Strongly proper dyadic subbases and their domain theoretic properties

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Abstract

We consider a second-countable Hausdorff space X with its subbase which induces a coding for each point of X. The topological space X has a subbase consisting of a countable collection of pairs of two disjoint open subsets. A dyadic subbase S is such a subbase with a fixed enumeration. For each point $x \in X$ and n-th pair of S, we assign 0 or 1 to n-th digit of the coding of x depending on which one of the pair contains x. Since x may belong to none of the pair, we allow undefinedness in the coding, and the bottom character is used in the sequence unless we know which one we should assign. Every sequence that contains the bottom character is called a bottomed sequence. When a dyadic subbase S is given, each point of X is represented by a bottomed sequence, and we can construct a domain D_S in which X is embedded.

A proper dyadic subbase is a dyadic subbase with an additional property. We study properties of the domain D_S induced by a proper dyadic subbase S of X. The limit (i.e. non-compact) elements of D_S form an upper set L_S, and the set of minimal elements of L_S is denoted by M_S. We show that X is homeomorphic to M_S if X is compact.

The domain D_S is not a Scott domain in general, i.e., it may not be consistently complete. Moreover, it is possible that D_S is consistently complete, but will not be consistently complete after the enumeration of S is changed. We introduce the strong properness of dyadic subbases and show that D_S is consistently complete regardless of the enumeration of S if and only if S is strongly proper. We also give a characterization of the regularity of spaces through strongly proper dyadic subbases.

If the space X is regular Hausdorff, then X is embedded in M_S. Therefore, M_S is not empty if X is regular. We construct an example of a Hausdorff but non-regular space with a dyadic subbase S such that M_S is empty. This example is a weakened prime integer topology and we show that its Hausdorff property can be deduced from a theorem of Sylvester and Schur.

We study a condition which ensures the existence of strongly proper dyadic subbases. It has been proved that every second-countable regular Hausdorff space has a proper dyadic subbase. We show that every locally compact separable metric space has a strongly proper dyadic subbase.

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Contents

Chapter 1

Introduction

A real number can be represented by an infinite sequence of finitely many digits in several ways such as binary expansion, signed digit representation and Gray expansion. In binary expansion, one number can have two different representations, e.g., $1 = 1.000\ldots = 0.111\ldots$. This duplication causes difficulties in computation. For example, when we add $0.1000\ldots$ and $0.0111\ldots$, we cannot determine the first digit from the finite prefixes of the two arguments. The modified Gray expansion of the unit interval, in contrast, avoids this disadvantage of binary expansion and provides a unique representation for each number by allowing undefinedness in a sequence [3, 9]. In this expansion, a real number is represented by a sequence of $\mathbb{T} = \{0, 1, \perp\}$, where the *bottom character* $\perp$ denotes undefinedness. Let G_n^a $(n < \omega, a \in \{0, 1\})$ denote the set of real numbers whose n-th digit in this coding is a. The family $\{G_n^a \mid n < \omega,\ a \in \{0, 1\}\}$ is a subbase of the unit interval, the *Gray subbase.*

A dyadic subbase can be considered as a generalization of the Gray subbase. Let X be a second-countable Hausdorff space. We can see immediately that X has a subbase that is the union of a countable collection of pairs of disjoint open sets. We call such a subbase with a fixed enumeration a *dyadic subbase* [10][1].

In Chapter 3, we study dyadic subbases and related domain representations. When a dyadic subbase S is given, every point of X is represented by a sequence of $\mathbb{T}$. We define a partial order in the set K_S of finite prefixes of the bottomed sequences given by S. We define a domain D_S as the ideal completion of K_S. The set of limit (i.e. non-compact) elements of D_S is denoted by L_S, and the set of minimal elements of L_S is called the *minimal limit set* M_S of D_S. A *proper* dyadic subbase is a dyadic subbase with an additional property. We prove that if the space X is regular Hausdorff and S is proper, then X is embedded in M_S. Moreover, if X is compact in addition, then X is homeomorphic to M_S.

In Chapter 4, we consider the consistent completeness of D_S. The dcpo D_S is not a Scott domain in general, i.e., D_S might not be consistently complete. Moreover, even if D_S is a Scott domain, changing the enumeration of some pairs in S might cause D_S not to be consistently complete. We define another poset $\hat{D}_S$ which is always a Scott domain, and we study the condition $D_S = \hat{D}_S$. We introduce the *strong properness* of dyadic subbases. We show that if a dyadic subbase S is strongly proper, then we have $D_S = \hat{D}_S$ and thus D_S is a Scott

[1] In some papers, the definition of dyadic subbases is different.

domain. Conversely, if D_S is a Scott domain regardless of the enumeration of S, then S is strongly proper. In fact, D_S can be a Scott domain even if S is not strongly proper. In Section 4.2, we give a characterization of the regularity of spaces through strongly proper dyadic subbases. As we have said, if X is regular, then X is embedded in M_S. Moreover, the image of each point of X is less than or equal to every consistent element in L_S. We prove that the converse holds if S is strongly proper.

In Chapter 5, we study a space with its dyadic subbase S such that the minimal limit set M_S is empty. Such a case happens only when X is not regular because every regular Hausdorff space is embedded in M_S. Moreover, if M_S is empty, then X is covered by subsets in which no pair of two points can be separated by closed neighborhoods. We say that a dyadic subbase S is *strongly independent* if D_S is equal to $\mathbb{T}^\omega$. If S is strongly independent, then M_S is empty. In Section 5.2, we construct an example of a Hausdorff space with a strongly independent dyadic subbase. This example is a weakened prime integer topology, and we show that the Hausdorff property of this space can be deduced from a theorem of Sylvester and Schur.

In Chapter 6, we study the existence of strongly proper dyadic subbases. It has been proved that every second-countable regular Hausdorff space has a proper dyadic subbase. We first give another proof of this fact that uses the metric induced by the Urysohn's metrization theorem. Then we show that every locally compact separable metric space has a strongly proper dyadic subbase.

Chapter 2

Preliminaries

2.1 Bottomed sequences

In this section, Σ is a finite set containing the bottom character $\perp$. Let ω be the first infinite ordinal. As usual, an element of ω is identified with the set of its predecessors.

For an ordinal number $n \leq \omega$, Σ^n denotes the set of sequences of elements of Σ of length n, i.e., the maps from n to Σ. We identify a sequence $\sigma \in \Sigma^n$ with its infinite extension

$$\sigma(k) := \begin{cases} \sigma(k) & (k < n) \\ \perp & (n \leq k < \omega) \end{cases} .$$

By this identification, we have $\Sigma^n \subseteq \Sigma^m$ if $n \leq m$. For a sequence $\sigma \in \Sigma^\omega$, its domain is defined as $\mathrm{dom}(\sigma) := \{k \mid \sigma(k) \neq \perp\}$ and its length as $\mathrm{len}(\sigma) := \min\{n \leq \omega \mid \mathrm{dom}(\sigma) \subseteq n\}$. The set of all sequences of finite length is denoted by $\Sigma^* := \bigcup_{n<\omega} \Sigma^n$.

Let σ and τ be sequences of Σ, $a \in \Sigma$ an element and $n < \omega$ a finite ordinal. $\sigma[n \mapsto a]$ is the sequence which maps n to a and equals σ elsewhere. $\sigma|_n$ is the first n prefix of σ, i.e., the restriction of σ to n. If the length $\mathrm{len}(\sigma)$ is finite, then a concatenation $\sigma\tau \in \Sigma^\omega$ is defined as

$$\sigma\tau(k) := \begin{cases} \sigma(k) & (k < \mathrm{len}(\sigma)) \\ \tau(k - \mathrm{len}(\sigma)) & (\mathrm{len}(\sigma) \leq k < \omega) \end{cases} .$$

The n times concatenation of σ is denoted by σ^n. Elements $a \in \Sigma$ are identified with sequences of length one.

2.2 Domain representations

In this section, $P = (P, \sqsubseteq)$ is a partially ordered set (poset).

Let A be a subset of P. If $p \in A$ and $p \sqsubseteq q$ implies $q \in A$, then A is called an *upper set*. We set $\uparrow A := \{q \mid (\exists p \in A)(p \sqsubseteq q)\}$. The dual notions of an upper set and $\uparrow A$ are a *lower set* and $\downarrow A$, respectively. Two elements $p, q \in P$ are *consistent* if they have a common upper bound in P. The set A is *directed* if A is not empty and any two elements of A are consistent in A. The least upper bound of a directed subset $A \subseteq P$ is denoted by $\bigsqcup A$ if it exists.

A *directed complete partial order (dcpo)* is a poset $D = (D, \sqsubseteq)$ that contains the least upper bound of every directed subset of D. A dcpo is *pointed* if it has a least element.

Let D be a pointed dcpo. An element $p \in D$ is called *compact* if $p \sqsubseteq \bigsqcup A$ implies $p \in \downarrow A$ for all directed subsets $A \subseteq D$. An element is called a *limit element* if it is not compact. The set of compact elements of D is denoted by $K(D)$. For an element $p \in D$, we set $\downarrow p := \downarrow\{p\}$, $\uparrow p := \uparrow\{p\}$ and $\mathrm{approx}(p) := \downarrow p \cap K(D)$. D is *algebraic* if $\mathrm{approx}(p)$ is directed and $p = \bigsqcup \mathrm{approx}(p)$ for all $p \in D$. A subset $A \subseteq D$ is *consistent* if it has an upper bound in D. The set D is *consistently complete* if every consistent subset of D has a least upper bound in D.

Definition 2.1. A *Scott domain* is an algebraic, consistently complete, pointed dcpo.

A poset P is a *conditional upper semilattice with least element (cusl)* if any two consistent elements have a least upper bound and if it has a least element.

An *ideal* of P is a directed lower set. The family of all ideals in P ordered by set inclusion is called the *ideal completion* of P, and denoted by $\mathrm{Idl}(P)$. Following are some known facts regarding cusls and Scott domains.

Proposition 2.2. If a poset P is a cusl, then $\mathrm{Idl}(P)$ is a Scott domain.

Proposition 2.3. Let D be a Scott domain. The set $K = K(D)$ is a cusl, and there is an isomorphism $\mathrm{Idl}(K) \cong D$.

Let D be a Scott domain. We set $K := K(D)$. The *Scott topology* of D is the topology on D generated by the family $\{\uparrow p \mid p \in K\}$. If two elements $p, q \in K$ are consistent, then their least upper bound $p \sqcup q$ exists in K, and we have $\uparrow p \cap \uparrow q = \uparrow(p \sqcup q)$. Therefore, the family $\{\uparrow p \mid p \in K\}$ forms a base. For more information on domains, see [4].

Example 2.4. We set $\mathbb{T} := \{0, 1, \bot\}$ ordered by $\bot \sqsubseteq 0, \bot \sqsubseteq 1$. $\mathbb{T}$ is a Scott domain, and its Scott topology is $\{\emptyset, \{0\}, \{1\}, \{0, 1\}, \mathbb{T}\}$. We equip $\mathbb{T}^\omega$ with the product order. The set of compact elements of $\mathbb{T}^\omega$ is $\mathbb{T}^*$, and $\mathbb{T}^\omega$ is a Scott domain. The family $\{\uparrow \sigma \mid \sigma \in \mathbb{T}^*\}$ forms a base of the Scott topology of $\mathbb{T}^\omega$. Two sequences $\sigma, \tau \in \mathbb{T}^\omega$ are consistent if and only if $\sigma(k) = \tau(k)$ for all $k \in \mathrm{dom}(\sigma) \cap \mathrm{dom}(\tau)$. For a pair of consistent sequences $\sigma, \tau \in \mathbb{T}^\omega$, their least upper bound $\sigma \sqcup \tau$ is given by $(\sigma \sqcup \tau)(k) = \sigma(k) \sqcup \tau(k)$. For other properties of $\mathbb{T}^\omega$, we refer the reader to [7].

Let D be a Scott domain with its Scott topology and X a topological space. Suppose that there is a quotient map μ from $D^R \subseteq D$ onto X. The triple (D, D^R, μ) is a *domain representation* of X [1]. In this paper, if we have an embedding $\varphi : X \to D$, we say that the pair (D, φ) is a domain representation of X, which corresponds to $(D, \varphi(X), \varphi^{-1})$.

Chapter 3

Proper dyadic subbases

This chapter reviews proper dyadic subbases [10, 11]. Throughout this chapter, $X = (X, \mathfrak{O})$ is a second-countable Hausdorff space. For a subset A of X, $\operatorname{cl} A$ denotes the closure of A, $\operatorname{int} A$ the interior of A.

Definition 3.1. A *dyadic subbase* of X is a map $S : \omega \times \{0, 1\} \to \mathfrak{O}$ such that

1. $\{S(n, a) \mid n < \omega,\ a \in \{0, 1\}\}$ is a subbase of X,
2. $S(n, 0) \cap S(n, 1) = \emptyset$ for all $n < \omega$.

For readability, $S(n, a)$ is denoted by S_n^a. Definition 3.1 does not imply that S_n^0 and S_n^1 are exteriors of each other, and this condition has been included in the definitions of dyadic subbases in other papers. Here we consider only the case in which the dyadic subbase is "proper" (Definition 3.2), from which it follows easily that S_n^0 and S_n^1 will be exteriors of each other.

Let S be a dyadic subbase of X. We use the notations

$$S(\sigma) := \bigcap_{k \in \operatorname{dom}(\sigma)} S_k^{\sigma(k)}, \tag{3.1}$$

$$\bar{S}(\sigma) := \bigcap_{k \in \operatorname{dom}(\sigma)} X \setminus S_k^{1-\sigma(k)}, \tag{3.2}$$

for all $\sigma \in \mathbb{T}^\omega$, where $\mathbb{T} = \{0, 1, \perp\}$ (Example 2.4).

Definition 3.2. A dyadic subbase S of X is

(i) *proper* if $\bar{S}(\sigma) = \operatorname{cl} S(\sigma)$ for all $\sigma \in \mathbb{T}^*$.

(ii) *independent* if $S(\sigma) \neq \emptyset$ for all $\sigma \in \mathbb{T}^*$.

We have the following characterization of proper dyadic subbases.

Proposition 3.3. A dyadic subbase S is proper if and only if

$$\forall \sigma \in \mathbb{T}^*.\ \bar{S}(\sigma) \neq \emptyset \Rightarrow S(\sigma) \neq \emptyset. \tag{3.3}$$

Proof. Suppose that S is proper. For any $\sigma \in \mathbb{T}^*$, if $\bar{S}(\sigma) = \operatorname{cl} S(\sigma)$ is not empty, then $S(\sigma)$ is not empty.

Conversely, suppose that (3.3) holds. For any $\sigma \in \mathbb{T}^*$, we always have $\operatorname{cl} S(\sigma) \subseteq \bar{S}(\sigma)$ because $\bar{S}(\sigma)$ is closed and contains $S(\sigma)$. Suppose that x belongs to $\bar{S}(\sigma)$. For any subset A, a point belongs to $\operatorname{cl} A$ if and only if all of its neighborhoods intersects with A. Let U be a neighborhood of x and we show $U \cap S(\sigma) \neq \emptyset$. Since the family $\{S(\tau) \mid \tau \in \mathbb{T}^*\}$ forms a base of X, there exists $\tau \in \mathbb{T}^*$ such that $x \in S(\tau) \subseteq U$. If σ and τ are not consistent, then we have $\bar{S}(\sigma) \cap S(\tau) = \emptyset$, but this contradicts the fact that x belongs to both $\bar{S}(\sigma)$ and $S(\tau)$. Therefore, σ and τ are consistent and there exists their least upper bound $\sigma \sqcup \tau \in \mathbb{T}^*$. We can see

$$\emptyset \neq \bar{S}(\sigma) \cap S(\tau) \subseteq \bar{S}(\sigma) \cap \bar{S}(\tau) = \bar{S}(\sigma \sqcup \tau).$$

By (3.3), we get $S(\sigma \sqcup \tau) \neq \emptyset$. Therefore, we obtain

$$\emptyset \neq S(\sigma \sqcup \tau) = S(\sigma) \cap S(\tau) \subseteq S(\sigma) \cap U.$$

Hence, we obtain $x \in \operatorname{cl} S(\sigma)$. □

By Proposition 3.3, we can see that an independent dyadic subbase is always proper. Let S be a dyadic subbase of X. We define a map $\varphi_S : X \to \mathbb{T}^\omega$ as

$$\varphi_S(x)(n) := \begin{cases} 0 & (x \in S_n^0) \\ 1 & (x \in S_n^1) \\ \perp & (\text{otherwise}) \end{cases} \tag{3.4}$$

for $x \in X$ and $n < \omega$. For all $\sigma \in \mathbb{T}^\omega$ and $x \in X$, we have

$$x \in S(\sigma) \iff \sigma \sqsubseteq \varphi_S(x), \tag{3.5}$$

$$x \in \bar{S}(\sigma) \iff \sigma \text{ and } \varphi_S(x) \text{ are consistent in } \mathbb{T}^\omega. \tag{3.6}$$

By (3.5), we have $\varphi_S^{-1}(\uparrow\sigma) = S(\sigma)$ for all $\sigma \in \mathbb{T}^*$. Therefore, the map φ_S is continuous. We can see that φ_S is injective because X is a Hausdorff space. Hence, the map φ_S is a topological embedding, and the pair $(\mathbb{T}^\omega, \varphi_S)$ is a domain representation of X.

Suppose that $\varphi_S(x)$ can be obtained on an infinite tape as follows. The output on the tape starts from $\perp^\omega$, and if we get "x is in S_n^a" as a result of computation, then the contents of the n-th cell of the tape is replaced by a. In such a computation, the n-th cell could be filled with 0 or 1 after the m-th cell is filled for some $m > n$. We want to obtain $\varphi_S(x)$ as the least upper bound of all finite time states of this output. However, the least upper bound of a strictly increasing sequence in $\operatorname{approx}(\varphi_S(x))$ could be less than $\varphi_S(x)$. Hence, we restrict the finite time states of this output to a subset K_S of $\mathbb{T}^\omega$ as follows. We set

$$K_S := \left\{\varphi_S(x)|_n \;\middle|\; x \in X,\, n < \omega\right\}, \tag{3.7}$$

$$D_S := \{\sigma \in \mathbb{T}^\omega \mid (\forall n < \omega)(\sigma|_n \in K_S)\}. \tag{3.8}$$

The set D_S is an algebraic pointed dcpo that is the ideal completion of K_S.

Clearly, $\varphi_S(X)$ is a subset of D_S. Moreover, as the next proposition shows, the map $\varphi_S : X \to D_S$ is continuous. Hence, if D_S is a Scott domain, we have another domain representation (D_S, φ_S) of X.

Proposition 3.4. Let S be a dyadic subbase of X. The family $\{S(\sigma) \mid \sigma \in K_S\}$ forms a base of X.

Lemma 3.5. Let S be a dyadic subbase of X. For all point $x \in X$, the family $\{S(\varphi_S(x)|_n) \mid n < \omega\}$ forms a neighborhood base at x, i.e., each subset $U \subseteq X$ is a neighborhood of x if and only if there exists $n < \omega$ such that $S(\varphi_S(x)|_n) \subseteq U$.

Proof. Let $x \in X$ be a point, $U \subseteq X$ a subset. Suppose that U is a neighborhood of x. Since the family $\{S(\sigma) \mid \sigma \in \mathbb{T}^*\}$ forms a base of X, there exists $\tau \in \mathbb{T}^*$ such that $x \in S(\tau) \subseteq U$. By (3.5), we obtain $\tau \sqsubseteq \varphi_S(x)$. Therefore, we get $S(\varphi_S(x)|_n) \subseteq S(\tau) \subseteq U$ for all $n \geq \mathrm{len}(\tau)$. $\square$

Proof of Proposition 3.4. Suppose that U is an open subset of X. By Lemma 3.5, for all $x \in U$, there exists $V \in \{S(\sigma) \mid \sigma \in K_S\}$ such that $x \in V \subseteq U$. $\square$

Suppose that S is a proper dyadic subbase of a Hausdorff space X. Since the space X is Hausdorff and S is proper, for any two distinct points $x, y \in X$, there exists $\sigma \in K_S$ such that $x \in S(\sigma)$ and $y \notin \mathrm{cl}\, S(\sigma) = \bar{S}(\sigma)$. By (3.5) and (3.6), $\varphi_S(x)$ and $\varphi_S(y)$ are not consistent in $\mathbb{T}^\omega$, i.e., x and y can be separated by S_n^0 and S_n^1 for some $n < \omega$. Therefore, we obtain $y \notin \bar{S}(\varphi_S(x))$, and hence

$$\forall x \in X.\ S(\varphi_S(x)) = \bar{S}(\varphi_S(x)) = \{x\}. \tag{3.9}$$

Suppose that $\varphi_S(x) \in K_S$ for some $x \in X$. Since $\varphi_S(x)$ is compact, $S(\varphi_S(x))$ is an open set. By (3.9), we have $S(\varphi_S(x)) = \{x\}$, and therefore, x is an isolated point. Since $\mathrm{dom}(\varphi_S(x))$ is finite, there exist $n \in \omega \setminus \mathrm{dom}(\varphi_S(x))$ and $a \in \{0, 1\}$ such that x is on the boundary of S_n^a, but this contradicts the fact that x is an isolated point. Therefore, X is embedded in the space of limit elements in D_S, i.e.,

$$L_S := D_S \setminus K_S. \tag{3.10}$$

We define the *minimal limit set* M_S of D_S as the set of minimal elements of L_S. We have the following [11].

Theorem 3.6. *Suppose that S is a proper dyadic subbase of a Hausdorff space X. If X is regular, then we have $\varphi_S(x) \sqsubseteq \sigma$ for all $\sigma \in L_S$ and $x \in \bar{S}(\sigma)$.*

Proof. See the proof of Proposition 4.12. $\square$

Theorem 3.7. *If X is compact Hausdorff and S is proper, then $\bar{S}(\sigma)$ is not empty for all $\sigma \in L_S$ and we have $\varphi_S(X) = M_S$.*

Proof. Assume that there exists $\sigma \in L_S$ with $\bar{S}(\sigma) = \emptyset$. By De Morgan's law, we have

$$\bar{S}(\sigma) = \bigcap_{k \in \mathrm{dom}(\sigma)} X \setminus S_k^{1-\sigma(k)} = X \setminus \bigcup_{k \in \mathrm{dom}(\sigma)} S_k^{1-\sigma(k)}.$$

Since $\bar{S}(\sigma)$ is empty, we get an open covering $X = \bigcup_{k \in \mathrm{dom}(\sigma)} S_k^{1-\sigma(k)}$. However, we have $\bar{S}(\sigma|_n) \neq \emptyset$ for all $n < \omega$ because $\sigma|_n$ belongs to K_S. Therefore, there is no finite subcovering. Hence, X is not compact.

Suppose that X is compact Hausdorff. Note that a compact Hausdorff space is always regular. For $\sigma \in L_S$ and $x \in X$, if $\sigma \sqsubseteq \varphi_S(x)$, then we have $x \in \bar{S}(\sigma)$ by (3.6), and we get $\varphi_S(x) \sqsubseteq \sigma$ by Theorem 3.6. Since $\varphi_S(x)$ belongs to L_S for all $x \in X$, $\varphi_S(x)$ is minimal in L_S, i.e., $\varphi_S(X) \subseteq M_S$. Suppose that $\sigma \in M_S$.

As above, $\bar{S}(\sigma)$ is not empty, and we get $\varphi_S(x) \sqsubseteq \sigma$ for a point $x \in \bar{S}(\sigma)$ by Theorem 3.6. Since σ is minimal in L_S, we get $\sigma = \varphi_S(x) \in \varphi_S(X)$. Hence, we obtain $\varphi_S(X) = M_S$. □

Let S be a proper dyadic subbase of a compact Hausdorff space X. From Theorems 3.6 and 3.7, we have $L_S = \uparrow\varphi_S(X) \cap D_S$. For all $\sigma \in \uparrow\varphi_S(x)$, we have $x \in \bar{S}(\sigma)$ by (3.6), $\bar{S}(\sigma) \subseteq \bar{S}(\varphi_S(x))$ from $\varphi_S(x) \sqsubseteq \sigma$, and $\bar{S}(\varphi_S(x)) = \{x\}$ by (3.9). That is, we have

$$\forall x \in X.\ \forall \sigma \in \uparrow\varphi_S(x).\ x \in \bar{S}(\sigma) \subseteq \bar{S}(\varphi_S(x)) = \{x\}.$$

Therefore, $\bar{S}(\sigma)$ is a one point set for all $\sigma \in L_S$. Hence, we can define a map $\rho_S : L_S \to X$ as $\rho_S(\sigma) = x \in \bar{S}(\sigma)$. It has been proved that ρ_S is a quotient map. Therefore, if D_S is additionally a Scott domain, then (D_S, L_S, ρ_S) is also a domain representation of X.

Following is the case in which X is the unit interval.

Example 3.8 (Gray subbase of the unit interval)**.** The unit interval $[0,1] \subseteq \mathbb{R}$ has an independent dyadic subbase as follows. We set

$$G_n^0 := \bigcup_{k<\omega} \left(\frac{4k-1}{2^{n+1}}, \frac{4k+1}{2^{n+1}} \right) \cap [0,1], \tag{3.11}$$

$$G_n^1 := \bigcup_{k<\omega} \left(\frac{4k+1}{2^{n+1}}, \frac{4k+3}{2^{n+1}} \right) \cap [0,1]. \tag{3.12}$$

The family $\{G_n^a \mid n < \omega, a \in \{0,1\}\}$ is the *Gray subbase.* We obtain

$$K_G = \bigcup_{m<\omega} \{0,1\}^m \cup \bigcup_{m,n<\omega} \{0,1\}^m \bot 10^n,$$

$$L_G = \{0,1\}^\omega \cup \bigcup_{m<\omega} \{0,1\}^m \bot 10^\omega.$$

For every number $x \in [0,1]$, the sequence $\varphi_G(x)$ contains at most one bottom character. If x is a dyadic rational number on the common boundary of G_n^0 and G_n^1, then we have $\varphi_G(x) \in \{0,1\}^n \bot 10^\omega$. The unit interval $[0,1]$ is homeomorphic to the space M_G.

Chapter 4

Strongly proper dyadic subbases

Throughout this chapter, S is a proper dyadic subbase of a second-countable Hausdorff space X.

4.1 Consistent completeness of domains

As we have shown, S induces an algebraic pointed dcpo D_S. However, D_S need not always be consistently complete. Moreover, even if D_S is a Scott domain, changing the enumeration of S might cause D_S not to be consistently complete. On the other hand, we can construct a Scott domain as follows. We set

$$\hat{K}_S := \{\sigma|_n \in \mathbb{T}^* \mid \sigma \in \uparrow\varphi_S(X),\, n < \omega\}, \tag{4.1}$$

$$\hat{D}_S := \{\sigma \in \mathbb{T}^\omega \mid (\forall n < \omega)(\sigma|_n \in \hat{K}_S)\}. \tag{4.2}$$

A compact sequence $\sigma \in \mathbb{T}^*$ belongs to $\hat{K}_S$ if and only if there exists a point $x \in X$ such that $\varphi_S(x)|_{\mathrm{len}(\sigma)} \sqsubseteq \sigma$.

Proposition 4.1. $\hat{K}_S$ is a cusl. Moreover, if two sequences $\sigma, \tau \in \hat{K}_S$ are consistent in $\mathbb{T}^*$, then their least upper bound $\sigma \sqcup \tau$ in $\mathbb{T}^*$ belongs to $\hat{K}_S$.

Proof. Suppose that two sequences $\sigma, \tau \in \hat{K}_S$ are consistent in $\mathbb{T}^*$ and $\mathrm{len}(\sigma) \leq \mathrm{len}(\tau)$. Their least upper bound $\sigma \sqcup \tau$ exists in $\mathbb{T}^*$ and has the same length as τ has. Since τ belongs to $\hat{K}_S$, there exists a point $x \in X$ such that $\varphi_S(x)|_{\mathrm{len}(\tau)} \sqsubseteq \tau$. We have

$$\varphi_S(x)|_{\mathrm{len}(\sigma\sqcup\tau)} = \varphi_S(x)|_{\mathrm{len}(\tau)} \sqsubseteq \tau \sqsubseteq \sigma \sqcup \tau.$$

Therefore, $\sigma \sqcup \tau$ belongs to $\hat{K}_S$ and is a least upper bound of σ and τ in $\hat{K}_S$. □

We study a condition on S which ensures that $K_S = \hat{K}_S$. Note that we always have $K_S \subseteq \hat{K}_S$.

We introduce some notations. Let S_n^∂ be the common boundary of S_n^0 and S_n^1. Similarly to equations (3.1) and (3.2), for any $\sigma \in \{0, 1, \partial, \bot\}^\omega$, we set

$$S(\sigma) := \bigcap_{k \in \mathrm{dom}(\sigma)} S_k^{\sigma(k)}, \tag{4.3}$$

$$\bar{S}(\sigma) := \bigcap_{k \in \mathrm{dom}(\sigma)} \mathrm{cl}\, S_k^{\sigma(k)}. \tag{4.4}$$

Suppose that $\sigma \in \{0, 1, \partial, \bot\}^\omega$. We call every ordinal $n < \mathrm{len}(\sigma)$ with $\sigma(n) = \bot$ an *inner bottom* of σ. Let σ^∂ be the sequence that maps all inner bottoms of σ to ∂ and equals σ elsewhere. For all $x \in X$, since $\varphi_S(x)$ is a limit element, every $n \in \omega \setminus \mathrm{dom}(\varphi_S(x))$ is an inner bottom of $\varphi_S(x)$. Similarly to (3.5) and (3.6), for all $\sigma \in \{0, 1, \partial, \bot\}^\omega$, we have

$$x \in S(\sigma) \iff \forall k \in \mathrm{dom}(\sigma).\ \varphi_S(x)^\partial(k) = \sigma(k), \tag{4.5}$$

$$x \in \bar{S}(\sigma) \iff \forall k \in \mathrm{dom}(\varphi_S(x)) \cap \mathrm{dom}(\sigma).\ \varphi_S(x)(k) = \sigma(k). \tag{4.6}$$

Proposition 4.2. Let S be a proper dyadic subbase of X. We have the following.

$$K_S = \{\sigma \in \mathbb{T}^* \mid S(\sigma^\partial) \neq \emptyset\}, \tag{4.7}$$

$$\hat{K}_S = \{\sigma \in \mathbb{T}^* \mid \bar{S}(\sigma^\partial) \neq \emptyset\}. \tag{4.8}$$

Proof. Let $\sigma \in \mathbb{T}^*$ be a sequence. We have $x \in S(\sigma^\partial) \Leftrightarrow \varphi_S(x)|_{\mathrm{len}(\sigma)} = \sigma$, and get equation (4.7). Similarly, we have $x \in \bar{S}(\sigma^\partial) \Leftrightarrow \varphi_S(x)|_{\mathrm{len}(\sigma)} \sqsubseteq \sigma$, and hence we obtain equation (4.8). □

Definition 4.3. We say that a dyadic subbase S of X is *strongly proper* if $\bar{S}(\sigma) = \mathrm{cl}\, S(\sigma)$ for all $\sigma \in \{0, 1, \partial, \bot\}^*$.

If a dyadic subbase S is strongly proper, then for all $\sigma \in \mathbb{T}^*$, we have $S(\sigma^\partial) \neq \emptyset$ if and only if $\mathrm{cl}\, S(\sigma^\partial) = \bar{S}(\sigma^\partial) \neq \emptyset$. Therefore, by Proposition 4.2, we obtain $K_S = \hat{K}_S$ if S is strongly proper. We will show that the converse is true. Moreover, D_S is a Scott domain regardless of the enumeration of S if and only if S is strongly proper [12].

Theorem 4.4. *Suppose that S is a proper dyadic subbase of a Hausdorff space X. The following are equivalent.*

1. *S is strongly proper.*
2. *For all permutations $\pi : \omega \xrightarrow{\sim} \omega$, $K_{S\pi}$ is a cusl, where $S\pi$ is the dyadic subbase defined as $S\pi_n^a := S_{\pi(n)}^a$ for $n < \omega$ and $a \in \{0, 1\}$.*
3. *$K_S = \hat{K}_S$.*

Concerning proper dyadic subbases, we have the following lemma.

Lemma 4.5. If a dyadic subbase S is proper, then we have $\uparrow \sigma \cap \{0, 1\}^{\mathrm{len}(\sigma)} \subseteq K_S$ for all $\sigma \in K_S$.

Proof. Suppose that S is proper and $\sigma \in K_S$. Take any sequence $\tau \in \uparrow\sigma \cap \{0,1\}^{\mathrm{len}(\sigma)}$. By equation (4.7), $S(\sigma^\partial)$ is not empty, and we have only to show that $S(\tau^\partial)$ is not empty. Since τ contains no inner bottom, we have $\tau^\partial = \tau$. We get $\mathrm{cl}\, S(\tau) = \bar{S}(\tau)$ because S is proper, and $\bar{S}(\tau) \supseteq S(\sigma^\partial)$ because $\mathrm{cl}\, S_n^{\tau(n)} \supseteq S_n^{\sigma^\partial(n)}$ for all $n < \mathrm{len}(\sigma) = \mathrm{dom}(\tau)$. Therefore, we have

$$\mathrm{cl}\, S(\tau^\partial) = \mathrm{cl}\, S(\tau) = \bar{S}(\tau) \supseteq S(\sigma^\partial) \neq \emptyset.$$

Hence, $S(\tau^\partial)$ is not empty. □

As a corollary to this lemma, we obtain a lemma about least upper bounds of consistent pairs in K_S.

Lemma 4.6. Let S be a proper dyadic subbase of X. If two sequences $\sigma, \tau \in K_S$ are consistent in $\mathbb{T}^*$, then they are consistent also in K_S. Moreover, if there exists a least upper bound υ of σ and τ in K_S, then υ is their least upper bound in $\mathbb{T}^*$.

Proof. Suppose that $\sigma, \tau \in K_S$ are consistent in $\mathbb{T}^*$ and $\mathrm{len}(\sigma) \leq \mathrm{len}(\tau)$. Their least upper bound $\sigma \sqcup \tau$ in $\mathbb{T}^*$ has the same length as τ has. For $a \in \{0,1\}$, we set

$$\upsilon_a(k) := \begin{cases} a & \text{(if } k \text{ is an inner bottom of } \sigma \sqcup \tau\text{)} \\ (\sigma \sqcup \tau)(k) & \text{(otherwise)} \end{cases}.$$

We can see that υ_0 and υ_1 are upper bounds of σ and τ, and they have no inner bottom. Therefore, υ_0 and υ_1 belong to $\uparrow\tau \cap \{0,1\}^{\mathrm{len}(\tau)}$. By Lemma 4.5, we have $\uparrow\tau \cap \{0,1\}^{\mathrm{len}(\tau)} \subseteq K_S$. Hence, υ_0 and υ_1 belong to K_S and are upper bounds of σ and τ in K_S.

Suppose that υ is the least upper bound of σ and τ in K_S. Since υ is smaller than or equal to υ_0 and υ_1, we get $\upsilon(k) \sqsubseteq \upsilon_a(k)$ for all $k < \omega$ and $a \in \{0,1\}$. Therefore, we obtain $\upsilon = \sigma \sqcup \tau$. □

Proof of Theorem 4.4. (1 ⇒ 2) The first condition itself does not depend on the permutation. Therefore, we have only to show that K_S is a cusl. Assume that σ and τ are consistent in K_S and $\mathrm{len}(\sigma) \leq \mathrm{len}(\tau)$. Let υ be the least upper bound of σ and τ in $\mathbb{T}^*$. We will show that υ belongs to K_S. By equation (4.7), $S(\tau^\partial)$ is not empty, and we have only to show that $S(\upsilon^\partial)$ is not empty. We have $S(\tau^\partial) \subseteq \bar{S}(\upsilon^\partial)$ because $S_k^{\tau^\partial(k)} \subseteq \mathrm{cl}\, S_k^{\upsilon^\partial(k)}$ for all $k < \mathrm{len}(\tau) = \mathrm{len}(\upsilon)$. Since S is strongly proper, we have $\mathrm{cl}\, S(\upsilon^\partial) = \bar{S}(\upsilon^\partial)$. Therefore, we obtain

$$\emptyset \neq S(\tau^\partial) \subseteq \bar{S}(\upsilon^\partial) = \mathrm{cl}\, S(\upsilon^\partial).$$

Hence, $S(\upsilon^\partial)$ is not empty.

(2 ⇒ 3) Note that we have $S\pi(\sigma \circ \pi) = S(\sigma)$ for all $\sigma \in \mathbb{T}^*$ and $\pi : \omega \xrightarrow{\sim} \omega$. We will show $K_S = \hat{K}_S$. That is, for all $\sigma \in K_S$, every sequence $\upsilon \in \uparrow\sigma \cap \mathbb{T}^{\mathrm{len}(\sigma)}$ belongs to K_S. Since the cardinality of $\mathrm{dom}(\upsilon) \setminus \mathrm{dom}(\sigma)$ is finite, we have only to show $\sigma[n \mapsto a] \in K_S$, where n is an inner bottom of σ and $a \in \{0,1\}$. Let $\pi : \omega \xrightarrow{\sim} \omega$ be the transposition $(0\, n)$. Note that $K_{S\pi}$ is a cusl by the assumption. We have $S\pi_0^\partial = S_n^\partial \supseteq S(\sigma^\partial) \neq \emptyset$. Since $S\pi_0^\partial$ is the boundary of $S\pi_0^a$, we obtain $S\pi_0^a \neq \emptyset$. Hence, $a \in \{0,1\}^1$ belongs to $K_{S\pi}$. The sequence $\sigma' := \sigma \circ \pi$ belongs to $K_{S\pi}$ because $S\pi(\sigma'^\partial) = S(\sigma^\partial)$ is not empty. Since we have $\sigma'(0) = \sigma(n) = \bot$, σ' and $a \in \{0,1\}^1$ are consistent in $\mathbb{T}^*$. By Lemma 4.6,

they are consistent also in $K_{S\pi}$. Since $K_{S\pi}$ is a cusl, their least upper bound exists in $K_{S\pi}$, and by Lemma 4.6, the least upper bound is $\sigma' \sqcup a = \sigma'[0 \mapsto a]$. Since $\sigma'[0 \mapsto a]$ belongs to $K_{S\pi}$, the set $S\pi(\sigma'[0 \mapsto a]^{\partial}) = S(\sigma[n \mapsto a]^{\partial})$ is not empty. Therefore, we obtain $\sigma[n \mapsto a] \in K_S$.

(3 $\Rightarrow$ 1) Suppose that $\sigma \in \{0, 1, \partial, \bot\}^*$ and $K_S = \hat{K}_S$. We will show $\bar{S}(\sigma) = \mathrm{cl}\, S(\sigma)$. We have $\bar{S}(\sigma) \supseteq \mathrm{cl}\, S(\sigma)$ because $\bar{S}(\sigma)$ contains $S(\sigma)$ and is closed. Suppose that $x \in \bar{S}(\sigma)$ and U is a neighborhood of x. We have only to show $S(\sigma) \cap U \neq \emptyset$. Since $\varphi_S(x)$ is a limit element, there exists a finite ordinal $n > \mathrm{len}(\sigma)$ such that $\mathrm{len}(\varphi_S(x)|_n) = n$ and $S(\varphi_S(x)|_n) \subseteq U$. We decompose σ into two sequences $\sigma_{0,1} := \sigma|_{\sigma^{-1}(\{0,1\})}$ and $\sigma_\partial := \sigma|_{\sigma^{-1}(\partial)}$. Since we have $x \in \bar{S}(\sigma) \subseteq \bar{S}(\sigma_{0,1})$, $\varphi_S(x)|_n$ and $\sigma_{0,1}$ are consistent in $\mathbb{T}^*$. Let υ be their least upper bound $\varphi_S(x)|_n \sqcup \sigma_{0,1}$ in $\mathbb{T}^*$. Since $\upsilon \in {\uparrow}\varphi_S(x)|_n \cap \mathbb{T}^n$, we get $\upsilon \in \hat{K}_S$. By the assumption, υ belongs to K_S, and therefore $S(\upsilon^{\partial})$ is not empty. Every $k \in \mathrm{dom}(\sigma_\partial)$ is an inner bottom of υ because we have $\sigma_{0,1}(k) = \varphi_S(x)|_n(k) = \bot$ and $k < n$. Therefore, we get $S(\upsilon^{\partial}) \subseteq S(\upsilon) \cap S(\sigma_\partial)$. Note that we have $S(\upsilon) = S(\varphi_S(x)|_n) \cap S(\sigma_{0,1})$ and $S(\sigma) = S(\sigma_{0,1}) \cap S(\sigma_\partial)$. Therefore, we obtain

$$\begin{aligned}\emptyset \neq S(\upsilon^{\partial}) &\subseteq S(\upsilon) \cap S(\sigma_\partial)\\ &= S(\varphi_S(x)|_n) \cap S(\sigma_{0,1}) \cap S(\sigma_\partial)\\ &= S(\varphi_S(x)|_n) \cap S(\sigma)\\ &\subseteq U \cap S(\sigma).\end{aligned}$$

Hence, $U \cap S(\sigma)$ is not empty. □

We give some examples of regular Hausdorff spaces with independent dyadic subbases. Before that, we show a proposition for strongly proper dyadic subbases. We do not use the notation S_n^a and we write $S(n, a)$ if we distinguish some dyadic subbases by their subscripts.

Proposition 4.7. Suppose that S_0 and S_1 are dyadic subbases of X_0 and X_1, respectively.

1. Let X_2 be the disjoint union of X_0 and X_1. A dyadic subbase S_2 of X_2 can be obtained by $S_2(0, a) := X_a$ and $S_2(n, a) := S_0(n-1, a) \cup S_1(n-1, a)$ for $1 \leq n < \omega$ and $a \in \{0, 1\}$.
2. Let X_3 be the Cartesian product of X_0 and X_1. A dyadic subbase S_3 of X_3 can be obtained by $S_3(2n, a) := S_0(n, a) \times X_1$ and $S_3(2n+1, a) := X_0 \times S_1(n, a)$ for $n < \omega$ and $a \in \{0, 1\}$.

Moreover, if both S_0 and S_1 are strongly proper, then S_2 and S_3 are also strongly proper.

Proof. From the definition, S_2 and S_3 are dyadic subbases of X_2 and X_3, respectively.

The latter statement follows from the fact that the operator taking closure commutes with both taking disjoint union and taking Cartesian product. □

Proposition 4.8. The Gray subbase G (Example 3.8) is strongly proper.

Proof. Take any sequence $\sigma \in \{0, 1, \partial, \bot\}^*$. Since G is proper, we have $\bar{G}(\sigma) = \mathrm{cl}\, G(\sigma)$ if σ contains no ∂. Suppose that $\sigma(n) = \partial$ for some $n < \omega$. Since we have $G_k^\partial = \{(2l+1)/2^{k+1} \mid l = 0, \dots, 2^k - 1\}$ for all $k < \omega$, we get $G_k^\partial \cap G_n^\partial = \emptyset$ for all $k \neq n$. Therefore, we obtain

$$\begin{aligned} G(\sigma) &= \bigcap_{k \in \mathrm{dom}(\sigma) \setminus \{n\}} (G_k^{\sigma(k)} \cap G_n^\partial) \\ &= \bigcap_{k \in \mathrm{dom}(\sigma) \setminus \{n\}} \big((G_k^{\sigma(k)} \cup G_k^\partial) \cap G_n^\partial\big) \\ &= \bigcap_{k \in \mathrm{dom}(\sigma) \setminus \{n\}} (\mathrm{cl}\, G_k^{\sigma(k)} \cap G_n^\partial) = \bar{G}(\sigma). \end{aligned}$$

By taking closures of both sides, we obtain $\mathrm{cl}\, G(\sigma) = \bar{G}(\sigma)$. □

Example 4.9. Consider the unit interval $[0, 1]$ with the Gray subbase, eliminate two points $1/4$ and $3/4$, and let X_1 be its one-point compactification.

Let p be the added point. Since $\varphi_G(1/4) = 0\bot10^\omega$ and $\varphi_G(3/4) = 1\bot10^\omega$, an independent dyadic subbase G_1 of X_1 can be defined as

$$\varphi_{G_1}(x) = \begin{cases} \bot\bot10^\omega & (x = p) \\ \varphi_G(x) & (\text{otherwise}) \end{cases}$$

for $x \in X_1$. Note that $G_1(0, \partial) = \{1/2, p\}$ and $G_1(1, \partial) = \{p\}$. The set of sequences of length 3 in K_{G_1} is $\{0, 1\}^3 \cup \{\bot\bot1, \bot10\}$, where $\bot\bot1 = \varphi_{G_1}(p)|_3$ and $\bot10 = \varphi_{G_1}(1/2)|_3$. K_{G_1} contains $\bot\bot1$ and 0 but not $0\bot1$. Hence, there is no least upper bound of $\bot\bot1$ and 0 in K_{G_1}.

Let $\pi : \omega \xrightarrow{\sim} \omega$ be a permutation and $G_1\pi$ the corresponding dyadic subbase of X_1. Suppose that $n > \max\{\pi^{-1}(0), \pi^{-1}(1)\} + 1$ is a finite ordinal. The sequence $\sigma = \varphi_{G_1\pi}(p)|_n \in K_{G_1\pi}$ has two inner bottoms $\pi^{-1}(0)$ and $\pi^{-1}(1)$. We have $G_1\pi(\sigma^\partial) \subseteq G_1(\partial\partial) = \{p\}$. We set $\tau = \sigma|_m[m \mapsto 0]$, where $m = \min\{\pi^{-1}(0), \pi^{-1}(1)\}$. Since $G_1\pi$ is independent, τ belongs to K_{G_1}. We can see $\sigma \sqcup \tau = \sigma[m \mapsto 0]$. We have $G_1\pi(\sigma[m \mapsto 0]^\partial) \subseteq G_1(\partial\partial[\pi(m) \mapsto 0])$, where $\pi(m)$ is 0 or 1 by definition. Since $G_1(0\partial) = G_1(\partial 0) = \emptyset$, we get $G_1\pi(\sigma[m \mapsto 0]^\partial) = \emptyset$. Hence, $\sigma[m \mapsto 0]$ does not belong to $K_{G_1\pi}$, and there is no least upper bound of σ and τ in $K_{G_1\pi}$. Therefore, $K_{G_1\pi}$ is not a cusl for any permutation π.

Example 4.10. Let X_2 be the disjoint union of two unit squares $[0, 1] \times [0, 1] \times \{0, 1\}$ with an independent dyadic subbase G_2 given by

$$\varphi_{G_2}((x, y, a))(k) = \begin{cases} a & (k = 0) \\ \varphi_G(x)((k-1)/2) & (k \text{ is odd}) \\ \varphi_G(y)((k-2)/2) & (k > 0 \text{ and } k \text{ is even}) \end{cases}$$

for $k < \omega$ and $(x, y, a) \in [0, 1] \times [0, 1] \times \{0, 1\}$. By Proposition 4.7, G_2 is strongly proper and K_{G_2} is a cusl.

Example 4.11. Let X_2 and G_2 be the same as in Example 4.10. Eliminate two points $(1/2, 0, 0)$ and $(1/2, 0, 1)$ from X_2, and let X_3 be its one-point compactification.

Let p be the added point. An independent dyadic subbase G_3 of X_3 is given by

$$\varphi_{G_3}(x) = \begin{cases} \bot\bot010^\omega & (x = p) \\ \varphi_{G_2}(x) & (\text{otherwise}) \end{cases}$$

for $x \in X$. We have $p \in \bar{G}_3(\partial 0) \neq \operatorname{cl} G_3(\partial 0) = \emptyset$. Thus, G_3 is not strongly proper.

We will show that K_{G_3} is a cusl. Note that

$$K_{G_3} = K_{G_2} \cup \{(\bot\bot 010^\omega)|_n \mid n < \omega\}. \tag{4.9}$$

We get $\{p\} = G_3(\partial) = G_3(\partial\partial 010^\omega|_n)$ for all $n > 0$. Therefore, for all $\sigma \in K_{G_3}$ with $\sigma(0) = \bot$, we have $\sigma = \bot\bot 010^\omega|_{\mathrm{len}(\sigma)}$. Note that for all $a \in \{0,1\}$ and $n < \omega$, the sequence $a\bot 010^\omega|_n$ belongs to K_{G_3} because $G_3(a\partial 010^\omega|_n)$ contains a point $(1/2, \epsilon, a)$, where ϵ is a sufficiently small positive real number. Suppose that $\sigma, \tau \in K_{G_3}$ are consistent in K_{G_3}. If $\sigma(0) = \tau(0) \neq \bot$, then we have $\sigma, \tau \in K_{G_2}$. Since G_2 is strongly proper, K_{G_2} contains $\sigma \sqcup \tau$. By equation (4.9), K_{G_3} also contains $\sigma \sqcup \tau$. Moreover, if one of $\tau(0)$ and $\sigma(0)$ is not the bottom, then we obtain $\sigma \sqcup \tau = \sigma[0 \mapsto a] \sqcup \tau[0 \mapsto a]$, where $a = \sigma(0) \sqcup \tau(0) \neq \bot$. Therefore, we can deduce $\sigma \sqcup \tau \in K_{G_3}$ from the case $\sigma(0) = \tau(0) \neq \bot$. Suppose that $\sigma(0) = \tau(0) = \bot$ and $\mathrm{len}(\sigma) \leq \mathrm{len}(\tau)$. We obtain $\sigma = \bot\bot 010^\omega|_{\mathrm{len}(\sigma)}$ and $\tau = \bot\bot 010^\omega|_{\mathrm{len}(\tau)}$. By $\mathrm{len}(\sigma) \leq \mathrm{len}(\tau)$, we get $\sigma \sqcup \tau = \tau$, and it belongs to K_{G_3}.

For a transposition $\pi = (01)$, two sequences 0 and $\bot\bot 0$ belong to $K_{G_3\pi}$, and they are consistent. Since we have $G_3\pi((0\bot 0)^\partial) = G_3\pi(0\partial 0) = G_3(\partial 00) = \emptyset$, the sequence $0\bot 0$ does not belong to $K_{G_3\pi}$. Therefore, there exists no least upper bound of 0 and $\bot\bot 0$ in $K_{G_3\pi}$. Hence, $K_{G_3\pi}$ is not a cusl.

4.2 Regularity of spaces

In this section, we give a characterization of the regularity of spaces through strongly proper dyadic subbases.

Proposition 4.12. Suppose that S is a strongly proper dyadic subbase of a Hausdorff space X. The space X is regular if and only if

$$\forall \sigma \in L_S.\ \forall x \in \bar{S}(\sigma).\ \varphi_S(x) \sqsubseteq \sigma. \tag{4.10}$$

Theorem 3.6 states that if S is a proper dyadic subbase of a regular Hausdorff space X, then (4.10) holds. First, we show the following characterization of the regularity.

Lemma 4.13. Suppose that S is a proper dyadic subbase of a Hausdorff space X. The space X is regular if and only if

$$\forall x \in X.\ \forall n \in \mathrm{dom}(\varphi_S(x)).\ \exists m < \omega.\ \bar{S}(\varphi_S(x)|_m) \cap S_n^\partial = \emptyset. \tag{4.11}$$

Proof. Suppose that X is regular, $x \in X$ and $n \in \mathrm{dom}(\varphi_S(x))$. Note that S_n^∂ does not contain x and is closed. Since X is regular, there exists an open neighborhood U of x such that $\operatorname{cl} U \cap S_n^\partial = \emptyset$. Since U is a neighborhood of x, there exists $m < \omega$ such that $S(\varphi_S(x)|_m) \subseteq U$. By taking closures, we obtain $\operatorname{cl} S(\varphi_S(x)|_m) \subseteq \operatorname{cl} U$. Since S is proper, we have $\operatorname{cl} S(\varphi_S(x)|_m) = \bar{S}(\varphi_S(x)|_m)$. Therefore, we obtain

$$\emptyset = \operatorname{cl} U \cap S_n^\partial \supseteq \operatorname{cl} S(\varphi_S(x)|_m) \cap S_n^\partial = \bar{S}(\varphi_S(x)|_m) \cap S_n^\partial.$$

Hence, $\bar{S}(\varphi_S(x)|_m) \cap S_n^\partial$ is empty.

Conversely, suppose that (4.11) holds. Take any point $x \in X$ with its neighborhood U. Since U is a neighborhood of x, there exists $n' < \omega$ such that $S(\varphi_S(x)|_{n'}) \subseteq U$. By the assumption, for all $n \in \mathrm{dom}(\varphi_S(x))$, there exists $m_n < \omega$ such that $\bar{S}(\varphi_S(x)|_{m_n}) \cap S_n^{\partial} = \emptyset$. Note that $\bar{S}(\varphi_S(x)|_k) \cap S_n^{1-\varphi_S(x)(n)} = \emptyset$ if $k > n$. Therefore, we have $\bar{S}(\varphi_S(x)|_k) \subseteq S_n^{\varphi_S(x)(n)}$ for all $k > \max\{m_n, n\}$, where $n \in \mathrm{dom}(\varphi_S(x)|_{n'})$. By taking their intersection, for a finite ordinal $k' > \max\{m_n, n \mid n \in \mathrm{dom}(\varphi_S(x)|_{n'})\}$, we obtain

$$\bar{S}(\varphi_S(x)|_{k'}) \subseteq S(\varphi_S(x)|_{n'}) \subseteq U.$$

Since $\bar{S}(\varphi_S(x)|_{k'}) \subseteq U$ is a closed neighborhood of x, X is regular. $\square$

Lemma 4.14. Suppose that S is a proper dyadic subbase of a Hausdorff space X. Let $A \subseteq X$ be a subset and $\sigma \in \mathbb{T}^\omega$ a sequence. If we have $\bar{S}(\sigma|_n) \cap A \neq \emptyset$ for all $n < \omega$, then there exists $\tau \in \uparrow\sigma \cap \{0,1\}^\omega$ such that $\bar{S}(\tau|_n) \cap A \neq \emptyset$ for all $n < \omega$.

Proof. Suppose that $\sigma(m) = \bot$ for some $m < \omega$. Since $\mathrm{cl}\, S_m^0 \cup \mathrm{cl}\, S_m^1 = X$, there exists $a \in \{0,1\}$ such that $\bar{S}(\sigma[m \mapsto a]|_n) \cap A \neq \emptyset$ for all $n < \omega$. Therefore, we can obtain such a sequence $\tau \in \uparrow\sigma \cap \{0,1\}^\omega$ without the bottom inductively. $\square$

Proof of Proposition 4.12. Suppose that S is a proper dyadic subbase of a regular Hausdorff space X. Let $\sigma \in \mathbb{T}^\omega \setminus \mathbb{T}^*$ be a sequence with $x \in \bar{S}(\sigma)$ for a point $x \in X$. We assume $\varphi_S(x) \not\sqsubseteq \sigma$, and we show $\sigma \notin L_S$. By equation (4.7), we have only to show that $S((\sigma|_k)^\partial) = \emptyset$ for some $k < \omega$. Since σ and $\varphi_S(x)$ are consistent and $\varphi_S(x) \not\sqsubseteq \sigma$, there exists $n \in \mathrm{dom}(\varphi_S(x))$ such that $\sigma(n) = \bot$. By Lemma 4.13, there exists $m < \omega$ such that $\bar{S}(\varphi_S(x)|_m) \cap S_n^\partial = \emptyset$. Since σ is a limit element, there exists a finite ordinal $k > \max\{m, n\}$ such that $\mathrm{len}(\sigma|_k) = k$. We will show that $S((\sigma|_k)^\partial)$ is empty. Since we have $k = \mathrm{len}(\sigma|_k)$, every $i \in k \setminus \mathrm{dom}(\sigma)$ is an inner bottom of $\sigma|_k$. Therefore, we obtain $S((\sigma|_k)^\partial) = S(\sigma^\partial|_k)$. Since σ and $\varphi_S(x)$ are consistent, we have $S_i^{\sigma^\partial(i)} \subseteq \mathrm{cl}\, S_i^{\varphi_S(x)(i)}$ for all $i \in \mathrm{dom}(\varphi_S(x))$. Hence, we get $S(\sigma^\partial|_k) \subseteq \bar{S}(\varphi_S(x)|_k)$. We get $S(\sigma^\partial|_k) \subseteq S_n^\partial$ from $k > n$ and $\sigma(n) = \bot$, $\bar{S}(\varphi_S(x)|_k) \subseteq \bar{S}(\varphi_S(x)|_m)$ from $k > m$. Therefore, we obtain

$$\begin{aligned} S((\sigma|_k)^\partial) &= S(\sigma^\partial|_k) \\ &\subseteq \bar{S}(\varphi_S(x)|_k) \cap S_n^\partial \\ &\subseteq \bar{S}(\varphi_S(x)|_m) \cap S_n^\partial = \emptyset. \end{aligned}$$

Hence, $S((\sigma|_k)^\partial)$ is empty.

Conversely, assume that X is not regular and S is strongly proper. By Lemma 4.13, there exist $x \in X$ and $n \in \mathrm{dom}(\varphi_S(x))$ such that $\bar{S}(\varphi_S(x)|_k) \cap S_n^\partial \neq \emptyset$ for all $k < \omega$. By Lemma 4.14, there exists $\tau \in \uparrow\varphi_S(x) \cap \{0,1\}^\omega$ such that $\bar{S}(\tau|_k) \cap S_n^\partial \neq \emptyset$ for all $k < \omega$. For all $k < \omega$, we have

$$\emptyset \neq \bar{S}(\tau|_k) \cap S_n^\partial = \bar{S}(\tau|_k[n \mapsto \partial]) \subseteq \bar{S}(\tau[n \mapsto \partial]|_k) = \bar{S}(\tau[n \mapsto \bot]^\partial|_k),$$

and since S is strongly proper, we get

$$\bar{S}(\tau[n \mapsto \bot]^\partial|_k) = \mathrm{cl}\, S(\tau[n \mapsto \bot]^\partial|_k) \subseteq \mathrm{cl}\, S((\tau[n \mapsto \bot]|_k)^\partial).$$

We set $\sigma := \tau[n \mapsto \bot]$. Since σ is a limit element and $S((\sigma|_k)^\partial) \neq \emptyset$ for all $k < \omega$, we obtain $\sigma \in L_S$. The sequence σ satisfies $x \in \bar{S}(\sigma)$ and $\varphi_S(x) \not\sqsubseteq \sigma$. $\square$

The assumption that S is strongly proper cannot be avoided as the next example shows.

Example 4.15 (Example 5.6 of [11]). We set $A_0 := \{\sigma \in \{0,1\}^\omega \mid \sigma \text{ has infinitely many } 1\}$ and $A_1 := \{\perp^n 10^\omega \mid n < \omega\}$. We equip $A := A_0 \cup A_1 \cup \{0^\omega\} \subseteq \mathbb{T}^\omega$ with a subspace topology induced from $\mathbb{T}^\omega$. The space A is Hausdorff because no two points of A are consistent in $\mathbb{T}^\omega$. An independent dyadic subbase S of A is defined as $S_n^a := \{x \in A \mid x(n) = a\}$ for all $n < \omega$ and $a \in \{0,1\}$.

We obtain $L_S = \{0,1\}^\omega \cup A_1$ and $A = M_S$. Therefore, we get $\varphi_S(x) \sqsubseteq \sigma$ for all $\sigma \in L_S$ and $x \in \bar{S}(\sigma)$. However, for all $\sigma, \tau \in \mathbb{T}^*$ with $\mathrm{len}(\sigma) \leq \mathrm{len}(\tau)$, we have $\perp^{\mathrm{len}(\tau)} 10^\omega \in \bar{S}(\sigma) \cap \bar{S}(\tau)$. Hence, every two closures of non-empty open sets intersect, and thus A is not regular. In fact, S is not strongly proper because we have $S(\partial 0) = \emptyset$, whereas $\bar{S}(\partial 0) \supseteq S(\partial\partial) \ni \perp\perp 10^\omega$.

Chapter 5

Strongly independent dyadic subbases

In this chapter, we construct a Hausdorff space X with a proper dyadic subbase S such that $M_S = \emptyset$. By Theorem 3.6, X is not regular.

5.1 Existence of minimal limit elements

In this section, S is a proper dyadic subbase of a second-countable Hausdorff space X. We introduce some topological conditions which cannot hold if X is regular.

Definition 5.1. A space X is called *adhesive* if X has two or more points and no pair of two points of X can be separated by closed neighborhoods. We say that X is *locally adhesive* if for all $x \in X$ with its open neighborhood U, there exists an adhesive neighborhood $V \subseteq U$ of x.

Note that not all subspace of an adhesive space is adhesive, e.g., the union of two disjoint open subsets is not adhesive.

Proposition 5.2. Suppose that S is a proper dyadic subbase of X and M_S is empty. For all $x \in X$, there exists an adhesive neighborhood of x. Moreover, X is locally adhesive if S is strongly proper.

Proof. Suppose that M_S is empty and $x \in X$ is a point. We define a decreasing sequence $(\sigma_n)_{n<\omega}$ in L_S as follows. We set $\sigma_0 := \varphi_S(x) \in L_S$. For an ordinal $n < \omega$, if there exists $\tau \in \downarrow\sigma_n \cap L_S$ with $\tau(n) = \bot \neq \sigma_n(n)$, then we set $\sigma_{n+1} := \tau$, and we set $\sigma_{n+1} := \sigma_n$ otherwise. We can see

$$\forall n < \omega.\ \forall \tau \in \downarrow\sigma_n \cap L_S.\ \sigma_n|_n = \tau|_n. \tag{5.1}$$

Let σ' be the greatest lower bound of $(\sigma_n)_{n<\omega}$ in $\mathbb{T}^\omega$. We have $\sigma'|_n = \sigma_n|_n \in K_S$ for all $n < \omega$, and hence σ' belongs to D_S. By (5.1), for all $\tau \in L_S$, if $\tau \sqsubseteq \sigma'$, then we get $\tau|_n = \sigma'|_n$ for all $n < \omega$, i.e., $\sigma' = \tau$. Hence, if σ' belongs to L_S, then σ' is minimal in L_S, but this contradicts the assumption that M_S is empty. Therefore, σ' is compact. We have $S(\sigma'^{\partial}\partial^{n-\mathrm{len}(\sigma')}) = S(\sigma_n^{\partial}|_n) \neq \emptyset$ for all $n > \mathrm{len}(\sigma')$. For any pair of two compact sequences $\tau_0, \tau_1 \in \uparrow\sigma' \cap K_S$, we have $\bar{S}(\tau_0) \cap \bar{S}(\tau_1) \supseteq S(\sigma'^{\partial}\partial^{n-\mathrm{len}(\sigma)}) \neq \emptyset$ for $n > \max\{\mathrm{len}(\tau_0), \mathrm{len}(\tau_1)\}$. Since the family $\{S(\tau) \mid \tau \in \uparrow\sigma' \cap K_S\}$ forms a base of $S(\sigma')$, $S(\sigma')$ is an adhesive neighborhood of x.

Suppose that S is strongly proper and M_S is empty. As above, for all $x \in X$, there exists its adhesive neighborhood $S(\sigma')$, where $\sigma' \in \downarrow\varphi_S(x) \cap K_S$. For all neighborhood U of x, there exists $n < \omega$ such that $S(\varphi_S(x)|_n) \subseteq U$. We set $\tau := \varphi_S(x)|_n \sqcup \sigma'$. Note that we have $x \in S(\tau) \subseteq U \cap S(\sigma')$. Since S is strongly proper, we obtain

$$\mathrm{cl}\, S(\tau^{\partial}\partial^{n-\mathrm{len}(\tau)}) = \bar{S}(\tau^{\partial}\partial^{n-\mathrm{len}(\tau)}) \supseteq S(\sigma'^{\partial}\partial^{n-\mathrm{len}(\sigma')}) \neq \emptyset$$

for all $n > \max\{\mathrm{len}(\tau), \mathrm{len}(\sigma')\}$. We can see similarly that $S(\tau)$ is adhesive. □

As a corollary to this, we have the following.

Corollary 5.3. If no open subsets of X is adhesive, then we have $\uparrow M_S = L_S$.

Proof. Suppose that $\sigma_0 \in L_S \setminus \uparrow M_S$. Similarly to the proof of Proposition 5.2, there exists a decreasing sequence $(\sigma_n)_{n<\omega}$ in L_S such that its greatest lower bound σ' belongs to K_S. We can see that $S(\sigma')$ is a non-empty adhesive subspace of X. □

We give a sufficient condition for $M_S = \emptyset$.

Definition 5.4. A dyadic subbase S of X is called *strongly independent* if $S(\sigma) \neq \emptyset$ for all $\sigma \in \{0, 1, \partial, \perp\}^*$.

If a dyadic subbase S is strongly independent, then we have $S(\sigma^{\partial}) \neq \emptyset$ for all $\sigma \in \mathbb{T}^*$, and hence $K_S = \mathbb{T}^*$. Since $L_S = \mathbb{T}^\omega \setminus \mathbb{T}^*$ has no minimal elements, we get $M_S = \emptyset$. We have a characterization of strongly independent dyadic subbases as follows.

Proposition 5.5. Let S be a proper dyadic subbase of X. S is strongly independent if and only if all of the following hold.

1. S is independent.
2. S is strongly proper.
3. X is adhesive.

Proof. Suppose that the three conditions above hold. Let $\sigma \in \{0, 1, \partial, \perp\}^*$ be a sequence. We will show $S(\sigma) \neq \emptyset$. For $a \in \{0, 1\}$, we define a sequence $\sigma_a \in \mathbb{T}^*$ as

$$\sigma_a(k) := \begin{cases} a & (k \in \sigma^{-1}(\partial)) \\ \sigma(k) & (\text{otherwise}) \end{cases}.$$

Since S is independent, both $S(\sigma_0)$ and $S(\sigma_1)$ are not empty. Hence, we can take two points $x \in S(\sigma_0)$ and $y \in S(\sigma_1)$. Since x and y cannot be separated by closed neighborhoods, we obtain $\mathrm{cl}\, S(\sigma_0) \cap \mathrm{cl}\, S(\sigma_1) \neq \emptyset$. Since S is strongly proper, we have $\mathrm{cl}\, S(\sigma_0) \cap \mathrm{cl}\, S(\sigma_1) = \bar{S}(\sigma_0) \cap \bar{S}(\sigma_1) = \bar{S}(\sigma) = \mathrm{cl}\, S(\sigma)$. Therefore, we obtain $S(\sigma) \neq \emptyset$.

Conversely, let S be a strongly independent dyadic subbase. Trivially, S is independent. Since we have $\mathbb{T}^* = K_S \subseteq \hat{K}_S \subseteq \mathbb{T}^*$, we get $K_S = \hat{K}_S$. By Theorem 4.4, S is strongly proper. We will show the third condition. Suppose that $x \in U$ and $y \in V$ for some $x, y \in X$ and their neighborhoods U, V. There exist two sequences $\sigma, \tau \in \mathbb{T}^*$ such that $x \in S(\sigma) \subseteq U$ and $y \in S(\tau) \subseteq V$. We define a sequence $\upsilon \in \{0, 1, \partial, \perp\}^*$ as

$$\upsilon(k) = \begin{cases} \sigma(k) \sqcup \tau(k) & \text{(if } \sigma(k) \text{ and } \tau(k) \text{ are consistent)} \\ \partial & \text{(if } \{\sigma(k), \tau(k)\} = \{0, 1\}) \end{cases} .$$

We have $\operatorname{cl} U \cap \operatorname{cl} V \supseteq \bar{S}(\sigma) \cap \bar{S}(\tau) = \bar{S}(\upsilon) \supseteq S(\upsilon)$. Since S is strongly independent, $S(\upsilon)$ is not empty. Therefore, two closures $\operatorname{cl} U$ and $\operatorname{cl} V$ intersect. □

5.2 Weakened prime integer topology

In this section, we will construct a Hausdorff space with a strongly independent dyadic subbase [12].

Let $\mathbb{N}$ be the set of natural numbers, i.e., the set of positive integers. For two natural numbers $p, r \in \mathbb{N}$, let $U_p(r)$ be the congruence class of $r \mod p$:

$$U_p(r) := \{n \in \mathbb{N} \mid n \equiv r \pmod{p}\}.$$

The *prime integer topology* is the topology on $\mathbb{N}$ generated by the family

$$\{U_p(r) \mid p : \text{prime number}, 0 < r < p\}.$$

Since there exist arbitrarily large prime numbers, any two points are separated by open neighborhoods. However, using the Chinese Remainder Theorem, we can show that the space is adhesive. Hence, the prime integer topology is Hausdorff but non-Urysohn, in particular, non-regular [8].

We weaken the prime integer topology as follows. Let $(p_n)_{n<\omega} := (3, 5, 7, 11, \dots)$ be the sequence of odd prime numbers. We set

$$U_p^0 := \bigcup \{U_p(r) \mid 0 < r < p/2\}, \tag{5.2}$$

$$U_p^1 := \bigcup \{U_p(r) \mid p/2 < r < p\} \tag{5.3}$$

for $p \in \mathbb{N}$. The topology on $\mathbb{N}$ generated by $\{U_{p_n}^a \mid n < \omega,\ a \in \{0, 1\}\}$ is denoted by $\mathfrak{P}_2$. We will study the space $(\mathbb{N}, \mathfrak{P}_2)$.

Proposition 5.6. The space $(\mathbb{N}, \mathfrak{P}_2)$ has a strongly independent dyadic subbase $S : \omega \times \{0, 1\} \to \mathfrak{P}_2$ defined as $S_n^a := U_{p_n}^a$ for $n < \omega$ and $a \in \{0, 1\}$.

Proof. We will later show that the space $(\mathbb{N}, \mathfrak{P}_2)$ is a Hausdorff space.

From the definition, the family $\{S_n^a \mid n < \omega, a \in \{0, 1\}\}$ is a subbase of $(\mathbb{N}, \mathfrak{P}_2)$ and we have $S_n^0 \cap S_n^1 = \emptyset$ for all $n < \omega$. Note that $S_n^\partial = \{qp_n \mid q \in \mathbb{N}\}$. By the Chinese Remainder Theorem, S is strongly independent. □

Theorem 5.7. *The topological space* $(\mathbb{N}, \mathfrak{P}_2)$ *is Hausdorff.*

We show Theorem 5.7 from the following.

Theorem 5.8 (Sylvester, 1912; Schur, 1929; Erdős, 1934)**.** *Let m and n be two natural numbers. If $n \geq m$, then there exists a number containing a prime divisor greater than m in the sequence $n + 1, n + 2, \dots, n + m$.*

The case $n = m$ corresponds to Bertrand's postulate. This was first proved by Sylvester and Schur independently, and an elementary proof was given by Erdős [2].

Proof of Theorem 5.7. Let $m, n \in \mathbb{N}$ be natural numbers with $m < n$. There are two cases: $2m \leq n$ or $m < n < 2m$.

(1) For the case $2m \leq n$, we have $n \geq 2$. From Bertrand's postulate (or Theorem 5.8), there exists a prime number p such that $n < p < 2n$. Since $p \geq 3$, we can put $p = p_k$ with $k < \omega$. We have $m < p_k/2 < n < p_k$, and hence $m \in S_k^0$ and $n \in S_k^1$.

(2) For the other case $m < n < 2m$, we have $0 < 2n - 2m - 1 < 2m$. Note that $2m + (2n - 2m - 1) = 2n - 1$. From Theorem 5.8, there exists a number qp containing a prime divisor p such that

$$p > 2n - 2m - 1, \tag{5.4}$$

$$2m + 1 \leq qp \leq 2n - 1. \tag{5.5}$$

By equation (5.4), p is odd if $n > m + 1$. On the other hand, by equation (5.5), we have $2m + 1 = qp = 2n - 1$ if $n = m + 1$. Therefore, p is always odd, and hence we can put $p = p_k$ with $k < \omega$. By equations (5.4) and (5.5), we obtain

$$(q-1)p_k < 2m < qp_k < 2n < (q+1)p_k,$$

and hence,

$$\frac{q-1}{2} p_k < m < \frac{q}{2} p_k < n < \frac{q+1}{2} p_k.$$

Therefore, we obtain $m \in S_k^1$ and $n \in S_k^0$ if q is even, whereas $m \in S_k^0$ and $n \in S_k^1$ if q is odd. □

We can see that the set $\{n \in \mathbb{N} \mid n + 1 \in S_k^a\}$ is not open for all $k > 0$ and $a \in \{0, 1\}$. Therefore, the increment function is not continuous with respect to this topology.

Example 5.9. We discuss another example of a Hausdorff space with a strongly independent dyadic subbase. We define a topology on ω. For a non-negative integer $n < \omega$, let $\sum_{k<\omega} t_k(n)3^k$ ($t_k(n) \in \{0, 1, 2\}$) be the ternary expansion of n. If we interpret 2 as $\perp$, then we get a non-Hausdorff topology. Note that $t_k(n) = 0$ for all $n \leq k$ and every sequence in $\{0, 1, 2\}^k$ is obtained as $(t_0(n), t_1(n), \ldots, t_{k-1}(n))$ for some $n > k$. We set

$$S_n^0 := \{m < \omega \mid t_n(m) = 0\} \setminus \{n\}, \tag{5.6}$$

$$S_n^1 := \{m < \omega \mid t_n(m) = 1\} \cup \{n\}. \tag{5.7}$$

For any two numbers $m < n$, we have $m \in S_n^0$ and $n \in S_n^1$. Therefore, the topology on ω generated by the family $\{S_n^a \mid n < \omega, a \in \{0, 1\}\}$ is Hausdorff. If we interpret 2 as ∂, then we get $n \in S((t_0(n), t_1(n), \ldots, t_{k-1}(n)))$ for all $n > k$. Hence, S is a strongly independent dyadic subbase of ω with this topology.

Chapter 6

Existence of strongly proper dyadic subbases

In this chapter, we will show the following.

Theorem 6.1. *Every locally compact separable metric space has a strongly proper dyadic subbase.*

Every separable metric space is second-countable and regular Hausdorff. Urysohn's metrization theorem states that every second-countable regular Hausdorff space is metrizable. Therefore, Theorem 6.1 states that every locally compact second-countable regular Hausdorff space has a strongly proper dyadic subbase.

6.1 Existence of proper dyadic subbases

First, we show the following.

Proposition 6.2. Every separable metric space $X = (X, d)$ has a proper dyadic subbase.

Proposition 6.2 has been proved [5, 6]. Using the metric directly, we give another proof of this fact. Let $f : X \to \mathbb{R}$ be a function, c a real number. We use the notations

$$\begin{aligned} U^0(f, c) &:= \{x \in X \mid f(x) < c\}, \\ U^1(f, c) &:= \{x \in X \mid f(x) > c\}, \\ U^\partial(f, c) &:= f^{-1}(c). \end{aligned} \tag{6.1}$$

We will construct a dyadic subbase $S : \omega \times \{0, 1\} \to \mathfrak{O}$ of the form

$$S^a_n := U^a(f_n, c_n) \tag{6.2}$$

for all $n < \omega$ and $a \in \{0, 1\}$, where $f_n : X \to \mathbb{R}$ is a continuous function and c_n is a real number for all $n < \omega$.

We say that c is a *local maximum (resp. local minimum)* of f if c is a maximum (resp. minimum) value of $f|_V$ for some open subset V. Local maxima and local minima are collectively called *local extrema*. If c is a local maximum of $f : X \to \mathbb{R}$, then there exists a point $x \in U^{\partial}(f, c)$ with its neighborhood V such that $V \cap U^1(f, c)$ is empty. The point x belongs to neither $\mathrm{cl}\, U^1(f, c)$ nor $U^0(f, c)$. Therefore, $U^0(f, c)$ is not the exterior of $U^1(f, c)$. Similarly, if c a local minimum, then there exists $y \notin \mathrm{cl}\, U^0(f, c) \cup U^1(f, c)$. Hence, every local extremum should be avoided in order to obtain a proper dyadic subbase. We do not fix real numbers c_n first, but give open intervals I_n from which c_n will be taken.

Lemma 6.3. There exist a sequence $(f_n)_{n<\omega}$ of continuous functions and a sequence $(I_n)_{n<\omega}$ of open intervals such that the family $\{U^a(f_n, c_n) \mid n < \omega,\ a \in \{0, 1\}\}$ is a subbase of X if $c_n \in I_n$ for $n < \omega$.

Proof. Since X is separable, there exists a dense countable set $\{x_n \in X \mid n < \omega\}$. Suppose that $\{B_n \mid n < \omega\}$ is a countable base of the space $\mathbb{R}_{>0}$ of positive real numbers. Note that if $b_n \in B_n$ for all $n < \omega$, then the set $\{b_n \mid n < \omega\}$ is dense in $\mathbb{R}_{>0}$. Let $n \mapsto (n_0, n_1)$ be a map from ω onto $\omega \times \omega$. We define $f_n(x) := d(x_{n_0}, x)$ and $I_n := B_{n_1}$. If $c_n \in I_n$ for all $n < \omega$, then the family $\{U^a(f_n, c_n) \mid n < \omega,\ a \in \{0, 1\}\}$ is a subbase of X because $\{U^0(f_n, c_n) \mid n < \omega\}$ forms a base. By definition, $U^0(f_n, c_n)$ and $U^1(f_n, c_n)$ are disjoint for all $n < \omega$. $\square$

As the following lemma shows, the set of local extrema is countable.

Lemma 6.4. Every function $f : X \to \mathbb{R}$ has at most countably many local extrema.

Proof. Let $\{B_n \mid n < \omega\}$ be a countable base of X. Since each local extremum is an extremum of $f|_{B_n}$ for some B_n, the number of local extrema of f is countable. $\square$

In order to obtain the properness property, we have only to avoid local extrema of finitely many functions. Therefore, we can avoid them.

Lemma 6.5. Let A be a subset of X, $f : X \to \mathbb{R}$ a continuous function, c a real number. If $c \in \mathbb{R}$ is not a local extremum of $f|_{\mathrm{cl}\, A}$, then we have $\mathrm{cl}\, A \setminus U^{1-a}(f, c) = \mathrm{cl}(A \cap U^a(f, c))$ for $a \in \{0, 1\}$.

Proof. We can see $\mathrm{cl}\, A \setminus U^{1-a}(f, c) \supseteq \mathrm{cl}\, A \cap \mathrm{cl}\, U^a(f, c) \supseteq \mathrm{cl}(A \cap U^a(f, c))$. Suppose that $x \in \mathrm{cl}\, A \setminus U^{1-a}(f, c)$ and V is an open neighborhood of x. Since c is not a local extremum of $f|_{\mathrm{cl}\, A}$, there exists $y \in V \cap \mathrm{cl}\, A$ such that $y \in U^a(f, c)$, i.e., $V \cap \mathrm{cl}\, A \cap U^a(f, c) \neq \emptyset$. Since $U^a(f, c)$ and V are open, we have $V \cap A \cap U^a(f, c) \neq \emptyset$. Therefore, we obtain $x \in \mathrm{cl}(A \cap U^a(f, c))$. $\square$

Proof of Proposition 6.2. By Lemma 6.3 we can take a sequence $(f_n)_{n<\omega}$ of continuous functions and a sequence $(I_n)_{n<\omega}$ of open intervals, such that the family $\{U^a(f_n, c_n) \mid n < \omega,\ a \in \{0, 1\}\}$ is a subbase of X if $c_n \in I_n$ for $n < \omega$.

First, we take $c_0 \in I_0$ which is not a local extremum of f_0, and set $S_0^a := U^a(f_0, c_0)$ for $a \in \{0, 1\}$. By Lemma 6.5, S_0^0 and S_0^1 are exteriors of each other.

Let n be a finite ordinal. Suppose that we have obtained a family $\{S_k^a \mid k < n, a \in \{0, 1\}\}$ such that $\bar{S}(\sigma) = \mathrm{cl}\, S(\sigma)$ for all $\sigma \in \mathbb{T}^n$. We take a real number $c_n \in I_n$ which is not a local extremum of $f_n|_{\bar{S}(\sigma)}$ for all $\sigma \in \mathbb{T}^n$. We set $S_n^a := U^a(f_n, c_n)$ for $a \in \{0, 1\}$. For all $\sigma \in \mathbb{T}^n$ and $a \in \{0, 1\}$, we have

$$\bar{S}(\sigma[n \mapsto a]) = \bar{S}(\sigma) \setminus S_n^{1-a} = \operatorname{cl} S(\sigma) \setminus S_n^{1-a}$$

by the assumption. Since c_n is not a local extremum of $f_n|_{\operatorname{cl} S(\sigma)}$, by Lemma 6.5, we obtain

$$\operatorname{cl} S(\sigma) \setminus S_n^{1-a} = \operatorname{cl}(S(\sigma) \cap S_n^a) = \operatorname{cl} S(\sigma[n \mapsto a]).$$

Therefore, $\bar{S}(\sigma) = \operatorname{cl} S(\sigma)$ holds for all $\sigma \in \mathbb{T}^{n+1}$. Hence, we obtain a proper dyadic subbase inductively. □

6.2 Fat points

In the proof of Proposition 6.2, c_n could be a local extremum of $f_n|_{f_m^{-1}(c_m)}$ for some $m, n < \omega$. If c_n is a local extremum of $f_n|_{f_m^{-1}(c_m)}$, then $U^0(f_n, c_n) \cap U^\partial(f_m, c_m)$ and $U^1(f_n, c_n) \cap U^\partial(f_m, c_m)$ are not exteriors of each other in the space $U^\partial(f_m, c_m)$, and we fail to obtain the strong properness. We have to avoid c_m such that already chosen c_n will be a local extremum of $f_n|_{f_m^{-1}(c_m)}$. In Section 6.3, we show that such real numbers can be avoided if the space X is locally compact.

In the rest of this section, X is a locally compact separable metric space.

Definition 6.6. A subset $A \subseteq X$ is

(i) *codense* if $\operatorname{int} A = \emptyset$.

(ii) *nowhere dense* if $\operatorname{int} \operatorname{cl} A = \emptyset$.

(iii) *meagre* if A is a countable union of nowhere dense subsets.

Let r be a non-negative integer, $f : X \to \mathbb{R}^r$ a continuous map. We say that a point $x \in X$ is *fat* with respect to f if $f(V)$ has an interior point for every neighborhood V of x. The set of all fat points with respect to f is denoted by $\operatorname{fat}_f X$.

Lemma 6.7. Let $f : X \to \mathbb{R}^r$ be a continuous map. For all subset $A \subseteq X$, we have $\operatorname{fat}_f A \subseteq \operatorname{fat}_f X \cap A$.

Proof. Let $x \in \operatorname{fat}_f A$ be a point, $V \subseteq X$ its open neighborhood. We can see $x \in A$. Since x belongs to $\operatorname{fat}_f A$, $f(A \cap V)$ has an interior point. Therefore, $f(V) \supseteq f(A \cap V)$ also has an interior point, and hence $x \in \operatorname{fat}_f X$. □

By definition, X has no fat point with respect to f if $f(X)$ is codense. We show its converse.

Proposition 6.8. Let r be a non-negative integer, $f : X \to \mathbb{R}^r$ a continuous map. If $\operatorname{fat}_f X$ is empty, then $f(X)$ is codense.

We make a remark about the case in which r is zero. $\mathbb{R}^0$ is a one point set and every point $x \in X$ is mapped to the same point by f. Therefore, we have $\operatorname{fat}_f X = X$. Note that for all subset A of a one point set, we have

$$A \text{ is codense } \Leftrightarrow A \text{ is nowhere dense } \Leftrightarrow A \text{ is meagre } \Leftrightarrow A = \emptyset.$$

Therefore, we can easily see that Proposition 6.8 holds in this case.

Baire category theorem states that every meagre subset of a complete metric space is codense. Since $\mathbb{R}^r$ is a complete metric space, we have the following.

Lemma 6.9 (Baire category theorem)**.** Every meagre subset of $\mathbb{R}^r$ is codense.

Proof. Suppose that K_n is a nowhere dense subset of $\mathbb{R}^r$ for all $n < \omega$. For any non-empty open subset $V \subseteq \mathbb{R}^r$, there exists a sequence of non-empty open subsets $(V_n)_{n<\omega}$ such that $V_0 \subseteq V \setminus K_0$ and $\operatorname{cl} V_n \subseteq V_{n-1} \setminus K_n$ for $0 < n$. We can see that $\bigcap_{n<\omega} V_n$ is non-empty. □

Proof of Proposition 6.8. Suppose that X has no fat point and $\{B_n \mid n < \omega\}$ is a base of X. For all $x \in X$, there exists a compact neighborhood $\operatorname{cl} B_n$ of x such that $f(\operatorname{cl} B_n)$ is codense. Note that the image of a compact set by a continuous map is always compact and every compact codense subset is nowhere dense. Therefore, we have $X = \bigcup\{\operatorname{cl} B_n \mid f(\operatorname{cl} B_n) \text{ is nowhere dense}\}$. We can see that $f(X) = \bigcup\{f(\operatorname{cl} B_n) \mid f(\operatorname{cl} B_n) \text{ is nowhere dense}\}$ is meagre, and hence $f(X)$ is codense by Lemma 6.9. □

We will take real numbers that determine a strongly proper dyadic subbase inductively. At each step, we will choose the real number c_n so that for every $m > n$, the set of real numbers which we have to avoid as c_m is meagre. First, we show that the pairwise condition can be satisfied.

Proposition 6.10. Let $f : X \to \mathbb{R}$ be a continuous function, r a non-negative integer, $g : X \to \mathbb{R}^r$ a continuous map, c a real number which is not a local extremum of $f|_{\operatorname{fat}_g X}$. The set $\{p \in \mathbb{R}^r \mid c \text{ is a local extremum of } f|_{g^{-1}(p)}\}$ is meagre.

Lemma 6.11. Suppose that f, g and c are as above. For any compact subset K of X, $g(K \cap U^\partial(f,c)) \setminus g(U^a(f,c))$ is nowhere dense for $a \in \{0,1\}$.

Proof. For $a \in \{0,1\}$, we have

$$g(K \cap U^\partial(f,c)) \setminus g(U^a(f,c)) \subseteq g(K \cap U^\partial(f,c)) \setminus \operatorname{int} g(U^a(f,c)).$$

We can see that the right hand side is closed. Therefore, we have only to show that $g(K \cap U^\partial(f,c)) \setminus \operatorname{int} g(U^a(f,c))$ is codense for $a \in \{0,1\}$. Let $V \subseteq g(K \cap U^\partial(f,c))$ be a non-empty open subset. We can see that $K \cap U^\partial(f,c) \cap g^{-1}(V)$ is a locally compact separable metric space. Therefore, by Proposition 6.8, the set $\operatorname{fat}_g(K \cap U^\partial(f,c) \cap g^{-1}(V))$ is not empty. By Lemma 6.7, we obtain

$$\begin{aligned} \emptyset \neq \operatorname{fat}_g(K \cap U^\partial(f,c) \cap g^{-1}(V)) &\subseteq \operatorname{fat}_g X \cap K \cap U^\partial(f,c) \cap g^{-1}(V) \\ &\subseteq \operatorname{fat}_g X \cap U^\partial(f,c) \cap g^{-1}(V). \end{aligned}$$

Since c is not a local extremum of $f|_{\operatorname{fat}_g X}$, we obtain $\operatorname{fat}_g X \cap U^a(f,c) \cap g^{-1}(V) \neq \emptyset$ for $a \in \{0,1\}$, and thus $g(U^a(f,c))$ has an interior point in V. Hence, $g(K \cap U^\partial(f,c)) \setminus \operatorname{int} g(U^a(f,c))$ is codense for $a \in \{0,1\}$. □

Lemma 6.12. Suppose that f, g and c are as above. For any open subset B of X, $g(B \cap U^\partial(f,c)) \setminus (g(B \cap U^0(f,c)) \cap g(B \cap U^1(f,c)))$ is meagre.

Proof. Since B is locally compact separable metric space, there exists a covering $\bigcup_{n<\omega} K_n = B$ consisting of countably many compact subsets K_n. Since c is not a local extremum of $f|_{\mathrm{fat}_g B}$, by Lemma 6.11, $g(K_n \cap U^\partial(f,c)) \setminus g(B \cap U^a(f,c))$ is nowhere dense for all $n < \omega$ and $a \in \{0,1\}$. Therefore, their union

$$\bigcup_{n<\omega, a\in\{0,1\}} g(K_n \cap U^\partial(f,c)) \setminus g(B \cap U^a(f,c))$$
$$= g(B \cap U^\partial(f,c)) \setminus (g(B \cap U^0(f,c)) \cap g(B \cap U^1(f,c)))$$

is meagre. □

Proof of Proposition 6.10. Let $\{B_n \mid n < \omega\}$ be a countable base of X. By Lemma 6.12, $g(B_n \cap U^\partial(f,c)) \setminus (g(B_n \cap U^0(f,c)) \cap g(B_n \cap U^1(f,c)))$ is meagre for all $n < \omega$. Therefore, their union

$$M := \bigcup_{n<\omega} (g(B_n \cap U^\partial(f,c)) \setminus (g(B_n \cap U^0(f,c)) \cap g(B_n \cap U^1(f,c)))$$

is meagre.

Suppose that c is a local extremum of $f|_{g^{-1}(p)}$ for a point $p \in \mathbb{R}^r$. There exists a base element B_n such that $B_n \cap g^{-1}(p) \cap U^\partial(f,c)$ is non-empty, and either $B_n \cap g^{-1}(p) \cap U^0(f,c)$ or $B_n \cap g^{-1}(p) \cap U^1(f,c)$ is empty. That is, p belongs to $g(B_n \cap U^\partial(f,c)) \setminus (g(B_n \cap U^0(f,c)) \cap g(B_n \cap U^1(f,c)))$. Therefore, we obtain

$$\{p \in \mathbb{R}^r \mid c \text{ is a local extremum of } f|_{g^{-1}(p)}\} \subseteq M.$$

□

We will show that if the set of $(r+1)$-tuples that we should avoid is meagre, then we have only to avoid a meagre subset at each step. For a real number c, a hyperplane $H_c \subseteq \mathbb{R}^{r+1}$ is defined as $\{(x_0, \ldots, x_r) \mid x_r = c\}$.

Proposition 6.13. Suppose that $M \subseteq \mathbb{R}^{r+1}$ is meagre. The set $\{c \in \mathbb{R} \mid M \cap H_c \text{ is meagre in } H_c\}$ is comeagre, i.e., its complement is meagre.

Lemma 6.14. Suppose that $K \subseteq \mathbb{R}^{r+1}$ is nowhere dense. The set $\{c \in \mathbb{R} \mid \mathrm{cl}\, K \cap H_c \text{ is codense in } H_c\}$ is comeagre.

Proof. Let $\{B_n \mid n < \omega\}$ be a countable base of $\mathbb{R}^r$. For a real number c, $B_n \times \{c\}$ denotes the set $\{(x_0, \ldots, x_{r-1}, c) \mid (x_0, \ldots, x_{r-1}) \in B_n\}$. Note that if $\mathrm{cl}\, K \cap H_c$ is not codense in H_c, then an open subset of H_c is contained in $\mathrm{cl}\, K$, and therefore, $B_n \times \{c\} \subseteq \mathrm{cl}\, K$ for some $n < \omega$. For all $n < \omega$, the set $\{c \in \mathbb{R} \mid B_n \times \{c\} \subseteq \mathrm{cl}\, K\}$ is nowhere dense because $\mathrm{cl}\, K$ is nowhere dense. We can see that their union

$$\bigcup_{n<\omega} \{c \in \mathbb{R} \mid B_n \times \{c\} \subseteq \mathrm{cl}\, K\} = \{c \in \mathbb{R} \mid \mathrm{cl}\, K \cap H_c \text{ is not codense in } H_c\}$$

is meagre. Therefore, its complement $\{c \in \mathbb{R} \mid \mathrm{cl}\, K \cap H_c \text{ is codense in } H_c\}$ is comeagre. □

Lemma 6.15. Suppose that $K \subseteq \mathbb{R}^{r+1}$ is nowhere dense. The set $\{c \in \mathbb{R} \mid K \cap H_c \text{ is nowhere dense in } H_c\}$ is comeagre.

Proof. We have $\operatorname{cl}(K \cap H_c) \subseteq \operatorname{cl} K \cap H_c$. Since H_c is closed, $\operatorname{cl}(K \cap H_c)$ equals the closure of $K \cap H_c$ in the space H_c. Therefore, if $\operatorname{cl} K \cap H_c$ is codense in H_c, then $K \cap H_c$ is nowhere dense in H_c. That is, we have

$$\{c \in \mathbb{R} \mid \operatorname{cl} K \cap H_c \text{ is codense in } H_c\} \subseteq \{c \in \mathbb{R} \mid K \cap H_c \text{ is nowhere dense in } H_c\}.$$

By Lemma 6.14, the left hand side is comeagre. □

Proof of Proposition 6.13. Suppose that $M \subseteq \mathbb{R}^{r+1}$ is meagre. There exists a countable covering $M = \bigcup_{n<\omega} K_n$, where K_n is nowhere dense for $n < \omega$. By Lemma 6.15, $\{c \in \mathbb{R} \mid K_n \cap H_c \text{ is nowhere dense in } H_c\}$ is comeagre for all $n < \omega$. By De Morgan's law, the intersection of countably many comeagre subsets is comeagre. Therefore, their intersection

$$C := \bigcap_{n<\omega} \{c \in \mathbb{R} \mid K_n \cap H_c \text{ is nowhere dense in } H_c\}$$

is comeagre. If $c \in C$, then $K_n \cap H_c$ is nowhere dense in H_c for all $n < \omega$, and therefore, $\bigcup_{n<\omega} K_n \cap H_c = M \cap H_c$ is meagre in H_c. Hence, the set $\{c \in \mathbb{R} \mid M \cap H_c \text{ is meagre in } H_c\} \supseteq C$ is comeagre. □

6.3 Proof of Theorem 6.1

6.3.1 Construction of a strongly proper dyadic subbase

By Lemma 6.3, we can take a sequence $(f_n)_{n<\omega}$ of continuous functions and a sequence $(I_n)_{n<\omega}$ of open intervals, such that the family $\{U^a(f_n, c_n) \mid n < \omega,\ a \in \{0,1\}\}$ is a subbase of X if $c_n \in I_n$ for $n < \omega$.

Every subset of ω is represented by the domain of a sequence of $\{\top, \bot\}$, where the *top character* $\top$ means non-bottom. For a sequence $\upsilon \in \{\top, \bot\}^*$, f_υ denotes the map $f_\upsilon : X \to \mathbb{R}^{\operatorname{dom}(\upsilon)}$ given by $f_\upsilon(x) = (f_k(x))_{k \in \operatorname{dom}(\upsilon)}$.

Take c_0 which is not a local extremum of $f_0|_{\operatorname{fat}_{f_\upsilon} X}$ for all $\upsilon \in \{\top, \bot\}^*$ with $\upsilon(0) = \bot$. Set $S_0^a := U^a(f_0, c_0)$ for $a \in \{0, 1, \partial\}$. By Proposition 6.10, the set $\{p \in \mathbb{R}^{\operatorname{dom}(\upsilon)} \mid c_0 \text{ is a local extremum of } f_0|_{f_\upsilon^{-1}(p)}\}$ is meagre for all $\upsilon = \bot\upsilon' \in \{\top, \bot\}^*$.

Let $n > 0$ be a finite ordinal. Suppose that we have obtained a sequence $(c_i)_{i<n}$ of real numbers such that

$$\begin{aligned}&\forall \tau \in \{\partial, \bot\}^n.\ \forall \upsilon = \bot^n \upsilon' \in \{\top, \bot\}^*.\ \forall k \in n \setminus \operatorname{dom}(\tau).\\ &\qquad \{p \in \mathbb{R}^{\operatorname{dom}(\upsilon)} \mid c_k \text{ is a local extremum of } f_k|_{S(\tau) \cap f_\upsilon^{-1}(p)}\} \text{ is meagre.}\end{aligned} \tag{6.3}$$

We set $S_i^a = U^a(f_i, c_i)$ for all $i < n$ and $a \in \{0, 1, \partial\}$. Setting $\tau = \upsilon = \bot$, we can see that c_k is not a local extremum of f_k for all $k < n$. Therefore, by Lemma 6.5, $X \setminus S_k^{1-a} = \operatorname{cl} S_k^a$ for all $k < n$ and $a \in \{0, 1\}$, i.e., S_k^∂ is the common boundary of S_k^0 and S_k^1 for all $k < n$. Thus, we can justify the notation $S(\tau)$ for $\tau \in \{\partial, \bot\}^n$. By Proposition 6.13, if $n \in \operatorname{dom}(\upsilon)$,

$$\begin{aligned}&\{c' \in \mathbb{R} = \mathbb{R}^{\{n\}} \mid \\ &\quad \{p \in \mathbb{R}^{\operatorname{dom}(\upsilon)\setminus\{n\}} \mid c_k \text{ is a local extremum of } f_k|_{S(\tau) \cap f_n^{-1}(c') \cap f_{\upsilon[n\mapsto\bot]}^{-1}(p)}\}\\ &\qquad\qquad\qquad\qquad \text{is meagre}\}\end{aligned}$$

is comeagre for all $\tau \in \{\partial, \bot\}^n$, $\upsilon = \bot^n \upsilon' \in \{\top, \bot\}^*$ and $k \in n \setminus \mathrm{dom}(\tau)$. Therefore, their intersection C_n is comeagre. By Lemma 6.4, the set of local extrema of $f_n|_{\mathrm{fat}_{f_\upsilon} S(\tau)}$ is countable for all $\tau \in \{\partial, \bot\}^n$ and $\upsilon = \bot^{n+1}\upsilon' \in \{\top, \bot\}^*$. Their union E_n is countable. We can take c_n from $(I_n \cap C_n) \setminus E_n$, and we set $S_n^a := U^a(f_n, c_n)$ for $a \in \{0, 1, \partial\}$. Note that $S(\tau)$ is a locally compact separable metric space for all $\tau \in \{\partial, \bot\}^n$. Since $c_n \notin E_n$, by Proposition 6.10, the set

$$\{p \in \mathbb{R}^{\mathrm{dom}(\upsilon)} \mid c_n \text{ is a local extremum of } f_n|_{S(\tau)\cap f_\upsilon^{-1}(p)}\}$$

is meagre for all $\tau \in \{\partial, \bot\}^n$ and $\upsilon = \bot^{n+1}\upsilon' \in \{\top, \bot\}^*$. Therefore, we can obtain a sequence $(c_n)_{n<\omega}$ which satisfies (6.3) for all $n < \omega$ inductively.

6.3.2 Proof of strong properness

Suppose that $(f_n)_{n<\omega}$ is a sequence of functions, $(c_n)_{n<\omega}$ is a sequence of real numbers, (6.3) holds for all $n < \omega$ and $\{S_n^a \mid n < \omega,\, a \in \{0, 1\}\}$ forms a subbase of X, where $S_n^a = U^a(f_n, c_n)$ for $n < \omega$ and $a \in \{0, 1\}$. We can easily see that S is a dyadic subbase of X. As we have remarked, from (6.3), we can deduce the fact that S_k^∂ is the common boundary of S_k^0 and S_k^1 for all $k < n$. Since (6.3) holds for all $n < \omega$, S_n^∂ is the common boundary of S_n^0 and S_n^1 for all $n < \omega$. Therefore, we can use the notations $S(\sigma)$ and $\bar{S}(\sigma)$ for all $\sigma \in \{0, 1, \partial, \bot\}^*$.

Lemma 6.16. For all $n < \omega$, c_n is not a local extremum of $f_n|_{\bar{S}(\sigma)}$ for all $\sigma \in \{0, 1, \partial, \bot\}^*$ with $\sigma(n) = \bot$.

Proof. In (6.3), setting $\upsilon := \bot$, we can see that c_k is not a local extremum of $f_k|_{S(\tau)}$ for all $\tau \in \{\partial, \bot\}^n$ and $k \in n \setminus \mathrm{dom}(\tau)$. Since (6.3) holds for all $n < \omega$, c_n is not a local extremum of $f_n|_{S(\tau)}$ for all $\tau \in \{\partial, \bot\}^*$ and $n \in \omega \setminus \mathrm{dom}(\tau)$.

Suppose that c_n is a local extremum of $f_n|_{\bar{S}(\sigma)}$ for a sequence $\sigma \in \{0, 1, \partial, \bot\}^*$ with $\sigma(n) = \bot$. There exists an open subset $V \subseteq X$ such that c_n is an extremum of $f_n|_{V\cap\bar{S}(\sigma)}$. Take a point $x \in V \cap \bar{S}(\sigma) \cap f_n^{-1}(c_n)$ and set $W := V \cap S(\varphi_S(x)|_{\mathrm{dom}(\sigma)})$. Let $\tau \in \{\partial, \bot\}^*$ be a sequence whose domain is the set $\mathrm{dom}(\sigma) \setminus \mathrm{dom}(\varphi_S(x))$. By definition, we have $S(\varphi_S(x)|_{\mathrm{dom}(\sigma)}) \cap S(\tau) = S(\varphi_S(x)^\partial|_{\mathrm{dom}(\sigma)})$. Since we have $S_k^{\varphi_S(x)^\partial(k)} \subseteq \mathrm{cl}\, S_k^{\sigma(k)}$ for all $k \in \mathrm{dom}(\sigma)$, we get $S(\varphi_S(x)^\partial|_{\mathrm{dom}(\sigma)}) \subseteq \bar{S}(\sigma)$. Therefore, we have

$$\begin{aligned} W \cap S(\tau) &= V \cap S(\varphi_S(x)|_{\mathrm{dom}(\sigma)}) \cap S(\tau) \\ &= V \cap S(\varphi_S(x)^\partial|_{\mathrm{dom}(\sigma)}) \\ &\subseteq V \cap \bar{S}(\sigma). \end{aligned}$$

We can see that c_n is an extremum value of $f|_{W\cap S(\tau)}$ because $W \cap S(\tau)$ is a subset of $V \cap \bar{S}(\sigma)$ and contains the point $x \in f_n^{-1}(c_n)$. Since W is open, c_n is a local extremum of $f|_{S(\tau)}$, a contradiction. □

Proposition 6.17. S is a strongly proper dyadic subbase.

Proof. Let $\sigma \in \{0, 1, \partial, \bot\}^*$ be a sequence, $\tau_0 \in \{\partial, \bot\}^*$ a sequence with $\mathrm{dom}(\tau_0) = \sigma^{-1}(\partial)$. Since the cardinality of $\sigma^{-1}(\{0, 1\})$ is finite, we can take a finite sequence $(\tau_k)_{k<m}$ such that $\tau_{k+1} = \tau_k[n \mapsto \sigma(n)]$ for some $n \in \mathrm{dom}(\sigma) \setminus \mathrm{dom}(\tau_k)$ for all $k < m$, and $\tau_{m-1} = \sigma$. We show that $\bar{S}(\tau_k) = \mathrm{cl}\, S(\tau_k)$ for all

$k < m$ by induction. By definition, we have $\bar{S}(\tau_0) = S(\tau_0) = \operatorname{cl} S(\tau_0)$. Assume that we have $\bar{S}(\tau_k) = \operatorname{cl} S(\tau_k)$ and $\tau_{k+1} = \tau_k[n \mapsto a]$ for some $k < m$, $n \in \operatorname{dom}(\sigma) \setminus \operatorname{dom}(\tau_k)$ and $a = \sigma(n) \in \{0, 1\}$. By the assumption, we have $\bar{S}(\tau_k[n \mapsto a]) = \bar{S}(\tau_k) \cap \operatorname{cl} S_n^a = \operatorname{cl} S(\tau_k) \cap \operatorname{cl} S_n^a$. By Lemma 6.16, c_n is not a local extremum of $f_n|_{\bar{S}(\tau_k)}$. By Lemma 6.5, we obtain $\operatorname{cl} S(\tau_k) \cap \operatorname{cl} S_n^a = \operatorname{cl}(S(\tau_k) \cap S_n^a) = \operatorname{cl} S(\tau_{k+1})$. □

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