Hence we conclude that the leading term in $n(E)$ in the large E limit is given by the volume of this approaching polygon as

$$n(E) = \frac{2}{\pi\hbar}\sum_{a=1}^M \frac{|s'_{a+1}-s'_a|}{\sum_{b=1}^M |s'_{a+1}-s'_b| \sum_{c=1}^M |s'_a - s'_c|} E^2 + \cdots \tag{6.35}$$

where s'_a are equal to s_a but reordered so that $s'_a \le s'_{a+1}$ for $a = 1, 2, \cdots, M-1$. Thus

$$C = \frac{4}{\pi\hbar}\sum_{a=1}^M \frac{|s'_{a+1}-s'_a|}{\sum_{b=1}^M |s'_{a+1}-s'_b| \sum_{c=1}^M |s'_a - s'_c|}. \tag{6.36}$$

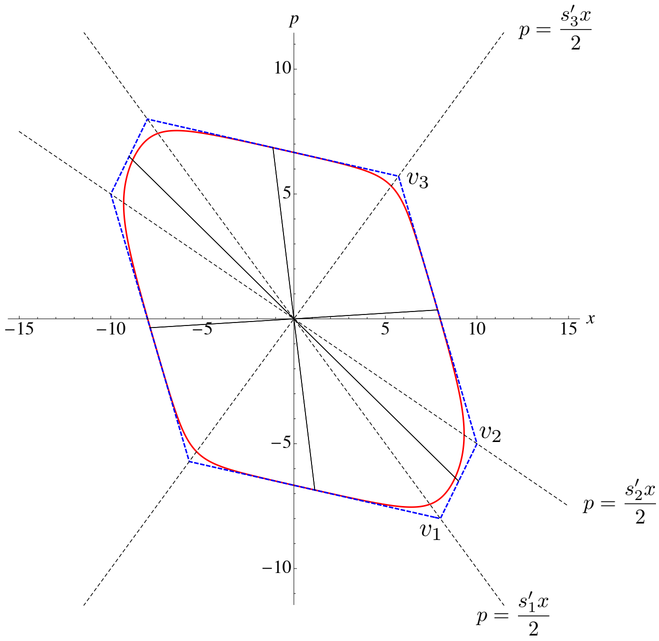

Figure 4: The solid red line is the classical Fermi surface for $\mathcal{N} = 3$ circular quiver with three vertices with $\{s_a\}_{a=1}^3 = \{-2, -1, 1\}$ at energy $E = 10$. The dashed blue line is the approaching polygon (6.34). We call the region located around the line $p = s_a' x/2$ $(x > 0)$ and surrounded by the lines connecting the midpoints of the edges of the polygon (solid black lines) as v_a.

For $\{s_a\} = \{(+1)^q, (-1)^p\}$, for example, we obtain

$$C = \frac{2}{\pi^2 qpk}, \tag{6.37}$$

which completely coincide with the volume of the radial section of $(\mathbb{C}^2/\mathbb{Z}_q \times \mathbb{C}^2/\mathbb{Z}_p)/\mathbb{Z}_k$. Though being complicated, the general expression is also shown to coincide with the volume of the hyperKahler cone obtained from the moduli space of $\mathcal{N} = 3$ superconformal quiver Chern-Simons theory argued in section 3 [31].

Before closing this subsection, we shall estimate the possible small k corrections in the semiclassical expansion. Roughly speaking, the effects of the commutators can be included through the derivatives of the c-numbered functions. Therefore the quantum corrections never grows as fast as linear in (x, p) as we send x or p infinity. This implies the approaching polygon (6.34) in the limit of $E \to \infty$ will not be modified by the quantum corrections and thus the expression of the constant C (6.37) is exact for finite k. On the other hand, the constant B, which is associated with the deviation from the polygon, should be modified. This estimation is justified from the explicit computation in the $\mathcal{N} = 4$ cases in the next subsection.

6.2 Semiclassically corrected Fermi surface and exact expression of B

In the last subsection we have found that the all order perturbative corrections to the Free energy add up to an Airy function. This is powerful prediction to the quantum corrections to the computation in supergravity. What we will encounter in an honest large N expansion of the free energy, however, is not the asymptotic expansion of the Airy function but rather an $\mathcal{O}(N^{\frac{1}{2}})$ correction due to the shift of the M2-brane charge N by a constant B in (6.1). Therefore it would be important to determine the explicit expression of B.

As we have argued above, the analysis in the classical limit $k \to 0$ is not enough to determine B. We need to extend our analysis to include the semiclassical corrections. Though this is difficult for general cases, recently we successfully evaluated such corrections [8] for the special cases where s_a take only two kinds of values which can be chosen as $s_a = \pm 1$, that is,

$$\begin{gathered} s_1 = s_2 = \cdots s_{q_1} = +1, \quad s_{q_1+1} = s_{q_1+2} = \cdots s_{q_1+p_1} = -1, \\ s_{q_1+p_1+1} = \cdots = s_{q_1+p_1+q_2} = +1, \quad s_{q_1+p_1+q_2+1} = \cdots s_{q_1+p_1+q_2+p_2} = -1, \\ \vdots \\ s_{q_1+p_1+\cdots+p_{m-1}} = \cdots = s_{q_1+p_1+\cdots p_{m-1}+q_m} = +1, \quad s_{q_1+p_1+\cdots q_m} = \cdots s_{q_1+p_1+\cdots+q_m+p_m} = -1 \end{gathered} \tag{6.38}$$

for some $q_a, p_a \in \mathbb{N}$ ($\sum_{a=1}^{M}(q_a + p_a) = M$). From now on we shall abbreviate this choice as $\{s_a\}_{a=1}^{M} = \{(+1)^{q_1}, (-1)^{p_1}, (+1)^{q_2}, (-1)^{p_2}, \cdots, (+1)^{q_m}, (-1)^{p_m}\}$. This is known as the most general choice for the supersymmetry enhancement from $\mathcal{N} = 3$ to $\mathcal{N} = 4$ [5]. Especially, we call the cases of minimal separation between $s_a = \pm 1$

$$\{s_a\}_{a=1}^{M} = \{(+1)^{q}, (-1)^{p}\} \tag{6.39}$$

as the "$(q, p)_k$ minimal models". In [8] we finally obtained the following expression for the coefficient B

$$B = \frac{1}{6k}\left(\frac{4}{qp} - \frac{p}{q} - \frac{q}{p}\right) + k\left(\frac{\Sigma(q, p^2)}{q} + \frac{\Sigma(q, p^2)}{p} - \frac{\Sigma(q, p)^2}{2qp} - \frac{\Sigma(q, p)}{2} + \frac{qp}{24}\right), \tag{6.40}$$

where

$$q = \sum_{a=1}^{m} q_m, \quad p = \sum_{a=1}^{m} p_m, \quad \Sigma(q^\alpha, p^\beta) = \frac{1}{\alpha!\beta!}\sum_{1\le a\le b\le m} q_a^\alpha p_b^\beta. \tag{6.41}$$

Remarkably, the semiclassical correction to B terminates at $\mathcal{O}(k)$, which implies that we have obtained the non-perturbatively exact expression of B although we have used the perturbation in k.

Below we first explain the derivation of (6.40). After that we try to determine B in more general $\mathcal{N} = 3$ theories, which will be successful only in the classical limit $k \to 0$.

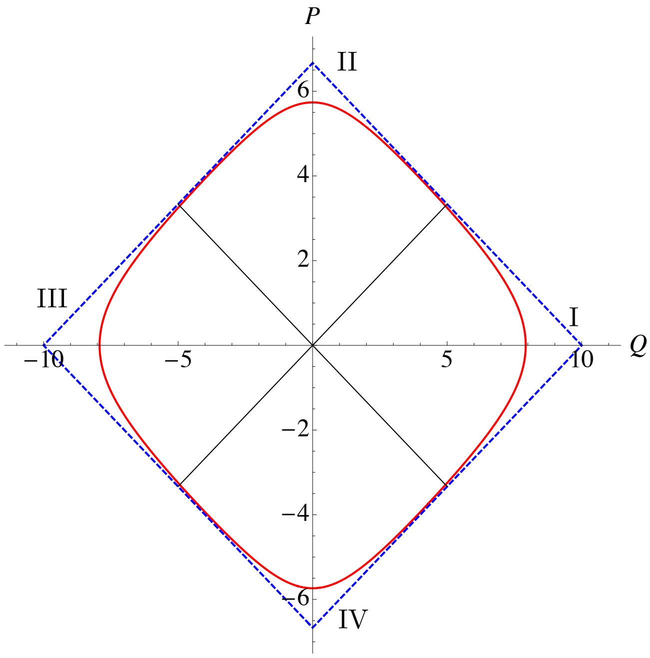

Figure 5: The solid red line is the Fermi surface for $\mathcal{N} = 4$ circular quiver with $\{s_a\}_{a=1}^5 = \{(+1)^2, (-1)^3\}$ at energy $E = 10$. The dashed blue line is the approaching diamond (6.45). We call the region located around the vertices $(Q, P) = (\pm 2E/q, 0)$ as region I,III and that around the vertices $(Q, P) = (0, \pm 2E/p)$ as region II,IV.

6.2.1 B for $\mathcal{N} = 4$ quivers

Since there are only two kinds of arguments $\widehat{p} \pm \frac{\widehat{x}}{2}$ in the density matrix (6.20), it is reasonable to redefine these two as the canonical operators

$$\widehat{Q} = -\widehat{p} + \frac{\widehat{x}}{2}, \quad \widehat{P} = \widehat{p} + \frac{\widehat{x}}{2}, \quad ([\widehat{Q}, \widehat{P}] = i\hbar) \tag{6.42}$$

and rewrite $\widehat{\rho}$ as

$$\widehat{\rho} = e^{-q_1 U(\widehat{Q})} e^{-p_1 T(\widehat{P})} e^{-q_2 U(\widehat{Q})} e^{-p_2 T(\widehat{P})} \cdots e^{-q_m U(\widehat{Q})} e^{-p_m T(\widehat{P})} \tag{6.43}$$

with

$$U(Q) = \log\left(2\cosh\frac{Q}{2}\right), \quad T(P) = \log\left(2\cosh\frac{P}{2}\right). \tag{6.44}$$

In this case the Fermi surface of energy E approaches in the limit $E \to \infty$ to a diamond

$$F_{\text{pol}} = \left\{(Q, P) \in \mathbb{R}^2 \middle| \frac{q|Q|}{2} + \frac{p|P|}{2} = E\right\}. \tag{6.45}$$

See figure 5.

To compute the quantum corrections systematically, we shall introduce the notion of the Wigner transformation [12, 38, 39, 40, 41] $\widehat{\mathcal{O}} \to \mathcal{O}_W$

$$\mathcal{O}_W \equiv \int \frac{dQ'}{2\pi} \left\langle Q + \frac{Q'}{2} \middle| \widehat{\mathcal{O}} \middle| Q - \frac{Q'}{2} \right\rangle e^{\frac{iQ'P}{\hbar}}. \tag{6.46}$$

The non-commutativity of operators $\widehat{\mathcal{O}}$ are encoded in the following non-commutative $\star$-product of the c-numbered functions $\mathcal{O}_W$ on the phase space

$$\star = \exp\left[\frac{i\hbar}{2}\left(\overleftarrow{\partial}_Q \overrightarrow{\partial}_P - \overleftarrow{\partial}_P \overrightarrow{\partial}_Q\right)\right] \tag{6.47}$$

as

$$(\widehat{A}\widehat{B})_W = A_W \star B_W, \tag{6.48}$$

which can be derived from the definition of the Wigner transformation (6.46). One can also easily show that the trace of a operator becomes the phase space integral of its Wigner transformation

$$\mathrm{Tr}\, \widehat{A} = \int \frac{dQdP}{2\pi\hbar} A_W, \tag{6.49}$$

hence the number of states $n(E)$ is given as[7]

$$n(E) = \mathrm{Tr}\, \theta(\widehat{H} - E) \approx \int_{H_W < E} \frac{dQdP}{2\pi\hbar}. \tag{6.50}$$

As advertised at the end of the previous section, the Wigner transformation precisely convert the commutators into the differentials. Now we can argue the (ir-)relevance of higher order commutators to the $1/N$ perturbative computation more concretely. Note that both $U(Q)$ and $T(P)$ in (6.44) satisfy

$$\partial_Q^2 U(Q) = \mathcal{O}(e^{-|Q|}), \quad \partial_P^2 T(P) = \mathcal{O}(e^{-|P|}) \tag{6.51}$$

for large arguments. Therefore, since we are concerned with the small deviation of the Fermi surface from the diamond (6.45) where at least one of Q and P is of order $\mathcal{O}(E)$, we can neglect all the terms proportional to $U^{(a)}T^{(b)}$ with $a, b \geq 2$. According to the Baker-Campbell-Hausdorff formula, the Hamiltonian operator $\widehat{H}$ can be expanded with the commutators $U, T, [U, T], [U, [U, T]], \cdots$. Here we would like to choose the bases of higher commutators as

$$P\left(\prod_{i=1}^{\ell} \mathrm{ad}\, L_i\right)[U, T] \tag{6.52}$$

with $L = (L_1, L_2, \cdots L_\ell)$ any finite sequence of U and T,[8] such as

$$[U, [U, T]], \quad [U, [U, [U, T]]], \quad [U, [T, [U, T]]], \quad [T, [T, [U, T]]], \quad \cdots. \tag{6.53}$$

[7]Precisely speaking, $f(\widehat{\mathcal{O}})_W$ does not necessarily coincide with $f(\mathcal{O}_W)$. This deviation is treated in detail in the next section, and indeed turns out to be irrelevant in the computation here.

[8]There are still some redundancy, e.g. $[T, [U, [U, T]]] = [U, [T, [U, T]]]$.

Let us look at the terms with the fewest number of differential in each commutator. Due to the definition of $\star$-product, these are obtained by the following replacements

$$\begin{aligned}
[U,T] &\to i\hbar U'T', \\
\operatorname{ad} U &\to i\hbar U'\partial_P, \\
\operatorname{ad} T &\to -i\hbar T'\partial_Q.
\end{aligned} \tag{6.54}$$

From these rules we can see that the commutator contains $U^{(a)}$ with $a \geq 2$ if the sequence L in (6.52) contains at least one U while the commutator contains $T^{(a)}$ with $a \geq 2$ if L contains at least one T. Therefore we conclude that the only commutators relevant to our computation are

$$\begin{aligned}
\widehat{H} = c_U U + c_T T + c_{UT}[U,T] &+ \sum_{\ell\geq 1} c_{U^{\ell+1}T}[\underbrace{U,[U,[\cdots,U}_{\ell},[U,T]\cdots] \\
&+ \sum_{\ell\geq 1} c_{T^\ell UT}[\underbrace{T,[T,[\cdots,T}_{\ell},[U,T]\cdots] + \cdots ,
\end{aligned} \tag{6.55}$$

where $c_{\dots}$ are constant coefficients depending on $\{q_a,p_a\}_{a=1}^m$. The relevant terms in the Wigner Hamiltonian are

$$\begin{aligned}
H_W = c_U U + c_T T + i\hbar c_{UT} U'T' &+ \sum_{\ell\geq 1} (i\hbar)^{\ell+1} c_{U^{\ell+1}T} (U')^{\ell+1} T^{(\ell+1)} \\
&- \sum_{\ell\geq 1} (-i\hbar)^{\ell+1} c_{T^\ell UT} (T')^{\ell+1} U^{(\ell+1)} + \cdots .
\end{aligned} \tag{6.56}$$

Below we assume that the Hamiltonian is hermitian, i.e. $c_{UT} = 0$. This is achieved by the similarity transformation

$$\widehat{\rho} \to e^{xU}\widehat{\rho}e^{-xU} \tag{6.57}$$

with some constant x. The values of $c_{\dots}$ and x will be displayed at the end of this section.

Now let us evaluate the small deviation of $n(E)$ from the result of diamond approximation (6.45), which we shall call δn. We divide the deviation into the contributions from the region close to each vertex of the diamond as depicted in figure 5

$$\begin{aligned}
\delta n &= -\frac{1}{2\pi\hbar}(\mathrm{vol(I)} + \mathrm{vol(II)} + \mathrm{vol(III)} + \mathrm{vol(IV)}) \\
&= -\frac{1}{\pi\hbar}(\mathrm{vol(I)} + \mathrm{vol(II)}).
\end{aligned} \tag{6.58}$$

In the second line we have used the facts vol(III) = vol(I) and vol(IV) = vol(II) which are obvious from the point symmetry of the Fermi surface. First we shall consider the volume of the region I. Since Q is positive and of order $\mathcal{O}(E)$ in this region we can replace $U \to Q/2$ and neglect the higher derivatives of U, $U^{(a)}$ with $a \geq 2$, to approximate the Fermi surface $H_W = E$ as

$$\frac{qQ}{2} + pT + \sum_{\ell\geq 1} c_{U^{\ell+1}T}\Big(\frac{i\hbar}{2}\Big)^{\ell+1} T^{(\ell+1)} = E. \tag{6.59}$$

Denoting the points on this Fermi surface as $(Q_{\rm FS}(P), P)$ while those on the diamond as $(Q_{\rm pol}(P), P)$ the volume of region I can be written as

$$\begin{aligned}\mathrm{vol(I)} &= \int_{-\frac{E}{p}}^{\frac{E}{p}} dP (Q_{\rm pol}(P) - Q_{\rm FS}(P)) \\ &= \frac{2}{q}\int_{-\frac{E}{p}}^{\frac{E}{p}} dP \Big[p\Big(T - \frac{|P|}{2}\Big) + \sum_{\ell\geq 1} c_{U^{\ell+1}T}\Big(\frac{i\hbar}{2}\Big)^{\ell+1} T^{(\ell+1)}\Big] \end{aligned} \tag{6.60}$$

where we have tentatively chosen the boundary of the integration domain as the midpoints of the edges. Since the integrand is exponentially suppressed for large P, we can push these boundaries to $\pm\infty$ without changing the perturbative result in $1/E$. Using the explicit expression of $T(P)$ (6.44) we can perform the integration over $P \in (-\infty, \infty)$ to obtain

$$\mathrm{vol(I)} = \frac{2}{q}\Big[\frac{p\pi^2}{6} - \frac{\hbar^2}{4} c_{UUT}\Big]. \tag{6.61}$$

Remarkably, the higher derivatives $T^{(\ell+1)}$ with $\ell \geq 2$ do not contribute. In the region II, on the other hand, we replace $T \to P/2$ and neglect $T^{(a)}$ with $a \geq 2$. By the similar manipulation as in the case of region I, we finally find that only $(T')^2 U^{(2)}$ contributes among the terms from higher commutators and

$$\mathrm{vol(II)} = \frac{2}{p}\Big[\frac{q\pi^2}{6} + \frac{\hbar^2}{4} c_{TUT}\Big]. \tag{6.62}$$

Putting these result together with the relation between $n(E)$ and B (6.26), we obtain B as

$$B = -\frac{1}{6k}\Big(\frac{4}{qp} - \frac{p}{q} - \frac{q}{p}\Big) + k\Big(\frac{c_{UUT}}{q} - \frac{c_{TUT}}{p}\Big), \tag{6.63}$$

where we have also used $\hbar = 2\pi k$.

Finally we shall display the explicit expression of the undetermined coefficients c_{UUT} and c_{TUT}. In appendix B we argue the expansion coefficients of

$$e^h = e^{q_1 U} e^{p_1 T} e^{q_2 U} e^{p_2 T} \cdots e^{q_m U} e^{p_m T} e^{q_{m+1} U}. \tag{6.64}$$

From the results (B.13) therein with replacement

$$\begin{aligned} q_1 \to -(q_1 - x), &\quad q_{m+1} \to -x, \\ q_a \to -q_a &\quad (a = 2, 3, \cdots, m), \\ p_a \to -p_a &\quad (a = 1, 2, \cdots, m) \end{aligned} \tag{6.65}$$

together with the multiplication of overall (-1), the coefficients c_{UT}, c_{UUT} and c_{TUT} can be written as

$$c_{UT} = xp - \Sigma(q, p) + \frac{qp}{2},$$

$$c_{UUT} = \frac{x^2 p}{2} + x\Big(\frac{qp}{2} - \Sigma(q,p)\Big) + \Sigma(q^2,p) - \frac{\Sigma(q,p)q}{2} + \frac{q^2 p}{12},$$
$$c_{TUT} = -\Sigma(q,p^2) + \frac{\Sigma(q,p)p}{2} - \frac{qp^2}{12}. \tag{6.66}$$

Fixing the value of x from the condition $c_{UT} = 0$ as

$$x = \frac{\Sigma(q,p)}{p} - \frac{q}{2}, \tag{6.67}$$

we obtain

$$c_{UUT} = \Sigma(q^2,p) - \frac{q^2 p}{24} - \frac{\Sigma(q,p)^2}{2p}. \tag{6.68}$$

Substituting the values of c_{UUT} and c_{TUT} into (6.63) we finally obtain the explicit expression of B (6.40).

6.2.2 B for $\mathcal{N}=3$ quivers in classical limit

Before closing this section we shall introduce the similar determination of B in more general $\mathcal{N}=3$ theories [8]. The semiclassical expansion of the Hamiltonian would be considered with the help of the Wigner transformation. However, the hermiticity of leading Hamiltonian, which is required for the arguments using the Fermi surface, is non-trivial in this case. Hence we shall focus on the classical limit

$$F = \bigg\{(x,p)\in\mathbb{R}^2 \bigg| \sum_{a=1}^{M} \log\bigg(2\cosh\frac{p-\frac{s'_a x}{2}}{2}\bigg) = E\bigg\}, \tag{6.69}$$

where s'_a are the reordered s_a (see below the equation (6.35)), and evaluate the deviation of the volume from the polygon approximation (6.34).

The computation is almost parallel as in the $\mathcal{N}=4$ cases. First we shall divide the Fermi surface into the pieces around each vertex of the polygon, as in figure 4. Since the Fermi surface is point symmetric about the origin, below we shall only consider the right half of them and call the region around the line $p - s'_a x/2 = 0$ as v_a. On the polygon (6.34) the arguments $|p - s'_a x/2|$ are always of order $\mathcal{O}(E)$, except around each vertex of the polygon where $|p-s'_a x/2|\sim 0$ only for the corresponding a. Hence it is reasonable to approximate the a-th piece of the original Fermi surface (6.69) as

$$\log 2\cosh\frac{p-\frac{s'_a x}{2}}{2} + \sum_{b\neq a}\frac{|p-\frac{s'_b x}{2}|}{2} = E. \tag{6.70}$$

We shall introduce a tilted coordinate $(x,\widetilde{p}_a) = (x, \frac{p - s'_a x/2}{2})$, as in figure 6. Since the Jacobian for the coordinate transformation $(x,p)\to(x,\widetilde{p})$ is trivial, we can compute the deviation of $n(E)$ on this piece as

$$\delta n(E) = -\frac{1}{2\pi\hbar}\int d\widetilde{p}_a dx. \tag{6.71}$$

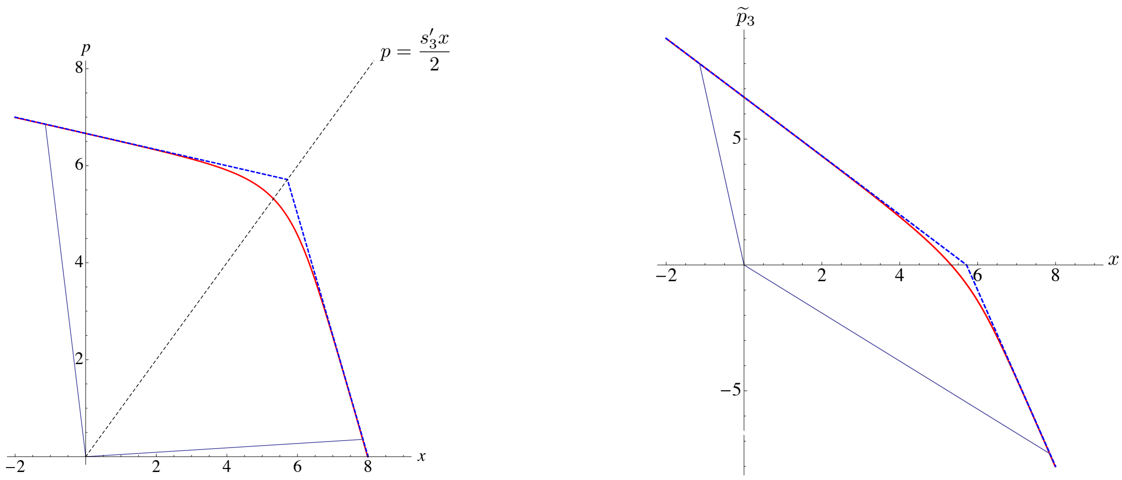

Figure 6: Left: v_3; right: v_3 in the tilted coordinates $(x, \widetilde{p}_3)$.

The points on the Fermi surface (6.70) can be expressed as $(x_{\mathrm{FS}}(\widetilde{p}_a), \widetilde{p}_a)$ with

$$x_{\mathrm{FS}}(\widetilde{p}_a) = \frac{4}{\sum_b |s'_a - s'_b|}\left[-\log 2\cosh\frac{\widetilde{p}_a}{2} + \frac{M+1-2a}{2}\widetilde{p}_a\right]. \tag{6.72}$$

Here we have used the facts

$$\begin{aligned} &p - \frac{s'_b x}{2} > 0 \quad (b < a), \\ &p - \frac{s'_b x}{2} < 0 \quad (b > a). \end{aligned} \tag{6.73}$$

Similarly, the points on the polygon (6.34) near the a-th vertex can be expressed as $(x_{\mathrm{pol}}(\widetilde{p}_a), \widetilde{p}_a)$ with

$$x_{\mathrm{pol}}(\widetilde{p}_a) = \frac{4}{\sum_b |s'_a - s'_b|}\left[-\frac{\widetilde{p}_a}{2} + \frac{M+1-2a}{2}\widetilde{p}_a\right]. \tag{6.74}$$

The deviation of the volume (6.71) is computed as

$$\delta n(E) = -\frac{1}{2\pi\hbar}\int d\widetilde{p}_a (x_{\mathrm{pol}} - x_{\mathrm{FS}}) = -\frac{2}{\pi\hbar\sum_{b\neq a}|s'_a - s'_b|}\int_{\widetilde{p}_{a-}}^{\widetilde{p}_{a+}} d\widetilde{p}_a\left[\log 2\cosh\frac{\widetilde{p}_a}{2} - \left|\frac{\widetilde{p}_a}{2}\right|\right]. \tag{6.75}$$

Here $\widetilde{p}_{a\pm}$ may correspond to the midpoints on the edge of the polygon, where $\widetilde{p}_{a\pm}$ are of order $\mathcal{O}(E)$. Since the integrand are of order $\mathcal{O}(e^{-E})$ for $\widetilde{p}_a \sim E$, we can replace the domain of integration from $(\widetilde{p}_{a-}, \widetilde{p}_{a+})$ to $(-\infty, \infty)$ without changing the results in the perturbative expansion in $1/E$. Therefore the deviation $\delta n(E)$ is computed as

$$\begin{aligned} \delta n(E) &= -\frac{2}{\pi\hbar\sum_{b\neq a}|s'_a - s'_b|}\int_{-\infty}^{\infty} d\widetilde{p}_a\left[\log 2\cosh\frac{\widetilde{p}_a}{2} - \left|\frac{\widetilde{p}_a}{2}\right|\right] \\ &= -\frac{\pi}{3\hbar\sum_b |s'_a - s'_b|}. \end{aligned} \tag{6.76}$$

Assembling the results together we conclude

$$B = \frac{\pi^2 C}{3} - \frac{1}{3k}\sum_{a=1}^{M}\frac{1}{\sum_b |s'_a - s'_b|} + \mathcal{O}(k). \tag{6.77}$$

6.3 Discussion and Comments

In this section we have determined the all order perturbative corrections in $1/N$ to the leading $N^{3/2}$ scaling behavior of the partition function. From the Fermi gas formalism it follows that these all order corrections universally add up to an Airy function as (6.1). Interestingly, the infinite series of perturbative corrections in the partition function are summarized into a simple cubic polynomial in terms of the grand potential $J(\mu)$ (6.23). Thus we may expect that the grand potential not only simplify the computation but also clarify beautiful structures of the large N expansion which are hidden in the original partition function $Z(N)$. In the following sections we analyze the non-perturbative effects $\mathcal{O}(e^{-\mu})$ in $J(\mu)$, which correspond to the effects of $\mathcal{O}(e^{-\sqrt{N}})$ in $Z(N)$, and indeed observe various interesting structures.

It will be very interesting future work to reproduce the large N expansion obtained in this section from the gravity side. The logarithm in the asymptotic expansion of the Airy function (6.3) can be understood as the 1-loop effect in the supergravity [42]. More recently, it was also proposed that the whole Airy function is possibly reproduced by applying the localization technique to the four dimensional supergravity on AdS_4 [43]. On the other hand, the coefficient B in (6.1) was not explained by neither of these ways. As commented at the beginning of section 6.2 the shift B is more relevant than the $\log N$ correction and thus may be in some sense classical effect. Indeed the same effect is caused by the modification of the charge quantization condition due to the orbifold on the background, as argued in [44] for $\mathrm{AdS}_4 \times S^7/\mathbb{Z}_k$. There is a discrepancy, however, between their result and the exact result in the dual theory, the ABJM theory [34]. We hope that our explicit results for the class of $\mathcal{N} = 4$ theories, where the pair of (q, p) (the total number of $s_a = \pm 1$) is the same while the ordering is different, will help us to resolve the mismatch, clarifying the relation between the shift B and the geometry.

Notice that in this section we have completely ignored the remaining coefficient, the overall constant A. Though the perturbative coefficients B and C in the grand potential $J(\mu)$ are determined from the perturbative coefficients of $n(E)$, we can not determine A in the same way, as the non-perturbative terms in $n(E)$ $\mathcal{O}(e^{-E})$ also contributes to the constant part of the grand potential as $(\alpha \in \mathbb{Z}_+)$

$$\int_0^\infty dE e^{-\alpha E}\frac{e^{\mu - E}}{1 + e^{\mu - E}} = \sum_{n=1}^{\infty}\frac{e^{n\mu}}{n + \alpha}(-1)^{n-1} = \frac{1}{\alpha} + \mathcal{O}(e^{-\mu}). \tag{6.78}$$

To determine A we need to compute $J(\mu)$ exactly with respect to the chemical potential μ, which we will explain in the next section.

7 WKB expansion and membrane instantons in $(q,p)_k$ models

In the last section we analyzed the grand potential $J(\mu)$ by considering the Fermi surface for large energy E perturbatively in $1/E$. Though this treatment is sufficient for the determination of the Airy function structure of the perturbative part of the partition function (6.1) and the coefficients C (6.37) and B (6.40) therein, we can not determine the overall constant A. It is also impossible to analyze the non-perturbative corrections in $1/N$, or $\mathcal{O}(e^{-\mu})$ in $J(\mu)$, at all. To determine these ingredients we need to analyze the grand potential exactly in μ. Fortunately in the $(q,p)_k$ minimal models (6.39) it is possible to determine the μ dependence of the grand potential completely exactly, order by order in the semiclassical expansion.

In this section we first explain how to perform such exact computation. To extract the constant A and the non-perturbative effects, we need to re-express the grand potential in the large μ expansion. After the exact computation, we explain how to do this. We also comment on the interpretation of the non-perturbative effects in the gravity side.

First recall the definition of the grand potential (6.19)

$$\mathcal{J}(\mu) = \operatorname{Tr}\log(1+e^{\mu-\widehat{H}}). \tag{7.1}$$

Expanding the logarithm, we can rewrite the grand potential as

$$\mathcal{J}(\mu) = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}e^{n\mu}\mathcal{Z}(n), \tag{7.2}$$

where

$$\mathcal{Z}(n) = \operatorname{Tr} e^{-n\widehat{H}} = \int\frac{dQdP}{2\pi\hbar}(e^{-n\widehat{H}})_W. \tag{7.3}$$

Here we have used the Wigner transformation introduced in the last section. Below we explain how to compute this quantity $\mathcal{Z}(n)$ exactly in the semiclassical expansion.

First let us consider the classical limit $k\to 0$, where $\mathcal{Z}(n)$ can be written as

$$\mathcal{Z}(n) = \frac{1}{\hbar}\mathcal{Z}_0(n). \tag{7.4}$$

In the classical limit we can simply replace the integrand as

$$(e^{-n\widehat{H}})_W \to e^{-nH_{\rm cl}} \tag{7.5}$$

with

$$H_{\rm cl} = qU + pT. \tag{7.6}$$

Here $U(Q)$ and $T(P)$ are c-functions (6.44). Therefore we can compute the $\mathcal{Z}_0(n)$ with the help of the following integration formula ($y\in\mathbb{R}$)

$$\int dx\Big(\frac{1}{2\cosh\frac{x}{2}}\Big)^y = \frac{\Gamma(\frac{y}{2})^2}{\Gamma(y)}, \tag{7.7}$$

as

$$\mathcal{Z}_0(n) = \frac{1}{2\pi}\frac{\Gamma(\frac{nq}{2})^2\Gamma(\frac{np}{2})^2}{\Gamma(nq)\Gamma(np)}. \tag{7.8}$$

Now let us consider the higher order corrections in the semiclassical expansion

$$\mathcal{Z}(n) = \sum_{\ell\geq 0}\hbar^{\ell-1}\mathcal{Z}_\ell(n). \tag{7.9}$$

The quantum corrections originates from the following two kinds of deviations

$$\begin{aligned} &\circ\ \text{The deviation of } H_W \text{ from } H_{\text{cl}} = q\widehat{U} + p\widehat{T}, \\ &\circ\ \text{The deviation of } (e^{-n\widehat{H}})_W \text{ from } e^{-nH_W}. \end{aligned} \tag{7.10}$$

The first effects are already argued in the previous section: we can compute them by first computing the higher commutators in the Hamiltonian operator and then computing the Wigner transformation of each commutator. On the other hand the second effects have been completely ignored in the previous section, so we would like to explain them here. Notice that the difference between the $\star$-product and the ordinary product (identity) is not only in the anti-symmetric part. Expanding the exponential in the definition of the $\star$-product (6.47) we also find the symmetric deviations as

$$A_W\star B_W = A_WB_W + \cdots - \frac{\hbar^2}{4}\{(\partial_Q^2A_W)(\partial_P^2B_W) - (\partial_Q\partial_PA_W)(\partial_Q\partial_PB_W)\} + \cdots, \tag{7.11}$$

which are the origin of the second deviation. Due to this fact it follows that $f(\widehat{\mathcal{O}})_W \neq f(\mathcal{O}_W)$ for a general function $f(x)$. An effective way to evaluate the deviation is to consider the Taylor expansion of $f(\widehat{\mathcal{O}})$ around $\widehat{\mathcal{O}} = \mathcal{O}_W$ as

$$f(\widehat{\mathcal{O}}) = \sum_{n=0}^{\infty}\frac{1}{n!}\frac{d^nf}{dx^n}\bigg|_{x=\mathcal{O}_W}(\widehat{\mathcal{O}} - \mathcal{O}_W)^n. \tag{7.12}$$

Since the Wigner transformation act only on operators, the Wigner transformation of both sides are

$$f(\widehat{\mathcal{O}})_W = \sum_{n=0}^{\infty}\frac{1}{n!}\frac{d^nf(x)}{dx^n}\bigg|_{x=\mathcal{O}_W}\mathcal{G}_n(\mathcal{O}_W), \tag{7.13}$$

with

$$\mathcal{G}_n(\mathcal{O}_W) \equiv (\widehat{\mathcal{O}} - \mathcal{O}_W)^n_W. \tag{7.14}$$

As $f(x) = e^{-nx}$ in our case we obtain

$$(e^{-n\widehat{H}})_W = e^{-nH_W}\sum_{\ell=0}^{\infty}\frac{(-n)^\ell}{\ell!}\mathcal{G}_\ell(H_W). \tag{7.15}$$

The semiclassical expansion of the grand potential is infinite series expansion. For tractability, the number of correction terms need to be finite at each order of $\hbar$ and also the correction terms need to be computable recursively order by order. For the first kind of the deviations in (7.10), this property is obviously granted. From the definition of the $\star$-product (6.47) we can immediately show that a commutation $[\cdot,\cdot]$ always raise the power of $\hbar$ by at least one. Therefore in the computation of $\mathcal{Z}_\ell(n)$ we only need to compute ℓ'-th commutators with $\ell' \le \ell$ in $\widehat{H}$. These higher commutators can be generated recursively by the Baker-Campbell-Hausdorff formula.

The second kind of the deviations also satisfy the requirements, if treated in terms of $\mathcal{G}_\ell$. First note that $\mathcal{G}_\ell(\mathcal{O}_W)$ exhibit the following $\hbar$-expansion

$$\mathcal{G}_\ell(\mathcal{O}_W) = \sum_{j \ge [\frac{\ell+2}{3}]} \hbar^{2j} \mathcal{G}_\ell^{(j)}(\mathcal{O}_W), \tag{7.16}$$

regardless of the explicit form of the operator $\widehat{\mathcal{O}}$. Therefore in the computation of $\mathcal{Z}_s(n)$ we only need to compute a finite number of $\mathcal{G}_\ell$. Moreover, we can also construct the following recursive structure of $\mathcal{G}$. For $\ell = 0, 1$, $\mathcal{G}_\ell$ are trivially given as

$$\mathcal{G}_0 = 1, \quad \mathcal{G}_1 = 0, \tag{7.17}$$

and for $\ell \ge 2$ there is a recursive system for $\mathcal{G}_\ell$

$$\begin{aligned} \underbrace{\mathcal{O}_W \star \mathcal{O}_W \star \cdots \star \mathcal{O}_W}_{\ell} &= \mathcal{O}_W \star (\underbrace{\mathcal{O}_W \star \mathcal{O}_W \star \cdots \star \mathcal{O}_W}_{\ell-1}), \\ \mathcal{G}_\ell(\mathcal{O}) &= \underbrace{\mathcal{O}_W \star \mathcal{O}_W \star \cdots \star \mathcal{O}_W}_{\ell} - \mathcal{O}_W^\ell - \sum_{m=2}^{\ell-1} \binom{\ell}{m} \mathcal{O}_W^{\ell-m} \mathcal{G}_m(\mathcal{O}). \end{aligned} \tag{7.18}$$

Plugging these results into the original $\mathcal{Z}(n)$ we obtain

$$\sum_{\ell \ge 0} \hbar^\ell \mathcal{Z}_\ell(n) = \int \frac{dQdP}{2\pi} e^{-nH_{\rm cl}} e^{-n(H_W - H_{\rm cl})} \left[1 + \sum_{\ell=2}^{\infty} \frac{(-n)^\ell}{\ell!} \mathcal{G}_\ell(H_W) \right]. \tag{7.19}$$

Now the $\hbar$ corrections to $\mathcal{Z}_\ell(n)$ will be systematically obtained by expanding the second and third factor in the integrand in the r.h.s.

7.1 Computation and universal structure of $\mathcal{Z}_\ell(n)$

The correction terms in the integrand (7.19) can be systematically integrated as follows. First note that the correction terms in the two factors in (7.19) are some polynomial of the derivatives of U and T

$$\mathcal{Z}_s(n) = \int \frac{dQdP}{2\pi} e^{-nH_{\rm cL}} \mathrm{Poly}(\{U^{(a)}\}, \{T^{(b)}\}). \tag{7.20}$$

Since $U(Q)$ and the prefactor $e^{-nH_{\rm cl}}$ are even functions and the domain of integration is symmetric under the reflection $Q \to -Q$, only the terms containing even number of Q-differentials ∂_Q in total contribute to $\mathcal{Z}_\ell(n)$. As a derivative is always accompanied with a $\hbar$ through the $\star$-product (6.47), this implies that $\mathcal{Z}_\ell(n) = 0$ for any odd ℓ.

Hence we have only to consider the $\mathcal{Z}_\ell(n)$ for even ℓ, where each term in the "Poly" takes the form of

$$U^{(a_1)}U^{(a_2)}\cdots T^{(b_1)}T^{(b_2)}\cdots . \qquad \Big(\sum_i a_i = \sum_i b_i = \ell \in 2\mathbb{N}\Big) \tag{7.21}$$

Due to the differential properties of the cosine hyperbolic functions, such a term can always be rewritten as a polynomial of

$$U^{(2)} = \frac{1}{(2\cosh\frac{Q}{2})^2}, \quad T^{(2)} = \frac{1}{(2\cosh\frac{P}{2})^2}. \tag{7.22}$$

For example,

$$\begin{aligned}
(U')^2 &= \frac{1}{4} - U^{(2)}, \quad U'U^{(3)} = -\frac{1}{2}U^{(2)} + 2(U^{(2)})^2, \quad U^{(4)} = U^{(2)} - 6(U^{(2)})^2, \\
U^{(3)}U^{(3)} &= (U^{(2)})^2 - 4(U^{(2)})^3, \quad U'U^{(5)} = -\frac{U^{(2)}}{2} + 8(U^{(2)})^2 - 24(U^{(2)})^3, \\
U^{(6)} &= U^{(2)} - 30(U^{(2)})^2 + 120(U^{(2)})^3
\end{aligned} \tag{7.23}$$

Therefore the computation of $\mathcal{Z}_\ell$ reduces to the following integrals

$$\int \frac{dQdP}{2\pi}\frac{1}{(2\cosh\frac{Q}{2})^{qn+2\alpha}}\frac{1}{(2\cosh\frac{P}{2})^{pn+2\beta}} = \frac{(\frac{nq}{2})_\alpha^2(\frac{np}{2})_\beta^2}{(nq)_{2\alpha}(np)_{2\beta}}\mathcal{Z}_0(n). \tag{7.24}$$

with some $\alpha, \beta \in \mathbb{Z}_+$. Here $(z)_\zeta$ are Pochhammer symbol $\Gamma(z+\zeta)/\Gamma(z)$ which are merely polynomial of degree ζ in z.

From these arguments we conclude that $\mathcal{Z}_\ell(n)$ have the following universal structure

$$\mathcal{Z}_\ell(n) = \begin{cases} 0 & (\ell\text{: odd}) \\ f_\ell(q,p;n)\mathcal{Z}_0(n) & (\ell\text{: even}) \end{cases}, \tag{7.25}$$

where $f_\ell(q,p;n)$ are some rational functions of n.

From the explicit computation we explicitly determined the rational functions f_ℓ for $\ell \le 8$ [9] as

$$\begin{aligned}
f_0(q,p;n) &= 1, \quad f_2(q,p;n) = \frac{q^2p^2(1-n^2)n^2}{384(1+qn)(1+pn)}, \\
f_4(q,p;n) &= \frac{q^3p^3(1-n^2)n^2}{92160(1+qn)(1+pn)}\left[-\frac{(9-n^2)(8+3qn)(8+3pn)}{16(3+qn)(3+pn)} + 4 - n^2\right], \\
f_6(q,p;n) &= \frac{q^3p^3(1-n^2)n^3}{3963617280(1+qn)(3+qn)(5+qn)(1+pn)(3+pn)(5+pn)}
\end{aligned}$$

$$(1920q^3 + 1920p^3 + 7168n + 5376q^2n + 14336qpn + 2944q^3pn + 5376p^2n$$
$$+ 2304q^2p^2n + 2944qp^3n + 14336qn^2 + 576q^3n^2 + 14336pn^2 + 17920q^2pn^2$$
$$+ 17920qp^2n^2 + 1656q^3p^2n^2 + 576p^3n^2 + 1656q^2p^3n^2 + 5376q^2n^3$$
$$+ 25088qpn^3 + 2272q^3pn^3 + 5376p^2n^3 + 12272q^2p^2n^3 + 2272qp^3n^3$$
$$+ 367q^3p^3n^3 + 576q^3n^4 + 8960q^2pn^4 + 8960qp^2n^4 + 1488q^3p^2n^4 + 576p^3n^4$$
$$+ 1488q^2p^3n^4 + 928q^3pn^5 + 3088q^2p^2n^5 + 928qp^3n^5 + 178q^3p^3n^5$$
$$+ 312q^3p^2n^6 + 312q^2p^3n^6 + 31q^3p^3n^7),$$

$$f_8(q,p;n) = -\frac{q^3p^3(1-n^2)}{30440580710400}$$
$$\frac{1}{(1+qn)(3+qn)(5+qn)(7+qn)(1+pn)(3+pn)(5+pn)(7+pn)}$$
$$(3225600q^5 + 3225600p^5 + 24576000n + 10321920q^2n + 9216000q^4n$$
$$+ 55050240qpn + 11796480q^3pn + 5406720q^5pn + 10321920p^2n$$
$$+ 22118400q^2p^2n + 1105920q^4p^2n + 11796480qp^3n - 1474560q^3p^3n$$
$$+ 9216000p^4n + 1105920q^2p^4n + 5406720qp^5n + 58982400qn^2$$
$$+ 18063360q^3n^2 + 153600q^5n^2 + 58982400pn^2 + 107347968q^2pn^2$$
$$+ 21872640q^4pn^2 + 107347968qp^2n^2 + 36495360q^3p^2n^2 + 3751680q^5p^2n^2$$
$$+ 18063360p^3n^2 + 36495360q^2p^3n^2 + 1234944q^4p^3n^2 + 21872640qp^4n^2$$
$$+ 1234944q^3p^4n^2 + 153600p^5n^2 + 3751680q^2p^5n^2 + 52101120q^2n^3$$
$$+ 2764800q^4n^3 + 127795200qpn^3 + 75792384q^3pn^3 + 952320q^5pn^3$$
$$+ 52101120p^2n^3 + 161206272q^2p^2n^3 + 23761920q^4p^2n^3 + 75792384qp^3n^3$$
$$+ 35307520q^3p^3n^3 + 1340992q^5p^3n^3 + 2764800p^4n^3 + 23761920q^2p^4n^3$$
$$+ 812160q^4p^4n^3 + 952320qp^5n^3 + 1340992q^3p^5n^3 + 18063360q^3n^4$$
$$+ 153600q^5n^4 + 104398848q^2pn^4 + 12840960q^4pn^4 + 104398848qp^2n^4$$
$$+ 90491904q^3p^2n^4 + 1443840q^5p^2n^4 + 18063360p^3n^4 + 90491904q^2p^3n^4$$
$$+ 13250304q^4p^3n^4 + 12840960qp^4n^4 + 13250304q^3p^4n^4 + 260784q^5p^4n^4$$
$$+ 153600p^5n^4 + 1443840q^2p^5n^4 + 260784q^4p^5n^4 + 2764800q^4n^5$$
$$+ 34504704q^3pn^5 + 737280q^5pn^5 + 79982592q^2p^2n^5 + 15375360q^4p^2n^5$$
$$+ 34504704qp^3n^5 + 43215616q^3p^3n^5 + 876992q^5p^3n^5 + 2764800p^4n^5$$
$$+ 15375360q^2p^4n^5 + 3419040q^4p^4n^5 + 737280qp^5n^5 + 876992q^3p^5n^5$$
$$+ 27859q^5p^5n^5 + 153600q^5n^6 + 5099520q^4pn^6 + 25334784q^3p^2n^6$$
$$+ 879360q^5p^2n^6 + 25334784q^2p^3n^6 + 7105536q^4p^3n^6 + 5099520qp^4n^6$$
$$+ 7105536q^3p^4n^6 + 220944q^5p^4n^6 + 153600p^5n^6 + 879360q^2p^5n^6$$
$$+ 220944q^4p^5n^6 + 276480q^5pn^7 + 3624960q^4p^2n^7 + 7757056q^3p^3n^7$$
$$+ 398528q^5p^3n^7 + 3624960q^2p^4n^7 + 1151040q^4p^4n^7 + 276480qp^5n^7$$

$$+ 398528q^3p^5n^7 + 14359q^5p^5n^7 + 192000q^5p^2n^8 + 1080576q^4p^3n^8$$
$$+ 1080576q^3p^4n^8 + 63696q^5p^4n^8 + 192000q^2p^5n^8 + 63696q^4p^5n^8$$
$$+ 56128q^5p^3n^9 + 147360q^4p^4n^9 + 56128q^3p^5n^9 + 3481q^5p^5n^9$$
$$+ 7536q^5p^4n^{10} + 7536q^4p^5n^{10} + 381q^5p^5n^{11}). \tag{7.26}$$

The explicit expression for higher ℓ were also obtained by the authors of [13] (note that $f_{2\ell}(q, p; \partial_\mu) = \frac{\mathcal{D}^{(\ell)}}{2\pi}$ in appendix B of [13]).

7.2 Large μ expansion of grand potential

Now that we have explicitly obtained $\mathcal{Z}_\ell(n)$, let us consider the large μ expansion of the grand potential $J(\mu)$. We shall define the grand potential in the small k expansion corresponding to the semiclassical expansion of $\mathcal{Z}(n)$ (7.9)

$$\mathcal{J}(\mu) = \sum_{\ell \geq 0} \hbar^{\ell-1} \mathcal{J}_\ell(\mu) \tag{7.27}$$

with

$$\mathcal{J}_\ell(\mu) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} e^{n\mu} \mathcal{Z}_\ell(n). \tag{7.28}$$

Below we mainly focus on the classical grand potential

$$\mathcal{J}_0(\mu) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} e^{n\mu} \mathcal{Z}_0(n). \tag{7.29}$$

Once we have obtained the large μ expansion of $J_0(\mu)$, the factorizing structure (7.25) together with the factor $e^{n\mu}$ in (7.2) implies that the semiclassical corrections $J_\ell(\mu)$ are obtained by replacing the n in $f_\ell(q, p; n)$ with the differential ∂_μ as

$$J_\ell(\mu) = f_\ell(q, p; \partial_\mu) J_0(\mu). \tag{7.30}$$

The large μ expansion of this series was first obtained in [8] by resumming the series with the help of the series expansion of the ratio of Gamma functions and the formal identities converting the series of e^μ into that of $e^{-\mu}$, like ($\mu \in \mathbb{R}$)

$$\sum_{\ell \in \mathbb{Z}} \frac{(-e^\mu)^\ell}{\ell + \alpha} = \frac{\pi}{\sin \pi\alpha} e^{-\alpha\mu}. \tag{7.31}$$

Later in [45] the large μ expansion of the grand potential was more generally argued by using a different method. In this section we would like to explain the latter method which is more systematic than the original derivation.

The key for the derivation is the following integration ($\mu \in \mathbb{R}$)

$$-\int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{ds}{2\pi i}\Gamma(s)\Gamma(-s)\mathcal{Z}_0(s)e^{s\mu}. \tag{7.32}$$

where $0 < \epsilon < 1$ is an arbitrary real parameter. There are two different ways to evaluate (7.32), depending on the sign of μ. First assume $\mu < 0$. In this case the integrand decays as $s \to \infty$, so we can evaluate the integration by pinching the contour right as in figure 7 and collecting the residues therein

$$\begin{aligned}-\int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{ds}{2\pi i}\Gamma(s)\Gamma(-s)\mathcal{Z}_0(s)e^{s\mu} &= \int_{C_+} \frac{ds}{2\pi i}\Gamma(s)\Gamma(-s)\mathcal{Z}_0(s)e^{s\mu} \\ &= \sum_{s_a^{(+)}(\mathrm{Re}(s_a^{(+)})>\epsilon)} \mathrm{Res}[\Gamma(s)\Gamma(-s)\mathcal{Z}_0(s)e^{s\mu}, s \to s_a^{(+)}]. \end{aligned} \tag{7.33}$$

Since $\mathcal{Z}_0(s)$ (7.8) have no poles in this region, the set of poles is only $s_a^{(+)} = 1, 2, \cdots$ coming from $\Gamma(-s)$. We find that the residues at this infinite series of poles precisely reproduce the original small e^μ expansion of the grand potential $J_0(\mu)$ (7.29).

In this sense, the integration (7.32) provides the analytic continuation of $J_0(\mu)$ from $\mu < 0$ ($e^\mu < 1$) to $\mu > 0$ ($e^\mu > 1$), i.e. the large μ expansion. The integration (7.32) is called the Mellin-Barnes representation of the series (7.29). Indeed, assuming $\mu > 0$, the integration can be evaluated by pinching the contour left

$$\begin{aligned} J_0(\mu) &= -\int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{ds}{2\pi i}\Gamma(s)\Gamma(-s)\mathcal{Z}_0(s)e^{s\mu} \\ &= \sum_{s_a^{(-)}(\mathrm{Re}(s_a^{(-)})\le 0)} \mathrm{Res}[-\Gamma(s)\Gamma(-s)\mathcal{Z}_0(s)e^{s\mu}, s \to s_a^{(-)}]. \end{aligned} \tag{7.34}$$

Due to the factor $e^{s\mu}$ in the integrand the residues are exponentially suppressed or at most polynomial in μ for $\mu \to \infty$, which immediately gives the desired large μ expansion of the grand potential.

7.3 Perturbative part and A

The residue at $s = 0$ is polynomial in μ, which produces the perturbative part of the grand potential. Indeed, since the integrand in (7.32) is $\mathcal{O}(s^{-4})$ as $s \to 0$ we obtain a cubic polynomial in μ by expanding the factor $e^{s\mu}$. Taking into account the semiclassical corrections (7.30) as well, we find that the coefficients of the cubic term, the quadratic term (vanishing) and the linear term determined by the Fermi surface analysis are precisely reproduced.

On the other hand the small k expansion of the remaining constant A do not terminate at finite order and found to be [8][9]

$$A = \frac{2\zeta(3)}{\pi\hbar}\frac{p^3+q^3}{qp} - \frac{\hbar qp(q+p)}{48\pi} - \frac{\hbar^3 q^2p^2(q+p)}{69120\pi} + \frac{\hbar^5 q^2p^2(q^3+p^3)}{58060800\pi} - \frac{\hbar^7 q^2p^2(q^5+p^5)}{13005619200\pi}$$

[9]Precisely speaking, the small k expansion of A was computed up to $\mathcal{O}(\hbar^3)$ in [8] and then extended up to $\mathcal{O}(\hbar^7)$ while preparing [10].

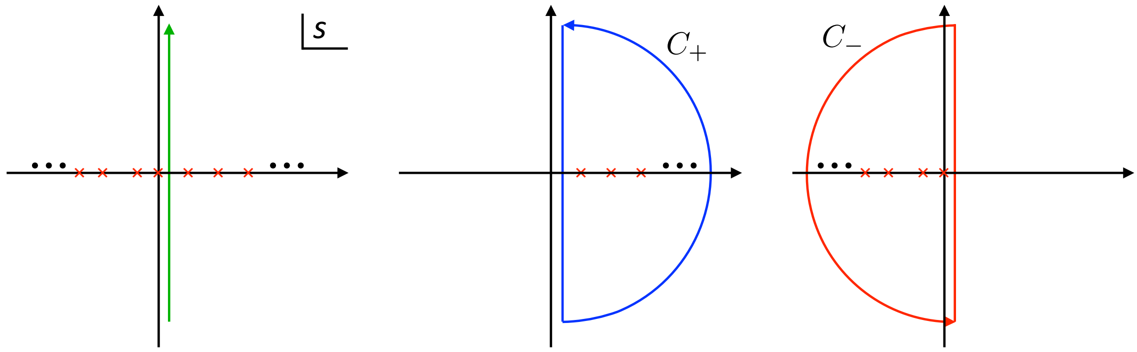

Figure 7: The original contour (left), the contour for $\mathrm{Re}(\mu) < 0$ (center) and the contour for $\mathrm{Re}(\mu) > 0$ of the integral (7.32).

$$+\,\mathcal{O}(\hbar^9). \tag{7.35}$$

At first sight the expression seems complicated. However, we figure out the following interesting decomposition structure at each order:

$$\begin{aligned}
\frac{2\zeta(3)}{\pi\hbar}\frac{p^3+q^3}{qp} &= \frac{\zeta(3)}{\pi^2}\left[\frac{p^2}{qk}+\frac{q^2}{pk}\right],\\
-\frac{\hbar qp(q+p)}{48\pi} &= -\frac{1}{24}(p^2\cdot(qk)+q^2\cdot(pk)),\\
-\frac{\hbar^3 q^2p^2(q+p)}{69120\pi} &= -\frac{\pi^2}{8640}(p^2\cdot(qk)^3+q^2\cdot(pk)^3),\\
&\vdots
\end{aligned} \tag{7.36}$$

that is

$$A = \frac{p^2 f(qk)+q^2 f(pk)}{2}. \tag{7.37}$$

with $f(k)$ given by the small k expansion as

$$f(k) = \frac{2\zeta(3)}{\pi^2 k} - \frac{k}{12} - \frac{\pi^2 k^3}{4320} + \frac{\pi^4 k^5}{907200} - \frac{\pi^6 k^7}{50803200} + \mathcal{O}(k^9). \tag{7.38}$$

This $f(k)$ coincide with the small k expansion of the constant map in the ABJM theory $A_{\mathrm{ABJM}}(k)$ [46]. Indeed if we postulate the decomposition (7.37), we can deduce $f(k) = A_{\mathrm{ABJM}}(k)$ by taking the limit $q,p\to 1$ where our theory reduce to the ABJM theory. Hence we conjecture

$$A = \frac{p^2 A_{\mathrm{ABJM}}(qk)+q^2 A_{\mathrm{ABJM}}(pk)}{2}. \tag{7.39}$$

This expression is also confirmed for small $k\in\mathbb{N}$, as we will explain in section 8.

7.4 Non-perturbative part

Now let us consider the residues at $\mathrm{Re}(s) < 0$. From the explicit expression $\mathcal{Z}_0(n)$ (7.8) we find three infinite series of poles

$$
\begin{aligned}
s &= -\frac{2l}{q}, \quad (l = 1, 2, \cdots) \\
s &= -\frac{2m}{p}, \quad (m = 1, 2, \cdots) \\
s &= -n, \quad (n = 1, 2, \cdots).
\end{aligned} \tag{7.40}
$$

As the rational prefactor $f_\ell(q,p;n)$ in the higher order corrections have at most finite number of poles,[10] we conclude that there are three corresponding series of non-perturbative effects in the grand potential

$$
J_{\mathrm{np}} = J_{\mathrm{np}}^{(q)} + J_{\mathrm{np}}^{(p)} + J_{\mathrm{np}}^{(2)}, \tag{7.41}
$$

with

$$
J_{\mathrm{np}}^{(q)} = \sum_{\ell=1}^{\infty} c_l^{(q)}(k) e^{-\frac{2l\mu}{q}}, \quad J_{\mathrm{np}}^{(p)} = \sum_{m=1}^{\infty} c_m^{(p)}(k) e^{-\frac{2m\mu}{p}}, \quad J_{\mathrm{np}}^{(2)} = \sum_{n=1}^{\infty} c_n^{(2)}(k) e^{-n\mu}. \tag{7.42}
$$

Here $c_l^{(q)}$, $c_m^{(p)}$ and $c_n^{(2)}$ are some k-dependent constants given in the small k expansion as

$$
c_l^{(q)} = \sum_{\ell=0}^{\infty} \hbar^{2\ell-1} c_{\ell,l}^{(q)}, \quad c_m^{(p)} = \sum_{\ell=0}^{\infty} \hbar^{2\ell-1} c_{\ell,m}^{(p)}, \quad c_n^{(2)} = \sum_{\ell=0}^{\infty} \hbar^{2\ell-1} c_{\ell,n}^{(2)}. \tag{7.43}
$$

The leading part $c_{0,l}^{(q)}$, $c_{0,m}^{(p)}$ and $c_{0,n}^{(2)}$ can be read off from the explicit expression of $\mathcal{Z}_0(s)$ as

$$
\begin{aligned}
c_{0,\ell}^{(q)} &= \frac{1}{\pi\ell \sin\frac{2\pi\ell}{q}} \binom{2\ell}{\ell} \frac{\Gamma(-\frac{p\ell}{q})^2}{\Gamma(-\frac{2p\ell}{q})}, \quad c_{0,m}^{(p)} = \frac{1}{\pi m \sin\frac{2\pi m}{p}} \binom{2m}{m} \frac{\Gamma(-\frac{qm}{p})^2}{\Gamma(-\frac{2qm}{p})}, \\
c_{0,n}^{(2)} &= \frac{(-1)^{n-1}}{2\pi n} \frac{\Gamma(-\frac{qn}{2})^2}{\Gamma(-qn)} \frac{\Gamma(-\frac{pn}{2})^2}{\Gamma(-pn)},
\end{aligned} \tag{7.44}
$$

where we have used the following formula

$$
\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z} \tag{7.45}
$$

to rewrite some of the Gamma functions. The higher order corrections are obtained by multiplying $f_{2s}(q,p;\partial_\mu)$ with replacement $\partial_\mu \to -2\ell/q$, $-2m/p$ or $-n$, respectively.

[10] At least up to $\mathcal{O}(k^9)$, all of these finite poles are precisely cancelled with the zeroes coming from the Gamma functions in the numerator of $\mathcal{Z}_0(n)$.

7.5 Gravitational interpretation of non-perturbative effects as instantons

Here we shall briefly comment on the interpretation of the non-perturbative corrections $\mathcal{O}(e^{-\mu})$ to the grand potential $J(\mu)$ in the gravity side. The dual eleven dimensional geometry to the $(q,p)_k$ model we are considering is $\mathrm{AdS}_4 \times Y_7$,

$$ds_{11}^2 = \frac{R_{\mathrm{AdS}}^2}{4} ds_{\mathrm{AdS}_4}^2 + R_{\mathrm{AdS}}^2 ds_{Y_7}^2, \tag{7.46}$$

where Y_7 is the radial section of $(\mathbb{C}^2/\mathbb{Z}_q, \times\mathbb{C}^2/\mathbb{Z}_p)/\mathbb{Z}_k$ according to the analysis of the moduli space [3] commented in section 3. The AdS radius R_{AdS} is given by

$$R_{\mathrm{AdS}} = (32\pi^2 qpkN)^{\frac{1}{6}} \ell_p^{(11)} \tag{7.47}$$

with $\ell_p^{(11)}$ the eleven dimensional Planck length. Below we provide an evidence that the non-perturbative effects can be interpreted as fundamental membranes winding on Y_7.

Suppose the perturbative grand potential (6.23) is corrected with a non-perturbative effect $e^{-\omega\mu}$ with some constant ω as

$$J(\mu) = J^{\mathrm{pert}} + e^{-\omega\mu}. \tag{7.48}$$

Substituting this into the inversion relation $J(\mu) \to Z(N)$ (6.25), we obtain

$$\begin{aligned} Z(N) &= \sum_{n=0}^{\infty} \frac{1}{n!} \int_{-i\infty}^{i\infty} \frac{d\mu}{2\pi i} e^{J^{\mathrm{pert}}(\mu) - \mu(N+n\omega)} \\ &= e^A C^{-\frac{1}{3}} \sum_{n=0}^{\infty} \frac{1}{n!} \mathrm{Ai}[C^{-\frac{1}{3}}(N - B + n\omega)], \end{aligned} \tag{7.49}$$

where we have expanded the exponential of the non-perturbative correction and used the integration formula defining the Airy function (6.2). In terms of the free energy $F = \log Z(N)$, the term $n = 0$ correspond to the perturbative part (6.1) while the terms $n > 0$ gives the non-perturbative free energy

$$F^{\mathrm{np}} = \log\left[1 + \sum_{n=1}^{\infty} \frac{1}{n!} \frac{\mathrm{Ai}[C^{-\frac{1}{3}}(N - B + n\omega)]}{\mathrm{Ai}[C^{-\frac{1}{3}}(N - B)]}\right]. \tag{7.50}$$

With the help of the asymptotic expansion formula of the Airy function (6.3) we find that the leading behavior of F^{np} is

$$F^{\mathrm{np}} \sim e^{-\omega\sqrt{N/C}}. \tag{7.51}$$

From the explicit expressions of the constant C (6.37) and R_{AdS} (7.47) the exponent is found to coincide, up to numerical factors, with

$$\omega\sqrt{\frac{N}{C}} \propto \omega R_{\mathrm{AdS}}^3 T_{\mathrm{M2}} \tag{7.52}$$

Hence the exponent can be interpreted as the energy of closed Euclidean M2-branes winding on some non-trivial 3-cycle inside Y_7.

Such winding effects are called the membrane instantons [36]. In current case Y_7 is the radial section of $(\mathbb{C}^2/\mathbb{Z}_q \times \mathbb{C}^2/\mathbb{Z}_p)/\mathbb{Z}_k$. As it would be natural to guess that the M2-branes can wind on the cycle orbifolded by $\mathbb{Z}_q$ or $\mathbb{Z}_p$, this way of interpretation indeed seems to be consistent with the variety of the non-perturbative effects (7.42).[11] For these reasons, from now on we shall call the non-perturbative effects $\mathcal{O}(e^{-\mu})$ as the "instantons".

7.6 Instanton coefficients and pole cancellation

So far we have formally regarded q and p as arbitrary real parameters in the computations, where the instanton coefficient are generically finite. However, some of the instanton coefficients diverge for the special values of q and p. Actually, the divergence always occurs for physical parameters, i.e. $q, p \in \mathbb{N}$. At first sight the occurrence of the divergences seems to be a contradiction, since the partition function itself is finite even for these special values. However, as we shall see in detail below, a divergence is always accompanied with the degeneracy of the instanton exponents over the sectors. The divergence of the individual coefficient actually occurs pairwisely in these degenerating instantons, and interestingly, they precisely cancel each other's divergence. The remaining coefficients are always finite and thus the grand potential is well defined.

To see the structure of divergences, it is convenient to rewrite the coefficients of the three species of instantons in a symmetric manner by using the formula (7.45)

$$J_{0,\text{np}}^{(z_i)}(\mu) = \sum_{\ell_i=1}^{\infty} \frac{F(\frac{\ell_i}{z_i};\mu)}{\ell_i} \prod_{j=1(\neq i)}^{3} \cot \frac{\pi z_j \ell_i}{z_i}, \tag{7.53}$$

where we have introduced $z_i = (q, p, 2)$, $\ell_i = (l, m, n)$ and

$$F(r;\mu) = -\frac{2\pi}{\cos 2\pi r} \frac{\Gamma(2qr+1)}{\Gamma(qr+1)^2} \frac{\Gamma(2pr+1)}{\Gamma(pr+1)^2} e^{-2r\mu}. \tag{7.54}$$

In this expression all the Gamma functions in the instanton coefficients are finite. The divergent structure are expressed only with the cotangent factors. Explicitly speaking, the divergence appears at $l \in \frac{q}{\gcd(q,p)}\mathbb{N} \cup \frac{q}{\gcd(q,2)}\mathbb{N}$ in $J_{0,\text{np}}^{(q)}(\mu)$, at $m \in \frac{p}{\gcd(p,2)}\mathbb{N} \cup \frac{p}{\gcd(p,q)}\mathbb{N}$ in $J_{0,\text{np}}^{(p)}(\mu)$ and at $n \in \frac{2}{\gcd(2,q)}\mathbb{N} \cup \frac{2}{\gcd(2,p)}\mathbb{N}$ in $J_{0,\text{np}}^{(2)}(\mu)$. As the factor $F(r;\mu)$ from different sectors share the same instanton exponent at these special instanton numbers, however, the divergences are possibly cancelled among those terms with the degenerating exponent. In [9] we have explicitly checked the cancellation of divergence by regularizing the divergence by the replacement $(q, p) \to (q(1+$

[11] Different from the $\mathbb{Z}_k$, there are the fixed point sets on S^7 for the orbifold action of $\mathbb{Z}_q$ and $\mathbb{Z}_p$. Therefore the volume of the 3-cycle containing these directions can be zero (vanishing cycle) and the volume interpretation of the instanton exponents seems to collapse. This problem would be avioded if we consider the appropriate blow up of the conical singularity.

$\varepsilon_1), p(1+\varepsilon_2))$. After summing all the relevant contributions we can see the instanton coefficient is indeed finite in the undeformed limit $\varepsilon_1, \varepsilon_2 \to 0$.

To see the explicit coefficients remaining after the cancellation, however, it is more convenient to go back to the Mellin-Barnes integration representation (7.32). The integrand have double/triple pole at the degeneracy of the instanton exponents. The finite part of the instanton coefficient is simply given by the residue at this pole. Here we shall demonstrate the computation for the case of triple degeneracy, which occurs when the three instanton numbers satisfy

$$\frac{l}{q} = \frac{m}{p} = \frac{n}{2} = \frac{\ell}{r}, \quad (\ell \in \mathbb{N}) \tag{7.55}$$

where $r = \gcd(q, p, 2)$. To obtain the residue it is convenient again to eliminate the singular Gamma functions from integrand in (7.32) with the help of the formula (7.45) as

$$-\Gamma(s)\Gamma(-s)\mathcal{Z}_0(s)e^{s\mu} = \frac{1}{\sin(\pi s)\tan\frac{\pi q s}{2}\tan\frac{\pi p s}{2}} \frac{2\pi^2\Gamma(1-qs)\Gamma(1-ps)}{s\Gamma(1-\frac{qs}{2})^2\Gamma(1-\frac{ps}{2})^2} e^{s\mu}. \tag{7.56}$$

Now only the first factor is singular at $s = -2\ell/r$

$$\frac{1}{\sin(\pi s)\tan\frac{\pi q s}{2}\tan\frac{\pi p s}{2}}\bigg|_{s=-\frac{2\ell}{r}+\epsilon} = \frac{4}{\pi^3 q p \epsilon^3}\left(1 + \frac{\pi^2\epsilon^2(2-q^2-p^2)}{12} + \cdots\right) \tag{7.57}$$

while the remaining terms are expanded as

$$\begin{aligned}
&\frac{2\pi^2\Gamma(1-qs)\Gamma(1-ps)}{s\Gamma(1-\frac{qs}{2})^2\Gamma(1-\frac{ps}{2})^2} e^{s\mu}\bigg|_{s=-\frac{2\ell}{r}+\epsilon} \\
&= -\frac{\pi^2 r\Gamma(1+\frac{2q\ell}{r})\Gamma(1+\frac{2p\ell}{r})}{\ell\Gamma(1+\frac{q\ell}{r})^2\Gamma(1+\frac{p\ell}{r})^2} \\
&\quad \left[1 + \epsilon\left\{\mu + \frac{r}{2\ell} - H_1\left(\frac{\ell}{r}\right)\right\}\right. \\
&\qquad + \epsilon^2\left\{\frac{\mu^2}{2} + \left(\frac{r}{2\ell} - H_1\left(\frac{\ell}{r}\right)\right)\mu\right. \\
&\qquad\qquad \left.\left. + \frac{r^2}{4\ell^2} - \frac{r}{2\ell}H_1\left(\frac{\ell}{r}\right) - \frac{1}{2}H_1\left(\frac{\ell}{r}\right)^2 + H_2\left(\frac{\ell}{r}\right) + \frac{\pi^2(q^2+p^2)}{24}\right\}\right] e^{-\frac{2\ell\mu}{r}}.
\end{aligned} \tag{7.58}$$

Here $H_s(r)$ is defined with the harmonic numbers h_s

$$h_s(m) = \sum_{\ell=1}^{m} \frac{1}{\ell^s}, \tag{7.59}$$

as

$$H_s(r) = q^s\bigl(2^{s-1}h_s(2qr) - h_s(qr)\bigr) + p^s\bigl(2^{s-1}h_s(2pr) - h_s(pr)\bigr). \tag{7.60}$$

The appearances of $H_s(r)$ are due to the derivatives of the Gamma functions in $F(r;\mu)$, through the formula

$$\psi^{(0)}(m) = -\gamma + h_1(m-1), \quad \psi^{(1)}(m) = \frac{\pi^2}{6} - h_2(m-1), \tag{7.61}$$

where γ is the Euler-Mascheroni constant. The polygamma functions are defined as

$$\psi^{(s-1)}(x) = \left(\frac{d}{dx}\right)^s \log\Gamma(x). \tag{7.62}$$

Plugging these results together we finally obtain the instanton coefficient of $e^{-2\ell/r}$ as

$$(a_{0,\ell}\mu^2 + b_{0,\ell}\mu + c_{0,\ell})e^{-\frac{2\ell}{r}} \tag{7.63}$$

with

$$\begin{aligned} a_{0,\ell} &= -\frac{2r}{\pi q p \ell}\binom{\frac{2q\ell}{r}}{\frac{q\ell}{r}}\binom{\frac{2p\ell}{r}}{\frac{p\ell}{r}}, \quad b_{0,\ell} = a_{0,\ell} \times \left[\frac{r}{\ell} - 2H_1\Big(\frac{\ell}{r}\Big)\right], \\ c_{0,\ell} &= a_{0,\ell} \times \left[\frac{r^2}{2\ell^2} - \frac{r}{\ell}H_1\Big(\frac{\ell}{r}\Big) - H_1\Big(\frac{\ell}{r}\Big)^2 + 2H_2\Big(\frac{\ell}{r}\Big) + \frac{\pi^2(4-q^2-p^2)}{12}\right]. \end{aligned} \tag{7.64}$$

It is important to notice that an instanton coefficient is generally not constant but polynomial in μ after the pole cancellation. The degree of the polynomial coefficient is given as

$$(\#\text{ of species contributing to the instanton}) - 1. \tag{7.65}$$

7.7 Effective chemical potential for triple degeneracy

As we have seen above, the instanton coefficient is quadratic polynomial in μ when all the three species of the instantons contribute to the pole cancellation

$$J^{\mathrm{np}} = J_a\mu^2 + J_b\mu + J_c + \cdots, \tag{7.66}$$

where

$$J_a = \sum_{\ell\ge1} a_\ell e^{-\frac{2\ell\mu}{r}}, \quad J_b = \sum_{\ell\ge1} b_\ell e^{-\frac{2\ell\mu}{r}}, \quad J_c = \sum_{\ell\ge1} c_\ell e^{-\frac{2\ell\mu}{r}}, \tag{7.67}$$

and we have abbreviate the other instanton effects as "$\cdots$". a_ℓ, b_ℓ, and c_ℓ are some μ independent constants whose explicit expressions in the limit $k \to 0$ are given as (7.64).

Note that the quadratic part J_a can be absorbed into the perturbative part of the grand potential by shifting the chemical potential μ,

$$\mu_{\mathrm{eff}} = \mu + \frac{J_a}{C}, \tag{7.68}$$

as

$$\begin{aligned} J(\mu) &= \frac{C}{3}\mu^3 + B\mu + A + J_a\mu^2 + J_b\mu^2 + J_c + \cdots \\ &= \frac{C}{3}\mu_{\mathrm{eff}}^3 + B\mu_{\mathrm{eff}} + A + \widetilde{J}_b\mu_{\mathrm{eff}} + \widetilde{J}_c + \cdots, \end{aligned} \tag{7.69}$$

where

$$\widetilde{J}_b = J_b - \frac{J_a^2}{C}, \quad \widetilde{J}_c = J_c + \frac{2J_a^3}{3C^2} - \frac{BJ_a}{C} - \frac{J_aJ_b}{C}. \tag{7.70}$$

Now $\widetilde{J}_b$ and $\widetilde{J}_c$ should be regarded as the instanton effects with exponents $e^{-2\ell\mu_{\text{eff}}/r}$:

$$\widetilde{J}_b = \sum_{\ell\geq 1} \widetilde{b}_\ell e^{-\frac{2\ell\mu_{\text{eff}}}{r}}, \quad \widetilde{J}_c = \sum_{\ell\geq 1} \widetilde{c}_\ell e^{-\frac{2\ell\mu_{\text{eff}}}{r}}. \tag{7.71}$$

Interestingly, a little simplification occurs in the instanton coefficients by the introduction of this "effective chemical potential" μ_{eff} (**??**). Speaking concretely,

- The instanton coefficients (7.64) are somehow asymmetric for

$$a_{0,\ell}, b_{0,\ell} \in \mathbb{Q}/\pi, \quad \text{while} \quad c_{0,\ell} \in \mathbb{Q}/\pi + \pi\mathbb{Q}. \tag{7.72}$$

  On the other hand, the instanton coefficients in terms of μ_{eff}, $\widetilde{b}$ and $\widetilde{c}$ are both in $\mathbb{Q}/\pi$.

- The same rationality is satisfied also for the higher order in small $\hbar$ expansion. This implies that the Taylor expansions of the instanton coefficients on the triple degeneracy a_ℓ, $\widetilde{b}_\ell(k)$ and $\widetilde{c}_\ell(k)$ schematically take the following form

$$\frac{1}{\pi} \sum_{m\geq 0} \alpha_m \hbar^{2m-1}, \quad (\alpha_m \in \mathbb{Q}). \tag{7.73}$$

Below we shall prove these properties.

Let us start with the classical limit. Remember the rationality of the perturbative coefficients (6.37) and (6.40),

$$C = \frac{1}{\hbar} C_0, \quad B = \frac{1}{\hbar} B_0 + \hbar B_2 \tag{7.74}$$

with

$$\begin{aligned} C_0 &= \frac{4}{\pi q p} \in \mathbb{Q}/\pi, \\ B_0 &= \frac{\pi(4 - q^2 - p^2)}{3qp} \in \pi\mathbb{Q}, \\ B_2 &= \frac{qp}{48\pi} \in \mathbb{Q}/\pi. \end{aligned} \tag{7.75}$$

First note that

$$\frac{J_a}{C} = \sum_{\ell\geq 1} \frac{a_{0,\ell}}{C_0} e^{-\frac{2\ell\mu}{r}} \tag{7.76}$$

have always rational coefficients. This ensures that the coefficients in $\widetilde{J}_b$ and $\widetilde{J}_c$ computed before expanding $e^{-2\ell\mu/r}$ with $e^{-2\ell\mu_{\text{eff}}/r}$ have the same rationality as those after the expansion. Therefore we would like to argue the rationality of the coefficients before expanding the exponents. The linear part $\widetilde{J}_b$ can be written explicitly, in the classical limit, as

$$\widetilde{J}_b = \frac{1}{\hbar}\sum_{\ell\geq 1}\Bigg(b_{0,\ell} - \frac{1}{C_0}\sum_{\ell_1,\ell_2\geq 1(\ell_1+\ell_2=\ell)} a_{0,\ell_1}a_{0,\ell_2}\Bigg)e^{-\ell\mu}. \tag{7.77}$$

Since $a_{0,\ell}, b_{0,\ell}, C_0 \in \mathbb{Q}/\pi$, we conclude $\widetilde{b}_{0,\ell} \in \mathbb{Q}/\pi$. Similarly we can see that the following combination in the constant part $\widetilde{J}_c$

$$\frac{2J_a^3}{3C^2} - \frac{J_aJ_b}{C} \tag{7.78}$$

have rational coefficients up to $\hbar$ in the classical limit. The only non-trivial parts are the remaining terms in $\widetilde{J}_c$

$$J_c - \frac{BJ_a}{C}. \tag{7.79}$$

Using the explicit expression of the coefficients we find the $\pi\mathbb{Q}$ part precisely cancels as

$$a_{0,\ell}\cdot\frac{\pi^2(4-q^2-p^2)}{12} - \frac{\pi(4-q^2-p^2)}{3qp}\cdot a_{0,\ell}\cdot\frac{\pi qp}{4} = 0. \tag{7.80}$$

Therefore we also conclude $\widetilde{c}_\ell \in \mathbb{Q}/\pi$.

Next let us consider the higher order corrections in the small k expansion. The higher order corrections are generated by operating

$$\hbar^{\ell-1}f_\ell(q,p;\partial_\mu) \tag{7.81}$$

($f(q,p;n)$ are rational functions of n with rational coefficients, given as (7.26)) on the instantons in the classical limit

$$\frac{1}{\hbar}\sum_{\ell\geq 1}(a_{0,\ell}\mu^2 + b_{0,\ell}\mu + c_{0,\ell})e^{-\frac{2\ell\mu}{r}}. \tag{7.82}$$

It is easy to see that the differential ∂_μ converts the coefficients as

$$\partial_\mu : \begin{pmatrix} a_\ell \\ b_\ell \\ c_\ell \end{pmatrix} \to M\begin{pmatrix} a_\ell \\ b_\ell \\ c_\ell \end{pmatrix} \tag{7.83}$$

with

$$M = \begin{pmatrix} -\frac{2\ell}{r} & 0 & 0 \\ 2 & -\frac{2\ell}{r} & 0 \\ 0 & 1 & -\frac{2\ell}{r} \end{pmatrix}. \tag{7.84}$$

Since the transformation matrix M and the power of M have only rational components and always satisfies $M^n_{i,j} = 0$ for $i < j$, the coefficients $a_{0,\ell}$ take the form of (7.73). Therefore in the purpose of the derivation of the schematic structure (7.73) we can again neglect the corrections coming from the exponents $e^{-2\ell\mu/r} \neq e^{-2\ell\mu_{\mathrm{eff}}/r}$. Since $b_{0,\ell}$ also take the form of (7.73), we can see $\widetilde{b}_\ell$ take the form of (7.73) as well by the similar manipulations as in the classical limit. Moreover, since $M^n_{1,1} = M^n_{3,3}$ and $B_2 \in \mathbb{Q}/\pi$ the cancellation of the $\pi\mathbb{Q}$ irrational part in (7.80) still holds. This ensures that $\widetilde{c}_\ell$ also satisfies the structure (7.73).

7.8 Discussion and Comments

In this section we have analyzed the grand potential $J(\mu)$ in small k expansion but exactly with respect to the chemical potential μ. The analysis provides the small k expansion of the constant $A(k)$ in the perturbative partition function (6.1) which we could not determine in the last section. Though the small k expansion does not terminate at finite order, we have achieved to conjecture its exact k dependence by discovering a simple decomposition structure (7.37).[12]

We have also obtained quantitative results on the non-perturbative effects in μ. In the gravity side these effects can be understood as the Euclidean M2-branes winding on some non-trivial three cycles, called the instantons. In this picture the exponents of the non-perturbative effects are related to the volume of the winding, and our observation of the variety of exponents modded by q and p nevertheless matches with the variety of orbifolds in the moduli space of the theory $(\mathbb{C}^2/\mathbb{Z}_q \times \mathbb{C}^2/\mathbb{Z}_p)/\mathbb{Z}_k$.

As we will see later there also exist the instantons of $\mathcal{O}(e^{-\mu/k})$ which are completely invisible in the small k expansion. In the case of the ABJM theory, there are two kinds of instantons, $\mathcal{O}(e^{-4\mu/k})$ and $\mathcal{O}(e^{-2\mu})$. The former corresponds to the M2-branes winding on the cycle modded by $\mathbb{Z}_k$ while the latter the M2-branes winding in some other three directions. Regarding the modded cycle as the M-theory direction and adopt the terminology of IIA picture, the former instantons correspond to the winding of the fundamental strings while the latter that of the D2-branes. In this sense the former instantons $\mathcal{O}(e^{-4\mu/k})$ are called the worldsheet instantons [50] while the latter $\mathcal{O}(e^{-2\mu})$ are called the membrane instantons [51].

Obtaining the exact instanton coefficients (7.44) we have discovered an interesting pole structure of the instantons: regarding q and p as continuous parameters the individual instanton coefficient diverges at some special values of q and p while the divergences completely cancel among the different species of the instantons. The physical implication of this pole cancellation mechanism is still obscure. Since q and p are always integers in physical theories as they are associated with the orbifold (or the number of vertices in the quiver), it is unclear whether there should be some counterpart of these phenomena in the gravity side or in the original field theory.

Nevertheless, for the purpose to reveal the exact structures of the instanton effects the pole cancellation provides great hint to guess the instanton coefficients. Suppose we only know of one series of the instanton $e^{-2\mu/q}$ but with the explicit expression of their instanton coefficients $c_{0,\ell}^{(q)}$ (7.44). The pole cancellation implies that in this case the pole structure of the explicit coefficient predict the other two series of the instantons $e^{-2\mu/p}$ and $e^{-\mu}$. Conversely, once we recognize all the instanton species $e^{-2\mu/q}$, $e^{-2\mu/p}$ and $e^{-\mu}$ but without their coefficients, the pole cancellation requires that the coefficient, say $c_\ell^{(q)}$ must have the following infinite series of poles: $c_\ell^{(q)} \to \infty$ at $\ell p/q \in \mathbb{N}$ or $2\ell/q \in \mathbb{N}$, which strongly restrict the (q,p) dependence of the coefficients. In

[12]Interestingly, the similar decomposition structure as well as the correlation to the subdivision of the membrane instantons is observed also in other theories [47, 48, 49]. We hope to provide some physical interpretation to these structure in future.

section 9 we will determine exact k dependence of the instanton coefficients by applying these ideas between the worldsheet instantons and the membrane instantons.

Lastly, though we have focused on the theory with minimal separation of $s_a = \pm 1$ in the levels (6.39), the similar techniques are also applicable to the non-minimal cases. Since they are identical to the minimal theory in the classical limit, we obtain the completely same series of instanton exponents and the pole cancellation mechanism.

8 Instanton effects for finite k

As we explain in this section, in the $(q,p)_k$ minimal models we can compute the exact values of the partition function $Z(N)$ for small integers $k,q,p \in \mathbb{N}$ iteratively in N, using the techniques developed in [52, 53, 54, 10].[13] These results provide distinct data for the exact expression of the grand potential from the small k expansion in the section 7.

Using the exact values we can confirm our conjectural expression of the constant A (7.37) in the perturbative partition function. Moreover, since the obtained values are exact, even the "small errors" from the $Z^{\text{pert}}(N) = e^A C^{-\frac{1}{3}} \operatorname{Ai}[C^{-\frac{1}{3}}(N-B)]$ are meaningful quantities as well. We can read off the instanton effects for finite k from these deviations. Interestingly we encounter new kind of instanton effects $\mathcal{O}(e^{-\mu/k})$ which were completely invisible in the small k expansion.

8.1 Exact partition function for $(q,p)_k$ models

Here we shall explain a systematic method to compute the quantities $\operatorname{Tr}\widehat{\rho}^n$. Once these quantities are determined for $1 \le n \le N$, we can read off the exact partition function $Z(N)$ according to the definitions (6.11) and (6.19).

8.1.1 $p = 1$

First we would like to explain the computation in the $(q,p)_k$ models with $p=1$ [54]. Notice that the matrix element of the density matrix $\widehat{\rho}$ is written as

$$\rho(Q_1,Q_2) = \frac{1}{2\pi}\langle Q_1|\widehat{\rho}|Q_2\rangle = \frac{1}{2\pi k}\frac{1}{\big(2\cosh\frac{Q_1}{2}\big)^{q/2}}\frac{1}{2\cosh\frac{Q_1-Q_2}{2k}}\frac{1}{\big(2\cosh\frac{Q_2}{2}\big)^{q/2}}, \tag{8.1}$$

where we have used the following Fourier transformation formula

$$\langle Q_1|\frac{1}{2\cosh\frac{\widehat{P}}{2}}|Q_2\rangle = \int\frac{dP}{2\pi k}\frac{e^{i(Q_1-Q_2)P/\hbar}}{2\cosh\frac{P}{2}} = \frac{1}{k}\frac{1}{2\cosh\frac{Q_1-Q_2}{2k}}. \tag{8.2}$$

The exact computation is achieved due to the following structure of this matrix element

$$\rho(Q_1,Q_2) = \frac{E(Q_1)E(Q_2)}{M(Q_1)+M(Q_2)}, \tag{8.3}$$

where individual ingredients are given explicitly as

$$M(Q) = 2\pi k e^{\frac{Q}{k}}, \quad E(Q) = \frac{e^{\frac{Q}{2k}}}{\big(2\cosh\frac{Q}{2}\big)^{q/2}}. \tag{8.4}$$

To see this fact let us express (8.3) schematically as

$$\{M,\rho\} = E\otimes E. \tag{8.5}$$

[13] See also [55] for the early development for the exact computation of $Z(N)$ with $N=1$ and 2.

Here we have regarded ρ, M and E respectively as a symmetric matrix with components $(\rho)_{Q,Q'} = \rho(Q,Q')$, $(M)_{Q,Q'} = M(Q)\delta(Q-Q')$ and $(E)_Q = E(Q)$, respectively. The integrations over Q in the power of the density matrix or the trace are regarded as the operations of matrix product. Using (8.5) repetitively, it is easy to derive similar (anti-)commutation relation for ρ^n with $n \geq 2$

$$M\rho^n - (-1)^n \rho^n M = \sum_{m=0}^{n-1} (-1)^m (\rho^m \cdot E) \otimes (\rho^{n-1-m} \cdot E). \tag{8.6}$$

These (anti-)commutation relations imply that, if we define a new set of vectors ψ_m by

$$\begin{aligned} \psi_m(Q) &= \frac{(\rho^m \cdot E)(Q)}{E(Q)} \\ &\left(\equiv \frac{1}{E(Q)} \prod_{i=1}^{n} \int dQ_i \rho(Q,Q_1)\rho(Q_1,Q_2)\cdots\rho(Q_{n-1},Q_n)E(Q_n)\right), \end{aligned} \tag{8.7}$$

the powers ρ^n can be written as

$$\rho^n(Q_1,Q_2) = \frac{E(Q_1)E(Q_2)}{M(Q_1)-(-1)^n M(Q_2)} \sum_{m=0}^{n-1} (-1)^m \psi_m(Q_1)\psi_{n-1-m}(Q_2). \tag{8.8}$$

The efficiency of this formula is worth stressing. When we compute a power of some matrix we need to compute the product of matrices at each step. In our formula, however, we can compute ρ^n just by picking up a specific vector E and multiplying ρ to it. Hence it is expected that our formula substantially simplifies the computation.

Moreover, for $k,q \in \mathbb{N}$ the vectors $\psi_m(Q)$ can be computed by simple iterative steps. From the definition (8.7) we obtain the following recursion relation

$$\psi_{m+1}(Q) = \frac{1}{E(Q)} \int dQ' \rho(Q,Q')E(Q')\psi_m(Q') \tag{8.9}$$

with the initial condition

$$\psi_0(Q) = 1. \tag{8.10}$$

When $k,q \in \mathbb{N}$, we can introduce new integration variable $u = e^{Q/(2k)}$ and rewrite the integration as

$$\psi_{m+1}(u) = \frac{1}{\pi} \int_0^\infty dv \frac{1}{v^2+u^2} \frac{v^{qk+1}}{(v^{2k}+1)^q} \psi_m(v). \tag{8.11}$$

This integration can be simplified by expanding $\psi_m(u)$ in the series of $\log u$

$$\psi_m(u) = \sum_{j\geq 0} \psi_m^{(j)}(u)(\log u)^j, \tag{8.12}$$

where $\psi_m^{(j)}(u)$ are rational functions in u, as [53]

$$\psi_m(u) = -\frac{1}{\pi} \sum_{j\geq 0} \frac{(2\pi i)^{j+1}}{j+1} \sum_{v_a \in \mathbb{C}\backslash\mathbb{R}^+} \text{Res}\left[\frac{1}{v^2+u^2} \frac{v^{qk+1}}{(v^{2k}+1)^q} \psi_m^{(j)}(v) B_{j+1}\left[\frac{\log^{(+)} v}{2\pi i}\right], v \to v_a\right]. \tag{8.13}$$

Here $\log^{(+)}$ is the logarithm function with the branch cut on $\mathbb{R}^+$, i.e.

$$\log^{(+)} z = \begin{cases} \log z & (0 < \mathrm{Arg}(z) < \pi \\ \log z + 2\pi i & (-\pi < \mathrm{Arg}(z) < 0) \end{cases} . \tag{8.14}$$

with log the usually used logarithm with branch cut on $\mathbb{R}^-$, and $B_j(z)$ are the Bernoulli polynomials defined like (6.31). We shall assume $u \in \mathbb{R}^+$ to collect the residue at $v = \pm iu$.

Once ψ_m are computed in this manner, we can obtain $\mathrm{Tr}\,\widehat{\rho}$ by integrating (8.8) with $Q_2 = Q_1$ by the same technique. For odd n we obtain

$$\mathrm{Tr}\widehat{\rho}^n = -\frac{1}{2\pi}\sum_{j\ge 0}\frac{(2\pi i)^{j+1}}{j+1}\sum_{v_a\in\mathbb{C}\setminus\mathbb{R}^+}\mathrm{Res}\Big[\frac{v^{qk-1}}{(v^{2k}+1)^q}f_n^{(j)}(v)B_{j+1}\Big[\frac{\log^{(+)} v}{2\pi i}\Big], v\to v_a\Big], \tag{8.15}$$

while if n is even the result is

$$\mathrm{Tr}\widehat{\rho}^n = -\frac{1}{2\pi}\sum_{j\ge 0}\frac{(2\pi i)^{j+1}}{j+1}\sum_{v_a\in\mathbb{C}\setminus\mathbb{R}^+}\mathrm{Res}\Big[\frac{v^{qk}}{(v^{2k}+1)^q}g_n^{(j)}(v)B_{j+1}\Big[\frac{\log^{(+)} v}{2\pi i}\Big], v\to v_a\Big]. \tag{8.16}$$

Here we have defined the rational functions $f_n^{(j)}(u)$ and $g_n^{(j)}(u)$ as

$$\begin{aligned} \sum_{j\ge 0} f_n^{(j)}(u)(\log u)^j &= \sum_{m=0}^{n-1}(-1)^m\psi_m(u)\psi_{n-m-1}(u), \\ \sum_{j\ge 0} g_n^{(j)}(u)(\log u)^j &= \sum_{m=0}^{n-1}(-1)^m\partial_u\psi_m(u)\psi_{n-m-1}(u). \end{aligned} \tag{8.17}$$

8.1.2 $p = 2$

Similar computation is possible also for the $(q,p)_k$ models with $p = 2$ [10]. In this case we use the Fourier transformation formula

$$\langle Q_1|\frac{1}{(2\cosh\frac{\widehat{P}}{2})^2}|Q_2\rangle = \int\frac{dP}{2\pi k}\frac{e^{i(Q_1-Q_2)P/\hbar}}{\big(2\cosh\frac{P}{2}\big)^2} = \frac{1}{2\pi k^2}\frac{Q_1-Q_2}{2\sinh\frac{Q_1-Q_2}{2k}} \tag{8.18}$$

to rewrite the density matrix as

$$\rho(Q_1,Q_2) = \frac{1}{2\pi}\langle Q_1|\widehat{\rho}|Q_2\rangle = \frac{1}{(2\pi k)^2}\frac{1}{\big(2\cosh\frac{Q_1}{2}\big)^{q/2}}\frac{Q_1-Q_2}{2\sinh\frac{Q_1-Q_2}{2k}}\frac{1}{\big(2\cosh\frac{Q_2}{2}\big)^{q/2}}. \tag{8.19}$$

which has the following structure

$$\rho(Q_1,Q_2) = \frac{(Q_1-Q_2)E(Q_1)E(Q_2)}{M(Q_1)-M(Q_2)} \tag{8.20}$$

with

$$M(Q) = (2\pi k)^2 e^{\frac{Q}{k}}, \quad E(Q) = \frac{e^{\frac{Q}{2k}}}{\big(2\cosh\frac{Q}{2}\big)^{q/2}}. \tag{8.21}$$

Schematically, this result can be rewritten as

$$[M,\rho]=(EQ)\otimes E-E\otimes(EQ). \tag{8.22}$$

This implies that we can follow almost similar iterative steps as in the case of $p=1$; the only difference is that we need to introduce two series of vectors corresponding to E and EQ

$$\phi_m(Q)=\frac{(\rho^m\cdot E)(Q)}{E(Q)},\quad \psi_m(Q)=\frac{(\rho^m\cdot EQ)(Q)}{E(Q)}, \tag{8.23}$$

which are computed recursively as

$$\begin{aligned}\phi_m(Q)&=\int dQ'\frac{1}{E(Q)}\rho(Q,Q')E(Q')\phi_{m-1}(Q'),\quad \phi_0(Q)=1,\\ \psi_m(Q)&=\int dQ'\frac{1}{E(Q)}\rho(Q,Q')E(Q')\psi_{m-1}(Q'),\quad \psi_0(Q)=Q.\end{aligned} \tag{8.24}$$

The powers of the density matrix are written as

$$\rho^n(Q_1,Q_2)=\frac{E(Q_1)E(Q_2)}{M(Q_1)-M(Q_2)}\sum_{m=0}^{n-1}\big[\psi_m(Q_1)\phi_{n-1-m}(Q_2)-\phi_m(Q_1)\psi_{n-1-m}(Q_2)\big], \tag{8.25}$$

and the traces can be computed as

$$\mathrm{Tr}\,\widehat{\rho}^n=\int dQ\frac{E(Q)^2}{dM/dQ}\sum_{m=0}^{n-1}\Big[\frac{d\psi_m(Q)}{dQ}\phi_{n-1-m}(Q)-\frac{d\phi_m(Q)}{dQ}\psi_{n-1-m}(Q)\Big]. \tag{8.26}$$

8.1.3 $p\geq 3$

Now the generalization to the $(q,p)_k$ models with $p\geq 3$ is straightforward [10], with the help of following Fourier transformation formula

$$\begin{aligned}\langle Q_1|\frac{1}{(2\cosh\frac{\widehat{P}}{2})^p}|Q_2\rangle&=\int\frac{dP}{2\pi k}\frac{e^{i(Q_1-Q_2)P/\hbar}}{\big(2\cosh\frac{P}{2}\big)^p}\\ &=\begin{cases}\dfrac{1}{(p-1)!k}\dfrac{1}{2\cosh\frac{Q_1-Q_2}{2k}}\displaystyle\prod_{j=1}^{\frac{p-1}{2}}\Big[\Big(\frac{Q_1-Q_2}{2\pi k}\Big)^2+\frac{(2j-1)^2}{4}\Big] & \text{(for odd } p)\\ \dfrac{Q_1-Q_2}{2\pi(p-1)!k^2}\dfrac{1}{2\sinh\frac{Q_1-Q_2}{2k}}\displaystyle\prod_{j=1}^{\frac{p}{2}-1}\Big[\Big(\frac{Q_1-Q_2}{2\pi k}\Big)^2+j^2\Big] & \text{(for even } p)\end{cases}.\end{aligned} \tag{8.27}$$

They coincide with the Fourier transformation for $p=1,2$ respectively up to some factor of polynomial in Q_1-Q_2. Therefore, with $M(Q)\propto e^{\frac{Q}{k}}$, the quantity

$$M\rho-(-1)^p\rho M \tag{8.28}$$

is written as a linear combination of $(EQ^\ell)\otimes(EQ^{\ell'})$ with $\ell,\ell'\geq 0$ and $\ell+\ell'\leq p-1$. This ensures that $\mathrm{Tr}\,\widehat{\rho}^n$ can be calculated as in the cases of $p=1,2$ by introducing the series of vectors

$$\phi_m^{(\ell)}(Q)=\frac{(\rho^m\cdot EQ^\ell)(Q)}{E(Q)}, \tag{8.29}$$

with $\ell=0,1,\cdots,p-1$.

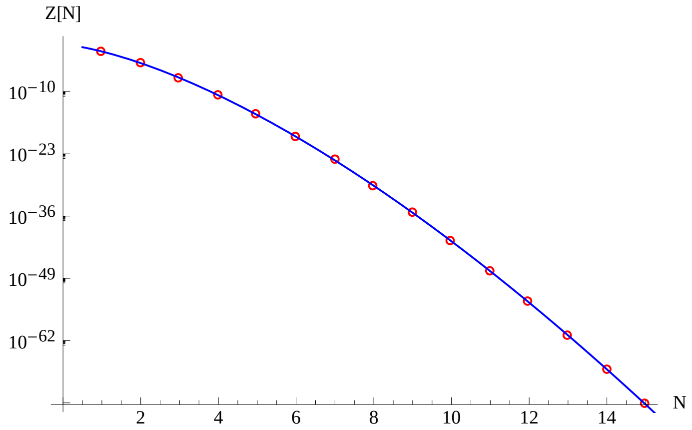

Figure 8: The exact values of $Z(N)$ for $(q,p)_k = (2,2)_1$ model (red circles) and the plot of the Airy function $Z^{\text{pert}}(N)$ (6.1) (blue line).

8.2 Numerical support for A

With the help of the methods explained in the last section we can compute the exact values of the partition function $Z(N)$ for a variety of the $(q,p)_k$ models recursively in N. We have computed the partition function $Z(N)$ with $N = 1, 2, \cdots, N_{\text{max}}$ for various (q,p), where the values of N_{max} are listed in table 1. In appendix C we display the explicit results for $q = p = 2$.

First, let us focus on the constant A in the Airy function expression of the perturbative partition function (6.1). As we can see in figure 8 the exact partition functions can be approximated with high accuracy by the Airy function, and hence we can read off the value of A numerically by fitting. On the other hand, it is known that the A in the ABJM theory is related to the constant map in the topological string and the explicit expression for finite k is available [46, 54] as

$$A_{\text{ABJM}}(k) = \frac{2\zeta(3)}{\pi^2 k}\left(1 - \frac{k^3}{16}\right) + \frac{k^2}{\pi^2}\int_0^\infty dx \frac{x}{e^{kx}-1}\log(1-e^{-2x}), \tag{8.30}$$

with which we can extrapolate the conjectural expression of A (7.37) to finite k. Comparing the values of A obtained in these two ways we find considerable agreement of the results (see table 1).

8.3 Instanton exponents for finite k

Now let us go on to the instanton effects. Subtracting the perturbative part as

$$\delta Z(N) = \frac{Z(N)}{Z^{\text{pert}}(N)} - 1, \tag{8.31}$$

where $Z(N)$ are the exact values and $Z^{\text{pert}}(N)$ is the perturbative part given by (6.1)

$$Z^{\text{pert}}(N) = e^A C^{-\frac{1}{3}}\,\text{Ai}[C^{-\frac{1}{3}}(N-B)] \tag{8.32}$$

$(q,p)_k$	$N_{\max}$	$A((7.37))$	$A(\text{fitting})$	fitting/(7.37)	$\omega(\text{fitting})$	$\frac{2}{p}$	$\frac{4}{kqp}$
$(1,2)_1$	20	0.28567667616	0.28567667513	0.99999999990	1.82	1	2
$(1,2)_2$	13	-0.3103049	-0.3103040	0.999997	0.940	1	1
$(1,3)_2$	12	-0.650390	-0.650343	0.99993	0.616	0.667	0.667
$(2,2)_1$	15	-0.2435877	-0.2435843	0.999986	0.887	1	1
$(2,2)_2$	13	-1.508088	-1.508040	0.99997	0.4992	1	0.5
$(2,2)_3$	6	-3.011	-0.299947	0.996	0.3330	1	0.3333
$(2,2)_4$	7	-4.91	-4.89	0.994	0.2519	1	0.25
$(2,2)_6$	6	-10.06	-9.80	0.97	0.189	1	0.167

Table 1: The values A(fitting) are determined by comparing the perturbative partition function (8.32) with the exact value at $N = N_{\max}$. The values of ω are determined by fitting the exact values of $\log|Z(N_{\max}-1)|$ and $\log|Z(N_{\max})|$ with (8.35).

with the coefficients C (6.37), B (6.40) and A (7.37), we observe that the exact values of the deviation $\delta Z(N)$ behaves as

$$\delta Z(N) \sim e^{-\omega\sqrt{N/C}} \tag{8.33}$$

with some positive constant ω. See figure 9. This behavior indeed corresponds to the instanton effects

$$J^{\mathrm{np}}(\mu) \sim e^{-\omega\mu}, \tag{8.34}$$

as argued in section 7.5. We can estimate the leading instanton exponents by fitting the exact values of $\delta Z(N)$ with the function

$$c_0 \exp\left[-\frac{2}{3}C^{-\frac{1}{2}}(N^{\frac{3}{2}} - (N+\omega)^{\frac{3}{2}})\right], \quad (c_0\text{: constant}) \tag{8.35}$$

as in table 1.

In some cases the exponent ω are found to be inversely proportional to k. These non-perturbative effects can be interpreted as the M2-branes winding on the cycle in $(\mathbb{C}^2/\mathbb{Z}_q \times \mathbb{C}^2/\mathbb{Z}_p)/\mathbb{Z}_k$ which is orbifolded by Z_k. Indeed the values of ω are somehow consistent with the value $4/(qpk)$ estimated by the orbifold [13].[14]

8.4 Instanton coefficients by fitting

After recognizing the instanton exponents

$$e^{-\frac{2\mu}{q}}, \quad e^{-\frac{2\mu}{p}}, \quad e^{-\frac{4\mu}{qpk}}, \tag{8.36}$$

[14]The case of mismatches can be understood to be due to the polynomial instanton coefficients.

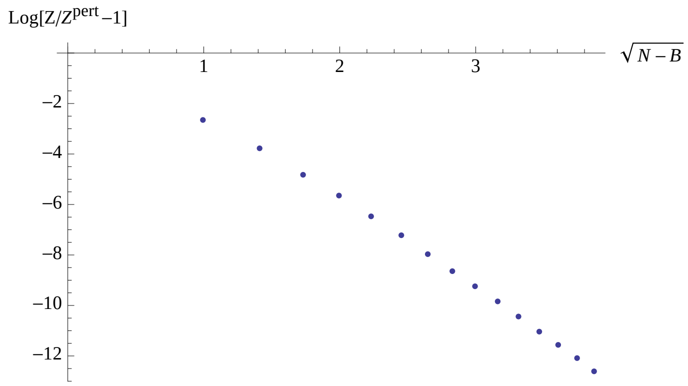

Figure 9: The deviation of the exact values from the Airy function for $(q,p)_k = (2,2)_1$ model.

we can try to read off the instanton coefficients. First we shall write down the more concrete relation between the grand potential and the partition function. Denote the non-perturbative effects in the grand potential $J^{\rm np}(\mu)$ as

$$J^{\rm np}(\mu) = f_1(\mu)e^{-\omega_1\mu} + f_2(\mu)e^{-\omega_2\mu} + \cdots . \tag{8.37}$$

Here $\{\omega_i\}_{i=1}^{\infty}$ $(\omega_1 < \omega_2 < \omega < 3\cdots)$ is the discrete set of positive numbers generated by the three fundamental exponents (8.36).

$$\{\omega_i\}_{i=1}^{\infty} = \left\{ \frac{2\ell_1}{q} + \frac{2\ell_2}{p} + \frac{4\ell_3}{qpk} \middle| \ell_1, \ell_2, \ell_3 \geq 0, \ell_1 + \ell_2 + \ell_3 \geq 1 \right\}. \tag{8.38}$$

The integrand in the inversion relation $J(\mu) \to Z(N)$ (6.25) can be expanded as

$$Z(N) = \int_{-i\infty}^{i\infty} \frac{d\mu}{2\pi i} e^{J^{\rm pert}(\mu) - \mu N}(1 + g_1(\mu)e^{-\omega_1\mu} + g_2(\mu)e^{-\omega_2\mu} + \cdots) \tag{8.39}$$

where $g_i(\mu)$ are again polynomials in μ generated by

$$e^{\sum_i f_i(\mu)e^{-\omega_i\mu}} = 1 + \sum_i g_i(\mu)e^{-\omega_i\mu}. \tag{8.40}$$

We can convert the μ dependence of each term in the expansion (8.39) through the term $e^{-\mu N}$ to obtain

$$Z(N) = Z^{\rm pert}(N) + g_1(-\partial_N)Z^{\rm pert}(N+\omega_1) + g_2(-\partial_N)Z^{\rm pert}(N+\omega_2) + \cdots . \tag{8.41}$$

We compare the logarithm of this expansion expression with the exact values. Omitting the terms with higher ω_i which are highly suppressed, we have only a small number of unknown coefficients, which we can determine by fitting.

Let us demonstrate the process of fitting for $(q,p)_k$ model with $q = p = 2$ and $k = 1$, where the exact values are available for $1 \leq N \leq N_{\rm max} = 15$. Since the three instanton exponents (8.36) coincide to $e^{-\mu}$, the non-perturbative part of the grand potential can be expanded as

$$J^{\rm np}(\mu) = f_1(\mu)e^{-\mu} + f_2e^{-2\mu} + \cdots . \tag{8.42}$$

The number of unknowns to be determined by fitting must be smaller than or equal to the number of data points. Assuming $f_i(\mu)$ are at most quadratic[15]

$$f_1(\mu) = \alpha_1\mu^2 + \beta_1\mu + \gamma_1, \tag{8.43}$$

we can take into account at most $[N_{\rm max}/3] = 5$ distinct instantons in (8.37).

First let us take into account $f_i(\mu)$ with $i \leq 2$ which generate $g_1(\mu)$ and $g_2(\mu)$ as

$$g_1(\mu) = f_1, \quad g_2(\mu) = f_2 + \frac{f_1^2}{2}. \tag{8.44}$$

Then we find by fitting the exact values of $\log|Z(N)|$ as[16]

$$\begin{aligned}
\alpha_1 &\to 0.4052855304342130\cdots, \\
\beta_1 &\to 0.4052620261973124\cdots, \\
\gamma_1 &\to 0.4054490736521232\cdots, \\
\alpha_2 &\to -2.6981751787051466\cdots, \\
\beta_2 &\to 0.8972468610591679\cdots, \\
\gamma_2 &\to -2.1703751084209262\cdots.
\end{aligned} \tag{8.45}$$

This value of α_1 is found to be $4/\pi^2$ with high precision:

$$\alpha_1/\left(\frac{4}{\pi^2}\right) = 1.00000196\cdots. \tag{8.46}$$

As the effects of higher instantons are suppressed for large N, the error of fitting would be expected to be minimized if we omit the data with small N. Indeed, if we use only the data with $N = 10, 11, 12, 13, 14, 15$ we obtain the coincidence (8.46) with improved precision

$$\alpha_1/\left(\frac{4}{\pi^2}\right) = 0.99999999976\cdots. \tag{8.47}$$

Hence we shall conclude α_1 is exactly $4/\pi^2$.

Fitting the remaining five unknowns with more restrictive data $N = 11, 12, 13, 14, 15$ we obtain

$$\begin{aligned}
\beta_1 &\to 0.4052847346628331\cdots, \\
\gamma_1 &\to 0.4052847326530695\cdots, \\
\alpha_2 &\to -2.6309461188679967\cdots, \\
\beta_2 &\to -0.2173534321981312\cdots, \\
\gamma_2 &\to 2.5502103476881299\cdots,
\end{aligned}$$

[15]We may tentatively include the cubic term. These coefficients are, however, fitted to be extremely small numbers, say $\mathcal{O}(10^{-6})$.

[16]We have used the "FindFit" command in Mathematica with the option "MaxExtraPrecision→300".

where the value of β_1 coincide with $4/\pi^2$ with higher precision

$$\beta_1/\left(\frac{4}{\pi^2}\right) = 1.00000000023\cdots . \tag{8.48}$$

After determining the coefficients of the lower order instantons we can include the higher order terms $f_i(\mu)e^{-\omega_i\mu}$ in (8.37) and continue the fitting process. The exact values are summarized in table 2 in the next section.

8.5 Discussion and Comments

In this section we have introduced the systematic computation of the exact values of the partition function. The exact values strongly support our conjecture for A (7.37). Once we accept the conjecture, then the exact values provide the information on the instanton effects for finite k. By the method of fitting we can determine the instanton coefficients in the grand potential order by order in the instanton expansion (8.37).

The determination of the instanton coefficients might seems to be ambiguous. However, if we substitute a wrong value to the coefficient at some step, the fitting in the next order requires the fine-tuning of huge values of the fitting parameter, which indicates the fitting is failed. The fitting would be successful only the lower coefficients hit the correct values, hence the determination is nevertheless plausible.

On the other hand there are also many disadvantages in our method. The determination of the exact coefficient become more difficult for higher order coefficients. Actually, in the case of $(q,p)_k = (2,2)_2$ the rational coefficients in table 2 are getting more complicated as we go to the higher order instantons. The determination becomes also difficult as q, p and k become larger, since the instantons are less suppressed in such cases. Especially for large values of q or p, each instanton coefficient is generically a complicated ratio of the Gamma functions even at the level of small k expansion, as obtained in section 7.4. In these cases it would be extremely difficult to guess the exact coefficients from the results of fitting.

In the next section we focus on the cases of $(q,p) = (2,2)$ and try to determine the exact k-dependence of the instanton coefficients. Fortunately these models avoid the problems listed above. In these cases the unit of the instanton exponents are not so small and the instanton coefficients in small k expansion are simple rational numbers modulo π (see (7.64), and (9.3) and (9.10) in the next section). We successfully determine the first few coefficients for $k = 1,2,3,4,6$. Moreover we observe a new structure of k-dependence in these leading coefficients. Once we have the idea of exact expression of the whole instanton expansion, we do not need to guess the coefficients from the numerical values obtained by fitting but rather can use them to check our conjecture.

9 Completely Exact results for $(2,2)$ model and topological string

In the preceding sections we have investigated the qualitative aspects of the instanton effects in various ways. In this section we try to determine the instanton effects for finite k quantitatively. Although we can analyze the grand potential for $k = 1, 2, \cdots$ or around $k = 0$, there are no general technique to study the grand potential directly for general finite k. Hence our method will be the interpolation from $k = 1, 2, \cdots$ and the extrapolation from the small k expansion.

It is reasonable to restrict ourselves to the *simplest* theories among the general $(q, p)_k$ models. From the observation in the small k expansion, the instanton effects seem to simplify at $q = p = 1$ or $q = p = 2$. In these cases the three kinds of the membrane instanton completely degenerate together. Therefore there is only one series of the membrane instanton whose coefficients are uniformly quadratic in μ.[17] Indeed the complete determination of the instanton coefficients was successful for $q = p = 1$ (ABJM theory) in [56] based on [57, 58, 59, 60] and for $q = p = 2$ in [10].

In this section we especially focus on the case $q = p = 2$. From the arguments in the last two sections, the membrane instanton exponent in the $(2,2)_k$ model is $e^{-\mu}$ and that of the worldsheet instanton is $e^{-\mu/k}$. Hence the whole non-perturbative part of the grand potential will be expanded as follows

$$J^{\text{np}}(\mu) = \sum_{\ell,m\geq 0((\ell,m)\neq(0,0))} f_{\ell,m}(\mu) e^{-(\ell+\frac{m}{k})\mu} \tag{9.1}$$

with $f_{\ell,m}(\mu)$ some polynomials in μ. Note that the non-perturbative effects consist not only of the pure membrane instantons ($m = 0$) and the pure worldsheet instanton ($\ell = 0$), but also of the possible bound states of them ($\ell, m \neq 0$). Below we will denote the coefficients of the pure membrane/worldsheet instantons as

$$f_{\ell,0}(\mu) = a_\ell \mu^2 + b_\ell \mu + c_\ell, \quad f_{0,m}(\mu) = d_m(\mu). \tag{9.2}$$

Once we postulate this expression of the instanton expansion we can determine the explicit values of the instanton coefficients at $k = 1, 2, 3, 4, 6$ for small ℓ, m by fitting the exact values of the partition functions $Z(N)$ (C.1) as table 2. In the following sections we use these discrete data together with the data of the small k expansion to determine the instanton coefficients as the functions of k

9.1 Membrane instanton by extrapolation

First let us consider pure membrane instantons. Since the worldsheet instantons and the bound states are the non-perturbative effects in k we can neglect both of them in the small k expansion

[17] One might think that the $q = p \geq 3$ are also simple at the same extent since the ghost instantons $e^{-\mu}$ never produce independent exponents. In these cases, however, the degrees of the instanton coefficients are not uniform: they can be linear or quadratic depending on whether the ghost instanton contribute or not. Actually the differential relation (9.12) in the membrane instanton coefficients do not hold in these cases and we can not determine the instanton coefficients with simple ansatz.

$$
\begin{aligned}
J^{\mathrm{np}}_{k=1} &= \frac{4\mu^2+4\mu+4}{\pi^2}e^{-\mu} + \left[-\frac{26\mu^2+\mu+9/2}{\pi^2}+2\right]e^{-2\mu} \\
&+ \left[\frac{736\mu^2-608\mu/3+616/9}{3\pi^2}-32\right]e^{-3\mu} \\
&+ \left[-\frac{2701\mu^2-13949\mu/12+11291/48}{\pi^2}+466\right]e^{-4\mu} \\
&+ \left[\frac{161824\mu^2-1268488\mu/15+1141012/75}{5\pi^2}-6720\right]e^{-5\mu} \\
&+ \left[-\frac{1227440\mu^2-10746088\mu/15+631257/5}{3\pi^2}+\frac{292064}{3}\right]e^{-6\mu} \\
&+ \left[\frac{37567744\mu^2-2473510336\mu/105+9211252832/2205}{7\pi^2}-1420800\right]e^{-7\mu} \\
&+ \mathcal{O}(e^{-8\mu}), \\
J^{\mathrm{np}}_{k=2} &= 4e^{-\frac{1}{2}\mu} + \left[\frac{2\mu^2+2\mu+2}{\pi^2}-6\right]e^{-\mu} + \frac{16}{3}e^{-\frac{3}{2}\mu} + \left[-\frac{13\mu^2+\mu/2+9/4}{\pi^2}-14\right]e^{-2\mu} \\
&+ \frac{544}{5}e^{-\frac{5}{2}\mu} + \left[\frac{368\mu^2-304\mu/3+308/9}{3\pi^2}-288\right]e^{-3\mu} - \frac{640}{7}e^{-\frac{7}{2}\mu} + \mathcal{O}(e^{-4\mu}), \\
J^{\mathrm{np}}_{k=3} &= \frac{16}{3}e^{-\frac{1}{3}\mu} - 4e^{-\frac{2}{3}\mu} + \left[\frac{4\mu^2+4\mu+4}{3\pi^2}+\frac{128}{9}\right]e^{-\mu} - \frac{613}{9}e^{-\frac{4}{3}\mu} + \frac{3536}{15}e^{-\frac{5}{3}\mu} \\
&+ \left[-\frac{26\mu^2+\mu+9/2}{3\pi^2}-\frac{7318}{9}\right]e^{2\mu} + \frac{544352}{189}e^{-\frac{7}{3}\mu} + \mathcal{O}(e^{-\frac{8}{3}\mu}), \\
J^{\mathrm{np}}_{k=4} &= 8e^{-\frac{1}{4}\mu} - 8e^{-\frac{1}{2}\mu} + \frac{80}{3}e^{-\frac{3}{4}\mu} + \left[\frac{\mu^2+\mu+1}{\pi^2}-96\right]e^{-\mu} + \frac{1888}{5}e^{-\frac{5}{4}\mu} - \frac{4736}{3}e^{-\frac{3}{2}\mu} \\
&+ \frac{44416}{7}e^{-\frac{7}{4}\mu} + \mathcal{O}(e^{-2\mu}), \\
J^{\mathrm{np}}_{k=6} &= 16e^{-\frac{1}{6}\mu} - \frac{52}{3}e^{-\frac{1}{3}\mu} + \frac{148}{3}e^{-\frac{1}{2}\mu} - 189e^{-\frac{2}{3}\mu} + \frac{4336}{5}e^{-\frac{5}{6}\mu} \\
&+ \left[\frac{2\mu^2+2\mu+2}{3\pi^2}-\frac{38102}{9}\right]e^{-\mu} + \frac{446032}{21}e^{-\frac{7}{6}\mu} + \mathcal{O}(e^{-\frac{4}{3}\mu}).
\end{aligned}
$$

Table 2: Instanton effects in the $(2,2)_k$ model found by fitting the exact values of the partition function (C.1)

of the grand potential. Hence the instanton coefficients obtained in section 7 directly correspond to the small k expansion of the membrane instanton coefficients $f_{\ell,0}(\mu)$.

From the results in section 7 we find the explicit small k expansion of the quadratic part a_ℓ

$$
\begin{aligned}
a_1 &= \frac{2}{\pi^2 k} + \mathcal{O}(k^9), \\
a_2 &= -\frac{9}{\pi^2 k} + 2k - \frac{2\pi^2 k^3}{3} + \frac{4\pi^4 k^5}{45} - \frac{2\pi^6 k^7}{315} + \mathcal{O}(k^9),
\end{aligned}
$$

$$
\begin{aligned}
a_3 &= \frac{200}{3\pi^2 k} - 32k + \frac{32\pi^2 k^3}{3} - \frac{64\pi^4 k^5}{45} + \frac{32\pi^6 k^7}{315} + \mathcal{O}(k^9), \\
a_4 &= -\frac{1225}{2\pi^2 k} + 500k - \frac{752\pi^2 k^3}{3} + \frac{704\pi^4 k^5}{9} - \frac{5792\pi^6 k^7}{315} + \mathcal{O}(k^9), \\
a_5 &= \frac{31752}{5\pi^2 k} - 7840k + \frac{17440\pi^2 k^3}{3} - \frac{26944\pi^4 k^5}{9} + \frac{10592\pi^6 k^7}{9} + \mathcal{O}(k^9), \\
a_6 &= -\frac{71148}{\pi^2 k} + 123480k - 128968\pi^2 k^3 + \frac{296272\pi^4 k^5}{3} - \frac{901624\pi^6 k^7}{3} + \mathcal{O}(k^9), \\
&\vdots
\end{aligned} \tag{9.3}
$$

As we have seen in section 7.7 the linear part and the constant part of the instanton coefficients are slightly simplified by the introduction of the effective chemical potential. First let us introduce the following notations

$$
J_a = \sum_{\ell=1}^{\infty} a_\ell e^{-\ell\mu}, \quad J_b = \sum_{\ell=1}^{\infty} b_\ell e^{-\ell\mu}, \quad J_c = \sum_{\ell=1}^{\infty} c_\ell e^{-\ell\mu}, \tag{9.4}
$$

to rewrite the semiclassical grand potential as

$$
J(\mu) = \frac{C}{3}\mu^3 + B\mu + A + J_a\mu^2 + J_b\mu + J_c. \tag{9.5}
$$

Then if we introduce the effective chemical potential as

$$
\mu_{\text{eff}} = \mu + \frac{1}{C}\sum_{\ell\geq 1} a_\ell e^{-\ell\mu}, \tag{9.6}
$$

then the quadratic part J_a of the membrane instantons is absorbed into the perturbative part

$$
J(\mu) = \frac{C}{3}\mu_{\text{eff}}^3 + B\mu_{\text{eff}} + A + \widetilde{J}_b\mu_{\text{eff}} + \widetilde{J}_c \tag{9.7}
$$

where $\widetilde{J}_b$ and $\widetilde{J}_c$ are slightly modified from the original components J_b and J_c as

$$
\widetilde{J}_b = J_b - \frac{J_a^2}{C}, \quad \widetilde{J}_c = J_c - \frac{BJ_a}{C} - \frac{J_aJ_b}{C} + \frac{2J_a^3}{C^2}. \tag{9.8}
$$

Introducing the instanton coefficients $\widetilde{b}_\ell$ and $\widetilde{c}_\ell$ for $\widetilde{J}_b$ and $\widetilde{J}_c$

$$
\widetilde{J}_b = \sum_{\ell\geq 1} \widetilde{b}_\ell e^{-\ell\mu_{\text{eff}}}, \quad \widetilde{J}_c = \sum_{\ell\geq 1} \widetilde{c}_\ell e^{-\ell\mu_{\text{eff}}}, \tag{9.9}
$$

we finally obtain the explicit small k expansion of these coefficients as

$$
\begin{aligned}
\widetilde{b}_1 &= -\frac{4}{\pi^2 k} + \frac{4k}{3} + \frac{4\pi^2 k^3}{45} + \frac{8\pi^4 k^5}{945} + \frac{4\pi^6 k^7}{4725} + \mathcal{O}(k^9), \\
\widetilde{b}_2 &= \frac{9}{\pi^2 k} - 6k + \frac{14\pi^2 k^3}{5} - \frac{44\pi^4 k^5}{105} + \frac{18\pi^6 k^7}{175} + \mathcal{O}(k^9), \\
\widetilde{b}_3 &= -\frac{328}{9\pi^2 k} + \frac{184k}{3} - \frac{152\pi^2 k^3}{5} + \frac{752\pi^4 k^5}{105} + \frac{8\pi^6 k^7}{525} + \mathcal{O}(k^9),
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{b}_4 &= \frac{777}{4\pi^2 k} - 598k + \frac{9704\pi^2 k^3}{15} - \frac{16448\pi^4 k^5}{45} + \frac{202304\pi^6 k^7}{1575} + \mathcal{O}(k^9), \\
\widetilde{b}_5 &= -\frac{30004}{25\pi^2 k} + \frac{18004k}{3} - \frac{96700\pi^2 k^3}{9} + \frac{1957000\pi^4 k^5}{189} - \frac{169748\pi^6 k^7}{27} + \mathcal{O}(k^9), \\
\widetilde{b}_6 &= \frac{8146}{\pi^2 k} - 60732k + \frac{835836\pi^2 k^3}{5} - \frac{26743288\pi^4 k^5}{105} + \frac{18972788\pi^6 k^7}{75} + \mathcal{O}(k^9), \\
&\vdots
\end{aligned}
\tag{9.10}
$$

and

$$
\begin{aligned}
\widetilde{c}_1 &= -\frac{8}{\pi^2 k} - \frac{8\pi^2 k^3}{45} - \frac{32\pi^4 k^5}{945} - \frac{8\pi^6 k^7}{1575} + \mathcal{O}(k^9), \\
\widetilde{c}_2 &= \frac{9}{\pi^2 k} - \frac{14\pi^2 k^3}{5} + \frac{88\pi^4 k^5}{105} - \frac{54\pi^6 k^7}{175} + \mathcal{O}(k^9), \\
\widetilde{c}_3 &= -\frac{656}{27\pi^2 k} + \frac{304\pi^2 k^3}{15} - \frac{3008\pi^4 k^5}{315} - \frac{16\pi^6 k^7}{525} + \mathcal{O}(k^9), \\
&\vdots
\end{aligned}
\tag{9.11}
$$

Interestingly, we observe that the $\widetilde{c}_\ell$ are always written as the derivative of $\widetilde{b}_\ell$ as

$$
\widetilde{c}_\ell = -\frac{k^2}{\ell}\frac{d}{dk}\left(\frac{\widetilde{b}_\ell}{k}\right). \tag{9.12}
$$

This relation hold at least up to $\mathcal{O}(k^9)$ while can be shown for arbitrary ℓ from the exact grand potential at each order in small k expansion. If we believe that this relation exact for finite k, the unknown membrane instanton coefficients are only a_ℓ and $\widetilde{b}_\ell$.

Let us try to determine the instanton coefficients $a_\ell(k)$ and $\widetilde{b}_\ell(k)$ for finite k by the extrapolation. First note that our explicit result of the small k expansion indicates that πa_ℓ and $\pi\widetilde{b}_\ell$ always have rational coefficients in terms of $\hbar = 2\pi k$, as argued in section 7.7. To decide the ansatz for $\widetilde{b}_\ell$ we further consult the pole cancellation mechanism among the instanton effects. In section 7.6 we have discovered that the coefficients of the membrane instantons are singular as the function of (q,p) at the points where the instanton exponent degenerate with each other. It would be reasonable to expect that the similar structure also exist between the membrane instanton and the worldsheet instantons. That is, the membrane instanton coefficients should be singular at k where the membrane instanton exponent $e^{-\ell\mu}$ degenerates with that of some worldsheet instanton $e^{-\frac{m\mu}{k}}$. Since the worldsheet instanton number m can be arbitrary positive integer, we conclude that the ℓ-th membrane instanton coefficient should be singular at $k \in \mathbb{N}/\ell$. Reflecting this infinite series of poles, we shall postulate the following "trigonometric" ansatz for $\widetilde{b}_\ell$

$$
\widetilde{b}_\ell(k) = \frac{1}{\pi \sin \ell\pi k}\sum_{a\geq 0}\alpha_a \cos a\pi k, \tag{9.13}
$$

with α_a some constants.

Next consider the quadratic part $a_\ell(k)$. In section 7.6 we have also observed that the degree of the instanton coefficient increases at the pole cancellation. Hence the instanton coefficient would be a cubic polynomial in μ, if a_ℓ would also diverges at $k \in \mathbb{N}/\ell$. Since we find at most quadratic coefficients in the exact coefficients at $k \in \mathbb{N}$ obtained by fitting table 2, we shall suppose $a_\ell(k)$ to be finite at $k \in \mathbb{N}/\ell$ and pose the following ansatz

$$a_\ell(k) = \frac{1}{\pi^2 k} \sum_{a \geq 0} \alpha_a \cos a\pi k \tag{9.14}$$

again with some constants (independent of those in $\widetilde{b}_\ell$ (9.13)).

Now we can find the following expression for $a_\ell(k)$ from the results of the small k expansion (9.3) as

$$\begin{aligned}
a_1 &= \frac{2}{\pi^2 k},\\
a_2 &= -\frac{8+\cos 2\pi k}{\pi^2 k},\\
a_3 &= \frac{152+48\cos 2\pi k}{3\pi^2 k},\\
a_4 &= -\frac{788+416\cos 2\pi k+21\cos 4\pi k}{2\pi^2 k},\\
a_5 &= \frac{17352+12800\cos 2\pi k+1520\cos 4\pi k+80\cos 6\pi k}{5\pi^2 k},\\
a_6 &\overset{?}{=} -\frac{100538+92604\cos 2\pi k+18264\cos 4\pi k+1864\cos 6\pi k+174\cos 8\pi k}{3\pi^2 k},
\end{aligned} \tag{9.15}$$

(annotation "?" for a_6 will be commented shortly) while for $\widetilde{b}_\ell$ we can determine

$$\begin{aligned}
\widetilde{b}_1 &= \frac{-4\cos \pi k}{\pi \sin \pi k},\\
\widetilde{b}_2 &= \frac{9+8\cos 2\pi k+\cos 4\pi k}{\pi \sin 2\pi k},\\
\widetilde{b}_3 &= \frac{-4(45\cos \pi k+28\cos 3\pi k+9\cos 5\pi k)}{3\pi \sin 3\pi k}.
\end{aligned} \tag{9.16}$$

Remarkably, we find that these expression $\widetilde{b}_\ell$ enjoy the following multi-covering structures

$$\widetilde{b}_1 = \beta_1(k), \quad \widetilde{b}_2 = \frac{1}{2}\beta_1(2k)+\beta_2(k), \quad \widetilde{b}_3 = \frac{1}{3}\beta_1(3k)+\beta_3(k). \tag{9.17}$$

Here β_ℓ are trigonometric and even simpler functions rather than the original coefficients $\widetilde{b}_\ell$

$$\beta_1(k) = -\frac{4}{\pi \sin \pi k}, \quad \beta_2(k) = -\frac{9\cos \pi k+\cos 3\pi k}{\pi \sin \pi k}, \quad \beta_3(k) = -\frac{24\cos \pi k+12\cos 3\pi k}{\pi \sin \pi k}. \tag{9.18}$$

If we further postulate similar structure for $\widetilde{b}_4$, $\widetilde{b}_5$ and $\widetilde{b}_6$

$$\begin{aligned}
&\widetilde{b}_4 = \frac{1}{4}\beta_1(4k)+\frac{1}{2}\beta_2(2k)+\beta_4(k), \quad \widetilde{b}_5 = \frac{1}{5}\beta_1(5k)+\beta_5(k),\\
&\widetilde{b}_6 = \frac{1}{6}\beta_1(6k)+\frac{1}{3}\beta_2(3k)+\frac{1}{2}\beta_3(2k)+\beta_6(k),
\end{aligned} \tag{9.19}$$

and the following ansatz for β_ℓ

$$\beta_\ell = \frac{1}{\pi \sin \pi k} \sum_{a \geq 1} \alpha_a \cos(2a-1)\pi k, \tag{9.20}$$

we can also determine β_4 and β_5 as

$$\begin{aligned}
\beta_4(k) &= \frac{96 \cos \pi k + 78 \cos 3\pi k + 18 \cos 5\pi k}{\pi \sin \pi k}, \\
\beta_5(k) &= -\frac{480 \cos \pi k + 440 \cos 3\pi k + 240 \cos 5\pi k + 40 \cos 7\pi k}{\pi \sin \pi k}, \\
\beta_6(k) &\overset{?}{=} \frac{2841 \cos \pi k + 2307 \cos 3\pi k + 2166 \cos 5\pi k + 672 \cos 7\pi k + 168 \cos 9\pi k}{\pi \sin \pi k}.
\end{aligned} \tag{9.21}$$

In the above results the extrapolation of each instanton coefficients is achieved with a few number of the profile functions as (9.13), (9.14) and (9.20). The number of the profile coefficients α_a required for each $a_\ell(k)$ and $\beta_\ell(k)$ (except $a_6(k)$ and $\beta_6(k)$) are always smaller than 5, the number of the data. We also observe that the coefficients are quite simple: $\alpha_a \in (-1)^{\ell-1}\mathbb{Z}_+/\ell$ in a_ℓ and $\alpha_a \in (-1)^\ell \mathbb{Z}_+$ for β_ℓ. These observations would be nevertheless nontrivial evidences to support our conjecture (9.15), (9.18) and (9.21). Especially, the uniform signs would imply that we have not relied on the "fine tuning" to make up the data of the small k expansion.

Unfortunately, however, we can not continue our determination for higher instanton coefficients. The number of profile functions required for extrapolation increases as ℓ becomes larger. Indeed we have found that a_6 and β_6 requires five nonzero coefficients α_a. Since the small k expansion only provides five linear equations for each set of α_a these determination are less plausible in such cases. For higher instantons our strategy completely collapse as the data never fix the profile coefficients uniquely.

Also notice that the determination of the instanton coefficients above are completely heuristic. Of course our ansatz (9.13) and (9.14) are not the unique choice for the requirements of the pole cancellations at all. Indeed there is another reason for our choice: we expect that the whole structure of the instanton is an natural, not so complicated, generalization of that in the ABJM case priorly determined in [56]. Even after accepting the restrictive ansatz (9.13) and (9.14) one may still doubt whether it is reasonable to determine the entire functions in k just from the first five coefficients in the small k expansions. As we press on, however, we will encounter an increasing number of non-trivial checks in our analysis.

9.2 Effective chemical potential for finite k

If we replace $\mu \to \mu_{\mathrm{eff}}$ in the worldsheet instanton effect $e^{-\mu/k}$, it produces the contribution in the form of the bound states

$$e^{-\frac{m\mu_{\mathrm{eff}}}{k}} = e^{-\frac{m\mu}{k}} + \sum_{\ell \geq 1} \left[\frac{m}{C} a_\ell + \frac{m^2}{2C^2} \sum_{\ell', \ell'' \geq 1 (\ell' + \ell'' = \ell)} a_\ell a_{\ell'} + \cdots \right] e^{-(\ell + \frac{m}{k})\mu}. \tag{9.22}$$

Hence the effective chemical potential possibly explains the bound states in the instanton effects of the grand potential (9.1). In the ABJM theory all the effective chemical potential indeed incorporates all the bound states in the grand potential [60, 56]. It motivate us to use the effective chemical potential μ_{eff} as the fundamental variable of the grand potential for finite k, not only in the small k expansion.

If we consider only the instantons effects $e^{-\omega\mu}$ with exponent $\omega \leq 5 + \frac{1}{k}$, our determination of a_ℓ for $\ell \leq 5$ (9.15) are enough to rewrite the data of the exact instanton expansion in table 2 in terms of μ_{eff}. For $k \in \mathbb{N}$, however, we can completely determine μ_{eff} including all order nonperturbative effects by a similar trick as in the ABJM case [60]. In (9.15) we have found that not only the trigonometric ansatz (9.14) works but also that the arguments of the cosine functions in the numerator are always in $2\pi k\mathbb{Z}_+$. If these structures hold in the higher membrane instantons as well, it follows from the periodicity of the cosine function that

$$a_\ell = \frac{(ka_\ell)\big|_{k=0}}{k} \quad (k \in \mathbb{N}). \tag{9.23}$$

We know the r.h.s. exactly from the classical limit of the grand potential (7.64). After the substitution of $q = p = 2$ we obtain

$$\sum_{\ell\geq 1}(ka_\ell)|_{k=0}e^{-\ell\mu} = J^{\text{np}}_{0,a} = \frac{2e^{-\mu}}{\pi^2}{}_4F_3\Big(1,1,\frac{3}{2},\frac{3}{2};2,2,2;-16e^{-\mu}\Big). \tag{9.24}$$

Hence we obtain the exact expression of the effective chemical potential for integral k

$$\mu_{\text{eff}} = \mu + 4e^{-\mu}{}_4F_3\Big(1,1,\frac{3}{2},\frac{3}{2};2,2,2;-16e^{-\mu}\Big). \tag{9.25}$$

If we express the instanton expansion in table 2 with μ_{eff}, redefining the non-perturbative part of the grand potential as

$$J^{\text{np}}(\mu_{\text{eff}}) = J - \frac{C}{3}\mu_{\text{eff}}^3 - B\mu_{\text{eff}} - A, \tag{9.26}$$

the instanton coefficients of $1/\pi^2$ become somewhat simpler

$$J^{\text{np}}_k = \sum_{\ell=1}^{\infty}\frac{f_{k,\ell}}{\pi^2}\bigg(\frac{\ell^2}{2}\mu_{\text{eff}}^2 + \ell\mu_{\text{eff}} + 1\bigg)e^{-\ell\mu_{\text{eff}}} + \sum_{m=1}^{\infty}g_{k,m}e^{-m\frac{\mu_{\text{eff}}}{k}}, \tag{9.27}$$

with $f_{k,\ell}$ and $g_{k,m}$ being rational numbers (see table 3).

9.3 Worldsheet instanton

Now let us try to determine the k dependence of the worldsheet instanton coefficients $\widetilde{d}_\ell$ in

$$J^{\text{np}}(\mu_{\text{eff}}) = \sum_{\ell\geq 1}(\widetilde{b}_\ell\mu_{\text{eff}} + \widetilde{c}_\ell)e^{-\ell\mu_{\text{eff}}} + \sum_{m\geq 1}\widetilde{d}_m e^{-\frac{\ell\mu_{\text{eff}}}{k}} + \cdots. \tag{9.28}$$

$$
\begin{aligned}
J^{\rm np}_{k=1} &= \frac{2(\mu_{\rm eff}^2+2\mu_{\rm eff}+2)}{\pi^2}e^{-\mu_{\rm eff}} + \left[-\frac{9(2\mu_{\rm eff}^2+2\mu_{\rm eff}+1)}{2\pi^2}+2\right]e^{-2\mu_{\rm eff}} \\
&\quad + \left[\frac{164(9\mu_{\rm eff}^2+6\mu_{\rm eff}+2)}{27\pi^2}-16\right]e^{-3\mu_{\rm eff}} \\
&\quad + \left[-\frac{777(8\mu_{\rm eff}^2+4\mu_{\rm eff}+1)}{16\pi^2}+138\right]e^{-4\mu_{\rm eff}} \\
&\quad + \left[\frac{15002(25\mu_{\rm eff}^2+10\mu_{\rm eff}+2)}{125\pi^2}-1216\right]e^{-5\mu_{\rm eff}} \\
&\quad + \left[-\frac{4073(18\mu_{\rm eff}^2+6\mu_{\rm eff}+1)}{3\pi^2}+\frac{32852}{3}\right]e^{-6\mu_{\rm eff}} \\
&\quad + \left[\frac{1445404(49\mu_{\rm eff}^2+14\mu_{\rm eff}+2)}{343\pi^2}-100272\right]e^{-7\mu_{\rm eff}} + \mathcal{O}(e^{-8\mu_{\rm eff}}), \\
J^{\rm np}_{k=2} &= 4e^{-\frac{1}{2}\mu_{\rm eff}} + \left[\frac{\mu_{\rm eff}^2+2\mu_{\rm eff}+2}{\pi^2}-7\right]e^{-\mu_{\rm eff}} + \frac{40}{3}e^{-\frac{3}{2}\mu_{\rm eff}} \\
&\quad + \left[-\frac{9(2\mu_{\rm eff}^2+2\mu_{\rm eff}+1)}{4\pi^2}-\frac{75}{2}\right]e^{-2\mu_{\rm eff}} + \frac{724}{5}e^{-\frac{5}{2}\mu_{\rm eff}} \\
&\quad + \left[\frac{82(9\mu_{\rm eff}^2+6\mu_{\rm eff}+2)}{27\pi^2}-\frac{1318}{3}\right]e^{-3\mu_{\rm eff}} + \frac{7704}{7}e^{-\frac{7}{2}\mu_{\rm eff}} + \mathcal{O}(e^{-4\mu_{\rm eff}}), \\
J^{\rm np}_{k=3} &= \frac{16}{3}e^{-\frac{1}{3}\mu_{\rm eff}} - 4e^{-\frac{2}{3}\mu_{\rm eff}} + \left[\frac{2(\mu_{\rm eff}^2+2\mu_{\rm eff}+2)}{3\pi^2}+\frac{112}{9}\right]e^{-\mu_{\rm eff}} - 61e^{-\frac{4}{3}\mu_{\rm eff}} \\
&\quad + \frac{3376}{15}e^{-\frac{5}{3}\mu_{\rm eff}} + \left[-\frac{3(2\mu_{\rm eff}^2+2\mu_{\rm eff}+1)}{2\pi^2}-\frac{2266}{3}\right]e^{-2\mu_{\rm eff}} + \frac{52880}{21}e^{-\frac{7}{3}\mu_{\rm eff}} \\
&\quad + \mathcal{O}(e^{-\frac{8}{3}\mu_{\rm eff}}), \\
J^{\rm np}_{k=4} &= 8e^{-\frac{1}{4}\mu_{\rm eff}} - 8e^{-\frac{1}{2}\mu_{\rm eff}} + \frac{80}{3}e^{-\frac{3}{4}\mu_{\rm eff}} + \left[\frac{\mu_{\rm eff}^2+2\mu_{\rm eff}+2}{2\pi^2}-\frac{197}{2}\right]e^{-\mu_{\rm eff}} + \frac{1928}{5}e^{-\frac{5}{4}\mu_{\rm eff}} \\
&\quad - \frac{4784}{3}e^{-\frac{3}{2}\mu_{\rm eff}} + \frac{44976}{7}e^{-\frac{7}{4}\mu_{\rm eff}} + \mathcal{O}(e^{-2\mu_{\rm eff}}), \\
J^{\rm np}_{k=6} &= 16e^{-\frac{1}{6}\mu_{\rm eff}} - \frac{52}{3}e^{-\frac{1}{3}\mu_{\rm eff}} + \frac{148}{3}e^{-\frac{1}{2}\mu_{\rm eff}} - 189e^{-\frac{2}{3}\mu_{\rm eff}} + \frac{4336}{5}e^{-\frac{5}{6}\mu_{\rm eff}} \\
&\quad + \left[\frac{\mu_{\rm eff}^2+2\mu_{\rm eff}+2}{3\pi^2}-\frac{38137}{9}\right]e^{-\mu_{\rm eff}} + \frac{148752}{7}e^{-\frac{7}{6}\mu_{\rm eff}} + \mathcal{O}(e^{-\frac{4}{3}\mu_{\rm eff}}).
\end{aligned}
$$

Table 3: Instanton expansion in the $(2,2)_k$ model in table 2 rewritten in terms of the effective chemical potential $\mu_{\rm eff}$.

from the interpolation of the data in table 3. Here we denote the possibly remaining bound states as "$\cdots$". We can use the coefficients of $e^{-\mu_{\rm eff}/k}$ in $J^{\rm np}_k(\mu_{\rm eff})$ for $k=2,3,4,6$ to determine $\widetilde{d}_1$, those of $e^{-\frac{2}{k}\mu_{\rm eff}}$ in $J^{\rm np}_k(\mu_{\rm eff})$ for $k=3,4,6$ to determine $\widetilde{d}_2$, and so on. First of all, from these data we conclude that $\widetilde{d}_\ell$ are $\mu_{\rm eff}$-independent constants.

Again counting on the pole cancellation mechanism, we expect that the worldsheet instanton coefficients $\widetilde{d}_\ell$ are singular at $k = \ell/\mathbb{N}$ and postulate the following ansatz

$$\widetilde{d}_m(k) = \frac{1}{\sin^2\frac{\pi}{k}}\sum_{a\geq 0}\alpha_a\cos\frac{a\pi}{k}. \tag{9.29}$$

where we assume double pole since to compensate the double pole of $\widetilde{c}_\ell$ (9.12).

In the determination of $\widetilde{d}_1(k)$ we can use the four coefficients of $e^{-\mu_{\mathrm{eff}}/k}$ in $J_k^{\mathrm{np}}(\mu_{\mathrm{eff}})$ for $k = 2, 3, 4, 6$. We can use the term $e^{-\mu_{\mathrm{eff}}}$ in $J_{k=1}^{\mathrm{np}}(\mu_{\mathrm{eff}})$ as well, which consist of the membrane instanton $e^{-\mu_{\mathrm{eff}}}$ and the worldsheet instanton $e^{-\mu_{\mathrm{eff}}/k}|_{k=1}$. Now that the membrane instanton coefficient is determined, this data also provide new constraint on $\widetilde{d}_1(k)$. The pole cancellation occurs on this coefficient. The membrane instanton effect is expanded at $k \to 1$ as

$$(\widetilde{b}_1\mu_{\mathrm{eff}} + \widetilde{c}_1)e^{-\mu_{\mathrm{eff}}} = \left[-\frac{4}{\pi^2(k-1)^2} - \frac{8+4\mu_{\mathrm{eff}}}{\pi^2(k-1)} - \frac{4}{3} + \mathcal{O}(k-1)\right]e^{-\mu_{\mathrm{eff}}}, \tag{9.30}$$

while the worldsheet instanton (be careful of the k dependent exponent)

$$\begin{aligned} d_1 e^{-\frac{\mu_{\mathrm{eff}}}{k}} = &\Bigg[\sum_{a\geq 0}\alpha_a(-1)^a\left(\frac{1}{\pi^2(k-1)^2} + \frac{2+\mu_{\mathrm{eff}}}{\pi^2(k-1)} + \frac{3\mu_{\mathrm{eff}}^2 + 6\mu_{\mathrm{eff}} + 6 + 2\pi^2}{6\pi^2}\right) \\ &- \frac{\pi^2}{2}\sum_{a\geq 0}a^2\alpha_a(-1)^a + \mathcal{O}(k-1)\Bigg]e^{-\mu_{\mathrm{eff}}}, \end{aligned} \tag{9.31}$$

From the requirements that the singular part completely cancels and that the finite part reproduces the coefficient in table 3, we obtain the following two constraints for α_a

$$\sum_{a\geq 0}\alpha_a(-1)^a = 4, \quad \sum_{a\geq 0}a^2\alpha_a(-1)^a = 0. \tag{9.32}$$

Hence six equations are available in total, with which we can determine

$$\widetilde{d}_1 = \frac{4}{\sin^2\frac{\pi}{k}}. \tag{9.33}$$

By the same steps, we can also determine

$$\widetilde{d}_2 = \frac{2}{\sin^2\frac{2\pi}{k}}\left(-9 - 10\cos\frac{2\pi}{k}\right), \quad \widetilde{d}_3 = \frac{1}{3\sin^2\frac{3\pi}{k}}\left(112 + 144\cos\frac{2\pi}{k} + 72\cos\frac{4\pi}{k}\right). \tag{9.34}$$

We find two interesting properties (analogous to the ABJM case):

- Our result $\widetilde{d}_2$ together with $\widetilde{b}_2$ (9.16) and $\widetilde{c}_2$ (9.12) also reproduces the non-perturbative effects of $e^{-2\mu_{\mathrm{eff}}}$ in $J_{k=1}^{\mathrm{np}}(\mu_{\mathrm{eff}})$ in table 3, where originally the "bound state" $e^{-(\ell+m/k)\mu_{\mathrm{eff}}}$ with $(\ell, m) = (1, 1)$ could also contribute. Similarly $\widetilde{d}_3$ in (9.34) also reproduces the term $e^{-3\mu_{\mathrm{eff}}}$ in $J_{k=1}^{\mathrm{np}}(\mu_{\mathrm{eff}})$ and $e^{-3\mu_{\mathrm{eff}}/k}$ in $J_{k=2}^{\mathrm{np}}(\mu_{\mathrm{eff}})$, both of which could contain the bound state of $(\ell, m) = (1, 1)$. This implies that the bound state are completely explained by the nonperturbative shift of the chemical potential $\mu \to \mu_{\mathrm{eff}}$ (9.6).

- The worldsheet instanton coefficients have the same multicovering structure as that observed in the membrane instanton coefficients (9.17)

$$\widetilde{d}_1 = \delta_1(k), \quad \widetilde{d}_2 = \frac{1}{2}\delta_1\Big(\frac{k}{2}\Big) + \delta_2(k), \quad \widetilde{d}_3 = \frac{1}{3}\delta_1\Big(\frac{k}{3}\Big) + \delta_3(k), \tag{9.35}$$

with

$$\delta_1(k) = \frac{4}{\sin^2\frac{\pi}{k}}, \quad \delta_2(k) = -\frac{5}{\sin^2\frac{\pi}{k}}, \quad \delta_3(k) = \frac{12}{\sin^2\frac{\pi}{k}}. \tag{9.36}$$

The new coefficients are again extremely simple. Moreover, the decomposition in the multicovering structure is consistent with the pole cancellation: all the divergences cancel only among the δ_m, β_ℓ and corresponding contribution in $\widetilde{c}_\ell$ with the same multicovering level n in (9.17) and (9.35).

The assumption that there are no explicit bound states in $J^{\rm np}(\mu_{\rm eff})$ increase the number of constraint available for the determination of $\widetilde{d}_\ell$, as it allows us to use the higher order terms in table 3. On the other hand the assumption of the multicovering structure reduces the arbitrariness of $\widetilde{d}_\ell$. As a result we can determine the higher worldsheet instanton coefficients. Assuming the multicovering structure

$$\widetilde{d}_4 = \frac{1}{4}\delta_1\Big(\frac{k}{4}\Big) + \frac{1}{2}\delta_2\Big(\frac{k}{2}\Big) + \delta_4(k), \quad \widetilde{d}_5 = \frac{1}{5}\delta_1\Big(\frac{k}{5}\Big) + \delta_5(k). \tag{9.37}$$

with ansatz

$$\delta_m(k) = \frac{1}{\sin^2\frac{\pi}{k}}\sum_{a\geq 0}\alpha_a\Big(\sin\frac{\pi}{k}\Big)^{2a}, \tag{9.38}$$

we obtain

$$\delta_4(k) = -\frac{48}{\sin^2\frac{\pi}{k}} + 5, \quad \delta_5(k) = \frac{240}{\sin^2\frac{\pi}{k}} - 96. \tag{9.39}$$

In the determination of d_6 and d_7 we need $\widetilde{b}_6$ and $\widetilde{b}_7$ to make use of the data at $k=1$ in table 3 which we could not determine from the WKB expansion. Once we accept the trigonometric ansatz and the multicovering structure for them, however, the explicit expression of them is not required for our purpose. Let us assume $\widetilde{b}_6$ to be written as

$$\widetilde{b}_6 = \frac{1}{6}\beta_1(6k) + \frac{1}{3}\beta_2(3k) + \frac{1}{2}\beta_3(2k) + \beta_6(k), \tag{9.40}$$

with

$$\beta_6(k) = \frac{\sum_{n=1}^{n_{\rm max}} m_{6,n}\sin 2\pi k n}{\pi\sin^2\pi k}. \tag{9.41}$$

What we will use in the determination of $\widetilde{d}_6$ is the expansion of $\widetilde{b}_6$ around $k=1$, which is given as

$$\widetilde{b}_6(1+\epsilon) = \frac{1}{6}\beta(6\epsilon) + \frac{1}{3}\beta_2(3\epsilon) + \frac{1}{2}\beta_3(2\epsilon) + \beta_6(\epsilon) = \widetilde{b}_6(\epsilon). \tag{9.42}$$

d	1	2	3	4	5	6	7
n_0^d	16	-20	48	-192	960	-5436	33712
n_1^d	0	0	0	5	-96	1280	-14816
n_2^d	0	0	0	0	0	-80	2512
n_3^d	0	0	0	0	0	0	-160
n_4^d	0	0	0	0	0	0	0

Table 4: The diagonal Gopakumar-Vafa invariants identified for the $(2,2)_k$ model.

Here we have used the periodicity of the sine functions in β. Therefore we already know of the expansion, which is just given by the WKB expansion. Now we have six constraint for $\widetilde{d}_6$ from $k=1,2,3,4,6$ in hand, with which we can determine the explicit expression of $\widetilde{d}_6$ as

$$\widetilde{d}_6 = \frac{1}{6}\delta_1\Big(\frac{k}{6}\Big) + \frac{1}{3}\delta_2\Big(\frac{k}{3}\Big) + \frac{1}{2}\delta_3\Big(\frac{k}{2}\Big) + \delta_6(k), \tag{9.43}$$

with

$$\delta_6(k) = -\frac{1359}{\sin^2\frac{\pi}{k}} + 1280 - 320\sin^2\frac{\pi}{k}. \tag{9.44}$$

Similarly, once we assume

$$\widetilde{b}_7 = \frac{1}{7}\beta_1(7k) + \beta_7(k), \quad \beta_7(k) = \frac{\sum_{n=1}^{n_{\max}} m_{7,n}\sin 2\pi kn}{\pi\sin^2\pi k}, \tag{9.45}$$

we can determine $\widetilde{d}_7$ as

$$\widetilde{d}_7 = \frac{1}{7}\delta_1\Big(\frac{k}{7}\Big) + \delta_7(k), \tag{9.46}$$

with

$$\delta_7(k) = \frac{8428}{\sin^2\frac{\pi}{k}} - 14816 + 10048\sin^2\frac{\pi}{k} - 2560\sin^4\frac{\pi}{k}. \tag{9.47}$$

9.4 Mysterious correspondence to topological string

The multicovering structure of the worldsheet instanton (9.35) together with the trigonometric ansatz (9.38) can be summarized into the following expansion form of the worldsheet instantons

$$\sum_{m\ge 1}\widetilde{d}_m e^{-\frac{\mu_{\rm eff}}{k}} = \sum_{n,d\ge 1}\sum_{g\ge 0} n_g^d \frac{1}{n}\Big(2\sin\frac{\pi n}{k}\Big)^{2g-2} e^{-\frac{nd\mu_{\rm eff}}{k}}. \tag{9.48}$$

where we have renamed the index a and the profile coefficients α_a in δ_d as $a\to g$ and $\alpha_a \to 2^{2a-2}n_g^d$.

Surprisingly,

- The worldsheet instanton series (9.48) completely coincide with the Gopakumar-Vafa formula [11] for the free energy of the topological string theory

$$F_{\rm GV}(g) = \sum_{\vec{d},n} n_g^{\vec{d}} \frac{1}{n} (2\sinh \pi n g_s)^{2g-2} e^{-n\vec{d}\cdot\vec{T}} \tag{9.49}$$

  with the string coupling constant g_s and the set of the Kahler parameters $\vec{T}$ identified as $g_s = 1/k$ and $\vec{T} = \mu_{\rm eff}/k(1,1,\cdots,1)$.

- We can read off the leading "diagonal" Gopakumar-Vafa invariants

$$n_g^d = \sum_{\vec{d}\,(\sum_i d_i = d)} n_g^{\vec{d}} \tag{9.50}$$

  from the worldsheet instanton coefficients (9.36), (9.39), (9.44) and (9.47) as in table 4. We found that they completely coincide with the indices of a known Calabi-Yau threefold, the local D_5 del Pezzo (see table 6 in [61]).

Moreover, rewriting the membrane instanton series as

$$\sum_{\ell\ge 1}(\widetilde{b}_\ell \mu_{\rm eff} + \widetilde{c}_\ell) e^{-\ell\mu_{\rm eff}} = \frac{\partial}{\partial g_s} \sum_{n=1}^{\infty} \sum_{\vec{d}} \sum_{j_L,j_R} N_{j_L,j_R}^{\vec{d}} g_s \frac{-\sin\frac{\pi n(2j_L+1)}{g_s}\sin\frac{\pi n(2j_R+1)}{g_s}}{4\pi n^2 \sin^3\frac{\pi n}{g_s}} e^{-\frac{n}{g_s}\vec{d}\cdot\vec{T}}. \tag{9.51}$$

This completely coincide with the Nekrasov-Shatashvili limit of the free energy of the refined topological string theory. The BPS indices $N_{j_L,j_R}^{\vec{d}}$ are related to the Gopakumar-Vafa invariants through

$$\sum_{j_L,j_R} N_{j_L,j_R}^{\vec{d}} \frac{s_R \sin 2\pi g_s s_L}{\sin 2\pi g_s} = \sum_{g=0}^{\infty} n_g^{\vec{d}} (2\sin \pi g_s)^{2g}, \tag{9.52}$$

and partly determined from the membrane instanton coefficients (9.18) and (9.21) as in table 5.

9.5 Discussion and Comments

In this section we have challenged the complete determination of the instanton coefficients for finite k. Starting from the most leading part, we have discovered various beautiful structure at each step: the trigonometric expression, multicovering structure and the bound state incorporation through the effective chemical potential, which help us to determine the higher order instantons. We have finally obtained a non-trivial correspondence with the (refined) topological string theory. In the determination, the pole cancellation mechanism have played the key role. Our final finding that structure that both the worldsheet instantons and the membrane instantons are related to the topological string with the same Calabi-Yau threefold is the manifestation of the pole cancellation mechanism.

d	$\sum_{j_L,j_R}(-1)^{d-1}N^d_{j_L,j_R}(j_L,j_R)$
1	$8(0,\frac{1}{2})$
2	$8(0,\frac{1}{2})+(0,\frac{3}{2})$
3	$8(0,\frac{1}{2})+8(0,\frac{3}{2})$
4	$(4+2m_1+5m_2)(0,\frac{1}{2})+(30-m_1-m_2)(0,\frac{3}{2})+(9-m_2)(0,\frac{5}{2})$ $+(5-3m_1-5m_2)(\frac{1}{2},0)+m_1(\frac{1}{2},1)+m_2(\frac{1}{2},2)$
5	$(-80+2m_3+5m_4+7m_5)(0,\frac{1}{2})+(80-m_3-m_4)(0,\frac{3}{2})$ $+(80-m_4-m_5)(0,\frac{5}{2})+(16-m_5)(0,\frac{7}{2})$ $+(96-3m_3-5m_4-7m_5)(\frac{1}{2},0)+m_3(\frac{1}{2},1)+m_4(\frac{1}{2},2)+m_5(\frac{1}{2},3)$

Table 5: The diagonal BPS indices identified for the $(2,2)_k$ model. $m_1, m_2, \cdots, m_5$ are some numbers which cannot be fixed from (9.18) nor (9.21). The overall sign $(-1)^{d-1}$ is introduced to make the numbers in table non-negative.

It is also remarkable that the coefficients in (9.48) actually coincide with the topological invariants of known CY_3, called the local D_5 del Pezzo. This implies we can predict the instanton effects up to arbitrarily higher order in principle.

Actually, the similar correspondence was also discovered in the ABJM theory [56]. In that case the worldsheet instantons are already related to the topological string theory on local $\mathbb{P}^1\times\mathbb{P}^1$ at first stage [35]. The membrane instantons were determined with the help of that relation and the exact computation. Nevertheless, our example is the first one without the highest maximal supersymmetry and would suggest that the correspondence is not completely accidental.

There are several problem to be addressed in future. First, though the diagonal Gopakumar-Vafa invariants coincide with the literature, there are discrepancies between the BPS indices determined from the membrane instantons in table 5 and those listed in [62] (see section 5.4 therein). Second, also note that there are more than one Kahler parameters in general while in the correspondence above we have to choose a special "diagonal" slice of them. In the case of the ABJM theory and it is understood that the non-diagonal choice are realized by changing the rank of the gauge group on each vertex. Roughly speaking we can assign the individual chemical potential μ_a dual to each rank N_a, thus the number of parameters 2 indeed coincide with the number of Kahler parameter for $\mathbb{P}^1\times\mathbb{P}^1$. In the case of the $(2,2)_k$ model, however, the number of chemical potential we can introduce is 4, which is smaller than 6 the number of Kahler parameters of the D_5 del Pezzo. It is non-trivial whether we can introduce additional two deformations in the matrix model.[18]

[18] An another evidence for $\mathrm{CY}_3 = $ local dP_5 argued in [63, 13] is the similarity between the approaching polygon (6.45) for Fermi surface and polygon dual to the toric diagram of dP_5 in [62]. In this sense, one may expect we can introduce the new parameters by the ordering of $\{s_a\}$ as

$$\{(+1)^2,(-1)^2\}\to\{(+1)^{q_1},(-1)^{p_1},\cdots\},\quad \Big(\sum_a q_a=\sum_a p_a=2\Big) \tag{9.53}$$

10 Summary of thesis and future directions

In this thesis we have reviewed our recent works, where we have computed the partition function of the $\mathcal{N} \geq 3$ $U(N)$ superconformal Chern-Simons theory in large N expansion. With the help of the Fermi gas formalism, we can show that the perturbative corrections to the partition function in $1/N$ universally add up to an Airy function which is characterized by three parameters depending on the detail of the theory. For a particular class of theories with $\mathcal{N} = 4$ supersymmetry, which we have called the $(q,p)_k$ models, we achieved to determine these parameters exactly.

We also analyzed the non-perturbative effects in $1/N$, which can be interpreted as the instanton effects in the dual eleven dimensional geometry, and discovered an interesting singular structure of the instanton effects. Based on this structure, in a special case $q = p = 2$, we successfully determined the k dependence of the instanton effects for finite k. We finally figured out in this theory the complete formula generating the whole instanton series, which coincide with the Gopakumar-Vafa formula in the topological string theory.

It would be interesting whether we can solve the other theories and how the above correspondence will be generalized. Also, though we have introduced only the result for the special class of the theories characterized by circular quivers, recently the Fermi gas formalism was found to be applicable to the more general theories which include the theories with non-circular quivers [47, 65], the ones with non-unitary gauge groups [66, 48, 55] and the ones without conformal symmetry [49]. In particular, the last set of theories is an example which contains continuous deformation parameters. As we can compute various observables other than the partition function by the differentiation with respect to the deformation parameters, the study of these theories will be interesting also in purpose of the applications.

The theories we have considered can be interpreted as the worldvolume theories of the M2-branes. If we believe that the $\mathrm{AdS}_4/\mathrm{CFT}_3$ correspondence hold also in quantum level, our results must be reproduced as the quantum effects in the gravity side. It was indeed suggested that the Airy function structure of the perturbative might be obtained by applying the localization technique to the AdS_4 supergravity [43]. Though that proposal does not includes the non-perturbative corrections in $1/N$, according to their interpretation as the instantons, they would be incorporated by the dynamical effects in the compactified seven dimensions. Through the efforts in these directions we hope to shed new light to the M-theory.

which do not modify the Fermi surface under the polygon approximation. According to the IIB brane construction, however, such deformation will be related to the rank deformation through the Hanany-Witten transitions [64].

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A Fredholm determinant formula: $Z(N) \to \mathcal{J}(\mu)$

In this appendix we explain the derivation of (6.12)

$$\mathcal{J}(\mu) = \mathrm{Tr}\log(1 + e^{\mu}\rho_0). \tag{A.1}$$

The derivation is achieved by the rearrangement of the sum over the permutations in the partition function (6.9)

$$Z(N) = \frac{1}{N!}\prod_{i=1}^{N}\int\frac{dx_i}{2\pi}\det_{i,j}\rho_0(x_i, x_j) = \frac{1}{N!}\prod_{i=1}^{N}\int\frac{dx_i}{2\pi}\sum_{\sigma\in S_N}(-1)^{\sigma}\prod_{i=1}^{N}\rho_0(x_i, x_{\sigma}(i)). \tag{A.2}$$

First note that a permutation can be always decomposed into a product of cyclic permutations. For example in S_3 and S_6

$$\begin{aligned}
&\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \to (1)(2,3), \quad \sigma = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \to (1,2,3), \\
&\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 6 & 1 & 3 & 5 \end{pmatrix} \to (1,2,4)(3,6,5), \\
&\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 5 & 4 & 2 & 6 & 1 \end{pmatrix} \to (1,3,4,2,5,6).
\end{aligned} \tag{A.3}$$

Now let us suppose σ consists of p_ℓ pieces of ℓ-th cyclic permutation ($\ell = 1, 2, \cdots$)

$$\sigma \to \prod_{\ell\geq 1}\prod_{\alpha=1}^{p_\ell}(n_1^{(\ell,\alpha)}, n_2^{(\ell,\alpha)}, \cdots, n_\ell^{(\ell,\alpha)}), \tag{A.4}$$

with the constraint

$$\sum_{\ell\geq 1}\ell p_\ell = N. \tag{A.5}$$

Then the sign of σ is

$$(-1)^{\sigma} = \prod_{\ell\geq 1}(-1)^{(\ell-1)p_\ell} \tag{A.6}$$

while the integrations in (A.2) are decomposed into those in each cyclic permutation as

$$\prod_{i=1}^{\ell}\int\frac{dx_{n_i^{(\ell,\alpha)}}}{2\pi}\rho_0(x_{n_1^{(\ell,\alpha)}}, x_{n_2^{(\ell,\alpha)}})\rho_0(x_{n_2^{(\ell,\alpha)}}, x_{n_3^{(\ell,\alpha)}})\cdots\rho(x_{n_\ell^{(\ell,\alpha)}}, x_{n_1^{(\ell,\alpha)}}) = \mathrm{Tr}\,\rho_0^{\ell}. \tag{A.7}$$

Hence each contribution to (A.2) depends only on the sizes of cyclic permutations or $\{p_\ell\}$, but independent of the permuted numbers $n_i^{(\ell,\alpha)}$ themselves. Counting the number of independent permutations for each set of $(p_1, p_2, \cdots)$ which is found to be

$$\prod_{\ell\geq 1}\frac{1}{\ell^{p_\ell}p_\ell!}, \tag{A.8}$$

we can finally rewrite the sum (6.9) as the sum over $\{p_\ell\}_{\ell\ge 1}$

$$Z(N)=\sum_{p_1,p_2,\cdots\ge 0\,(\sum_{\ell\ge 1}\ell p_\ell=N)}\prod_{\ell\ge 1}\frac{1}{\ell^{p_\ell}p_\ell!}(-1)^{(\ell-1)p_\ell}(\operatorname{Tr}\rho_0^\ell)^{p_\ell}. \tag{A.9}$$

The sum over $\{p_\ell\}_{\ell\ge 1}$ is complicated due to the constraint (A.5). However, we can convert it into the unconstrained sum by introducing the chemical potential μ and switching to the generating function (grand partition function) with respect to N (6.11)

$$\sum_{N\ge 0}e^{\mu N}Z(N)=\prod_{\ell\ge 0}\sum_{p\ge 0}e^{\mu\ell p}\frac{1}{\ell^p p!}(-1)^{(\ell-1)p}(\operatorname{Tr}\rho_0^\ell)^p \tag{A.10}$$

$$=\prod_{\ell\ge 0}\exp\left[\frac{(-1)^{\ell-1}}{\ell}e^{\mu\ell}\operatorname{Tr}\rho_0^\ell\right], \tag{A.11}$$

where we have performed the sum over p. Further performing the product over ℓ we finally obtain the simple expression for the grand potential $\mathcal{J}(\mu)$ (A.1).

B Recursive determination of $\mathcal{N}=4$ Hamiltonian

The one particle Hamiltonian in the Fermi gas formalism of the $U(N)$ $\mathcal{N}=4$ circular quiver Chern-Simons theory takes the following form.

$$e^h=e^{q_1U}e^{p_1T}e^{q_2U}e^{p_2T}\cdots e^{q_{m+1}U}. \tag{B.1}$$

Precisely speaking, the Hamiltonian $\widehat{H}$ is related to h as

$$\widehat{H}=-h(q_a\to -q_a,p_a\to -p_a). \tag{B.2}$$

According to the Baker-Campbell-Hausdorff formula h can be expanded with U, T and their commutators as

$$h=c_UU+c_TT+c_{UT}[U,T]+c_{UUT}[U,[U,T]]+c_{TUT}[T,[U,T]]+\cdots, \tag{B.3}$$

where obviously

$$c_U=\sum_{a=1}^{m+1}q_a,\quad c_T=\sum_{a=1}^{m}p_b. \tag{B.4}$$

In this section we show that the other coefficients can be determined recursively order by order with the help of the following quantities

$$\Sigma(q^\alpha,p^\beta,q^\gamma,\cdots)\equiv\frac{1}{\alpha!\beta!\gamma!\cdots}\sum_{a\le b<c\cdots}q_a^\alpha p_b^\beta q_c^\gamma. \tag{B.5}$$

First let us rewrite the r.h.s of (B.1) as

$$e^h=(e^{q_1U}e^{p_1T}e^{q_2U}e^{p_2T}\cdots e^{q_{m+1}U})(e^{q_1U}e^{p_1T}e^{q_2U}e^{p_2T}\cdots e^{q_{m+1}U})$$

$$(e^{-q_{m+1}U}\cdots e^{-p_2T}e^{-q_2U}e^{-p_1T}\cdots e^{-q_1U}), \tag{B.6}$$

where the last two products completely cancel each other. Since the middle set of factors in this expression is e^h, we obtain the following relation for h

$$(e^{q_1 \operatorname{ad} U}e^{p_1 \operatorname{ad} T}e^{q_2 \operatorname{ad} U}e^{p_2 \operatorname{ad} T}\cdots e^{q_{m+1} \operatorname{ad} U} - 1)h = 0. \tag{B.7}$$

Here $\operatorname{ad}(\cdot)$ is a linear map $\operatorname{ad}\mathcal{O} : X \to [\mathcal{O}, X]$. Expanding the l.h.s. order by order we obtain the equations relating the coefficients $c_{\dots}$. Here we shall demonstrate the determination of c_{UT} from the terms of order $\mathcal{O}(U^2T)$. The contributions consist of the terms in h with the orders raised by the actions of $\operatorname{ad} U$ or $\operatorname{ad} T$:

$$\sum_a q_a c_{UT}[U,[U,T]] + \left(\sum_{a\neq b} q_a q_b + \sum_a \frac{1}{2}q_a^2\right)c_T[U,U,T] + \sum_{a\leq b} q_a p_b c_U[U,[T,U]] = 0 \tag{B.8}$$

Notice that each additional coefficient takes the form of (B.5)

$$\sum_a q_a = \Sigma(q), \quad \sum_{a\neq b} q_a q_b + \frac{1}{2}\sum_a q_a^2 = \Sigma(q^2), \quad \sum_{a\leq b} q_a b_b = \Sigma(q,p). \tag{B.9}$$

From this equation we obtain c_{UT} in terms of c_U and c_T as

$$c_{UT} = \frac{\Sigma(q,p)c_U - \Sigma(q^2)c_T}{\Sigma(q)}. \tag{B.10}$$

Similarly we obtain c_{UUT} and c_{TUT} as

$$\begin{aligned} c_{UUT} &= -\frac{1}{\Sigma(q)}(\Sigma(q^2)c_{UT} + \Sigma(q^3)c_T - \Sigma(q^2,p)c_U), \\ c_{TUT} &= -\frac{1}{\Sigma(p)}(\Sigma(p^2)c_{UT} - \Sigma(p^3)c_U + \Sigma(p^2,q)c_T). \end{aligned} \tag{B.11}$$

Note that the definition of $\Sigma(\cdots)$ are redundant and there are several relations among them. For example,

$$\begin{aligned} \Sigma(q^\alpha) &= \frac{1}{\alpha!}\Sigma(q)^\alpha, \quad \Sigma(p^\alpha) = \frac{1}{\alpha!}\Sigma(p)^\alpha, \\ \Sigma(p,q) &= \Sigma(q)\Sigma(p) - \Sigma(q,p), \\ \Sigma(p^2,q) &= \frac{\Sigma(q)\Sigma(p)^2}{2} - \Sigma(p)\Sigma(q,p) + \Sigma(q,p^2). \end{aligned} \tag{B.12}$$

We can write down the explicit expression of the leading coefficients as

$$\begin{aligned} c_{UT} &= \Sigma(q,p) - \frac{\Sigma(q)\Sigma(p)}{2}, \\ c_{UUT} &= -\frac{\Sigma(q)\Sigma(q,p)}{2} + \Sigma(q^2,p) + \frac{\Sigma(q)^2\Sigma(p)}{12}, \\ c_{TUT} &= \frac{\Sigma(p)\Sigma(q,p)}{2} - \Sigma(q,p^2) - \frac{\Sigma(q)\Sigma(p)^2}{12}. \end{aligned} \tag{B.13}$$

We can determine the higher order coefficients in the same way, though we have to be careful about the independent commutators.

C Exact values of the partition function for $(2,2)_k$ models $Z_k^{(2,2)}(N)$

The first few exact values of the partition function of the $(2,2)_k$ model are computed by the technique explained in section 8, as

$$
\begin{aligned}
&Z_1^{(2,2)}(1)=\frac{1}{4\pi^2},\quad Z_1^{(2,2)}(2)=\frac{15-\pi^2}{576\pi^4},\quad Z_1^{(2,2)}(3)=\frac{855+75\pi^2-16\pi^4}{518400\pi^6},\\
&Z_2^{(2,2)}(1)=\frac{1}{8\pi^2},\quad Z_2^{(2,2)}(2)=\frac{528-136\pi^2+9\pi^4}{73728\pi^4},\\
&Z_2^{(2,2)}(3)=\frac{67680-31200\pi^2+22454\pi^4-2025\pi^6}{265420800\pi^6},\\
&Z_3^{(2,2)}(1)=\frac{1}{12\pi^2},\quad Z_3^{(2,2)}(2)=\frac{4131-1593\pi^2-128\sqrt{3}\pi^3+192\pi^4}{1259712\pi^4},\\
&Z_3^{(2,2)}(3)=(22537035-19628325\pi^2-1296000\sqrt{3}\pi^3+15828048\pi^4+2188800\sqrt{3}\pi^5\\
&\qquad -2560000\pi^6)/(275499014400\pi^6),\\
&Z_4^{(2,2)}(1)=\frac{1}{16\pi^2},\quad Z_4^{(2,2)}(2)=\frac{552-272\pi^2-72\pi^3+45\pi^4}{294912\pi^4},\\
&Z_4^{(2,2)}(3)=\frac{152640-184800\pi^2-43200\pi^3+167482\pi^4+77400\pi^5-38475\pi^6}{4246732800\pi^6},\\
&Z_6^{(2,2)}(1)=\frac{1}{24\pi^2},\quad Z_6^{(2,2)}(2)=\frac{136080-92232\pi^2-25088\sqrt{3}\pi^3+21801\pi^4}{161243136\pi^4},\\
&Z_6^{(2,2)}(3)=(1565192160-2799360000\pi^2-711244800\sqrt{3}\pi^3+2988770238\pi^4\\
&\qquad +1550649600\sqrt{3}\pi^5-1090902475\pi^6)/(141055495372800\pi^6)\quad .
\end{aligned}
\tag{C.1}
$$

To display the complete data, we shall adopt the plain-text style so that the readers who try to confirm our results can easily put these data into Mathematica. Here each Z22k stands for the list $\{Z_k^{(2,2)}(1), Z_k^{(2,2)}(2),\cdots, Z_k^{(2,2)}(N_{\max})\}$.

```
Z221 = {1/(4 \[Pi]^2), -((-15 + \[Pi]^2)/(576 \[Pi]^4)), (
   855 + 75 \[Pi]^2 - 16 \[Pi]^4)/(518400 \[Pi]^6), (
   60165 + 69090 \[Pi]^2 - 43463 \[Pi]^4 + 3632 \[Pi]^6)/(
   812851200 \[Pi]^8), (
   5608575 + 24900750 \[Pi]^2 - 31048605 \[Pi]^4 + 10115600 \[Pi]^6 -
    732672 \[Pi]^8)/(
   2194698240000 \[Pi]^10), (2731650075 + 33679748025 \[Pi]^2 -
      70150543155 \[Pi]^4 + 48479305215 \[Pi]^6 -
      12507386672 \[Pi]^8 +
      838964736 \[Pi]^10)/(38240422133760000 \[Pi]^12), \
(2119933690875 + 61136013863925 \[Pi]^2 - 192479256844875 \[Pi]^4 +
      227858160084555 \[Pi]^6 - 121910335426880 \[Pi]^8 +
      27429824344832 \[Pi]^10 -
      1745071865856 \[Pi]^12)/(1266675742758666240000 \[Pi]^14),
```

```
   1/(182401306957247938560000000 \[Pi]^16) (614658246827625 +
      37211876234032500 \[Pi]^2 - 165704461466994450 \[Pi]^4 +
      300314300500714500 \[Pi]^6 - 271611766174283535 \[Pi]^8 +
      123836221634288800 \[Pi]^10 - 25323384980987136 \[Pi]^12 +
      1547093849702400 \[Pi]^14),
   1/(569310959274962265833472000000 \[Pi]^18) (338298787177225875 +
      39871029262403839500 \[Pi]^2 - 239415641043753324150 \[Pi]^4 +
      619599595267431078300 \[Pi]^6 - 850018115192454653445 \[Pi]^8 +
      645047772730512409440 \[Pi]^10 -
      262012693656247569152 \[Pi]^12 +
      49813477980198027264 \[Pi]^14 - 2945074585927680000 \[Pi]^16),
   1/(822085025193045511863533568000000000 \[Pi]^20) \
(765621683210169410625 + 166062633159099004453125 \[Pi]^2 -
      1296997005244105169754750 \[Pi]^4 +
      4572669583655871879431250 \[Pi]^6 -
      8899684658812204257075075 \[Pi]^8 +
      10119575319053733909455625 \[Pi]^10 -
      6742436565811685754883040 \[Pi]^12 +
      2503309857017096864096000 \[Pi]^14 -
      448876279110137070993408 \[Pi]^16 +
      25818684401823252480000 \[Pi]^18), (2303477077177729663835625 +
      880027764146777853547565625 \[Pi]^2 -
      8690371157670945742834584750 \[Pi]^4 +
      40368672123285212086316186250 \[Pi]^6 -
      106565948142012635084257524075 \[Pi]^8 +
      170109742285438685942415328125 \[Pi]^10 -
      167739240307876225120652575440 \[Pi]^12 +
      100821204198157435732126956800 \[Pi]^14 -
      34794190696827750548244369408 \[Pi]^16 +
      5943095518870268728096849920 \[Pi]^18 -
      333862667882982109347840000 \
\[Pi]^20)/(1754691161173044062342000588881920000000 \[Pi]^22), \
(44922868769554423085093906250 +
      29177579194014104191247497087500 \[Pi]^2 -
      356076616058351590618269835031250 \[Pi]^4 +
      2127206089098239707460400002505000 \[Pi]^6 -
      7378456616297829890709334459256250 \[Pi]^8 +
      15821536138546787418834134053483500 \[Pi]^10 -
      21655667680052162940333930630138750 \[Pi]^12 +
      19043796432700825100000836287799200 \[Pi]^14 -
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      1051687827591894017128702640326400 \[Pi]^16 +
      341403395703886072903587365228544 \[Pi]^18 -
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      3079075624747251953219665920000 \
\[Pi]^22)/(26733070778703560898592847371733827584000000000 \[Pi]^24), \
(4448584040297369309652193003125 +
      4769863402554401198067329527968750 \[Pi]^2 -
      70594342302205792357812920710940625 \[Pi]^4 +
      532909377734580331865869877951062500 \[Pi]^6 -
      2373194390617844893057015458584308125 \[Pi]^8 +
      6622829860438203731846226010179618750 \[Pi]^10 -
      12045192327124569447436429954950198375 \[Pi]^12 +
      14548091692535526094611790225407330000 \[Pi]^14 -
      11634904346739726928960509478551097600 \[Pi]^16 +
      5982804036260434340273345197881856000 \[Pi]^18 -
      1843224125874746027995843625905422336 \[Pi]^20 +
      291466982574153983300048391241728000 \[Pi]^22 -
      15749165340694202177783070720000000 \
\[Pi]^24)/(225894448080045089593109560291150843084800000000000 \
\[Pi]^26), (3051803535907858703033108449528125 +
      5268908152311874169936996064652621875 \[Pi]^2 -
      93086842398874316393881992295280371875 \[Pi]^4 +
      876886569456044433151878064944363346875 \[Pi]^6 -
      4927508637637263727811919266560386560625 \[Pi]^8 +
      17475439355937735393734592311075789570625 \[Pi]^10 -
      40913967137846750070950447460873269861625 \[Pi]^12 +
      65000152227761714776681166325963154844625 \[Pi]^14 -
      70732735000697479821983645244899320292400 \[Pi]^16 +
      52205312372415824928514733244944089465600 \[Pi]^18 -
      25251757245793417253990753791236995149824 \[Pi]^20 +
      7434638350328956545170481114463497486336 \[Pi]^22 -
      1139271286450925065304408425758720000000 \[Pi]^24 +
      60561690915695198597801500999680000000 \
\[Pi]^26)/(1434520103087518336952082951672924313925713920000000000 \
\[Pi]^28), (23137407456809612391454351085216578125 +
      62954229635321050575163339190798441484375 \[Pi]^2 -
      1309941573799988455483805232788098117096875 \[Pi]^4 +
      15268186040900301480202826750556336795234375 \[Pi]^6 -
      106832754164533132727288247242710361809490625 \[Pi]^8 +
      472580252338941135993059689678828904724328125 \[Pi]^10 -
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```
      1390168248662692186179877009622320394777330625 \[Pi]^12 +
      2815133306189432412358921616126980481987353125 \[Pi]^14 -
      3994692175349508807371135938089071463886404000 \[Pi]^16 +
      3979783013945165918923095709064436004748160000 \[Pi]^18 -
      2741904362221974405434817031786724710050734080 \[Pi]^20 +
      1257645940814959433539447857684291851319705600 \[Pi]^22 -
      355810400761262299389892970480636009417539584 \[Pi]^24 +
      53024334172261922336242288102326927360000000 \[Pi]^26 -
      2777398435926318104088460698499153920000000 \
\[Pi]^28)/(\
10857882660269426292390315861212364132103728660480000000000000 \
\[Pi]^30)};
```

```
Z222 = {1/(8 \[Pi]^2), (528 - 136 \[Pi]^2 + 9 \[Pi]^4)/(
   73728 \[Pi]^4), (
   67680 - 31200 \[Pi]^2 + 22454 \[Pi]^4 - 2025 \[Pi]^6)/(
   265420800 \[Pi]^6), (
   85881600 - 12983040 \[Pi]^2 + 55295520 \[Pi]^4 -
    63338032 \[Pi]^6 + 5854275 \[Pi]^8)/(
   13317754060800 \[Pi]^8), (8990956800 + 12289536000 \[Pi]^2 -
      16340073120 \[Pi]^4 - 21239357200 \[Pi]^6 +
      20153965077 \[Pi]^8 -
      1808375625 \[Pi]^10)/(71915871928320000 \[Pi]^10), \
(156552412262400 + 799484432947200 \[Pi]^2 -
      2370598432392960 \[Pi]^4 + 759104950897920 \[Pi]^6 +
      1917623480954896 \[Pi]^8 - 1343578259469672 \[Pi]^10 +
      115898191014375 \[Pi]^12)/(8019597775865905152000 \[Pi]^12),
   1/(1062564626911128969019392000 \[Pi]^14) (269997793396531200 +
      337526969542963200 \[Pi]^2 - 1517698598772964608 \[Pi]^4 +
      1778722441856680320 \[Pi]^6 + 350226523122632956 8 \[Pi]^8 -
      1963411061557504832 \[Pi]^10 + 107054161771652601 30 \[Pi]^12 -
      888481148528098125 \[Pi]^14),
   1/(489629780080648228924135833600000 \[Pi]^16) \
(138462159627561369600 0 + 353440019638805299200 00 \[Pi]^2 -
      212495559558761480601600 \[Pi]^4 +
      405893735329776325632000 \[Pi]^6 -
      162167991395052325562880 \[Pi]^8 -
      453291486612233871347200 \[Pi]^10 +
      661962025500724840347072 \[Pi]^12 -
      297606457449165047911200 \[Pi]^14 +
      23842627179995514515625 \[Pi]^16), (838481028426223902720000 +
      3928933061787348821606400 0 \[Pi]^2 -
      28992189097948109877043200 0 \[Pi]^4 +
      70175863432575217644687360 0 \[Pi]^6 -
      34550946263336903857416960 0 \[Pi]^8 -
      1445881178560991398471365120 \[Pi]^10 +
      3195722106661710197912741440 \[Pi]^12 -
      2902223406682130096671398432 \[Pi]^14 +
      1123231781667534159396656475 \[Pi]^16 -
      87181723128690845748843750 \
\[Pi]^18)/(30564649391754385042360255276646400 0000 \[Pi]^18), \
(66528275066241892655431680000 +
      532648345382667851576279040 0000 \[Pi]^2 -
```

```
      4460863977928374462071584358400 0 \[Pi]^4 +
      103868043482475177416771420160000 \[Pi]^6 +
      1220808098813835952865914140672 00 \[Pi]^8 -
      1051271621523299016434901146112000 \[Pi]^10 +
      2262184008121886009635025297149440 \[Pi]^12 -
      26412590863669488170616530771072 00 \[Pi]^14 +
      1829911076924045340147848927295408 \[Pi]^16 -
      6287033191471885084272424550550 00 \[Pi]^18 +
      474351360879815302974993246093 75 \
\[Pi]^20)/(282466263818837324807476535164655370240000000 0 \[Pi]^20), \
(436912150740920502085323325440000 +
      5672757859486202509749241380864 0000 \[Pi]^2 -
      49283340578045809666676172722 9952000 \[Pi]^4 +
      3519342712492999087954554431815 68000 \[Pi]^6 +
      11356400347163558260547093186011545600 \[Pi]^8 -
      5968207939477641574033027666537021 4400 \[Pi]^10 +
      14436378940374370243781862124687176 1920 \[Pi]^12 -
      20244012424972544495263522717771333 8880 \[Pi]^14 +
      18049269541183815838559504128858956 4704 \[Pi]^16 -
      10440782521675480216437832806807413 0016 \[Pi]^18 +
      32558742999830421534051420349193966 250 \[Pi]^20 -
      23949260346753338364531298108792968 75 \
\[Pi]^22)/(24116291685819165582482808628673880338202624000000 00 \
\[Pi]^22), (1482827248756757218064882061017088 0000 +
      2999574211071546036460636269623377920000 \[Pi]^2 -
      236950734705831440425782720470243 08224000 \[Pi]^4 -
      1218740706655603682245875036778152591 36000 \[Pi]^6 +
      2396019969373408736990056276462428148531200 \[Pi]^8 -
      1363201192657848267039719206042091542609 9200 \[Pi]^10 +
      4137862611081128570756486425044318251556 8640 \[Pi]^12 -
      7550194426416028533369329597970133238943 7440 \[Pi]^14 +
      8703318641561155640852326596146832252751 6928 \[Pi]^16 -
      6602641250136997274598982776009035003656 0128 \[Pi]^18 +
      3350725873495987319279613410147617077624 0096 \[Pi]^20 -
      9641052639367613838177103066820846342130000 \[Pi]^22 +
      693126540034893267246815942104315140234375 \
\[Pi]^24)/(\
11757328866937348847431746752626313655314707749273600000000 \
\[Pi]^24), (15921650348825216567679087681680179200000 +
      48605878247781864721645802500286054400000 00 \[Pi]^2 -
```

```
      265414130113837972105157053960024031232000000 \[Pi]^4 -
      740810055204465752324299872110728200192000000 \[Pi]^6 +
      1179368337706779490988528598239978105868288000 0 \[Pi]^8 -
      7945226742076391171170270255983526537482240000 0 \[Pi]^10 +
      3077848194057552613578261350192331738605015040 00 \[Pi]^12 -
      7454086886310233022198839550979593523409356800 00 \[Pi]^14 +
      1162543883473094885990918112851367767855387961600 \[Pi]^16 -
      1183606853362388768608108812963252842345636416000 \[Pi]^18 +
      802994655701893608743639595119345211928466684256 \[Pi]^20 -
      368600748065516251067001372724623001864373394000 \[Pi]^22 +
      990831238985511548665718471637395302188358593 75 \[Pi]^24 -
      697731432275310030948500739412656122329101562 5 \
\[Pi]^26)/(\
198698857851241195521596520119384700774818560962723840000000000 0 \
\[Pi]^26)};
```

```
Z223 = {1/(12 \[Pi]^2), (
   4131 - 1593 \[Pi]^2 - 128 Sqrt[3] \[Pi]^3 + 192 \[Pi]^4)/(
   1259712 \[Pi]^4),
   1/(275499014400 \[Pi]^6) (22537035 - 19628325 \[Pi]^2 -
      1296000 Sqrt[3] \[Pi]^3 + 15828048 \[Pi]^4 +
      2188800 Sqrt[3] \[Pi]^5 - 2560000 \[Pi]^6), (153806997015 -
      163949745330 \[Pi]^2 - 11384997120 Sqrt[3] \[Pi]^3 +
      305029727667 \[Pi]^4 + 72216748800 Sqrt[3] \[Pi]^5 -
      319258457424 \[Pi]^6 - 54966352896 Sqrt[3] \[Pi]^7 +
      55705395200 \[Pi]^8)/(104971736462745600 \[Pi]^8),
   1/(765243958813415424000 \[Pi]^10) (154770778554525 -
      68699776650750 \[Pi]^2 - 12711123936000 Sqrt[3] \[Pi]^3 +
      195357994149465 \[Pi]^4 + 231276614688000 Sqrt[3] \[Pi]^5 -
      1229311921719600 \[Pi]^6 - 482840928460800 Sqrt[3] \[Pi]^7 +
      1291183043083776 \[Pi]^8 + 251698589286400 Sqrt[3] \[Pi]^9 -
      231500021760000 \[Pi]^10),
   1/(648013481884536586901913600 \[Pi]^12) (1460030404410807075 +
      2474487005320655325 \[Pi]^2 -
      123237815995324800 Sqrt[3] \[Pi]^3 -
      13702478434623715275 \[Pi]^4 +
      5769688110442502400 Sqrt[3] \[Pi]^5 -
      7023094125716803365 \[Pi]^6 -
      24222904922403411840 Sqrt[3] \[Pi]^7 +
      63307957689755781552 \[Pi]^8 +
      31713757725013816320 Sqrt[3] \[Pi]^9 -
      62035096973680074240 \[Pi]^10 -
      13147388255483412480 Sqrt[3] \[Pi]^11 +
      11255034191937536000 \[Pi]^12)};
```

```
Z224 = {1/(16 \[Pi]^2), (
   552 - 272 \[Pi]^2 - 72 \[Pi]^3 + 45 \[Pi]^4)/(294912 \[Pi]^4), (
   152640 - 184800 \[Pi]^2 - 43200 \[Pi]^3 + 167482 \[Pi]^4 +
    77400 \[Pi]^5 - 38475 \[Pi]^6)/(
   4246732800 \[Pi]^6), (13310640 - 24037440 \[Pi]^2 -
      5503680 \[Pi]^3 + 44641352 \[Pi]^4 + 32457600 \[Pi]^5 -
      50681332 \[Pi]^6 - 31171644 \[Pi]^7 +
      13593825 \[Pi]^8)/(26635508121600 \[Pi]^8),
   1/(2301307901706240000 \[Pi]^10) (12392956800 -
      22710240000 \[Pi]^2 - 6314112000 \[Pi]^3 +
      53976318480 \[Pi]^4 + 90366192000 \[Pi]^5 -
      174030757000 \[Pi]^6 - 233010993600 \[Pi]^7 +
      220886345682 \[Pi]^8 + 166341324600 \[Pi]^9 -
      66374083125 \[Pi]^10), -(1/(
     128313564413854482432000 0 \[Pi]^12)) (-60317238643200 +
      43909065446400 \[Pi]^2 + 34512065280000 \[Pi]^3 +
      235764285482880 \[Pi]^4 - 1068346008576000 \[Pi]^5 +
      718244435187840 \[Pi]^6 + 5351172561504000 \[Pi]^7 -
      4491355363103048 \[Pi]^8 - 9456677491852800 \[Pi]^9 +
      6369403190832576 \[Pi]^10 + 5653633803147000 \[Pi]^11 -
      2111768611981875 \[Pi]^12),
   1/(680041361223122540172410880000 \[Pi]^14) (233191874397081600 +
      514801922732697600 \[Pi]^2 - 141592190861721600 \[Pi]^3 -
      8789321000992371840 \[Pi]^4 + 8920712618737920000 \[Pi]^5 +
      23218762446896976960 \[Pi]^6 - 74856884046254069760 \[Pi]^7 -
      1059165605080719136 \[Pi]^8 + 241586730273496608000 \[Pi]^9 -
      106730926490725032496 \[Pi]^10 -
      342602951891211595584 \[Pi]^11 +
      177448845285662967558 \[Pi]^12 +
      181810853889070933800 \[Pi]^13 -
      64472565083164363125 \[Pi]^14)};
```

```
Z226 = {1/(24 \[Pi]^2), (
   136080 - 92232 \[Pi]^2 - 25088 Sqrt[3] \[Pi]^3 + 21801 \[Pi]^4)/(
   161243136 \[Pi]^4), (1565192160 - 2799360000 \[Pi]^2 -
      711244800 Sqrt[3] \[Pi]^3 + 2988770238 \[Pi]^4 +
      1550649600 Sqrt[3] \[Pi]^5 -
      1090902475 \[Pi]^6)/(141055495372800 \[Pi]^6),
   1/(171985693020562391040 \[Pi]^8) (183530640948480 -
      556534185903360 \[Pi]^2 - 137221312757760 Sqrt[3] \[Pi]^3 +
      1131265400205024 \[Pi]^4 + 861676733030400 Sqrt[3] \[Pi]^5 -
      1446129417081168 \[Pi]^6 - 1171549427586048 Sqrt[3] \[Pi]^7 +
      734020872735325 \[Pi]^8),
   1/(250755140423979966136320000 \[Pi]^10) (201088903991673600 -
      813776233338432000 \[Pi]^2 -
      206610867542016000 Sqrt[3] \[Pi]^3 +
      2164521400969203360 \[Pi]^4 +
      2704085561673216000 Sqrt[3] \[Pi]^5 -
      5073325561376895600 \[Pi]^6 -
      9315384492703564800 Sqrt[3] \[Pi]^7 +
      8355360744608825259 \[Pi]^8 +
      10289831957525612800 Sqrt[3] \[Pi]^9 -
      5964558896187478125 \[Pi]^10),
   1/(679491384780559836147260955033600000 \[Pi]^12) \
(333828167370597372211200 - 1468389904971364134144000 \[Pi]^2 -
      425460227326305583104000 Sqrt[3] \[Pi]^3 +
      2525073484417964968561920 \[Pi]^4 +
      1017166606430284387123200 Sqrt[3] \[Pi]^5 -
      4200831601041824440876800 \[Pi]^6 -
      6471830348816327996497920 Sqrt[3] \[Pi]^7 +
      2239522436780776483669396 \[Pi]^8 +
      1691423675902727399356416 Sqrt[3] \[Pi]^9 -
      8506898795960551216054548 \[Pi]^10 -
      1668240405347317738003584 Sqrt[3] \[Pi]^11 +
      9128039521484454620737187 \[Pi]^12)};
```

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