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“Sequential Auctions of Heterogeneous Objects”

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Sequential Auctions of Heterogeneous Objects*

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Abstract

We consider sequential second-price auctions in which heterogeneous objects are sold to bidders with unit demand and a single dimensional type. We show that a symmetric increasing equilibrium exists if objects are ordered in terms of dispersiveness of value distributions. Equilibrium price declines when objects are equivalent on average and additional conditions hold.

Keywords: sequential auctions, declining price anomaly, dispersiveness

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1 Introduction

Sequential auctions are a popular institution for selling multiple objects. In a canonical independent private values model of homogeneous objects, equilibrium price is martingale (Weber 1983). However, many empirical studies suggest that price tends to decrease in actual markets (Ashenfelter 1989, Ashenfelter and Genesove 1992, Beggs and Graddy 1997). To solve this "declining price anomaly," many theoretical studies examine various realistic extensions, which include risk aversion (McAfee and Vincent 1983, Mezzetti 2011), uncertainty about the future (Bernhardt and Scoones 1994, Engelbrecht-Wiggans 1994, Jeitschko 1999), and goods heterogeneity (Kittsteiner et al. 2004).

Beggs and Graddy (1997) empirically examine heterogeneous objects and declining price. Kittsteiner et al. (2004) formulate a declining valuation model and show declining price in equilibrium. However, as Beggs and Graddy point out, valuation or quality of objects does not always decline in many applications such as art and used cars.

This paper extends Kittsteiner et al. (2004) and incorporates a more general goods heterogeneity that allows increasing valuations. We consider sequential second-price auctions, with bidders having unit demand and a single dimensional type. We show that the dispersive order of sales is critical for both the existence of a symmetric increasing equilibrium and the declining price phenomenon. A symmetric increasing equilibrium exists if a more dispersive object is sold first. Under the dispersive order of sales, declining price in equilibrium is reasonable when the objects are equivalent on average, i.e., when they have the same mean value. Expected price declines in equilibrium when there are many bidders or when values are symmetrically distributed. Moreover, the declining price in the "deflated scale" is explained solely by the dispersive order of sales.

2 Model and Equilibrium

The seller allocates two heterogeneous objects A and B in sequential second-price auctions.[1] The seller allocates object A in the first auction and then object B in the second.

[1] The analysis is straightforwardly extended to the case with more than two objects.

There are $n \geq 3$ potential buyers, each of whom has unit demand. Each bidder i has private information about his *type* $z_i \in [\underline{z}, \bar{z}]$, which is independently and identically distributed. The cumulative distribution function of z_i is denoted by F, which has a density function $f > 0$. Bidder i's value of $k = A, B$ is denoted by $v^k(z_i)$, where $v^k(\cdot)$ is identical to all bidders. Both v^A and v^B are continuous and strictly increasing in z_i. Without loss of generality, let $v^A(z) \equiv z$ and $v^B(z) \equiv w(z)$.

We assume that the winner of the first auction does not participate in the second auction even if $z_i < w(z_i)$.[2] We focus on a symmetric increasing Bayesian Nash equilibrium, in which bidders take a symmetric increasing pure strategy. Note that all remaining bidders in the second auction submit their true values $w(z_i)$ in a weakly dominant strategy. Hence, we focus on the incentives in the first auction.

The key condition of the paper is the dispersiveness of value distributions.

Definition 1 The value distribution of object A is *more dispersive* than that of object B if $w'(z) < 1$ for all z. In short, we say that object A is *more dispersive* than B.

We assume that a more dispersive object is sold first. When we consider two random variables z and $y = w(z)$, our notion of dispersiveness is equivalent to the dispersive order in stochastic orders (Shaked and Shanthikumer 2007). Kittsteiner et al. (2004) and Elmaghraby (2003) assume the dispersive order of sales to guarantee an equilibrium. Bernhardt and Scoones (1994) also consider a similar notion of dispersion, which is weaker than ours.

Suppose that a bidding function $b^1(z_i)$ forms a symmetric increasing equilibrium in the first auction. Then, the equilibrium bid should satisfy the following Euler equation (first-order condition) of the payoff maximization:

$$z_i - b^1(z_i) = E[w(z_i) - w(z_{n-1}^{(2)}) | z_{n-1}^{(2)} < z_i], \tag{1}$$

where $z_m^{(j)}$ indicates the j-th highest order statistic of m IID random variables. The left-hand side is the marginal profit in the first auction by slightly increasing the first

[2] This is also assumed in the model of stochastically equivalent objects by Engelbrecht-Wiggans (1994) and Bernhardt and Scoones (1994).

bid $b^1(z_i)$. In a second-price auction, the marginal change in bid affects the bidder's own payoff only when his bid is very critical: $z_i = z_{n-1}^{(1)}$. Hence, the marginal increase in bid generates the ex-post gain of $z_i - b^1(z_i)$ at the first auction. The right-hand side is the option value of participating in the second auction. In the marginal case of $z_i = z_{n-1}^{(1)}$, the bidder would win the second auction with probability 1 after losing in the first auction. These two effects balance in equilibrium. Equation (1) immediately provides the equilibrium bidding function at the first auction. The dispersive order of sales guarantees the Euler equation to be sufficient for equilibrium.

Proposition 1 *Sequential second-price auctions have a symmetric increasing equilibrium if object A is more dispersive than B. The equilibrium bidding function at the first auction is*

$$b^1(z_i) = z_i - w(z_i) + E[w(z_{n-1}^{(2)})|z_{n-1}^{(2)} < z_i]. \tag{2}$$

Proof. All proofs are presented in the Appendix.

3 Equilibrium Price Trend

Let p^t be the equilibrium price at the t-th auction. Because $p^1 = b^1(z_n^{(2)})$ and $p^2 = w(z_n^{(3)})$, (2) implies

$$p^1 = z_n^{(2)} - w(z_n^{(2)}) + E[p^2|p^1]. \tag{3}$$

Hence, the expected price $E[p^t]$ declines if and only if $E[z_n^{(2)}] > E[w(z_n^{(2)})]$.

Hereafter, let us assume that objects are *equivalent on average*: i.e., both objects A and B have the same mean value.

Assumption 1 $E[z] = E[w(z)] = \mu$.

Under Assumption 1, the expected price tends to decline with reasonable additional conditions. We provide two cases in which equilibrium price declines.

3.1 Large Number of Bidders

The first scenario is such that a large number of bidders exist. When $w' < 1$, Assumption 1 indicates $z > w(z)$ for a sufficiently large z. As $n \to \infty$, we have $E[z_n^{(2)}] \to \bar{z}$, which leads to $E[z_n^{(2)}] > E[w(z_n^{(2)})]$.

Proposition 2 *Suppose that Assumption 1 holds and that object A is more dispersive than B. The expected price declines, i.e., $E[p^1] > E[p^2]$, if the number n of bidders is sufficiently large.*

3.2 Symmetric Distributions

Another scenario is such that the value distributions of A and B are symmetric. The value distribution of object A is *symmetric* if $\mu = (\underline{z}+\bar{z})/2$ and $f(\mu-z) = f(\mu+z)$ for all $z \leq \mu$. In addition, the distribution of object B is also symmetric if $w'(\mu-z) = w'(\mu+z)$ for $z \leq \mu$. Under symmetric distributions, the equilibrium price declines even for a small number of bidders.

Proposition 3 *Suppose that the number of bidders $n \geq 4$ and that object A is more dispersive than B. Further suppose that Assumption 1 holds and that both distributions are symmetric. Then, the equilibrium price declines: $E[p^1] > E[p^2]$.*

3.3 Deflated Price Trend

Kittsteiner et al. (2004) consider sequential auctions with declining valuations and assume an equivalent condition to $w'(z) \leq 1$ (Assumption A5, p. 93). Their main theorem states that if object A is (weakly) more dispersive and $w(z) \leq z$ for all z, then

$$p^1 \geq w(p^1) \geq E[p^2|p^1] \tag{4}$$

(Theorem 2, p. 94). That is, equilibrium price declines in devaluated or deflated scale. This result is independent of declining valuations $w(z) \leq z$ but derived only from the dispersive order of objects. Indeed, because $p^1 < z_n^{(2)}$ must hold in equilibrium, the dispersiveness indicates

$$z_n^{(2)} - p^1 > w(z_n^{(2)}) - w(p^1).$$

Substituting (3) for p^1, we have

$$w(p^1) > w(z_n^{(2)}) - z_n^{(2)} + p^1 = E[p^2|p^1].$$

The deflated price declines when a more dispersive object is sold first, regardless of level of valuation. Thus, the declining price anomaly observed in the case of heterogeneous objects by Beggs and Graddy (1997) is solved.

A Proofs

A.1 Proof of Proposition 1

We assume $w'(z) < 1$ for all z. Suppose that all bidders other than i take $b^1(z)$ in the first auction, which is defined by (2). The expected payoff for bidder i when he bids $b^1(\hat{z}_i)$ in the first auction and bids his true valuation $w(z_i)$ in the second is given by:

$$U_i(z_i, \hat{z}_i) = \int_{\underline{z}}^{\hat{z}_i} [z_i - b^1(y_1)] f^{(1)}(y_1) \mathrm{d}y_1 + \int_{\hat{z}_i}^{\overline{z}} \int_{\underline{z}}^{\min\{y_1, z_i\}} [w(z_i) - w(y_2)] f^{(1,2)}(y_1, y_2) \mathrm{d}y_2 \mathrm{d}y_1,$$

where $f^{(k)}$ denotes the density of $z_{n-1}^{(k)}$ and $f^{(1,2)}$ denotes the joint density of $z_{n-1}^{(1)}$ and $z_{n-1}^{(2)}$. We shall show that for all $\hat{z}_i > z_i$, $\frac{\partial}{\partial \hat{z}_i} U_i(z_i, \hat{z}_i) < 0$ and for all $\hat{z}_i < z_i$, $\frac{\partial}{\partial \hat{z}_i} U_i(z_i, \hat{z}_i) > 0$.

Suppose $\hat{z}_i > z_i$. We have

$$\frac{\partial}{\partial \hat{z}_i} U_i(z_i, \hat{z}_i) = [z_i - b^1(\hat{z}_i)] f^{(1)}(\hat{z}_i) - \int_{\underline{z}}^{z_i} [w(z_i) - w(y_2)] f^{(1,2)}(\hat{z}_i, y_2) \mathrm{d}y_2.$$

Then,

$$\begin{aligned} \frac{\frac{\partial}{\partial \hat{z}_i} U_i(z_i, \hat{z}_i)}{f^{(1)}(\hat{z}_i)} &= z_i - b^1(\hat{z}_i) - \int_{\underline{z}}^{z_i} [w(z_i) - w(y_2)] f^{(2)}(y_2 | z_{n-1}^{(1)} = \hat{z}_1) \mathrm{d}y_2 \\ &< z_i - b^1(\hat{z}_i) - \int_{\underline{z}}^{\hat{z}_i} [w(z_i) - w(y_2)] f^{(2)}(y_2 | z_{n-1}^{(1)} = \hat{z}_1) \mathrm{d}y_2 \qquad (5) \\ &= z_i - b^1(\hat{z}_i) - w(z_i) + E[w(z_{n-1}^{(2)}) | z_{n-1}^{(2)} < \hat{z}_i]. \end{aligned}$$

The inequality holds because $w(z_i) - w(y_2) < 0$ for $y_2 > z_i$. Substituting (2) for $b^1(\hat{z}_i)$, (5) yields

$$\begin{aligned}\frac{\frac{\partial}{\partial \hat{z}_i} U_i(z_i, \hat{z}_i)}{f^{(1)}(\hat{z}_i)} &< z_i - b^1(\hat{z}_i) - w(z_i) + E[w(z_{n-1}^{(2)}) | z_{n-1}^{(2)} < \hat{z}_i] \\ &= z_i - \hat{z}_i - (w(z_i) - w(\hat{z}_i)) < 0.\end{aligned}$$

The last inequality comes from $w'(z) < 1$.

Suppose $\hat{z}_i < z_i$. Similarly to the previous paragraph, we have the result as follows:

$$\begin{aligned}\frac{\frac{\partial}{\partial \hat{z}_i} U_i(z_i, \hat{z}_i)}{f^{(1)}(\hat{z}_i)} &= z_i - b^1(\hat{z}_i) - w(z_i) + E[w(z_{n-1}^{(2)}) | z_{n-1}^{(2)} < \hat{z}_i] \\ &= z_i - \hat{z}_i - (w(z_i) - w(\hat{z}_i)) > 0.\end{aligned}$$

■

A.2 Proof of Proposition 2

Let $g(z) = z - w(z)$. By (3), it suffices to show $E[g(z_n^{(2)})] > 0$ for a large n. Because g is increasing by the dispersive order of sales and $E[g(z)] = 0$, there exists $\hat{z}$ and $g(z) > 0$ for all $z > \hat{z}$. Let $F_n^{(2)}$ be the CDF of $z_n^{(2)}$; thus,

$$\begin{aligned}F_n^{(2)}(z) &= F^n(z) + n(1 - F(z))F^{n-1}(z) \\ &\leq F^{n-1}(z)(n + F(z)).\end{aligned} \tag{6}$$

Notice that for every $\delta \in (0, 1)$, $\delta^{n-1} n$ converges to 0 as $n \to \infty$. Therefore, for every $z \in (\underline{z}, \bar{z})$, $F_n^{(2)}(z)$ converges to 0 as $n \to \infty$. For an arbitrary $z \geq \hat{z}$, we have

$$\begin{aligned}E[g(z_n^{(2)})] &= \int_{\underline{z}}^{z} g(s) f_n^{(2)}(s) \mathrm{d}s + \int_{z}^{\bar{z}} g(s) f_n^{(2)}(s) \mathrm{d}s \\ &> \int_{\underline{z}}^{z} g(\underline{z}) f_n^{(2)}(s) \mathrm{d}s + \int_{z}^{\bar{z}} g(z) f_n^{(2)}(s) \mathrm{d}s \\ &= F_n^{(2)}(z) g(\underline{z}) + (1 - F_n^{(2)}(z)) g(z).\end{aligned} \tag{7}$$

Hence, we have $E[g(z_n^{(2)})] > 0$ for a sufficiently large n. ■

A.3 Proof of Proposition 3

Note that for every pair of generic realizations $(z_n^{(2)}, z_n^{(j)})$ with $j \geq 3$, dispersiveness implies

$$z_n^{(2)} - z_n^{(j)} > w(z_n^{(2)}) - w(z_n^{(j)}). \tag{8}$$

Case 1. The number of bidders n is odd; $n = 2m + 1$ and $m \geq 2$. By symmetric distributions, the sample median is an unbiased estimator of its mean: $E[z_n^{(m+1)}] = E[w(z_n^{(m+1)})] = \mu$. By taking expectation of (8) with $j = m + 1$, we have

$$\begin{aligned} E[z_n^{(2)} - z_n^{(m+1)}] &> E[w(z_n^{(2)}) - w(z_n^{(m+1)})] \\ \therefore\ E[z_n^{(2)} - w(z_n^{(2)})] &> 0. \end{aligned} \tag{9}$$

Case 2. The number of bidders n is even; $n = 2m$ and $m \geq 2$. By symmetric distributions, we have

$$E\Big[\frac{z_n^{(m)} + z_n^{(m+1)}}{2}\Big] = E\Big[\frac{w(z_n^{(m)}) + w(z_n^{(m+1)})}{2}\Big] = \mu.$$

In a similar manner to Case 1, we have

$$\begin{aligned} \frac{1}{2}\Big(E[z_n^{(2)} - z_n^{(m)}] + E[z_n^{(2)} - z_n^{(m+1)}]\Big) &> \frac{1}{2}\Big(E[w(z_n^{(2)}) - w(z_n^{(m)})] + E[w(z_n^{(2)}) - w(z_n^{(m+1)})]\Big) \\ \therefore\ E[z_n^{(2)} - w(z_n^{(2)})] &> 0. \end{aligned} \tag{10}$$

■

References

[1] Ashenfelter, O. (1989), "How Auctions Work for Wine and Art," *Journal of Economic Perspectives*, 3, 23–36.

[2] Ashenfelter, O., and D. Genesove (1992), "Testing for Price Anomalies in Real-Estate Auctions," *American Economic Review*, 80, 501–505.

[3] Beggs, A., and K. Graddy (1997), "Declining Values and the Afternoon Effect: Evidence from Art Auctions." *The Rand Journal of Economics*, 28, 544–565.

[4] Bernhardt, D., and D. Scoones (1994), "A Note on Sequential Auctions," *American Economic Review*, 84, 653–657.

[5] Elmaghraby, W. (2003), "The Importance of Ordering in Sequential Auctions," *Management Science*, 49, 673–682.

[6] Engelbrecht-Wiggans, R. (1994), "Sequential Auctions of Stochastically Equivalent Objects," *Economics Letters*, 44, 87–90.

[7] Jeitschko, T.D. (1999), "Equilibrium Price Paths in Sequential Auctions with Stochastic Supply," *Economics Letters*, 64, 67–72.

[8] Kittsteiner, T., J. Nikutta, and E. Winter (2004), "Declining Valuations in Sequential Auctions," *International Journal of Game Theory*, 33, 89–106.

[9] McAfee, R.P., and D. Vincent (1993), "The Declining Price Anomaly," *Journal of Economic Theory*, 60, 191–212.

[10] Mezzetti, C. (2011), "Sequential Auctions with Informational Externalities and Aversion to Price Risk: Decreasing and Increasing Price Sequences," *The Economic Journal*, 121, 990–1016.

[11] Shaked, M., and J.G. Shanthikumar (2007), *Stochastic Orders*, Springer, New York.

[12] Weber, R.J. (1983), "Multiple-Object Auctions," in Engelbrecht-Wiggans, R., M. Shubik, R.M. Stark (eds.), *Auctions, Bidding, and Contracting: Uses and Theory*, New York University Press, New York, 165–191.