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An alternative construction of Kontsevich-Kuperberg-Thurston's universal finite type invariant of homology 3-spheres

Tatsuro Shimizu

Research Institute for Mathematical Sciences, Kyoto University

1 Introduction

Kontsevich-Kuperberg-Thurston invariant is one variation of M. Kontsevich's Chern-Simons perturbation theoretic invariant. G. Kuperberg and D. Thurston ([5]) gave the construction of the invariant based on M. Kontsevich's idea in [4] and they showed that this invariant is a universal finite type invariant for integral homology 3-spheres as the LMO is.

Kontsevich-Kuperberg-Thurston invariant, denoted by z^{KKT}, is a sequence $\{z_n^{\mathrm{KKT}}\}_{n\in\mathbb{N}}$. z_n^{KKT} is a topological invariant of rational homology 3-spheres taking values in the finite dimensional rational vector space $\mathcal{A}_n(\emptyset)$. $\mathcal{A}_n(\emptyset)$ is the quotient space divided by some relations (called IHX, AS relations) from the vector space freely generated by oriented Jacobi diagrams with $2n$-vertexes. We don't give an explicit definition of this space and Jacobi diagrams (For example, see [5], [6]). In this article we treat only the case of $n=1$. In this case $\mathcal{A}_1(\emptyset)$ is isomorphic to the 1-dimensional vector space $\mathbb{Q}$. So we take and fix such an isomorphism and then we consider z_1^{KKT} as a $\mathbb{Q}$ valued invariant. It is known that z_1^{KKT} equals to $\frac{4}{3}$ times the Casson-Walker invariant.

2 Preliminary

In this article, all homology 3-spheres are oriented, smooth and with a metric. The assumptions "smooth","oriented" and "with a metric" are not usual. We will use these structures in the construction of the invariant. The invariant is, however, independent of the choices of these structure (i.e. topological invariant).

Let Y be a rational homology 3-sphere. Let $\infty\in Y$ be a base point. Take $N(\infty;Y)\subset Y$ a neighborhood of ∞ in Y and let $N(\infty;S^3)$ be a neighborhood of ∞ in $S^3=\mathbb{R}^3\cup\{\infty\}$. We take an orientation preserving diffeomorphism $\varphi^\infty:(N(\infty;Y),\infty)\xrightarrow{\cong}(N(\infty;S^3),\infty)$

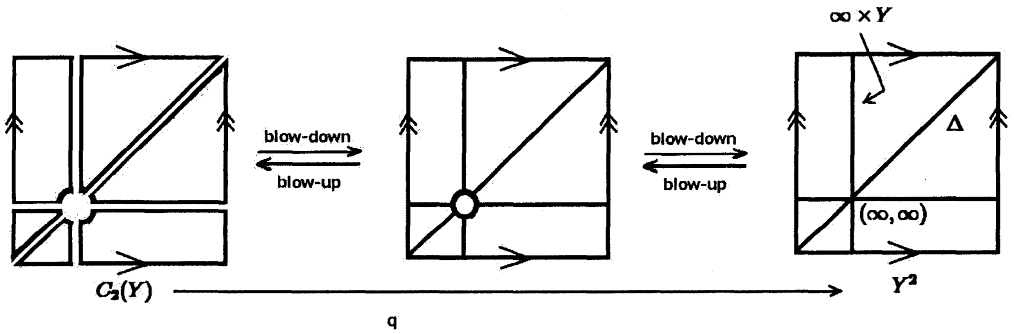
and then we identify $N(\infty;Y)$ with $N(\infty;S^3)$ via φ^∞. So we consider $N(\infty;Y)\setminus\infty\subset\mathbb{R}^3$ under this identification.

We introduce a compactification $C_2(Y)$ of the two point configuration space $(Y\setminus\infty)^2\setminus\Delta=\{(x,y)\mid x\neq y\}$. We denote by $B\ell(A,B)$ the real blowing-up of A along B for submanifold B in A: $B\ell(A,B)=A\setminus B\cup S\nu_B$. Here ν_B is the normal bundle of B in A and $S\nu_B$ is the unit sphere bundle of ν_B.

Let $q_1:B\ell(Y^2,\infty^2)\to Y^2$ be the blow-down map. There are three submanifolds $\overline{q_1^{-1}((Y\setminus\infty)\times\infty)}$, $\overline{q_1^{-1}(\infty\times(Y\setminus\infty))}$ and $\overline{q_1^{-1}(\Delta\setminus\infty^2)}$ of $B\ell(Y^2,\infty^2)$. The over-line means that the closure. We define

$$C_2(Y)=B\ell(B\ell(Y^2,\infty^2),\overline{q_1^{-1}((Y\setminus\infty)\times\infty)}\sqcup\overline{q_1^{-1}(\infty\times(Y\setminus\infty))}\sqcup\overline{q_1^{-1}(\Delta\setminus\infty^2)}).$$

We denote by $q:C_2(Y)\to Y^2$ the composition of blow-down maps. $C_2(Y)$ is a closed 6-manifold with boundary and corner. It is known that there is a natural smooth structure on $C_2(Y)$ ([6]).



3 The original construction of Kontsevich-Kuperberg-Thurston invariant

Take $a_1,a_2,a_3\in S^2\subset\mathbb{R}^3$ be unit vectors. Let $\tau:T(Y\setminus\infty)\xrightarrow{\cong}(Y\setminus\infty)\times\mathbb{R}^3$ be a framing such that $\tau|_{N(\infty;Y)\setminus\infty}$ coincides with the standard trivialization of $T\mathbb{R}^3=\mathbb{R}^3\times\mathbb{R}^3$.

To define the invariant, we will construct 4-cycles $W_1(\tau),W_2(\tau)$ and $W_3(\tau)$ of $(C_2(Y),\partial C_2(Y))$ by using τ.

We first construct 3-dimensional submanifolds $W_1^\partial(\tau),W_2^\partial(\tau)$ and $W_3^\partial(\tau)$ of $\partial C_2(Y)$. Let $p_{S^3}:C_2(S^3)\to S^2$ be the extended map of $\mathbb{R}^3\times\mathbb{R}^3\setminus\Delta\to S^2$, $(x,y)\mapsto\frac{y-x}{\|y-x\|}$. Let

$p_0 : \partial C_2(Y) \setminus q^{-1}(\Delta \setminus \infty^2) \to S^2$ be the smooth map defined as follows:

$$p_0|_{q^{-1}(\infty^2)} = p_{S^3}|_{q^{-1}(\infty^2)} : q^{-1}(\infty^2) \to S^2,$$
$$p_0|_{q^{-1}((Y\setminus\infty)\times\infty)} : q^{-1}(Y \setminus \infty) \times \infty = (Y \setminus \infty) \times ST_\infty Y \to ST_\infty Y = S^2 \overset{-1}{\to} S^2 \text{ and}$$
$$p_0|_{q^{-1}(\infty\times(Y\setminus\infty))} : q^{-1}(\infty \times (Y \setminus \infty)) = ST_\infty Y \times (Y \setminus \infty) \to ST_\infty Y = S^2.$$

Here the maps $ST_\infty Y \times (Y \setminus \infty) \to ST_\infty Y$, $ST_\infty Y \times (Y \setminus \infty) \to ST_\infty Y$ are projection and $S^2 \overset{-1}{\to} S^2$ is an involution defined by $x \mapsto -x$. Then $p_0^{-1}(a_1), p_0^{-1}(a_2)$ and $p_0^{-1}(a_3)$ are 3-dimensional submanifolds of $\partial C_2(Y) \setminus q^{-1}(\Delta \setminus \infty^2)$. There is a canonical bundle isomorphism $\nu_{\Delta\setminus\infty^2} \cong ST(Y \setminus \infty)$. We define $p(\tau) : q^{-1}(\Delta \setminus \infty^2) \to S^2$ as follows:

$$q^{-1}(\Delta \setminus \infty^2) = S\nu_{\Delta\setminus\infty^2} \cong ST(Y \setminus \infty) \overset{\tau}{\cong} (Y \setminus \infty) \times S^2 \to S^2.$$

Here the last map is the projection.

Definition 3.1. For $i = 1, 2, 3,$

$$W_i^\partial(\tau) = p_0^{-1}(a_i) \cup p(\tau)^{-1}(a_i).$$

We remark that $W_i^\partial(\tau)$ is a compact 3-manifold without boundary owing to the assumption of τ. $W_i^\partial(\tau)$ represents a cycle[1] of $\partial C_2(Y)$. We next extend it to a cycle of $(C_2(Y), \partial C_2(Y))$.

Lemma 3.2. *There exists a 4-cycle $W_i(\tau)$ of $(C_2(Y), \partial C_2(Y))$ such that $\partial W_i(\tau) = W_i^\partial(\tau)$, for $i = 1, 2, 3$.*

Proof. Since Y is a rational homology 3-sphere, the boundary map
$H_4(C_2(Y), \partial C_2(Y); \mathbb{Q}) \to H_3(\partial C_2(Y); \mathbb{Q})$ is an isomorphism. □

We take $W_i(\tau)$ as above.

Remark 3.3. This 4-cycle or the Poincaré dual of this 4-cycle is called a propagator.

For generic $W_1(\tau)$, $W_2(\tau)$ and $W_3(\tau)$, the intersection $W_1(\tau) \cap W_2(\tau) \cap W_3(\tau)$ is a compact oriented 0-dimensional manifold. So we can count it with sign.

Theorem 3.4 (Kuperberg and Thurston [5]). *For generic a_i and $W_i(\tau)$,*

$$z_1^{\mathrm{KKT}}(Y) = \sharp(W_1(\tau) \cap W_2(\tau) \cap W_3(\tau)) + \frac{1}{4}\sigma(\tau)$$

is a topological invariant of Y. In particular, $z_1^{\mathrm{KKT}}(Y)$ is independent of the choice of τ.

Here $\sigma(\tau) \in \mathbb{Z}$ is the signature defect of τ defined as follows. Let τ_{S^3} be a framing of S^3 satisfying $\tau_{S^3}|_{S^3\setminus N(\infty;S^3)} = \tau_{\mathbb{R}^3}$. Then $\tau \cup \tau_{S^3} = \tau|_{Y\setminus N(\infty;Y)} \cup \tau_{S^3}|_{N(\infty;S^3)}$ is a framing of Y. Let $\sigma_Y(\tau \cup \tau_{S^3})$ be the signature defect[2] of it and $\sigma_{S^3}(\tau_{S^3})$ be the signature defect of τ_{S^3}. We define $\sigma(\tau) = \sigma_Y(\tau \cup \tau_{S^3}) - \sigma_{S^3}(\tau_{S^3})$.

[1] In this article, all cycles are with rational coefficients.
[2] See [1] for the definition of the signature defect of an honest framing.

4 An alternative construction of Kontsevich-Kuperberg-Thurston invariant

In this section, we give an alternative construction of z_1^{KKT} We also construct propagators as 4-cycles.

Take $a_1, a_2, a_3 \in S^2 \subset \mathbb{R}^3$ as above. Let γ_1, γ_2 and γ_3 be vector fields on $Y \setminus \infty$ such that $\gamma_i|_{N(\infty;Y)\setminus\infty}$ coincides with the constant vector field a_i of $\mathbb{R}^3$ and γ_i transversally intersect to the zero-section of $T(Y \setminus \infty)$ for $i = 1, 2, 3$.

We first construct 3-dimensional submanifolds $W^\partial(\gamma_1)$, $W^\partial(\gamma_2)$ and $W^\partial(\gamma_3)$ of $\partial C_2(Y)$ as in the above section. Let

$$c_{\gamma_i} = \overline{\left\{ \frac{\gamma_i(x)}{\|\gamma_i(x)\|} \in ST_xY \;\middle|\; x \in Y \setminus (\infty \cup \gamma_i^{-1}(0)) \right\}}^{\text{closure}} \subset ST(Y \setminus \infty),$$

for $i = 1, 2, 3$. c_{γ_i} is a manifold with boundary and an end. Its boundary is in near $\gamma_i^{-1}(0)$.

Lemma 4.1. *$c(\gamma_i) = c_{\gamma_i} \cup c_{-\gamma_i}$ is a manifold without boundary*[3].

Outline of proof. c_{γ_i} and $c_{-\gamma_i}$ have same boundaries but their orientation are opposite. So these boundaries are cancel each other. □

Definition 4.2. $W^\partial(\gamma_i) = c(\gamma_i) \cup p_0^{-1}(\{a_i, -a_i\})$

We take a 4-cycle $W(\gamma_i)$ of $(C_2(Y), \partial C_2(Y))$ such that $\partial W(\gamma_i) = \frac{1}{2} W^\partial(\gamma_i)$.

Proposition 4.3. *For generic a_i and γ_i, $\sharp(W(\gamma_1) \cap W(\gamma_2) \cap W(\gamma_3))$ is independent of the choice of $W(\gamma_1), W(\gamma_2)$ and $W(\gamma_3)$* [4].

This proposition is proved by a homological argument similar to the argument to prove the well-definedness of the linking number of two component links.

We next define the correction term to cancel out the influence of the choice of γ_1, γ_2 and γ_3. Recall that $\tau_{S^3} : TS^3 \to S^3 \times \mathbb{R}^3$ is a framing of S^3 such that $\tau_{S^3}|_{S^3 \setminus N(\infty;S^3)} = \tau_{\mathbb{R}^3}$. We consider $a_i \in \mathbb{R}^3$ as a constant vector field of trivial $\mathbb{R}^3$ bundle. Then $\tau_{S^3}^* a_i$ is a constant vector filed of S^3. Let X be a compact oriented 4-manifold with $\chi(X) = 0$ and $\partial X = Y$. Take a non-vanishing vector field η_X on X such that $\eta_X|_Y$ is the outward normal vector field of $Y = \partial X$. Let $T^vX \to X$ be the normal bundle of η_X in TX. Then $T^vX|_Y = TY$. Let $ST^vX \to X$ be the unit sphere bundle of T^vX. Take β_i is a generic section of $T^vX \to X$ such that $\beta_i|_Y = \gamma_i|_{Y \setminus N(\infty;Y)} \cup \tau_{S^3}^* a_i|_{N(\infty;S^3)}$. Let

$$c_{\beta_i} = \overline{\left\{ \frac{\beta_i(x)}{\|\beta_i(x)\|} \in (ST^vX)_x \;\middle|\; x \in X \setminus \beta_i^{-1}(0) \right\}}^{\text{closure}} \subset ST^vX.$$

[3] $c(\gamma_i)$ has two ends near ∞.

[4] This number depends on the choice of γ_1, γ_2 and γ_3.

By a similar argument as in Lemma 4.1, $c(\beta_i) = c_{\beta_i} \cup c_{-\beta_i}$ is a 4-dimensional submanifold of ST^vX such that $\partial c(\beta_i) = c(\gamma_i \cup \tau^*_{S^3}a_i)$.

For generic β_1, β_2 and β_3, $c(\beta_1) \cap c(\beta_2) \cap c(\beta_3)$ is a compact oriented 0-dimensional manifold. Furthermore, this argument is extended to any closed 4-manifold whose Euler number is zero and we can check that $\sharp(c(\beta_1) \cap c(\beta_2) \cap c(\beta_3))$ is a cobordism invariant of closed 4-dimensional manifold whose Euler number is zero.

Lemma 4.4 ([8], [7]). *$\widetilde{I}(\gamma_1, \gamma_2, \gamma_3) = \frac{1}{8}\sharp(c(\beta_1) \cap c(\beta_2) \cap c(\beta_3)) - \frac{3}{4}\mathrm{Sign}X$ is independent of the choice of β_i and X.*

Remark 4.5. This correction term was first defined by T. Watanabe in [8] for integral homology 3-spheres to construct the Morse homotopy invariant. We modified his construction to extend to rational homology 3-spheres and we determined the number $-\frac{3}{4}$ before the term Sign$X$.

Theorem 4.6 ([7]).

$$\widetilde{z}_1(Y) = \sharp(W(\gamma_1) \cap W(\gamma_2) \cap W(\gamma_3)) - \widetilde{I}(\gamma_1, \gamma_2, \gamma_3)$$

is a topological invariant of Y.

Theorem 4.7 ([7]). *$\widetilde{z}_1(Y) = z_1^{\mathrm{KKT}}(Y)$ for any rational homology 3-sphere Y.*

Proof. Let τ be a framing of $Y \setminus \infty$ as above section. Then $\tau^* a_i$ is a non-vanishing vector field of $Y \setminus \infty$ by considering $a_i \in \mathbb{R}^3$ as a constant vector filed of trivial $\mathbb{R}^3$ bundle. By the definition, we have $\partial W(\tau^* a_i) = \partial W_i(\tau)$. Then,

$$\sharp(W(\tau^* a_1) \cap W(\tau^* a_2) \cap W(\tau^* a_3)) = \sharp(W_1(\tau) \cap W_2(\tau) \cap W_3(\tau)).$$

Let $\Omega_3^{\mathrm{Sign}=0}$ be the cobordism group generated by all 3-dimensional framed manifolds $\{(Y, \tau) \mid \tau : TY \overset{\cong}{\to} Y \times \mathbb{R}^3\}$ and dividing by a cobordism relation $\sim$: $(Y, \tau) \sim \emptyset$ if and only if there exists compact framed 4-manifold (X, T) such that

- $\mathrm{Sign}(X) = 0$,
- $T|_Y$ is isomorphic to the stable framing of τ.

We consider $\widetilde{I}(\tau^* a_1, \tau^* a_2, \tau^* a_3)$ and $\sigma(\tau)$ as an invariant of framed manifold $(Y, \tau|_{Y \setminus N(\infty;Y)} \cup \tau_{S^3}|_{N(\infty;S^3)})$. Then these two invariant factor through $\Omega_3^{\mathrm{Sign}=0}$. We can show that $\Omega_3^{\mathrm{Sign}=0} \otimes \mathbb{Q} \cong \mathbb{Q}$ and $\widetilde{I}(\tau_0^* a_1, \tau_0^* a_2, \tau_0^* a_3) = -\frac{1}{4}\sigma(\tau_0) \neq 0$ for a framing τ_0 of $S^3 \setminus \infty$. Then we have $\widetilde{I}(\tau^* a_1, \tau^* a_2, \tau^* a_3) = -\frac{1}{4}\sigma(\tau)$ for any τ and Y. $\square$

5 An application of our construction

In this section, we give an application of our construction of z^{KKT}.

5.1 Watanabe's invariant

In the 1990s, K. Fukaya constructed an invariant of a pair of two local systems on a 3-manifold by using three Morse functions in [2]. Fukaya's invariant is sum of principal term depending on Morse functions and the correction term to cancel out the influence of the choice of Morse functions. M. Futaki pointed out in [3] that Fukaya's invariant sometimes depends on the choice of Morse functions.

In 2012, T. Watanabe introduced a new type of correction term and then constructed a topological invariant of a integral homology 3-spheres taking values in $\mathcal{A}(\emptyset) = \Pi_n \mathcal{A}_n(\emptyset)$.

In this subsection, we review the degree 1-part, i.e. $A_1(\emptyset) \cong \mathbb{Q}$-valued part, of Watanabe's invariant with a little modification.

Take $a_1, a_2, a_3 \in S^2 \subset \mathbb{R}^3$ as above. Let $f_1, f_2, f_3 : Y \setminus \infty \to \mathbb{R}$ be Morse functions such that $f_i|_{N(\infty;Y)\setminus\infty}$ coincides with the projection $q_{a_i} : \mathbb{R}^3 \to \mathbb{R}$ to the a_i-direction and f_i has no critical points of index 0 or 3. Let $\mathrm{Crit}(f_i) = \{p^i_1, \ldots, p^i_{k_i}, q^i_1, \ldots, q^i_{k_i}\}$ be the set of critical points of f_i. We assume that $\mathrm{ind}(p^i_j) = 2, \mathrm{ind}(q^i_j) = 1$. Let $\partial^i : C_2(Y \setminus \infty; \mathbb{Q}) \to C_1(Y \setminus \infty; \mathbb{Q}), \partial^i[p^i_j] = \sum_k \partial^i_{jk}[q^i_k]$ be the boundary map of the Morse-Smale complex of f_i. Since $Y \setminus \infty$ is rational homology ball, the map ∂^i is an isomorphism. Then there exists the inverse map $g^i : C_1(Y \setminus \infty; \mathbb{Q}) \to C_2(Y \setminus \infty; \mathbb{Q}), g^i[q^i_j] = \sum_k g^i_{jk}[p^i_k]$. Let $\{\Phi^t_{f_i}\}_{t\in\mathbb{R}}$ be the 1-parameter family of diffeomorphisms associated to $-\mathrm{grad} f_i$. We denote by $\mathcal{A}_{q^j_k}$ and $\mathcal{D}_{p^i_j}$ the ascending manifold of q^i_k and the descending manifold of p^i_j respectively.

Definition 5.1.

$$W(f_i) = \{(x,y) \in (Y \setminus \infty)^2 \setminus \Delta \mid \exists t \in \mathbb{R} \setminus 0, \Phi^t_{f_i}(x) = y\}$$
$$+ \sum_{j,k} (g^i_{jk}(\mathcal{A}_{q^i_k} \times \mathcal{D}_{p^i_j}) - g^i_{jk}(\mathcal{D}_{p^i_j} \times \mathcal{A}_{q^i_k})).$$

$W(f_i)$ is a wighted sum of 4-manifold in $(Y \setminus \infty)^2$.

Theorem 5.2 (Watanabe [8]).

$$z_1^{\mathrm{FW}}(Y) = \frac{1}{8}\sharp(W(f_1) \cap W(f_2) \cap W(f_3)) - \widetilde{I}(\mathrm{grad} f_1, \mathrm{grad} f_2, \mathrm{grad} f_3)$$

is a topological invariant of Y.

Watanabe also defined $z_n^{\mathrm{FW}}(Y) \in \mathcal{A}_n(\emptyset)$ and he conjectured that

- Is z_n^{FW} trivial or not?
- $z_n^{\mathrm{FW}} = z^{\mathrm{KKT}}$?

5.2 An application of our construction to Watanabe's invariant

Theorem 5.3. $z_1^{\mathrm{FW}}(Y) = z_1^{\mathrm{KKT}}(Y)$.

Proof. Let $\overline{W}(f_i)$ be the closure of $W(f_i) \cap ((Y \setminus \infty)^2 \setminus \Delta)$ in $C_2(Y)$. Then $\overline{W}(f_i)$ is a 4-cycle of $(C_2(Y), \partial C_2(Y))$. By the definition of $W(f_i)$,

$$\begin{aligned}\partial\overline{W}(f_i) &= p_0^{-1}(\{a_i, -a_i\}) \cup c(\mathrm{grad} f_i) + \sum_{jk}(g^i_{jk}(\mathcal{A}_{q^i_k} \cap \mathcal{D}_{p^i_j}) - g^i_{jk}(\mathcal{D}_{p^i_j} \cap \mathcal{A}_{q^i_k}))\\ &= p_0^{-1}(\{a_i, -a_i\}) \cup c(\mathrm{grad} f_i)\\ &= W^{\partial}(\mathrm{grad} f_i).\end{aligned}$$

Then $z_1^{\mathrm{FW}}(Y) = \widetilde{z}_1(Y) = z_1^{\mathrm{KKT}}(Y)$. □

Remark 5.4. We can show that $z_n^{\mathrm{FW}}(Y) = \widetilde{z}_n(Y) = z_n^{\mathrm{KKT}}(Y)$ for any $n \geq 1$. See [7] for more detail.

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Research Institute for Mathematical Sciences
Kyoto University
Kyoto 606-8502
JAPAN
E-mail address: shimizu@kurims.kyoto-u.ac.jp

京都大学・数理解析研究所　清水　達郎