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# Fluctuations of the cosmic background radiation appearing in the 10-dimensional cosmological model

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We consider a cosmological model starting from (1) the $(1 + 3 + 6)$-dimensional space-times consisting of the outer space (the 3-dimensional expanding section) and the inner space (the 6-dimensional section) and reaching (2) the Friedmann model after the decoupling of the outer space from the inner space, and derive fluctuations of the background radiation appearing in the above 10-dimensional space-times. For this purpose we first derive the fluid-dynamical perturbations in the above 10-dimensional space-times, corresponding to two kinds of curvature perturbations (in the scalar mode) in the non-viscous case, and next study the quantum fluctuations in the scalar and tensor modes, appearing at the stage when the perturbations are within the horizon of the inflating outer space. Lastly we derive the wave-number dependence of fluctuations (the power spectrum) in the two modes, which formed at the above decoupling epoch and are observed in the Friedmann stage. It is found that this can be consistent with the observed spectra of the cosmic microwave background radiation.

Subject Index E80, E84

## 1. Introduction

In order to derive the observed fluctuations of cosmic microwave background radiation, we study the cosmological evolution of the $(1 + 3 + 6)$-dimensional space-times, in which it is assumed that our universe was born as an isotropic and homogeneous 10-dimensional space-time and evolved to the state consisting of the three-dimensional inflating outer space and the six-dimensional collapsing inner space. Our four-dimensional Friedmann universe appeared after the decoupling of the outer space from the inner space. This scenario is supported by the present superstring theory (Kim et al. [1,2] in a matrix model).

In a previous paper [3] we discussed the entropy production at the stage when the above inflation and collapse coexist, and showed how viscous processes help the increase of cosmological entropy. We also discussed the possibility that we satisfy, at the same time, the condition that the entropy in the Guth level [4] is obtained and the condition that the inner space decouples from the outer space. In the subsequent paper [5], we studied the evolution of cosmological perturbations in the non-viscous case, solving the equations for geometrical perturbations.

In this paper we treat the fluidal perturbations corresponding to the geometrical perturbations, the quantum fluctuations in the scalar and tensor modes, and the consistency with the observations of cosmic microwave background radiation (CMB) in the non-viscous case. In Sect. 2, we first review the background model and the perturbed quantities. In Sect. 3, we derive the perturbed fluid-dynamical

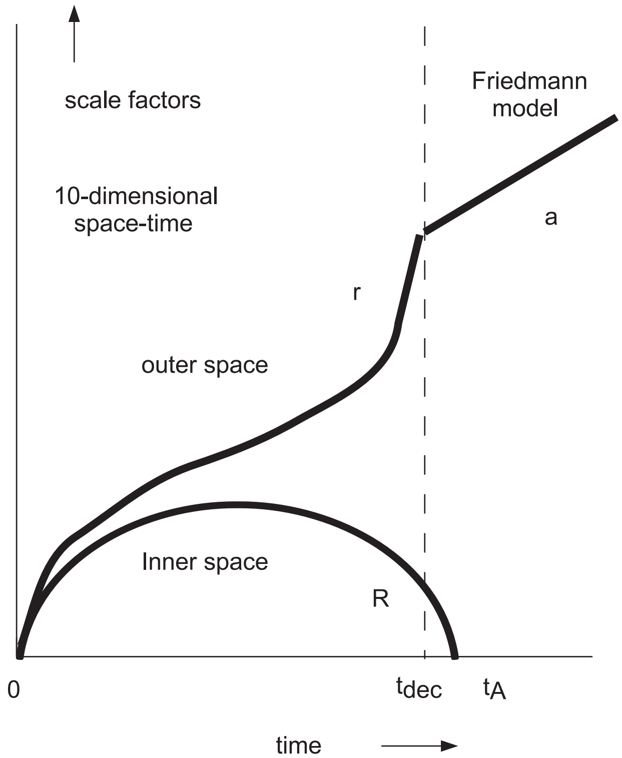


**Fig. 1.** Scale factors of outer and inner spaces in the 10-dimensional space-times and the Friedmann model. $t_{dec}$ and $t_A$ denote the decoupling epoch (when the Friedmann model starts) and the singular epoch of the inner space, respectively.

equations corresponding to the geometrical perturbations in the 10-dimensional space-times, and solve them. In Sect. 4, we consider the quantum fluctuations in the scalar and tensor modes at the stage when they were within the horizon of the outer space with the inflationary expansion, and derive the initial conditions for their perturbations appearing after this stage. In Sect. 5, we derive the spectra of perturbations in these two modes, and compare them with the observed ones. In Sect. 6, concluding remarks are given. In Appendix A, we derive the higher-order terms in the two curvature perturbations with respect to small wave-numbers in the outer space.

## 2. Review of the background model and the perturbed quantities

### *2.1. Our background model*

We consider a cosmological model starting from (1) the $(1 + 3 + 6)$-dimensional space-times consisting of the outer space (the 3-dimensional expanding section) and the inner space (the 6-dimensional section), and reaching (2) the Friedmann model after the decoupling of the outer space from the inner space, as shown schematically in Fig. 1.

#### *2.1.1. Background 10-dimensional model before the decoupling*

The background 10-dimensional space-time is expressed in the form of a product of two homogeneous spaces $\mathrm{M_d}$ and $\mathrm{M_D}$ as

$$ds^2 = -dt^2 + r^2(t)\ {}^d g_{ij}\left(x^k\right)\ dx^i dx^j + R^2(t)\ {}^D g_{ab}\left(X^c\right)\ dX^a dX^b, \tag{1}$$

where ${}^d g_{ij}$ and ${}^D g_{ab}$ are the metrics of the outer space $\mathrm{M_d}$ and the inner space $\mathrm{M_D}$ with constant curvatures $K_r$ and $K_R$, respectively. Here the dimensions of $\mathrm{M_d}$ and $\mathrm{M_D}$ are $d = 3$ and $D = 6$. The inner space $M_D$ expands initially and collapses after the maximum expansion with $K_R = 1$, while the outer space $M_d$ continues to expand with $K_r = 0$ or $-1$. Then the background metric is

$$\begin{aligned} g_{00} &= -1, \quad g_{01} = g_{0a} = g_{ia} = 0, \\ g_{ij} &= r^2\, {}^d g_{ij}, \quad g_{ab} = R^2\, {}^D g_{ab}, \end{aligned} \tag{2}$$

and the Ricci tensor is

$$\begin{aligned} R^0_0 &= -\left(d\frac{\ddot{r}}{r} + D\frac{\ddot{R}}{R}\right), \\ R^i_j &= -\delta^i_j \left[\left(\frac{\dot{r}}{r}\right)^{\cdot} + \frac{\dot{r}}{r}\left(d\frac{\dot{r}}{r} + D\frac{\dot{R}}{R}\right) + (d-1)\frac{K_r}{r^2}\right], \\ R^a_b &= -\delta^a_b \left[\left(\frac{\dot{R}}{R}\right)^{\cdot} + \frac{\dot{R}}{R}\left(d\frac{\dot{r}}{r} + D\frac{\dot{R}}{R}\right) + (D-1)\frac{K_R}{R^2}\right], \end{aligned} \tag{3}$$

where $i, j = 1, \dots, d,\ a, b = d+1, \dots, d+D$, and an overdot denotes $d/dt$. At the singular stage when $R$ is near 0, the curvature terms with $K_r/r^2$ and $K_R/R^2$ are negligible, compared with the main terms, and the curvatures can be treated approximately as $K_r = K_R = 0$. The background energy–momentum tensor is

$$T^\mu_\nu = p\delta^\mu_\nu + (\rho + p)u^\mu u_\nu, \tag{4}$$

where $u^\mu$ is the fluid velocity, $\rho$ the energy density, and $p$ the pressure. Here $\rho$ and $p$ are the common photon density and pressure in both spaces. The fluid is extremely hot and satisfies the equation of state $p = \rho/n$ of photon gas, where $n = d + D = 9$. The Einstein equations are expressed as

$$R^\mu_\nu = -8\pi\bar{G}\left(T^\mu_\nu - \tfrac{1}{2}\delta^\mu_\nu T^\lambda_\lambda\right), \tag{5}$$

where $\bar{G}$ is the $(1 + d + D)$-dimensional gravitational constant. In the following, we set $8\pi\bar{G} = 1$. The background equation of motion for the matter is

$$\frac{\dot{\rho}}{\rho + p} + d\frac{\dot{r}}{r} + D\frac{\dot{R}}{R} = 0. \tag{6}$$

The Einstein equations for $r$ and $R$ were solved numerically in the previous paper [3] and their behaviors were shown in Figs. 1–7 of [3]. At the early stage, the expansion of the total universe is nearly isotropic (i.e. $r \propto R$). At the later stage, the inner space collapses after the maximum expansion, and at the final stage we have an approximate solution

$$r = r_0\, \tau^\eta, \quad R = R_0\, \tau^\gamma \quad (r_0,\ R_0 : const) \tag{7}$$

with

$$\eta = \left\{1 - [D(n-1)/d]^{1/2}\right\}/n, \quad \gamma = \left\{1 + [d(n-1)/D]^{1/2}\right\}/n, \tag{8}$$

and $\tau = t_A - t$, where $t_A$ is the final time corresponding to $R = 0$. For $d = 3$ and $D = 6$, we have

$$\gamma = -\eta = 1/3. \tag{9}$$

For the solutions (7), Eqs. (4) and (5) lead to $R^0_0 = 0$ and $T^0_0 - \frac{1}{2}T^\mu_\mu \propto \rho$, so that we have

$$\rho = 0 \tag{10}$$

at the final stage. The curvature tensor is singular, on the other hand, in the limit $\tau \to 0$, like that in the four-dimensional Kasner space-time [6].

*2.1.2. Decoupling of the two spaces and the Friedmann stage*

As $\tau$ decreases, the inner space contracts and finally the size reaches the Planck length at the decoupling epoch. We discussed the decoupling condition at this quantum-gravitational epoch and the entropy production to this epoch in the previous paper [3].

At present we cannot analyze the process of decoupling accurately, because no quantum theory of gravitation has been established yet. However, it is expected that the inner space is so homogeneous and quietly evolves without violent phenomena. This is because in both the inner and outer spaces the perturbations are assumed to be caused by quantum fluctuations before the decoupling, grow gravitationally, and remain very small and at the linear stage, before the decoupling. Note here that gravitational instability in the outer and inner spaces was treated in the previous paper [5]. Thus the inner space separates quietly from the outer space and disappears, while in the outer space the Friedmann model appears after the decoupling.

After this decoupling epoch it is assumed here that the outer space is separated from the inner space and described using the Friedmann model with the metric

$$ds^2 = -dt_f{}^2 + a^2(t_f)\left[d\chi^2 + \sigma^2(\chi)\left(d\theta^2 + \sin^2\theta d\Omega^2\right)\right], \tag{11}$$

with the cosmic time $t_f$ and $\sigma(\chi) = \sin\chi, \chi, \sinh\chi$ for curvature $+, 0, -$ in the space, and the scale factor $a(t_f) \propto t_f{}^{1/2}$ at the radiation-dominated hot stage. At the decoupling epoch $t_{dec}$ and $(t_f)_{dec}$, the entropy is assumed to be conserved in the 10-dimensional space-time and the Friedmann model. The behavior of the scale factors is shown in Fig. 1.

*2.1.3. Horizon crossing*

The ratio of physical sizes of perturbations with the wavelength $1/k$ to the Hubble length $1/H$ is

$$r(\tau)H/k \propto \tau^{-4/3} = (t_A - t)^{-4/3} \tag{12}$$

before the decoupling, where $r \propto \tau^{-1/3}$ and $H \propto \tau^{-1}$, and it increases with time.

After the decoupling epoch we have the ratio at the Friedmann stage,

$$a(t_f)H/k \propto t_f{}^{-1/2}, \tag{13}$$

which decreases with time. The two ratios are nearly equal at the decoupling epoch.

These ratios are shown schematically in Fig. 2. They can take the value 1 in both sides before and after the decoupling epoch, that is, we can have the horizon crossing in both sides. Quantum fluctuations are created at the epoch of $r(\tau)H/k < 1$ in the outer space of the 10-dimensional space-time, and the fluctuations are observed as the fluctuations of the background radiation at the epochs of $a(t_f)H/k < 1$ at the Friedmann stage.

## 2.2. Perturbed quantities

The simplest treatment of perturbations of geometrical and fluidal quantities is to expand them using harmonics, and to find the gauge-invariant quantities. For the four-dimensional universe (in the Friedmann model) it was shown in Bardeen's theory on perturbations [7]. In the multi-dimensional universe consisting of the outer and inner homogeneous spaces $\mathrm{M_d}$ and $\mathrm{M_D}$ with different geometrical structures, we can have no harmonics in the $(d + D)$-dimensional space. Abbott et al. [8] considered separate expansions in $\mathrm{M_d}$ and $\mathrm{M_D}$ using the harmonics defined in the individual spaces, classified the perturbations in $\mathrm{M_d}$ and $\mathrm{M_D}$ individually as scalar (S), vector (V), and tensor (T), and classified the six types of perturbations in $\mathrm{M_d} + \mathrm{M_D}$ into three modes: the scalar mode (including SS), the

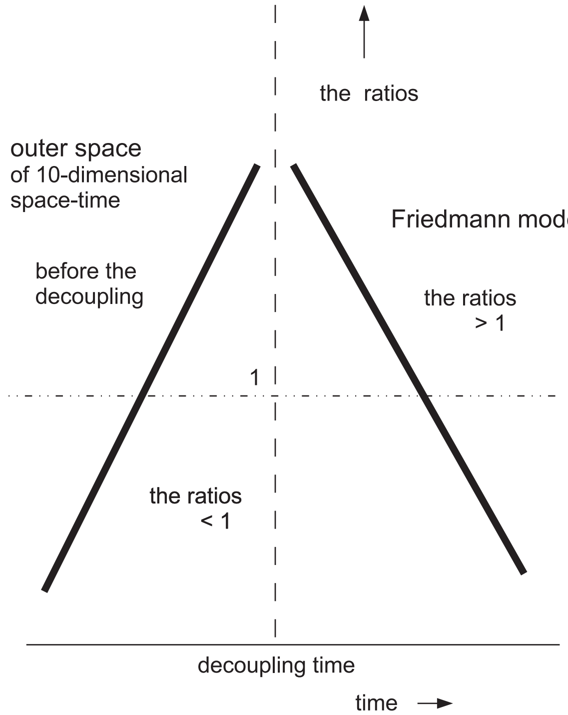


**Fig. 2.** Ratios of physical sizes of perturbations with the wave-number $k$ to the Hubble length $1/H$. The ratio $r(\tau)H/k$ in the outer space of the 10-dimensional space-times ($\tau \equiv t - t_A$) is shown as the solid line on the left-hand side, and the ratio $a(t_f)H/k$ in the Friedmann model is shown as the solid line on the right-hand side.

vector mode (including SV, VS, VV), and the tensor mode (including ST, TS). The left and right sides of signatures correspond to the types of perturbations in $M_d$ and $M_D$, respectively.

In this paper only scalar and tensor modes are considered in the 10-dimensional space-times. So quantities in these modes are shown here.

### 2.2.1. *The scalar mode*

The metric perturbations are expressed as

$$\begin{aligned} g_{00} &= -\left(1 + 2A\, q^{(0)} Q^{(0)}\right), \\ g_{0i} &= -rb^{(0)} q_i^{(0)} Q^{(0)}, \quad g_{0a} = -RB^{(0)} q^{(0)} Q_a^{(0)}, \\ g_{ij} &= r^2 \left[\left(1 + 2h_L q^{(0)} Q^{(0)}\right) {}^d g_{ij} + 2h_T^{(0)} q_{ij}^{(0)} Q^{(0)}\right], \\ g_{ab} &= R^2 \left[\left(1 + 2H_L q^{(0)} Q^{(0)}\right) {}^D g_{ab} + 2H_T^{(0)} q^{(0)} Q_{ab}^{(0)}\right], \\ g_{ia} &= 2rRG^{(0)} q_i^{(0)} Q_a^{(0)}, \end{aligned} \tag{14}$$

where $q^{(0)}$, $q_i^{(0)}$, $q_{ij}^{(0)}$ and $Q^{(0)}$, $Q_a^{(0)}$, $Q_{ab}^{(0)}$ are scalar harmonics in $M_d$ and $M_D$, respectively, and $A$, $b^{(0)}$, $B^{(0)}$, $h_L$, $H_L$, $h_T^{(0)}$, $H_T^{(0)}$, and $G^{(0)}$ are functions of $t$.

The perturbations of fluid velocities and the energy–momentum tensor are expressed as

$$u^0 = 1 - Aq^{(0)} Q^{(0)}, \quad u^i = \frac{v^{(0)}}{r} q^{(0)i} Q^{(0)}, \quad u^a = \frac{V^{(0)}}{R} q^{(0)} Q^{(0)a}, \tag{15}$$

and

$$\begin{aligned}
&T_0^0 = -\rho\left(1+\delta\, q^{(0)} Q^{(0)}\right),\\
&T_i^0 = r\,(\rho+p)\left(v^{(0)}-b^{(0)}\right) q_i^{(0)} Q^{(0)}, \quad T_a^0 = R\,(\rho+p)\left(V^{(0)}-B^{(0)}\right) q^{(0)} Q_a^{(0)},\\
&T_j^i = p\left(1+\pi_L q^{(0)} Q^{(0)}\right)\delta_j^i, \quad T_b^a = p\left(1+\Pi_L q^{(0)} Q^{(0)}\right)\delta_b^a,\\
&T_a^i = 0,
\end{aligned} \tag{16}$$

where we consider a perfect fluid, so that the anisotropic pressure terms vanish and we have

$$\pi_L = \Pi_L = \delta. \tag{17}$$

For the metric perturbations in Eq. (14) the following gauge-invariant quantities are defined:

$$\begin{aligned}
\Phi_h &= h_L + \frac{h_T^{(0)}}{d} + \frac{r}{k_r^{(0)}}\frac{\dot r}{r} b^{(0)} - \frac{r^2}{k_r^{(0)2}}\frac{\dot r}{r}\dot h_T^{(0)},\\
\Phi_H &= h_L + \frac{H_T^{(0)}}{D} + \frac{R}{k_R^{(0)}}\frac{\dot R}{R} B^{(0)} - \frac{R^2}{k_R^{(0)2}}\frac{\dot R}{R}\dot H_T^{(0)},
\end{aligned} \tag{18}$$

$$\begin{aligned}
\Phi_A^{(r)} &= A + \frac{r}{k_r^{(0)}}\dot b^{(0)} + \frac{r}{k_r^{(0)}}\left(\frac{\dot r}{r} + D\frac{\dot R}{R}\right) b^{(0)}\\
&\quad - \frac{r^2}{k_r^{(0)2}}\left[\ddot h_T^{(0)} + \left(2\frac{\dot r}{r} + D\frac{\dot R}{R}\right)\dot h_T^{(0)}\right] + D\left(H_L + \frac{H_T^{(0)}}{D}\right),\\
\Phi_A^{(R)} &= A + \frac{R}{k_R^{(0)}}\dot B^{(0)} + \frac{R}{k_R^{(0)}}\left(d\frac{\dot r}{r} + \frac{\dot R}{R}\right) B^{(0)}\\
&\quad - \frac{R^2}{k_R^{(0)2}}\left[\ddot H_T^{(0)} + \left(d\frac{\dot r}{r} + 2\frac{\dot R}{R}\right)\dot H_T^{(0)}\right] + d\left(h_L + \frac{h_T^{(0)}}{d}\right).
\end{aligned} \tag{19}$$

The gauge-invariant quantities $\Phi_h$ and $\Phi_A^{(r)}$ in the outer space correspond to the gauge-invariant perturbations defined by Bardeen [7] in the $(1+3)$-dimensional usual universes, and $\Phi_H$ and $\Phi_A^{(R)}$ in the inner space are similar to the above quantities. $\Phi_h$ and $\Phi_H$ represent the curvature perturbations in both spaces.

The gauge-invariant quantities for fluid velocity and energy density perturbations are given by

$$v_s^{(0)} = v^{(0)} - \frac{r}{k_r^{(0)}}\dot h_T^{(0)}, \quad V_s^{(0)} = V^{(0)} - \frac{R}{k_R^{(0)}}\dot H_T^{(0)}, \tag{20}$$

and

$$\epsilon_m = \delta + \frac{n+1}{n}\left[d\frac{\dot r}{k_r^{(0)}}\left(v^{(0)} - b^{(0)}\right) + D\frac{\dot R}{k_R^{(0)}}\left(V^{(0)} - B^{(0)}\right)\right]. \tag{21}$$

It should be noticed that $v_s^{(0)}$, $V_s^{(0)}$, and $\epsilon_m$ do not vanish, though $\rho = 0$ at the final stage as in Eq. (10). As a gauge-invariant quantity that has no counterpart in the usual universe, we have

$$\Phi_G = G^{(0)} - \frac{1}{2}\frac{k_R^{(0)}}{k_r^{(0)}}\frac{r}{R}h_T^{(0)} - \frac{1}{2}\frac{k_r^{(0)}}{k_R^{(0)}}\frac{R}{r}H_T^{(0)}. \tag{22}$$

Moreover, the auxiliary quantities ($\Phi_6$ and $\Phi_7$) and $\tilde{\Phi}_G$ are defined by

$$\frac{\dot r}{r}\frac{\dot R}{R}\Phi_6 \equiv \frac{\dot R}{R}\left(h_L + \frac{h_T^{(0)}}{d}\right) - \frac{\dot r}{r}\left(H_L + \frac{H_T^{(0)}}{D}\right), \tag{23}$$

$$\Phi_7 \equiv (r/\dot r)\Phi_h - (R/\dot R)\Phi_H - \Phi_6, \tag{24}$$

and

$$\tilde{\Phi}_G \equiv \frac{rR}{k_r^{(0)}k_R^{(0)}}\Phi_G. \tag{25}$$

*2.2.2. The tensor mode*

We have only metric perturbations given by

$$\begin{aligned} g_{00} &= -1, \quad g_{0i} = g_{0a} = g_{ia} = 0, \\ g_{ij} &= r^2\left({}^d g_{ij} + 2h_T^{(2)}q_{ij}^{(2)}Q^{(0)}\right), \\ g_{ab} &= R^2\left({}^D g_{ab} + 2H_T^{(2)}q^{(0)}Q_{ab}^{(2)}\right), \end{aligned} \tag{26}$$

and have no fluidal perturbations, where we have neglected anisotropic stresses. In this mode, $h_T^{(2)}$ and $H_T^{(2)}$ correspond to the TS and ST parts of curvature perturbations and they themselves are gauge invariant.

More details about perturbations can be seen in the previous paper [5].

## 3. Evolution of fluidal perturbations in the scalar mode

In the previous paper [5], we derived the equations for geometrical perturbations $\Phi_h$, $\Phi_H$, $\tilde{\Phi}_G$, and $\Phi_6$ in the 10-dimensional space-times, and found their behavior by solving them. In this section we derive the equations for gauge-invariant variables representing fluidal perturbations $\epsilon_m$, $v_s^{(0)}$, and $V_s^{(0)}$ from the equations $\delta T^{\nu}_{\mu;\nu} = 0$ with $\mu = 0$, $i$, and $a$, respectively, and derive their behaviors, where the suffices $\nu$, $i$, and $a$ take the values $0 \sim d + D$, $1 \sim d$, and $d + 1 \sim d + D$, respectively, in the outer and inner spaces with dimensions $d$ and $D$, respectively, where $d = 3$ and $D = 6$. In the following, $v_s^{(0)}$ and $V_s^{(0)}$ are expressed as $v_s$ and $V_s$ for simplicity.

First, we obtain the following equation for $\dot\epsilon_m$ from $\delta T^{A}_{0;A} = 0$:

$$\begin{aligned} &\frac{n}{n+1}\left[\dot\epsilon_m - \frac{1}{n}\left(d\frac{\dot r}{r} + D\frac{\dot R}{R}\right)\epsilon_m\right] \\ &\quad = -d\left\{\dot\Phi_h + \left[D\frac{\dot R}{R} + (d-2)\frac{\dot r}{r}\right]\Phi_h\right\} + \left[-\frac{k_r^{(0)}}{r} + \frac{d}{k_r^{(0)}}\left(\ddot r - \frac{\dot r^2}{r}\right)\right]v_s \\ &\qquad - D\left\{\dot\Phi_H + \left[d\frac{\dot r}{r} + (D-2)\frac{\dot R}{R}\right]\Phi_H\right\} + \left[-\frac{k_R^{(0)}}{R} + \frac{D}{k_R^{(0)}}\left(\ddot R - \frac{\dot R^2}{R}\right)\right]V_s \\ &\qquad - 2\left[d\frac{\dot r}{r}\left(\frac{k_R^{(0)}}{R}\right)^2 + D\frac{\dot R}{R}\left(\frac{k_r^{(0)}}{r}\right)^2\right]\tilde{\Phi}_G, \end{aligned} \tag{27}$$

where $n = d + D$, $p = \rho/n$, a dot denotes $d/dt$, $k_r^{(0)}$ and $k_R^{(0)}$ are the wave-numbers in the outer and inner spaces, respectively, $r = r_0\tau^{-1/3}$, $R = R_0\tau^{1/3}$, $\tau = t_0 - t$, and $t_0$ denotes the epoch of $r \to \infty$ and $R = 0$.

Equations for $\dot{v}_s$ and $\dot{V}_s$ are obtained from $\delta T^A_{i;A} = 0$ and $\delta T^A_{a;A} = 0$, respectively, as

$$\dot{v}_s + \left(\frac{\dot{r}}{r} - \frac{D}{n}\frac{\dot{R}}{R}\right) v_s$$
$$= -\frac{D}{n}\frac{k_r^{(0)}}{k_R^{(0)}}\frac{\dot{R}}{r} V_s + \frac{\epsilon_m}{n+1}\frac{k_r^{(0)}}{r} - \frac{k_r^{(0)}}{r}\left[(d-2)\,\Phi_h + D\Phi_H + 2\left(\frac{k_R^{(0)}}{R}\right)^2 \tilde{\Phi}_G + \frac{n+1}{n} D\frac{\dot{R}}{R}\Phi_7\right], \tag{28}$$

and

$$\dot{V}_s + \left(\frac{\dot{R}}{R} - \frac{d}{n}\frac{\dot{r}}{r}\right) V_s$$
$$= -\frac{d}{n}\frac{k_R^{(0)}}{k_r^{(0)}}\frac{\dot{r}}{R} v_s + \frac{\epsilon_m}{n+1}\frac{k_R^{(0)}}{R} - \frac{k_R^{(0)}}{R}\left[(D-2)\,\Phi_H + d\Phi_h + 2\left(\frac{k_r^{(0)}}{r}\right)^2 \tilde{\Phi}_G - \frac{n+1}{n} d\frac{\dot{r}}{r}\Phi_7\right]. \tag{29}$$

From the latter two equations we obtain

$$\left(\frac{r}{k_r^{(0)}} v_s - \frac{R}{k_R^{(0)}} V_s\right)^{\cdot} = 2\,(\Phi_h - \Phi_H) + \frac{1}{n}\left(d\frac{\dot{r}}{r} + D\frac{\dot{R}}{R}\right)\left(\frac{r}{k_r^{(0)}} v_s - \frac{R}{k_R^{(0)}} V_s\right)$$
$$- 2\left[\left(\frac{k_r^{(0)}}{r}\right)^2 - \left(\frac{k_R^{(0)}}{R}\right)^2\right]\tilde{\Phi}_G - \frac{n+1}{n}\left(d\frac{\dot{r}}{r} + D\frac{\dot{R}}{R}\right)\Phi_7, \tag{30}$$

and, integrating this equation, we have

$$\frac{r}{k_r^{(0)}} v_s - \frac{R}{k_R^{(0)}} V_s = -\tau^{1/9}\int d\tau\; \tau^{-1/9}\left\{2\,(\Phi_h - \Phi_H) - 2\left[\left(\frac{k_r^{(0)}}{r}\right)^2 - \left(\frac{k_R^{(0)}}{R}\right)^2\right]\tilde{\Phi}_G\right.$$
$$\left. - \frac{n+1}{n}\left(d\frac{\dot{r}}{r} + D\frac{\dot{R}}{R}\right)\Phi_7\right\} \;\equiv\; A_0\left(\Phi_h, \Phi_H, \tilde{\Phi}_G, \Phi_7\right). \tag{31}$$

From Eqs. (27) and (31), we obtain

$$\frac{n}{n+1}\dot{\epsilon}_m = \frac{1}{n+1}\left(d\frac{\dot{r}}{r} + D\frac{\dot{R}}{R}\right)\epsilon_m - A_1 + C_h v_s + C_H V_s, \tag{32}$$

and from Eq. (31)

$$V_s = \frac{k_R^{(0)}}{R}\left(\frac{r}{k_r^{(0)}} v_s - A_0\right), \tag{33}$$

where

$$A_1 \equiv d\left\{\dot{\Phi}_h + \left[D\frac{\dot{R}}{R} + (d-2)\,\frac{\dot{r}}{r}\right]\Phi_h\right\}$$
$$+ D\left\{\dot{\Phi}_H + \left[d\frac{\dot{r}}{r} + (D-2)\,\frac{\dot{R}}{R}\right]\Phi_H\right\} + 2\left[d\frac{\dot{r}}{r}\left(\frac{k_R^{(0)}}{R}\right)^2 + D\frac{\dot{R}}{R}\left(\frac{k_r^{(0)}}{r}\right)^2\right]\tilde{\Phi}_G, \tag{34}$$

$$C_h \equiv -\frac{k_r^{(0)}}{r} + \frac{d}{k_r^{(0)}}\left(\ddot{r} - \dot{r}^2/r\right), \tag{35}$$

$$C_H \equiv -\frac{k_R^{(0)}}{R} + \frac{D}{k_R^{(0)}}\left(\ddot{R} - \dot{R}^2/R\right). \tag{36}$$

Next, differentiating Eq. (32) with respect to $t$ and eliminating $v_s$ and $\dot{v}_s$ using Eqs. (28) and (32), we obtain the following equations for $\epsilon_m$ and $v_s$:

$$\ddot{\epsilon}_m + D_0\dot{\epsilon}_m + D_1\epsilon_m = \frac{n+1}{n}E \tag{37}$$

and

$$v_s = \frac{1}{\tilde{C}}\left\{\frac{1}{n+1}\left[n\dot{\epsilon}_m - \left(d\frac{\dot{r}}{r} + D\frac{\dot{R}}{R}\right)\epsilon_m\right] + A_1 + C_H\frac{k_R^{(0)}}{R}A_0\right\}, \tag{38}$$

where

$$\tilde{C} \equiv C_h + \frac{k_R^{(0)}}{k_r^{(0)}}\frac{r}{R}C_H, \tag{39}$$

$$D_0 \equiv \frac{D}{n}\left(\frac{\dot{r}}{r} - \frac{\dot{R}}{R}\right) - \frac{\dot{\tilde{C}}}{\tilde{C}}, \tag{40}$$

$$D_1 \equiv -\frac{1}{n}\left(d\frac{\dot{r}}{r} + D\frac{\dot{R}}{R}\right)^{\cdot} - \frac{1}{n}\left(d\frac{\dot{r}}{r} + D\frac{\dot{R}}{R}\right)\left(\frac{\dot{r}}{r} - \frac{\dot{\tilde{C}}}{\tilde{C}}\right) - \frac{k_r^{(0)}}{r}\frac{\tilde{C}}{n}, \tag{41}$$

$$A_2 \equiv (d-2)\,\Phi_h + D\Phi_H + 2\left(\frac{k_R^{(0)}}{R}\right)^2\tilde{\Phi}_G + \frac{n+1}{n}D\frac{\dot{R}}{R}\Phi_7. \tag{42}$$

$$E \equiv -\left(A_1 + \frac{k_R^{(0)}}{R}C_H A_0\right)\left(\frac{\dot{r}}{r} - \frac{\dot{\tilde{C}}}{\tilde{C}}\right) - \dot{A}_1 - k_R^{(0)}\left(\frac{C_H}{R}A_0\right)^{\cdot} + \left(\frac{D}{n}\frac{\dot{R}}{R}A_0 - A_2\right)\frac{k_r^{(0)}}{r}\tilde{C}. \tag{43}$$

### *3.1. Outside the horizons*

At epoch $t_{dec}$ when the outer space and the inner space decouple, $\tau$ is assumed to be so small that

$$x \equiv \frac{3}{4r_0}k_r^{(0)}\tau^{4/3} \ll 1 \tag{44}$$

and

$$y \equiv \frac{3}{2R_0}k_R^{(0)}\tau^{2/3} \ll 1. \tag{45}$$

Under these conditions the perturbations with wave-numbers $k_r^{(0)}$ and $k_R^{(0)}$ are outside the horizons in the outer and inner spaces, and we have the relations

$$\Phi_h = (\tau/\tau_i)^{-8/3}(\Phi_h)_i + \Delta\Phi_h, \tag{46}$$

$$\Phi_H = (\tau/\tau_i)^{-4/3}(\Phi_H)_i + \Delta\Phi_H, \tag{47}$$

where $\Delta\Phi_h$ and $\Delta\Phi_H$ consist of higher-order terms $O(x^2)$ and $O(y^2)$ with respect to $x$ and $y$, respectively, which are shown in Appendix A, and $\tau_i$ is an arbitrary epoch at the stage when Eqs. (44) and (45) are satisfied.

Now let us derive $\epsilon_m$, $v_s$, and $V_s$ corresponding to the above curvature perturbations, neglecting higher-order terms, such as $\Delta\Phi_h$, $\Delta\Phi_H$, $\tilde{\Phi}_G$, and $\Phi_6$. Here we must pay attention to $\Phi_7$.

Substituting Eqs. (46) and (47) into Eqs. (39)–(43), we obtain

$$C_h = -\frac{k_r^{(0)}}{r_0}\tau^{1/3} + \frac{r_0}{k_r^{(0)}}\tau^{-7/3}, \quad C_H = -\frac{k_R^{(0)}}{R_0}\tau^{-1/3} - \frac{2R_0}{k_R^{(0)}}\tau^{-5/3},$$
$$\tilde{C} \simeq -\frac{r_0}{k_r^{(0)}}\tau^{-7/3}, \quad \dot{\tilde{C}}/\tilde{C} = -\tilde{C}'/\tilde{C} = 7/(3\tau), \quad \Phi_7 \simeq 3\tau(\Phi_h + \Phi_H), \tag{48}$$

$$D_0 \simeq -17/(9\tau), \quad D_1 \simeq 0. \tag{49}$$

Moreover,

$$A_0 \simeq 3\tau(\Phi_h + \Phi_H), \quad A_1 = (3/\tau)(\Phi_h + 2\Phi_H),$$
$$A_2 \simeq -\tfrac{1}{3}(17\Phi_h + 2\Phi_H), \quad E \simeq 0. \tag{50}$$

From Eqs. (37), (38), and (33), we obtain in the lowest order

$$\epsilon_m = 0, \tag{51}$$

$$v_s = 4x\ \Phi_h, \tag{52}$$

and

$$V_s = -2y\ \Phi_H. \tag{53}$$

So, $\epsilon_m$ is of higher orders ($\sim O(x^2)$, $O(y^2)$). Here $\Phi_h$ and $\Phi_H$ are independent, because $\Phi_{hi}$ and $\Phi_{Hi}$ are given arbitrarily.

### 3.2. *Inside the horizons*

At earlier epochs of $\tau \gg \tau_{dec}$, $x$ and $y$ are comparable with 1 or larger than 1. It was shown in Sect. 3 of [5] that in the case of $x \gg 1$ and $y \gg 1$ under the condition

$$\mu/x \left(\equiv \left[(k_R^{(0)}/R(t))/(k_r^{(0)}/r(t))\right]^2\right) \ll 1, \tag{54}$$

the perturbations show wavy behaviors, depending on the wave-number $k_r^{(0)}$ (in the outer space) as $\exp(i\omega x)$, where $\omega$ is a constant. In the case of $y \gg 1$ and $x \gg 1$ under the condition $\mu/x \gg 1$, the perturbations show wavy behaviors, depending on the wave-number $k_R^{(0)}$ (in the inner space) as $\exp(i\omega y)$. In the former case ($\mu/x \ll 1$) the waves (depending on $k_r^{(0)}$) appear mainly in the outer space but do not appear in the inner space. In the latter case ($\mu/x \gg 1$), on the other hand, the waves (depending on $k_R^{(0)}$) appear mainly in the inner space but do not appear in the outer space.

At the stage of $x \gg 1$ and $y \gg 1$, the perturbations are inside the horizons and can be created by quantum fluctuations in both the outer space and the inner space. For these perturbations we assume that the values of $k_r^{(0)}$ and $k_R^{(0)}$ are smoothly distributed around the average values ($\bar{k}_r^{(0)}$ and $\bar{k}_R^{(0)}$). Here we pay attention to the perturbations with $x \gg 1$, $\bar{y}(\equiv (3/2R_0)\bar{k}_R^{(0)}) \gg 1$, and $\bar{\mu}/x \equiv [(\bar{k}_R^{(0)}/R(t)]/[k_r^{(0)}/r(t)]^2 \ll 1$. In this case, the perturbations with large $k_r^{(0)}$ have wavy behaviors in the outer space and the perturbations with $k_R^{(0)} \approx \bar{k}_R^{(0)}$ in the inner space are negligible. In the following we study their wavy behaviors proportional to $\exp(i\omega x)$. This is because such perturbations will survive and may be connected with the present observational information through the CMB radiation, after the decoupling of the outer space from the inner space. The other perturbations including the components [$\propto \exp(i\omega y)$] in the inner space will be disturbed or erased when the inner space is decoupled, and disappear.

For perturbations with $x \gg 1$, $\bar{y} \gg 1$, and $\bar{\mu}/x \ll 1$, it is found from Eqs. (35), (36), (39), (40), and (41) that

$$\begin{aligned} &C_h \simeq -k_r^{(0)}/r, \quad |C_H| \simeq \bar{k}_R^{(0)}/R \ll |C_h|, \\ &\tilde{C} \simeq C_h, \quad \dot{\tilde{C}}/\tilde{C} \simeq -\dot{r}/r, \end{aligned} \tag{55}$$

and

$$D_0 \simeq \frac{7}{9\tau}, \quad D_1 \simeq \left(\frac{k_r^{(0)}}{r}\right)^2 \frac{1}{9}\left(1 + \frac{15}{16}x^{-2}\right). \tag{56}$$

Here let us put $\epsilon_m$ and curvature perturbations $\Phi_h$ and $\Phi_H$ as

$$\epsilon_m = \epsilon_{m0} \exp i\omega x, \quad \Phi_h = \Phi_{h0} \exp i\omega x, \quad \Phi_H = \Phi_{H0} \exp i\omega x. \tag{57}$$

Then we obtain, for $x \gg 1$,

$$\begin{aligned} \ddot{\epsilon}_m + D_0\dot{\epsilon}_m + D_1\epsilon_m \simeq{}& \left(k_r^{(0)}/r\right)^2 \left[\left(\frac{1}{9} - \omega^2\right)\epsilon_{m0} + i\omega\left(2\epsilon_{m0,x} - \frac{1}{3}\epsilon_{m0}/x\right)\right. \\ &\left. + \epsilon_{m0,xx} - \frac{1}{3}\epsilon_{m0,x}/x + \frac{5}{48}\epsilon_{m0}/x^2\right] \exp i\omega x \\ ={}& \left(\frac{k_r^{(0)}}{r^2}\right)^2 \left(\frac{1}{9} - \omega^2\right) \epsilon_{m0} \ \exp i\omega x \ \left[1 + O\left(1/x\right)\right] \quad \text{for } \omega \neq 1/3. \end{aligned} \tag{58}$$

On the other hand, we get from Eqs. (31) and (34),

$$\begin{aligned} A_0 &= -\frac{3\tau/2}{i\omega x}\ \left(\Phi_h - \Phi_H\right)\ \left[1 + O\left(1/x\right)\right], \\ A_1 &= -\frac{k_r^{(0)}}{r} i\omega \left(d\Phi_h + D\Phi_H\right)\ \left[1 + O\left(1/x\right)\right], \\ \dot{A}_1 &= -{A_1}' = -\left(\frac{k_r^{(0)}}{r}\right)^2 \omega^2 \left(d\Phi_h + D\Phi_H\right)\ \left[1 + O\left(1/x\right)\right], \\ A_2 &= \left(\Phi_h + 6\Phi_H\right)\ \left[1 + O\left(1/x\right)\right]. \end{aligned} \tag{59}$$

Here it is noticed that $\tilde{\Phi}_G$ is of higher order with respect to $1/x$ ($\ll 1$) and is neglected. From Eqs. (78) and (81) of [5], it is found that $\Phi_6 = (2, 6)\tau\Phi_h$ for $\omega = 1, 1/3$, respectively, so that $\Phi_7 = 0$ for $\omega = 1, 1/3$. Moreover, using the condition $\bar{\mu}/x \ll 1$, we find that

$$\frac{\bar{k}_R^{(0)}}{R}|C_H A_0| \ll |A_1|. \tag{60}$$

Then, from Eqs. (43) and (59), we obtain

$$E \simeq \left(\frac{k_r^{(0)}}{r}\right)^2 \left[\left(1 + 3\omega^2\right)\Phi_{h0} + 6\left(1 + \omega^2\right)\Phi_{H0}\right]\exp i\omega x\ \left[1 + O\left(1/x\right)\right]. \tag{61}$$

So it is found from Eqs. (37), (58), and (61) that

$$\epsilon_{m0} \simeq \left(\tfrac{1}{9} - \omega^2\right)^{-1} \tfrac{10}{9}\left[\left(1 + 3\omega^2\right)\Phi_{h0} + 6\left(1 + \omega^2\right)\Phi_{H0}\right]\ \left[1 + O\left(1/x\right)\right] \quad \text{for } \omega \neq 1/3 \tag{62}$$

and

$$\epsilon_{m0} \simeq -ix\frac{40}{27}(\Phi_{h0} + 5\Phi_{H0}) \quad \text{for } \omega = 1/3. \tag{63}$$

In Sect. 3 of [5], it was found that the approximate wavy solutions of equations for curvature perturbations are given only for $\omega = 1$ and $1/3$, and in these cases the solutions have the following relations:

$$\Phi_{H0} = -\tfrac{1}{3}\Phi_{h0} \quad \text{for } \omega = 1, \tag{64}$$

and

$$\Phi_{H0} = \Phi_{h0} \quad \text{for } \omega = 1/3. \tag{65}$$

So we have the following expressions for $\epsilon_{m0}$:

$$\epsilon_{m0} = \; O(1/x)\;\Phi_{h0} \quad\; \omega = 1, \tag{66}$$

and

$$\epsilon_{m0} \simeq -\tfrac{80}{9} i x \Phi_{h0} \quad \text{for } \omega = 1/3. \tag{67}$$

## 4. Quantum fluctuations

In Sect. 2 and the previous paper [5], the perturbations were classified into three modes. In this paper we treat only their scalar and tensor modes and consider the perturbations created by the quantum effect in the comparably later stage of the 10-dimensional universe which is associated with the inflating outer space and collapsing inner space. Here Weinberg's procedure is used for the quantization [9].

### 4.1. *The scalar mode*

At the stage of $x \gg 1$ and $\bar{y} \gg 1$, the length of perturbations in the outer space can be smaller than the horizon size and they may be caused by the quantum effect, while at the later stage of $x < 1$ the length of perturbations is larger than the horizon size and they are frozen. So we should first consider the quantum fluctuations at earlier epochs of $x \gg 1$ and $\bar{y} \gg 1$. Additionally, moreover, we assume that $\bar{\mu}/x \ll 1$, corresponding to the perturbations in the inner space with the average value $(\bar{k}_R^{(0)})$, which was described in Sect. 3. Then these perturbations appear mainly in the outer space, and hence we can treat the perturbations as if they are in the four-dimensional space-time (consisting of the time $t$ and the outer space).

The energy density perturbation $\epsilon_m$ is expressed by Eq. (37) in connection with gravitational perturbations. This equation is also derived from the action principle as

$$I = \int dt\, d^3\boldsymbol{x}(\mathcal{L}_\epsilon + \mathcal{L}_g), \tag{68}$$

where $\mathcal{L}_\epsilon$ and $\mathcal{L}_g$ are the fluidal and gravitational parts of the total Lagrangian, and $\boldsymbol{x}$ is the coordinate in the outer space. Here $\mathcal{L}_\epsilon$ can be derived from Eq. (37) as follows. At the stage of $x \gg 1$, $\bar{y} \gg 1$, and $\bar{\mu}/x \ll 1$, we have $D_0$ and $D_1$ in Eq. (56), and then the fluidal part in the equation of motion is derived using the following Lagrangian:

$$\mathcal{L}_\epsilon = \frac{1}{2} r^{7/3}\left[\left(\frac{\partial \epsilon_m}{\partial t}\right)^2 + \left(\frac{1}{3r}\right)^2\left(\frac{\partial \epsilon_m}{\partial \boldsymbol{x}}\right)^2\right], \tag{69}$$

where $r = r_0\,(t_0 - t)^{-1/3}$, and $\partial\epsilon_m/\partial\boldsymbol{x} = i\boldsymbol{k}_r^{(0)}\epsilon_m$ for $\epsilon_m \propto \exp\left(i\boldsymbol{k}_r^{(0)}\boldsymbol{x}\right)$.

On the other hand, $\epsilon_m$ can be expanded as

$$\epsilon_m(\boldsymbol{x}, t) = \int d\boldsymbol{k}_r^{(0)} \left[\epsilon_m\left(k_r^{(0)}, t\right) \exp\left(i\boldsymbol{k}_r^{(0)}\boldsymbol{x}\right) \alpha\left(\boldsymbol{k}_r^{(0)}\right) + \epsilon_m^*\left(k_r^{(0)}, t\right) \exp\left(-i\boldsymbol{k}_r^{(0)}\boldsymbol{x}\right) \alpha^*\left(\boldsymbol{k}_r^{(0)}\right)\right], \tag{70}$$

and $\Phi_h$ and $\Phi_H$ also can be written as

$$\Phi_h(\boldsymbol{x}, t) = \int d\boldsymbol{k}_r^{(0)} \left[\Phi_h\left(k_r^{(0)}, t\right) \exp\left(i\boldsymbol{k}_r^{(0)}\boldsymbol{x}\right) \alpha\left(\boldsymbol{k}_r^{(0)}\right) + \Phi_h^*\left(k_r^{(0)}, t\right) \exp\left(-i\boldsymbol{k}_r^{(0)}\boldsymbol{x}\right) \alpha^*\left(\boldsymbol{k}_r^{(0)}\right)\right],$$
$$\Phi_H(\boldsymbol{x}, t) = \int d\boldsymbol{k}_r^{(0)} \left[\Phi_H\left(k_r^{(0)}, t\right) \exp\left(i\boldsymbol{k}_r^{(0)}\boldsymbol{x}\right) \alpha\left(\boldsymbol{k}_r^{(0)}\right) + \Phi_H^*\left(k_r^{(0)}, t\right) \exp\left(-i\boldsymbol{k}_r^{(0)}\boldsymbol{x}\right) \alpha^*\left(\boldsymbol{k}_r^{(0)}\right)\right], \tag{71}$$

where the reality of these fields requires them to take the above forms. The interaction of the photon field with the gravitational field makes the commutation relation of $\alpha(\boldsymbol{k}_r^{(0)})$ and $\alpha^*(\boldsymbol{k}_r^{(0)})$ complicated, but they become simple at very early times [9].

In many cases when quantum fluctuations have so far been treated in a system of a scalar (inflaton) field and the gravitational field, the quantization of the scalar field is tried first [9]. In the present case also, when we consider a system of a photon scalar field and the gravitational field, we try first the quantization of the photon scalar field in the following.

The canonical conjugate to $\epsilon_m(\boldsymbol{x}, t)$ is then

$$\pi_m(\boldsymbol{x}, t) = \partial\mathcal{L}_\epsilon/\partial\left(\frac{\partial\epsilon_m}{\partial t}\right) = r^{7/3}\frac{\partial\epsilon_m}{\partial t}. \tag{72}$$

The commutator of $\epsilon_m$ and $\pi_m$ is

$$[\epsilon_m(\boldsymbol{x}, t), \epsilon_m(\boldsymbol{y}, t)] = 0, \quad [\epsilon_m(\boldsymbol{x}, t), \partial\epsilon_m(\boldsymbol{y}, t)/\partial t] = i r^{-7/3}\delta^3(\boldsymbol{x} - \boldsymbol{y}). \tag{73}$$

These commutation relations imply that $\alpha(\boldsymbol{k})$ and $\alpha^\star(\boldsymbol{k})$ behave as conventionally normalized annihilation and creation operators:

$$[\alpha(\boldsymbol{k}), \alpha(\boldsymbol{k}')] = 0, \quad [\alpha(\boldsymbol{k}), \alpha^*(\boldsymbol{k}')] = \delta^3(\boldsymbol{k} - \boldsymbol{k}'), \tag{74}$$

when $\epsilon_m(k_r^{(0)}, t)$ is normalized at $r \to 0$ as

$$\epsilon_m(k_r^{(0)}, t) \propto [r(t)]^{-2/3}[k_r^{(0)}]^{-1/2} \exp\left(i\omega k_r^{(0)} \int_{t_*}^{t} \frac{dt'}{r(t')}\right), \tag{75}$$

where $t_*$ is arbitrary and $\omega$ is a constant ($= 1$ or $1/3$). This expression for $\epsilon_m(k_r^{(0)}, t)$ is used as the initial condition for created fields of energy density $\epsilon_m$. Here we choose the quantum state during the inflation of the outer space under the simple assumption that the state of the universe is the vacuum state $|0\rangle$, defined so that

$$\alpha(\boldsymbol{k})|0\rangle = 0 \quad \text{and} \quad \langle 0|0\rangle = 1. \tag{76}$$

This corresponds to the Bunch–Davies vacuum [10] in the outer space within the 10-dimensional universe. As described in Sect. 3, $\epsilon_m$ and the curvature perturbations as quantum fluctuations are proportional to each other. So the behavior of $\epsilon_m$ in Eq. (75) is common to that of $\Phi_h$ and $\Phi_H$, and, using Eqs. (64)–(67), we obtain

$$\Phi_H(k_r^{(0)}, t) = -\frac{1}{3}\Phi_h(k_r^{(0)}, t) \propto x\, [r(t)]^{-2/3} \left[k_r^{(0)}\right]^{-1/2} \exp\left(i\omega k_r^{(0)} \int_{t_*}^{t} \frac{dt'}{r(t')}\right) \tag{77}$$

for $\omega = 1$, and

$$\Phi_H(k_r^{(0)}, t) = \Phi_h(k_r^{(0)}, t) \propto x^{-1}\, [r(t)]^{-2/3} \left[k_r^{(0)}\right]^{-1/2} \exp\left(i\omega k_r^{(0)} \int_{t_*}^{t} \frac{dt'}{r(t')}\right) \tag{78}$$

for $\omega = 1/3$.

### 4.2. *The tensor mode*

In the tensor mode, there are two types (ST) and (TS), as described in Sect. 2 and [5]. (ST) has the three-dimensional scalar and the six-dimensional tensor, while (TS) has the three-dimensional tensor and the six-dimensional scalar. Here we take up (TS) with the amplitude $h_T^{(2)}$, and neglect (ST) with the amplitude $H_T^{(2)}$, which may not be connected with the observation in the three-dimensional outer space, after the decoupling of the inner space.

In [5], we studied the behavior of tensor perturbations $h_T^{(2)}$. They satisfy

$$\ddot{h}_T^{(2)} + \left(d\frac{\dot{r}}{r} + D\frac{\dot{R}}{R}\right)\dot{h}_T^{(2)} + \left[\left(\frac{k_r^{(2)}}{r}\right)^2 + \left(\frac{k_R^{(0)}}{R}\right)^2\right] h_T^{(2)} = 0. \tag{79}$$

Here we consider the case of

$$\left(\frac{\bar{k}_R^{(0)}}{R}\right) \Big/ \left(\frac{k_r^{(2)}}{r}\right) \ll 1, \tag{80}$$

where $k_r^{(2)}$ and $k_R^{(0)}$ are the wave-numbers in the outer and inner spaces, respectively, and $\bar{k}_R^{(0)}$ is the average wave-number in the inner space. Then we have

$$\ddot{h}_T^{(2)} - 3\frac{\dot{r}}{r}\dot{h}_T^{(2)} + \left(\frac{k_r^{(2)}}{r}\right)^2 h_T^{(2)} = 0, \tag{81}$$

where we used the relation $R \propto 1/r \propto \tau^{1/3}$ and $d = D/2 = 3$. This equation can also be derived from the action principle as

$$I = \int dt\, d^3x\, \mathcal{L}_t, \tag{82}$$

where

$$\mathcal{L}_t = \frac{1}{2}r^{-3}\left[\left(\frac{\partial h_T^{(2)}}{\partial t}\right)^2 + \frac{1}{r^2}\left(\frac{\partial h_T^{(2)}}{\partial x^i}\right)^2\right] \tag{83}$$

for $\partial h_T^{(2)} \propto \exp\left(i\boldsymbol{k}_r^{(2)}\boldsymbol{x}\right)$.

On the other hand, the amplitude $\partial h_T^{(2)}$ takes the form

$$h_T^{(2)}(\boldsymbol{x}, t) = \int d\boldsymbol{k}_r^{(2)} \left[ h_T^{(2)}\left(k_r^{(2)}, t\right) \exp\left(i\boldsymbol{k}_r^{(2)}\boldsymbol{x}\right) \alpha\left(\boldsymbol{k}_r^{(2)}\right) + h_T^{(2)*}\left(k_r^{(2)}, t\right) \exp\left(-i\boldsymbol{k}_r^{(2)}\boldsymbol{x}\right) \alpha^*\left(\boldsymbol{k}_r^{(2)}\right)\right], \tag{84}$$

and the canonical conjugate to $h_T^{(2)}(\boldsymbol{x}, t)$ is then

$$\pi_T(\boldsymbol{x}, t) = \partial\mathcal{L}_t / \left(\partial h_T^{(2)}/\partial t\right) = r^{-3}\frac{\partial h_T^{(2)}}{\partial t}. \tag{85}$$

The commutator of $h_T^{(2)}$ and $\pi_T$ is

$$\left[h_T^{(2)}(\boldsymbol{x}, t), h_T^{(2)}(\boldsymbol{y}, t)\right] = 0, \ \left[h_T^{(2)}(\boldsymbol{x}, t), \partial h_T^{(2)}(\boldsymbol{y}, t)/\partial t\right] = ir^3\delta^3(\boldsymbol{x}-\boldsymbol{y}). \tag{86}$$

These commutation relations imply that $\alpha(\boldsymbol{k})$ and $\alpha^*(\boldsymbol{k})$ behave as conventionally normalized annihilation and creation operators, in the same way as Eq. (73), when $h_T^{(2)}\left(k_r^{(2)}, t\right)$ is normalized at $r \to 0$ as

$$h_T^{(2)}\left(k_r^{(2)}, t\right) \propto [r(t)]^2 \left[k_r^{(2)}\right]^{-1/2} \exp\left(ik_r^{(2)} \int_{t_*}^t \frac{dt'}{r(t')}\right). \tag{87}$$

This expression of $h_T^{(2)}(k_r^{(2)}, t)$ is used as the initial condition of created fields in the tensor mode $h_T^{(2)}$. Here we choose the quantum state during the inflation of the outer space, so that the state of the universe may satisfy the relation in Eq. (76).

## 5. Spectra of fluctuations and their comparison with CMB observation

### 5.1. *The scalar mode*

The information about the perturbations which are created by the quantum fluctuations inside the horizon can be used to make an initial condition for the evolution of perturbations which re-enter the horizon after the long inflation. For this purpose, we use the quantities which are conserved outside the horizon. In the four-dimensional universe with three-dimensional space-section, we have a gauge-invariant curvature perturbation, represented as

$$\mathcal{R}_4 \equiv \Phi_H, \tag{88}$$

which is a conserved quantity [7]. In the 10-dimensional universe, on the other hand, we have the following two independent similar quantities as the candidates:

$$\mathcal{R}_h \equiv (\tau/\tau_{dec})^{8/3}\Phi_h \quad \text{and} \quad \mathcal{R}_H \equiv (\tau/\tau_{dec})^{4/3}\Phi_H, \tag{89}$$

where $\tau_{dec}$ represents the epoch when the inner space decouples from the outer space. For $x\left(\equiv (3/4r_0)k_r^{(0)}\tau^{4/3}\right) < 1$ and $y\left(\equiv (3/2R_0)k_R^{(0)}\tau^{2/3}\right) < 1$, $\mathcal{R}_h$ and $\mathcal{R}_H$ are nearly constant, and so these can be regarded as quantities conserved outside the horizon.

As other candidates for conserved quantities, we may consider

$$\mathcal{R}_v \left( \equiv \frac{\dot{r}}{k_r^{(0)}} v_s \right) \quad \text{and} \quad \mathcal{R}_V \left( \equiv \frac{\dot{R}}{k_R^{(0)}} V_s \right), \tag{90}$$

but, for $x \ll 1$, they are not independent of $\mathcal{R}_h$ and $\mathcal{R}_H$,

$$\mathcal{R}_v = -\frac{23}{32}\Phi_h + \frac{19}{4}\Phi_H, \tag{91}$$

$$\mathcal{R}_V = -\frac{11}{32}\Phi_h + \frac{13}{4}\Phi_H \tag{92}$$

with respect to the main terms. For $x \gg 1$ and $y \gg 1$, moreover, we find that $v_s$ and $V_s$ are comparable with $\Phi_h$ and $\Phi_H$, respectively, and $\mathcal{R}_v$ and $\mathcal{R}_V$ are $\sim v_s/x$ and $\sim V_s/y$, respectively, which are small compared with $\Phi_h$ and $\Phi_H$. This means that the roles of $\mathcal{R}_v$ and $\mathcal{R}_V$ are small, compared with those of $\mathcal{R}_h$ and $\mathcal{R}_H$, respectively. In this paper, therefore, we adopt $\mathcal{R}_h$ and $\mathcal{R}_H$ as the conserved quantities in the 10-dimensional universe. Neither of them, however, is necessarily a conserved quantity which is directly connected at epoch $\tau_{dec}$ with $R_4$ in the four-dimensional universe.

Here we construct the 10-dimensional gauge-invariant conserved quantity $\mathcal{R}_{10}$ using $\mathcal{R}_h$ and $\mathcal{R}_H$, by imposing the following two conditions:

(1) $\mathcal{R}_{10} = \mathcal{R}_4$ at epoch ($\tau_{dec}$) of the decoupling of the outer space from the inner space, and
(2) $\mathcal{R}_{10}$ is consistent with the spectral constraint given by the CMB observation.

As the first candidate of $\mathcal{R}_{10}$, we consider a linear combination of $\Phi_h$ and $\Phi_H$ as

$$\mathcal{R}_{10} = \lambda_0 \mathcal{R}_H + \lambda_1 \mathcal{R}_h, \tag{93}$$

where constants $\lambda_0$ and $\lambda_1$ are determined so as to satisfy the above two conditions (1) and (2).

At epoch $\tau$ when $x\ \big(= (3/4r_0)k_r^{(0)}\tau^{4/3}\big) \gg 1$ and $y\big(= (3/4r_0)k_R^{(0)}\tau^{2/3}\big) \gg 1$, $\Phi_h$ and $\Phi_H$ are created by quantum fluctuations and are expressed using Eqs. (77) and (78) as

$$\begin{aligned} \Phi_h &= \tau^{14/9}\left[k_r^{(0)}\right]^{1/2} \exp(ix) + \alpha\tau^{-10/9}\left[k_r^{(0)}\right]^{-3/2} (4r_0/3)^2 \exp(ix/3), \\ \Phi_H &= -\tfrac{1}{3}\tau^{14/9}\left[k_r^{(0)}\right]^{1/2} \exp(ix) + \alpha\tau^{-10/9}\left[k_r^{(0)}\right]^{-3/2} (4r_0/3)^2 \exp(ix/3), \end{aligned} \tag{94}$$

where we used $r \propto \tau^{-1/3}$ and $\alpha$ is an arbitrary constant. Inserting Eq. (94) into Eq. (89), we obtain

$$\begin{aligned} \mathcal{R}_h &= \tau_{dec}{}^{-8/3}\left[k_r^{(0)}\right]^{1/2}\left[\tau^{38/9}\exp(ix) + \alpha\tau^{14/9}\left(\frac{4r_0}{3k_r^{(0)}}\right)^2 \exp(ix/3)\right] \\ &= \left[k_r^{(0)}\right]^{-8/3} \tau_{dec}{}^{-8/3} (4r_0\, x/3)^{19/6}\left[\exp(ix) + \alpha x^{-2}\exp(ix/3)\right], \end{aligned} \tag{95}$$

$$\begin{aligned} \mathcal{R}_H &= \tau_{dec}{}^{-4/3}\left[k_r^{(0)}\right]^{1/2}\left[-\frac{1}{3}\tau^{26/9}\exp(ix) + \alpha\tau^{2/9}\left(\frac{4r_0}{3k_r^{(0)}}\right)^2 \exp(ix/3)\right] \\ &= \left[k_r^{(0)}\right]^{-5/3} \tau_{dec}{}^{-4/3} (4r_0\, x/3)^{13/6}\left[-\frac{1}{3}\exp(ix) + \alpha x^{-2}\exp(ix/3)\right]. \end{aligned} \tag{96}$$

As $x$ decreases and becomes smaller than 1, the $x$ dependence of $\mathcal{R}_h$ and $\mathcal{R}_H$ changes from the wavy behavior ($\propto \exp(ix)$ and $\exp(ix/3)$) to the constant ones. At epoch $\tau_{eq}$ with $x = 1$, we have, therefore,

$$\mathcal{R}_h\left(\tau_{eq}\right) = \Xi\ \left(\zeta_h + \alpha\zeta_h'\right), \tag{97}$$

$$\mathcal{R}_H\left(\tau_{eq}\right) = \Xi\ \frac{x_{dec}}{x}\left(-\frac{1}{3}\zeta_H + \alpha\zeta_H'\right), \tag{98}$$

where

$$\Xi \equiv \left[k_r^{(0)}\right]^{-8/3} \tau_{dec}{}^{-8/3}\,(4r_0/3)^{19/6}, \tag{99}$$

where $x_{dec} \equiv (3/4r_0)\,k_r^{(0)}\tau_{dec}{}^{4/3}$, and $\zeta_h$, $\zeta_h'$, $\zeta_H$, and $\zeta_H'$ are constants. The exact values of these constants are determined by solving dynamical equations for $\Phi_h$ and $\Phi_H$ given in [5], but they are estimated to be $\approx 1$, because $\mathcal{R}_h$ and $\mathcal{R}_H$ are nearly constant for $x < 1$ and $y < 1$.

Now we assume that the CMB spectrum is determined at epoch $\tau_{eq}$ when $x = 1$ (indicating the horizon exit), and consider the $k_r^{(0)}$ dependence of $\mathcal{R}_{10}$ at this epoch. Here $\mathcal{R}_{10}$ at epoch $\tau_{eq}$ is expressed as

$$\mathcal{R}_{10} = \mathcal{R}_0 z^{-5/3} + \mathcal{R}_1 z^{-8/3} \tag{100}$$

around the observed wave-number $\left(k_r^{(0)}\right)_{obs}$, where $z \equiv k_r^{(0)}/\left(k_r^{(0)}\right)_{obs}$,

$$\mathcal{R}_0 \equiv \lambda_0 \mathrm{Re}\left(-\tfrac{1}{3}\zeta_H + \alpha\zeta_H'\right)\ (x_{dec}\ \Xi)_{z=1}, \tag{101}$$

$$\mathcal{R}_1 \equiv \lambda_1 \mathrm{Re}\left(\zeta_h + \alpha\zeta_h'\right)\ \Xi_{z=1}, \tag{102}$$

where Re means the real part, and $\lambda_0$ and $\lambda_1$ are coefficients in Eq. (93).

The CMB observation shows that the $k_r^{(0)}$ dependence of $\mathcal{R}_{10}$ is

$$z^{(-4+n_s)/2} = z^{-1.517} \tag{103}$$

for $\left(k_r^{(0)}\right)_{obs} = 0.002\,\mathrm{Mpc}^{-1}$, where $n_s = 0.966$ according to the WMAP seven-year result [11]. The condition that Eq. (100) and Eq. (103) should be consistent in the neighborhood of $z = 1$ is

$$z^{-5/3} + \delta_1 z^{-8/3} = (1 + \delta_1) z^{-1.517}, \tag{104}$$

where $\delta_1 \equiv \mathcal{R}_1/\mathcal{R}_0 = (x_{dec})^{-1}(\lambda_1/\lambda_0)\mathrm{Re}(\zeta_h + \alpha\zeta_h')/\mathrm{Re}(-\frac{1}{3}\zeta_H + \alpha\zeta_H')$. From the continuity of this equation and its first derivative at $z = 1$, it is found that

$$\delta_1 = -0.130. \tag{105}$$

That is, the observational spectrum (103) can be reproduced when $\mathcal{R}_H$ is main and $\mathcal{R}_h$ is about 10% of the total $\mathcal{R}_{10}$.

The above definition of $\mathcal{R}_{10}$ satisfies the condition of continuity of Eq. (104) in the first derivative, but not in the second derivative. In order to also satisfy the condition of continuity in the second derivative, we consider the second candidate of $\mathcal{R}_{10}$ at $\tau_{eq}$ expressed as

$$\mathcal{R}_{10} = \lambda_0 \mathcal{R}_H + \lambda_1 \mathcal{R}_h + \lambda_2 [\mathcal{R}_h]^2/\mathcal{R}_H, \tag{106}$$

where constants $\lambda_0$, $\lambda_1$, and $\lambda_2$ are determined so as to satisfy the above two conditions (1) and (2). Here, $\mathcal{R}_{10}$ at epoch $\tau_{eq}$ is rewritten as

$$\mathcal{R}_{10} = \mathcal{R}_0 z^{-5/3} + \mathcal{R}_1 z^{-8/3} + \mathcal{R}_2 z^{-11/3}, \tag{107}$$

where $\mathcal{R}_0$ and $\mathcal{R}_1$ are defined by Eqs. (101) and (102), and

$$\mathcal{R}_2 \equiv \lambda_2 \left[\mathrm{Re}\left(\zeta_h + \alpha\zeta_h'\right)^2 / \mathrm{Re}\left(-\tfrac{1}{3}\zeta_H + \alpha\zeta_H'\right)\right]\left[(x_{dec})^{-1}\,\Xi\right]_{z=1}. \tag{108}$$

Then, from the condition that Eqs. (107) and (103) should be consistent in the neighborhood of $z = 1$, we have

$$z^{-5/3} + \delta_1 z^{-8/3} + \delta_2 z^{-11/3} = (1 + \delta_1 + \delta_2)\, z^{-1.517}, \tag{109}$$

where $\delta_1 \equiv \mathcal{R}_1/\mathcal{R}_0$ and $\delta_2 \equiv \mathcal{R}_2/\mathcal{R}_0$. From the continuity of this equation and its first and second derivatives at $z = 1$, it is found that

$$\delta_1 = -0.260 \quad \text{and} \quad \delta_2 = 0.0696. \tag{110}$$

Now let us define the power spectrum of curvature perturbations as [11,12]

$$\mathcal{P}_s \equiv \frac{4\pi\left(k_r^{(0)}\right)^3}{(2\pi)^3}|\mathcal{R}_{10}|^2. \tag{111}$$

Then for $R_{10}$, $\delta_1$, and $\delta_2$ in Eqs. (106) and (110),

$$\mathcal{P}_s = \frac{4\pi\left[\left(k_r^{(0)}\right)_{obs}\right]^3}{(2\pi)^3}|\mathcal{R}_0|^2\,(1 + \delta_1 + \delta_2)^2\,. \tag{112}$$

The WMAP seven-year normalization [11] gives

$$(\mathcal{P}_s)_{obs} = 2.42 \times 10^{-9} \tag{113}$$

on the scale $\left(k_r^{(0)}\right)_{obs} = 0.002\,\mathrm{Mpc}^{-1}$. Then we obtain

$$\mathcal{R}_0 = \lambda_0|\mathcal{R}_H\left(\tau_{eq}\right)| = \frac{\sqrt{2}\pi}{1 + \delta_1 + \delta_2}\left[\mathcal{P}_s/\left(k_r^{(0)}\right)^3\right]_{obs}^{1/2}. \tag{114}$$

On the other hand, we have

$$\begin{aligned}
\frac{\lambda_1}{\lambda_0} &= \delta_1 \cdot x_{dec}\mathrm{Re}\left(-\frac{1}{3}\zeta_H + \alpha\zeta_H'\right)/\mathrm{Re}\left(\zeta_h + \alpha\zeta_h'\right),\\
\frac{\lambda_2}{\lambda_0} &= \delta_2 \cdot \left[x_{dec}\mathrm{Re}\left(-\frac{1}{3}\zeta_H + \alpha\zeta_H'\right)/\mathrm{Re}\left(\zeta_h + \alpha\zeta_h'\right)\right]^2,
\end{aligned} \tag{115}$$

where $x_{dec} = \left(\tau_{dec}/\tau_{eq}\right)^{4/3} = \left(r_{eq}/r_{dec}\right)^4 \ll 1$, and the factor $\mathrm{Re}\left(-\frac{1}{3}\zeta_H + \alpha\zeta_H'\right)/\mathrm{Re}\left(\zeta_h + \alpha\zeta_h'\right)$ is of the order of 1. Since $\delta_2 \simeq (\delta_1)^2$, we have $\lambda_2/\lambda_0 \simeq (\lambda_1/\lambda_0)^2$.

It is concluded that $\mathcal{R}_{10}$ is consistent with the observed spectra of CMB radiation under the conditions of (114) and (115).

Thus we could derive the condition that the parameters $\lambda_0$, $\lambda_1$, and $\lambda_2$ in $\mathcal{R}_{10}$ should satisfy for consistency with the CMB observation. From their ratios the role of the curvature perturbation in the inner space is found to be larger than that in the outer space. This condition and its consequeces are concerned with the condition at the earlier stage and the initial condition of the universe, which should be expressed as a perturbation model with the theoretical model parameters, and the above three parameters should be related to the latter parameters. They may be influenced through $\mathcal{R}_{10}$ by the process of decoupling, which has not been discussed here, because its quantum-gravitational process cannot be treated at present. This situation in the observational aspect is compared in subsection 5.3 with the situations in other inflation models.

### 5.2. *The tensor mode*

In the limit of $x\left(\equiv \left(3/4r_0 big\right)k_r^{(2)}\tau^{4/3}\right) \to 0$, the gauge-invariant perturbation $h_T^{(2)}$ tends to $a + b\ln\tau$, as seen from the analyses in Abbott et al. [8] and the previous paper [5], where $a$ and $b$ are constants. So, as the quantity $\mathcal{R}_t$ which is conserved outside the horizon, we adopt

$$\mathcal{R}_t \equiv h_T^{(2)}\left[a + b\ln\tau_{dec}\right] / \left[a + b\ln\tau\right], \tag{116}$$

so that $\mathcal{R}_t$ leads to a constant in the limit of $x \to 0$.

At the epoch $\tau_{eq}$ of $x = 1$, we have the relation

$$\tau \propto \left[k_r^{(2)}\right]^{-3/4}, \tag{117}$$

so that $r^2\left[k_r^{(2)}\right]^{-1/2} \propto \tau^{-2/3}\left[k_r^{(2)}\right]^{-1/2} = const.$, and from Eq. (87)

$$h_T^{(2)}\left(k_r^{(2)}, \tau_{eq}\right) = \lambda_t \cdot \exp(ix), \tag{118}$$

where $\lambda_t$ is a constant. Then it is found from Eq. (116) that

$$\mathcal{R}_t\left(\tau_{eq}\right) = \lambda_t\left[a + b\ln\tau_{dec}\right]\left[a - \tfrac{3}{4}b\ln k_r^{(2)} + const.\right]^{-1}\exp(ix). \tag{119}$$

As $x$ decreases and becomes $<1$, the $x$ dependence of $\mathcal{R}_t$ changes from the wavy behavior to the stationary constant one. But the $k_r^{(2)}$ dependence does not change, so that the spectrum in the tensor mode has the form

$$\left[a - \tfrac{3}{4}b\ln k_r^{(2)} + const.\right]^{-1}. \tag{120}$$

The corresponding power spectrum is

$$\mathcal{P}_t \equiv \frac{4\pi\left(k_r^{(2)}\right)^3}{(2\pi)^3}\left|\mathcal{R}_t\left(\tau_{eq}\right)\right|^2. \tag{121}$$

The amplitude of $\mathcal{R}_t\left(\tau_{eq}\right)$ should be determined, corresponding to the observation, which has not been given yet. At present, we have the condition $r \equiv \mathcal{P}_t/\mathcal{P}_s < 0.24$ for $k_r^{(0)} = k_r^{(2)} = 0.002$ Mpc$^{-1}$ [11].

### 5.3. *Comparison with the spectral analyses in other inflation models*

In the four-dimensional universe due to the Einstein theory, the quantity conserved outside the horizon is uniquely defined using one of the curvature perturbations [7]. But in hypothetical inflation models with inflaton scalar fields (including the non-minimal coupling with the Ricci scalar), the values of parameters such as slow-roll parameters ($\epsilon$, $\eta$) and the number $N$ of inflationary e-folds, [9,13] and the coupling parameter $\xi$ in the scalar field equation [14–17] are not unique. The observed spectral index $n_s$ ($\approx 0.97$) is, therefore, obtained by adjusting the above parameters $\epsilon$, $\eta$, $N$, and $\xi$.

In the $R + R^2$ modified gravitational theory, we have an inflation model associated with the de Sitter type solution which was derived first by Nariai and Tomita [18,19] and rederived later by Starobinsky [20]. Mukhanov and Chibisov [21] derived the quantum fluctuations generated at the de Sitter stage, and it was found that the spectral index $n_s$ of these fluctuations can be expressed as

$$n_s - 1 = -1/\left[1 + \tfrac{1}{2}\ln\left(k_{obs}/aH\right)\right] = -1/\left(1 + \tfrac{1}{2}N\right), \tag{122}$$

where $k_{obs}$ is the observed wave-number, $N$ is the inflationary e-fold, and $a$ and $H$ are the scale factor and the Hubble constant at the epoch when the de Sitter expansion ends. This number $N$ is determined to be 70, so that we may have $n_s \simeq 0.97$ (the observed value).

In the present case of a photon scalar field in the 10-dimensional universe, the inflation of the outer space is unique, because the scale factor $r$ of the outer space is $\propto \tau^{-1/3}$ (in the non-viscous case). On the other hand, the conserved quantity is not unique, because there are two independent curvature perturbations $\Phi_h$ and $\Phi_H$ before the decoupling of the outer and inner spaces. It is, therefore, a key point to determine how to combine them in this case, to derive $\mathcal{R}_{10}$ (connecting the two epochs outside the horizon). To obtain the observed spectral index $n_s$, we made the examples of the combination of $\Phi_h$ and $\Phi_H$ as the conserved quantity, so that the theoretical spectral index $n_s$ may be consistent with the observed one.

## 6. Concluding remarks

In this paper I showed the possibility of deriving the observed fluctuation of CMB radiation from the quantum fluctuations which appeared at the inflating stage of the outer space in the 10-dimensional universe. In contrast to the rapid inflation in the inflaton scalar field, our inflation is a power type, but we have two independent curvature perturbations which make possible the consistency with the observed spectra.

For simplicity, on the other hand, I neglected the viscosity which may play important roles in dynamics between the outer space and the inner space. If we take the viscosity into account, not only is much entropy produced (as shown in the previous paper [3]), but also the severe condition such as $\lambda_2/\lambda_0 \simeq (\lambda_1/\lambda_0)^2 \ll 1$ for producing the observed CMB fluctuations in the scalar mode may be softened. The next step is to study the perturbations and quantum fluctuations to derive the condition in the case with viscous processes due to the transport of 10-dimensional gravitational waves [3,22].

## Appendix A. Higher-order terms $\Delta\Phi_h$ and $\Delta\Phi_H$ of curvature perturbations $\Phi_h$ and $\Phi_H$

The higher-order terms $\Delta\Phi_h$ and $\Delta\Phi_H$, with respect to $x$ and $y$ ($\ll 1$), in Eqs. (46) and (47) are derived from the part of the Einstein equations in the scalar mode:

$$\delta R^0_i = -\delta T^0_i, \tag{A1}$$

$$\delta R^0_a = -\delta T^0_a, \tag{A2}$$

where the perturbed components of energy–momentum tensors are

$$\delta T^0_i = r\,(\rho + p)\left(v_s^{(0)} - b^{(0)}\right) q_i\, Q, \tag{A3}$$

$$\delta T^0_a = R\,(\rho + p)\left(V_s^{(0)} - B^{(0)}\right) q\, Q_a. \tag{A4}$$

Using the expressions of $\delta R_i^0$ and $\delta R_a^0$ in the appendix of Ref. [8], we obtain

$$-\frac{r}{k_r^{(0)}}(\rho+p)\,v_s^{(0)} = -(d-1)\,\dot{\Phi}_h + \left[-dD\frac{\dot{R}}{R} - (d-1)(d-2)\frac{\dot{r}}{r}\right]\Phi_h$$
$$- D\dot{\Phi}_H + \left[(2-d)\,D\frac{\dot{r}}{r} - D(D-1)\frac{\dot{R}}{R}\right]\Phi_H$$
$$- \left(\frac{k_R^{(0)}}{R}\right)^2 \dot{\tilde{\Phi}}_G + \left[2(2-d)\frac{\dot{r}}{r}\left(\frac{k_R^{(0)}}{R}\right)^2 - D\frac{\dot{R}}{R}\left(\frac{k_r^{(0)}}{r}\right)^2\right]\tilde{\Phi}_G$$
$$+ \left\{\frac{1}{2}\left(\frac{k_R^{(0)}}{R}\right)^2 + (d-1)\left[\frac{\ddot{r}}{r} - \left(\frac{\dot{r}}{r}\right)^2\right] + D\frac{\dot{r}}{r}\frac{\dot{R}}{R}\right\}\left(-\Phi_6 + \frac{r}{\dot{r}}\Phi_h - \frac{R}{\dot{R}}\Phi_H\right), \tag{A5}$$

$$-\frac{R}{k_R^{(0)}}(\rho+p)\,V_s^{(0)} = -(D-1)\,\dot{\Phi}_H + \left[-dD\frac{\dot{r}}{r} - (D-1)(D-2)\frac{\dot{R}}{R}\right]\Phi_H$$
$$- d\dot{\Phi}_h + \left[(2-D)\,d\frac{\dot{R}}{R} - d(d-1)\frac{\dot{r}}{r}\right]\Phi_h$$
$$- \left(\frac{k_r^{(0)}}{r}\right)^2 \dot{\tilde{\Phi}}_G + \left[2(2-D)\frac{\dot{R}}{R}\left(\frac{k_r^{(0)}}{r}\right)^2 - D\frac{\dot{r}}{r}\left(\frac{k_R^{(0)}}{R}\right)^2\right]\tilde{\Phi}_G$$
$$+ \left\{\frac{1}{2}\left(\frac{k_r^{(0)}}{r}\right)^2 + (D-1)\left[\frac{\ddot{R}}{R} - \left(\frac{\dot{R}}{R}\right)^2\right] + d\frac{\dot{r}}{r}\frac{\dot{R}}{R}\right\}\left(\Phi_6 + \frac{R}{\dot{R}}\Phi_H - \frac{r}{\dot{r}}\Phi_h\right), \tag{A6}$$

where a dot denotes $d/dt$, and $\Phi_h$, $\Phi_H$, $\tilde{\Phi}_G$, and $\Phi_6$ are defined in Eqs. (15), (33), and (36) of [5]. At the final stage of the inflating outer space and the collapsing inner space, we have $\rho = p = 0$, as shown in Eq. (10) of [5]. So, the left-hand sides of Eqs. (A5) and (A6) vanish.

Now let us consider the stage of $x \ll 1$ and $y \ll 1$, where $x$ and $y$ are defined in Eqs. (44) and (45). Then, the lowest-order terms in $\Phi_h$ and $\Phi_H$ with respect to $x$ and $y$ are expressed as

$$\Phi_h = (\tau/\tau_i)^{-8/3}\,\Phi_{hi}, \quad \Phi_H = (\tau/\tau_i)^{-4/3}\,\Phi_{Hi}, \tag{A7}$$

and $\Phi_6 = \tilde{\Phi}_G = 0$. To derive the next-order terms, let us put

$$\Phi_h = (\tau/\tau_i)^{-8/3}\,\Phi_{hi} + \Delta\Phi_h, \tag{A8}$$

$$\Phi_H = (\tau/\tau_i)^{-4/3}\,\Phi_{Hi} + \Delta\Phi_H. \tag{A9}$$

From Eq. (32) of [5], we can derive

$$\tilde{\Phi}_G = \tfrac{3}{2}\left[-\tau^{-2/3}\tau_i^{8/3}\Phi_{hi} + \tau^{2/3}\tau_i^{4/3}\Phi_{Hi}\right]\ln(\tau/\tau_i), \tag{A10}$$

$$\left(\tilde{\Phi}_G\right)' = \tfrac{3}{2}\left[-\tau^{-5/3}\tau_i^{8/3}\Phi_{hi} + \tau^{-1/3}\tau_i^{4/3}\Phi_{Hi}\right] + \left[\tau^{-5/3}\tau_i^{8/3}\Phi_{hi} + \tau^{-1/3}\tau_i^{4/3}\Phi_{Hi}\right]\ln(\tau/\tau_i), \tag{A11}$$

where $\tau \equiv t_0 - t$ and a dash denotes $d/d\tau$. For $\Phi_6$, we have

$$\Phi_6' + \frac{1}{\tau}\Phi_6 = 3\tau\left(\Delta\Phi_h' + \frac{8/3}{\tau}\Delta\Phi_h + \Delta\Phi_H' + \frac{4/3}{\tau}\Delta\Phi_H\right) + 2\left[\left(k_r^{(0)}/r\right)^2 - \left(k_R^{(0)}/R\right)^2\right]\tilde{\Phi}_G, \tag{A12}$$

which is derived from Eq. (58) of [5]. Here we define the auxiliary quantities $X$ and $Y$ by

$$X \equiv \Delta\Phi_h' + \frac{8}{3\tau}\Delta\Phi_h \quad \text{and} \quad Y \equiv \Delta\Phi_H' + \frac{4}{3\tau}\Delta\Phi_H. \tag{A13}$$

Then, from Eqs. (A5), (A8), (A9), (A11), and (A12), we obtain the following equations for $X$ and $Y$:

$$2X + 6Y = A, \tag{A14}$$

and

$$3X + 5Y = B + \frac{2}{\tau^2}\Phi_6, \tag{A15}$$

where $A$ and $B$ are expressed as

$$\begin{aligned} A = &-3\left(k_R^{(0)}/R_0\right)^2\tau^{-1}{\tau_i}^{4/3}\Phi_{Hi} + \Bigg\{\left[-2\left(k_R^{(0)}/R_0\right)^2\tau^{-7/3} + 3\left(k_r^{(0)}/r_0\right)^2\tau^{-1}\right]{\tau_i}^{8/3}\Phi_{hi} \\ &- 3\left(k_r^{(0)}/r_0\right)^2\tau^{1/3}{\tau_i}^{4/3}\Phi_{Hi}\Bigg\}\ln\left(\tau/\tau_i\right), \end{aligned} \tag{A16}$$

and

$$\begin{aligned} B = &\,3\left(k_r^{(0)}/r_0\right)^2\tau^{-1}{\tau_i}^{8/3}\Phi_{hi} + \Bigg\{\left[3\left(k_r^{(0)}/r_0\right)^2\tau^{-1} - \tfrac{3}{2}\left(k_R^{(0)}/R_0\right)^2\tau^{-7/3}\right]{\tau_i}^{8/3}\Phi_{hi} \\ &+ \left[-5\left(k_r^{(0)}/r_0\right)^2\tau^{1/3} + \tfrac{3}{2}\left(k_R^{(0)}/R_0\right)^2\tau^{-1}\right]{\tau_i}^{4/3}\Phi_{Hi}\Bigg\}\ln\left(\tau/\tau_i\right). \end{aligned} \tag{A17}$$

Eliminating $\Phi_6$ from Eq. (A15) by use of (A12), we have

$$3X' + 5Y' + \frac{3}{\tau}\left(X + 3Y\right) = B' + \frac{3}{\tau}B + \frac{2}{\tau^2}C, \tag{A18}$$

where $C$ is

$$C \equiv 2\left[\left(k_r^{(0)}/r\right)^2 - \left(k_R^{(0)}/R\right)^2\right]\tilde{\Phi}_G \tag{A19}$$

with $\tilde{\Phi}_G$ defined by Eq. (A10).

Integrating Eqs. (A14) and (A18) with respect to $X$ and $Y$, we obtain $X$ and $Y$, expressed as

$$\begin{aligned} X = \tau^{-1}&\left\{\left[\frac{3}{2}\left(\frac{k_r^{(0)}}{r_0}\right)^2 - \frac{37}{28}\tau^{-4/3}\left(\frac{k_R^{(0)}}{R_0}\right)^2\right]\ln\left(\tau/\tau_i\right)\right. \\ &\left.+\left[\frac{23}{24}\left(\frac{k_r^{(0)}}{r_0}\right)^2 + \frac{243}{392}\tau^{-4/3}\left(\frac{k_R^{(0)}}{R_0}\right)^2\right]\right\}{\tau_i}^{8/3}\Phi_{hi} \\ &+\tau^{1/3}\left\{\left[-12\left(\frac{k_r^{(0)}}{r_0}\right)^2 + \frac{9}{4}\tau^{-4/3}\left(\frac{k_R^{(0)}}{R_0}\right)^2\right]\ln\left(\tau/\tau_i\right)\right. \\ &\left.+\left[\frac{243}{8}\left(\frac{k_r^{(0)}}{r_0}\right)^2 - \frac{9}{4}\tau^{-4/3}\left(\frac{k_R^{(0)}}{R_0}\right)^2\right]\right\}{\tau_i}^{4/3}\Phi_{Hi}, \end{aligned} \tag{A20}$$

and

$$Y = \tau^{-1} \left\{ \frac{3}{28} \tau^{-4/3} \left( \frac{k_R^{(0)}}{R_0} \right)^2 \ln(\tau/\tau_i) - \left[ \frac{23}{72} \left( \frac{k_r^{(0)}}{r_0} \right)^2 + \frac{81}{392} \tau^{-4/3} \left( \frac{k_R^{(0)}}{R_0} \right)^2 \right] \right\} {\tau_i}^{8/3} \Phi_{hi}$$
$$+ \tau^{1/3} \left\{ \left[ \frac{7}{2} \left( \frac{k_r^{(0)}}{r_0} \right)^2 - \frac{3}{4} \tau^{-4/3} \left( \frac{k_R^{(0)}}{R_0} \right)^2 \right] \ln(\tau/\tau_i) \right.$$
$$\left. + \left[ -\frac{81}{8} \left( \frac{k_r^{(0)}}{r_0} \right)^2 + \frac{1}{4} \tau^{-4/3} \left( \frac{k_R^{(0)}}{R_0} \right)^2 \right] \right\} {\tau_i}^{4/3} \Phi_{Hi}. \quad \text{(A21)}$$

Integrating Eq. (A13) with respect to $\Delta\Phi_h$ and $\Delta\Phi_H$, moreover, we obtain the following expressions:

$$\Delta\Phi_h = \left\{ \left[ \ln(\tau/\tau_i) + \frac{19}{72} \right] x^2 + \left[ -\frac{37}{28} \ln(\tau/\tau_i) + \frac{939}{14 \times 56} \right] y^2 \right\} (\tau/\tau_i)^{-8/3} \Phi_{hi}$$
$$+ \left\{ -\frac{16}{3} \left[ \ln(\tau/\tau_i) + \frac{89}{6} \right] x^2 + \left[ \frac{3}{8} \ln(\tau/\tau_i) - \frac{1089}{64 \times 49} \right] y^2 \right\} (\tau/\tau_i)^{-4/3} \Phi_{Hi}, \quad \text{(A22)}$$

and

$$\Delta\Phi_H = \left\{ -\frac{23}{54} x^2 + \left[ \frac{1}{42} (\ln \tau/\tau_i)^2 - \frac{9}{112} \right] y^2 \right\} (\tau/\tau_i)^{-8/3} \Phi_{hi}$$
$$+ \left\{ \left[ \frac{7}{3} \ln \tau/\tau_i - \frac{61}{8} \right] x^2 - \left[ \frac{1}{12} \ln(\tau/\tau_i) + \frac{1}{48} \right] y^2 \right\} (\tau/\tau_i)^{-4/3} \Phi_{Hi}. \quad \text{(A23)}$$

For $x(\ll 1)$ and $y(\ll 1)$, therefore, $\Delta\Phi_h$ and $\Delta\Phi_H$ are small, compared with the main terms $(\tau/\tau_i)^{-8/3}\Phi_{hi}$ and $(\tau/\tau_i)^{-4/3}\Phi_{Hi}$.

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