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Recent advances on 1-cocycles in the space of knots

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Abstract

This is a survey of two recent papers [8, 13] in which were introduced new methods for constructing 1-cocycles in the space of knots. The construction from [13] is a natural adaptation of Polyak-Viro's formulas for finite-type knot invariants; it is conjectured to give the first combinatorial formulas for $\mathbb{Z}$-valued Vassiliev 1-cocycles. In [8], the cocycles take values in skein modules associated with quantum knot invariants; conjecturally, the examples produced detect information regarding the geometry of knots.

For a broader panorama on this topic, see also [6, 7, 15, 16, 19], Sections 1.6–1.8 of [5], and [9] which is a sequel to [8].

1 Finite-type 1-cocycles of knots given by Polyak-Viro formulas [13]

Vassiliev's cohomology classes were introduced in 1990, at a time when the cohomology of the space of knots had barely been studied. In the years that followed, two kinds of explicit formulas were proved to describe all of Vassiliev's 0-cocycles, best known as finite-type invariants: an integral formula, due to Kontsevich [12], and purely combinatorial formulas due to Polyak-Viro [14] (see also [10]).

In the meantime, in higher degree, only one example was proved to exist, by Teiblum and Turchin (at the time a student of Vassiliev), and this 1-cocycle v_3^1 waited until 2001 before Vassiliev [19] found how to actually evaluate it, over $\mathbb{Z}_2$, with a combinatorial formula involving differential geometry. Ten years later, Sakai [15] described a realization of v_3^1 over $\mathbb{R}$ by means of an integral formula.

The purpose of [13] is to fill the gap of a missing combinatorial formula for v_3^1 over $\mathbb{Z}$, to remove geometric conditions from the computation process, and to find more examples of 1-cocycles. Just like the original works of Vassiliev [17, 18], this article considers the space of smooth long knots – i.e., embeddings $\mathbb{R} \hookrightarrow \mathbb{R}^3$ that are standard outside of $[0, 1]$.

1.1 Preliminary: Gauss diagram formulas (after [14])

The spirit of Polyak-Viro's Gauss diagram formulas is to count the subdiagrams of a knot, with weights. Here, knot diagrams are represented combinatorially using Gauss diagrams – see Fig.1, and a subdiagram is the result of removing a (possibly empty) set of arrows.

One of the origins of that idea lies in the well-known formula for computing the linking number of a 2-component oriented link: given a link diagram, $lk(L_1, L_2)$ is the sum of writhes of all crossings where L_1 goes over L_2. In Polyak-Viro's language, each such crossing is a subdiagram with only one arrow remaining, oriented from L_1 to L_2, and the weight given to each subdigram is the writhe of the crossing.

In [14], Polyak and Viro give such formulas for computing (among others) the first two Vassiliev invariants (Theorems 1 and 2). In both cases (as well as in the linking number formula), the weight given to a subdiagram is equal to the product of its writhe numbers, times a constant which depends only on the underlying unsigned diagram. Such particular choices of weights proved to be extremely common in further literature (see for instance [3, 2, 4]), and yield what is often called *arrow diagram formulas* (an *arrow diagram* is a Gauss diagram deprived from its decorating signs).

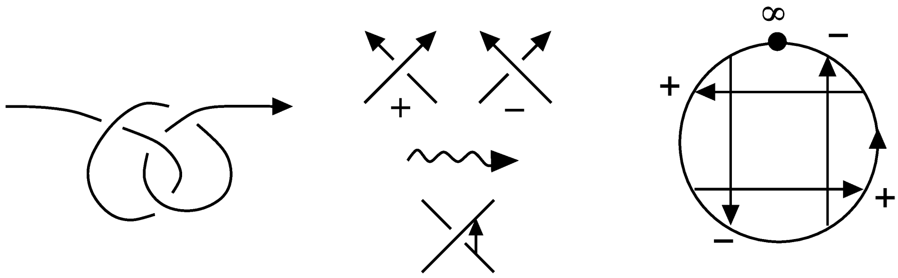


Figure 1: A long figure eight knot diagram and its Gauss diagram

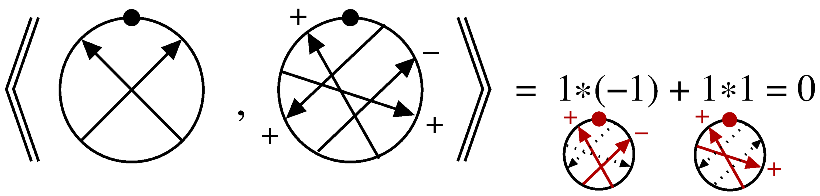


Figure 2: Example of computation of the Casson invariant on a random Gauss diagram

The main interest of these formulas lies in the following result.

Theorem 1 (Polyak-Viro [14], Goussarov [10]). *A knot invariant admits a Gauss diagram formula if and only if it is a Vassiliev invariant.*

Although the proof given by the authors of this theorem does not mention the original definition of Vassiliev invariants, it is possible to obtain such formulas by computing weight systems (see [1]) and integrating them via the homological calculus presented in [19]. This process highlights the deep origin of the writhe numbers in these formulas, as co-orientations of singular strata in the space of all (including singular) knots. It is also one reason to believe that computing products of writhes at a higher level could yield formulas for Vassiliev 1-cocycles.

1.2 Arrow germ formulas

The idea in [13] is to copy Polyak-Viro's construction using as raw material not a Gauss diagram, but a Reidemeister move, which is here regarded as an elementary path in the space of knots.

Definition 1. *An i-*germ *$(i = 1, 2, 3)$ is a couple of Gauss diagrams that differ only by a Reidemeister i move.*

A partial 3-germ *is a 3-germ with one arrow removed (in both diagrams) from the Reidemeister triple.*

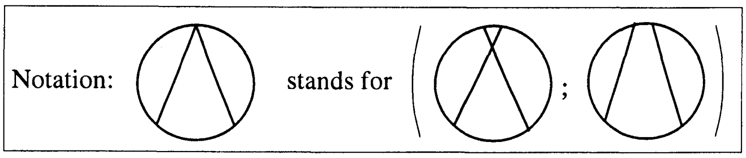
A subgerm *is the result of removing a set of arrows from a (possibly partial) germ,* consistently in both diagrams. *Arrows involved in the Reidemeister move cannot be removed, except in 3-germs where at most one arrow from the triple can be forgotten.*

As before, these notions have a counterpart with no sign decorations, called *(partial) arrow germs* – and as before, the latter are meant to count subgerms, weighted with the product of writhes of the arrows involved.

The main result of [13] is the following.

Theorem 2. *The formal sum α_3^1 of (partial) arrow germs on Fig.3 defines a 1-cocycle in the space of long knots, over $\mathbb{Z}$. Its reduction mod 2 coincides with Teiblum-Turchin's cocycle v_3^1.*

The second statement is proved by showing directly that α_3^1 mod 2 is "of finite type", using Vassiliev's homological calculus presented in [19]. The proof of the first statement relies on a *higher order Reidemeister theorem*, that is, an exhaustive list of all 2-codimensional strata corresponding to the most generic degeneracies of Reidemeister moves (by definition, a 1-cocycle should vanish on the meridians of such strata). A complete description of those strata and their meridians is given in [13]; see also [7].



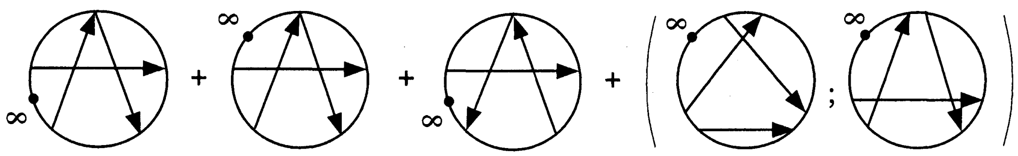


Figure 3: The first non-trivial arrow germ formula α_3^1: three partial arrow 3-germs and one arrow 3-germ

As one can see, no 1- or 2-germs are involved in the formula of Theorem 2, and this is actually a general fact: in any cohomology class represented by germ formulas, there is a formula which contains only (possibly partial) 3-germs ([13], Proposition 2.8). As a result, the linear system with germs as variables and 2-meridians as equations is reduced to a reasonable size ([13], Theorem 2.11).

Conjecture 1. *Every 1-cohomology class with an arrow germ presentation is of finite type in the sense of Vassiliev.*

So far, only one way is known for proving that a 1-cocycle is of finite-type over $\mathbb{Z}$; it consists in defining orientations on the varieties involved in Vassiliev's homological calculus [19]. It has not been done yet, to the best of our knowledge.

1.3 Evaluation of α_3^1 on canonical cycles

One interest of 1-cocycles is that evaluating them on loops that are defined canonically for all knots produces knot invariants. The easiest example of such a loop, for long knots, is the loop $\mathrm{rot}(K)$ which consists of a full positive rotation of K around its axis – see Figs.4 and 5. It is proved in [13] that $\alpha_3^1(\mathrm{rot}(K))$ is equal to minus the Casson invariant of K:

$$\alpha_3^1(\mathrm{rot}(K)) = -v_2(K).$$

This equality was conjectured to hold for the Teiblum-Turchin cocycle in [16]. Turchin's conjecture would follow from the above result and Conjecture 1.

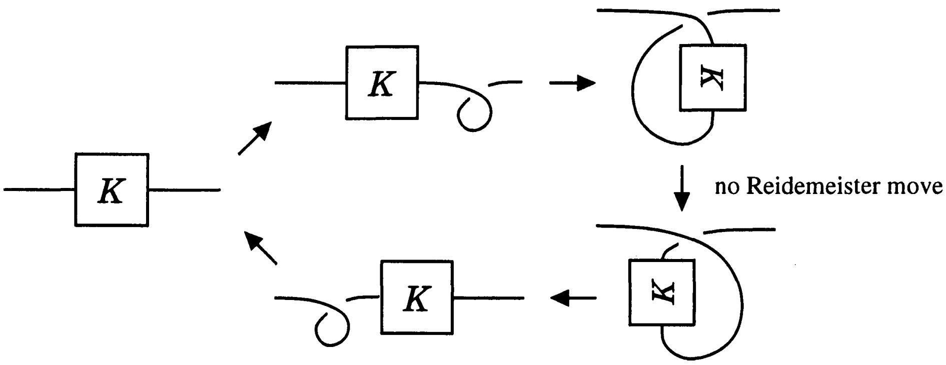


Figure 4: One realization of the loop $\mathrm{rot}(K)$ as a sequence of Reidemeister moves

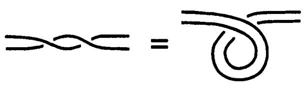


Figure 5: The railway followed by K on Fig.4 is isotopic to a full rotation around the axis

A result of Hatcher [11] states that besides $\mathrm{rot}(K)$, there is essentially only one other interesting loop in the moduli space of all long knots equivalent to a given prime knot. This second loop, often called the *Hatcher loop* $\mathrm{Hat}(K)$, consists of sliding a little "ball at infinity" along a fixed parametrization of the knot in $\mathbb{S}^3$ (a framing convention should be made, because every time the ball makes a full spin around itself, it adds $\pm\mathrm{rot}(K)$ to the loop). Although it does not seem easy to evaluate α_3^1 on $\mathrm{Hat}(K)$, further investigation showed that there is another arrow germ formula $\tilde{\alpha}_3^1$, with the same properties as α_3^1 so far, and such that if K has framing 0, then

$$\tilde{\alpha}_3^1(\mathrm{Hat}(K)) = -6v_3(K).$$

Here v_3 denotes the only Vassiliev invariant of order 3 with value 1 on the positive trefoil, -1 on the negative trefoil and 0 on the unknot. This result is to appear in a subsequent article.

2 Quantum one-cocycles for knots [8]

In this article, Fiedler defines a new family of 1-cocycles in the topological moduli space of long knots, which are then extended to cocycles in the space of string links. As in the previous article (Section 1), the formulas are constructed as algebraic intersection forms with the variety of Reidemeister moves, purely combinatorially: follow a loop in the space of knot diagrams, and every time you meet a Reidemeister move, count something. Only this time what you count does not live in a "small" group like $\mathbb{Z}$, but in a more complicated structure that stays closer to topology and keeps more information.

One of the motivations here is to detect geometry-related information, with the following result in mind (see the definitions of *rot* and *Hat* in the previous section).

Theorem 3 (Hatcher [11]). *Let K be a long knot which is not a satellite. Then the loops $\mathrm{rot}(K)$ and $\mathrm{Hat}(K)$ are linearly independent over $\mathbb{Q}$ if and only if K is hyperbolic.*

It means that if one can create a 1-cocycle powerful enough to detect exactly when $\mathrm{rot}(K)$ and $\mathrm{Hat}(K)$ are linearly dependent, then it gives a simple criterion to evaluate the geometry of prime knots. The ultimate goal in that direction is to find a cocycle v such that the quotient $v(\mathrm{rot})/v(\mathrm{Hat})$, when not a rational number, is related to the hyperbolic volume.

2.1 The space of cochains

There are a lot of combinatorial tools required to define properly the objects in Fiedler's theory; we begin with a simplified version, which will be refined gradually in the next subsections. In particular, we first set aside orientation and sign issues.

Let us describe the elementary cochains from which one tries to obtain 1-cocycles. They are based on a simple surgery operation that generalizes knot mutations. Indeed, let T be a tangle diagram in a 2-disc, with 6 boundary points. One defines an elementary cochain c_T as follows. Pick a generic path γ in the space of long knot diagrams. For each Reidemeister III move involved in γ, notice that the local Reidemeister picture is a tangle diagram in a disc, with 6 boundary points; remove that disc from your diagram, and replace it with T, making sure that the boundary points match; there are six ways to do that, and we will explain later how to choose one canonically. The evaluation $c_T(\gamma)$ is by definition the formal sum of all tangle diagrams so obtained.

By considering tangles with 4 (respectively, 2) boundary points, one can similarly construct cochains that detect Reidemeister II (respectively, I) moves.

Now if one stops here, the space of cochains is indexed by tangles in a disc, so that it is infinite-dimensional. Moreover, a brief examination reveals that there are very few chances to ever find a cocycle in that space: indeed, a 1-cocycle should vanish on 2-meridians, and it is not easy for a formal sum of tangle diagrams to vanish (even with proper signs defined). The natural answer to both these issues is to push the values of the cochains in a skein module. In [8], two modules are separately considered: the HOMFLYPT and Kauffman polynomials, which both yield finitely generated cochain spaces. Note that in the former case, all graft tangles T should be oriented and surgeries performed consistently.

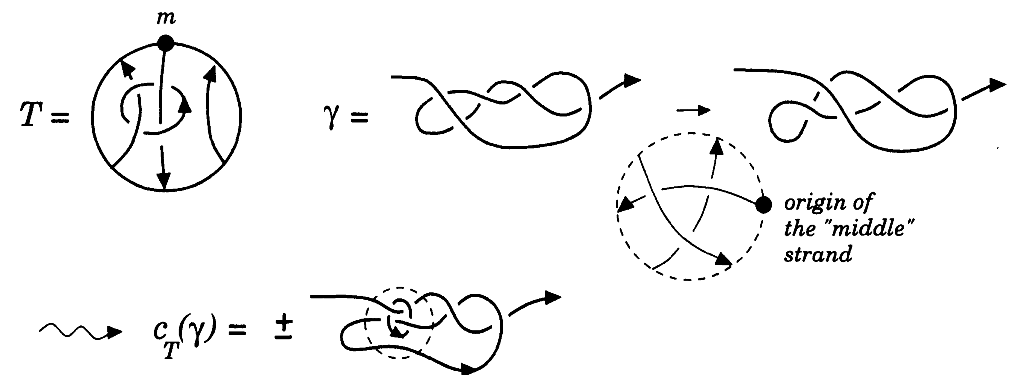


Figure 6: Elementary cochain associated with an oriented tangle T. The middle strand issue is explained in Subsection 2.1.3.

2.1.1 Local and global types

An empirical fact in this theory is that 1-cocycle formulas tend to contain a lot of symmetry, and the more symmetry there is in a formula, the more likely it is to be cohomologically trivial. There are mainly two ideas in [8] to get round this difficulty: first, use parameters and decorations so as to break the symmetry as much as possible; second, find a way to extract interesting information even from a cocycle that is cohomologically trivial (see Section 2.2.1).

Let us assume for now that we work with long knot diagrams (see example on Fig.1). There are a number of data that one can read given a Reidemeister move in such a diagram, which together define the *local* and *global types* of the move. Then, if l denotes one particular local type, and g a global type, one can enrich the previous settings by defining an elementary cochain $c_{T,l,g}$: it makes the same computation as c_T but **only if the Reidemeister move has type** (l, g) (so that c_T is the sum of $c_{T,l,g}$ over all types l and g).

Local types are more conveniently read on the local knot diagram picture, while global types require to look at how the involved crossings are arranged on the Gauss diagram, with respect to each other and to the point at infinity; the point at infinity appears to be the key to break the symmetry and allow non-trivial solutions to exist.

Figure 7 sums up the definitions of all types, and where to find them in [8].

Terminology. The notation "$c_{T,l,g}$", although convenient for an overview of the article, is not the one used in [8]. There, cochains are systematically defined by giving, for each couple of types (l, g), which tangles T (called *partial smoothings*) contribute. For instance, "for the cochain R, the partial smoothing T_{r_c} for type 3 is " means in the current terminology that R can be written

$$R = c_{\;,3,r_c} + \text{other terms.}$$

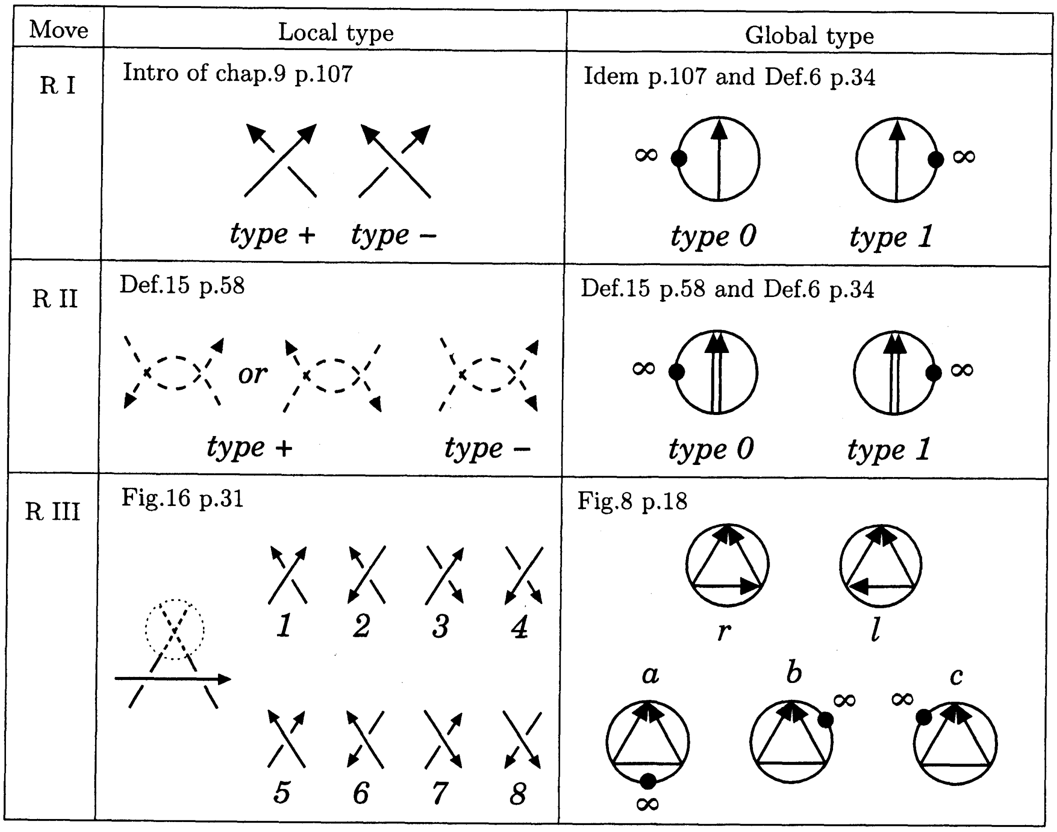


Figure 7: All local and global types with their reference pages in [8] (version 2)

2.1.2 Signs

One obvious necessary condition for a cochain c to define a 1-cocycle is that c should vanish on any little loop in which one Reidemeister move is performed and then undone. As a result, one must either work over $\mathbb{Z}/2$, or define "co-orientations" of Reidemeister moves. The latter choice is made in [8], as follows.

In the case of Reidemeister I and II moves, the evaluation of an elementary cochain $c_{T,l,g}$ comes with an additional minus sign if and only if the move *destroys* crossings.

In the case of Reidemeister III, the co-orientation is defined separately for each local type. That is the meaning of the + and − signs in [8] Fig.16 p.31, they indicate on which side of the move each picture is. When evaluating a cochain, count an additional minus sign if and only if the move goes from the positive to the negative side. Note that the signs here were not chosen arbitrarily. The easiest way to show it might be to point out that in Fig.8, where every local type appears exactly once, all eight triangles are on their respective positive side.

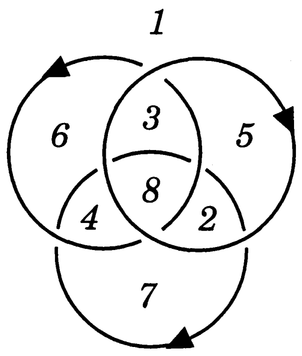


Figure 8: All eight local types on their positive side, in one picture

2.1.3 Gluing conventions

We now explain how Fiedler chooses one out of six ways of gluing a graft tangle after removing a Reidemeister III tangle (for R I there is no choice since the skein modules are 1-dimensional, and for R II it is directly made clear on each picture, like [8] Fig.41 p.59, which in our current notations would read $-(v - v^{-1})c_{\rangle\langle,+,0}$).

First, notice that for local types 2 and 6 (called *star-like*), sources and sinks alternate along the boundary of the Reidemeister tangle; for the remaining (*braid-like*) types 1, 3, 4, 5, 7 and 8, the sources can be grouped together on one side of the disk. In the latter case, the convention is that the three boundary points at the *left* of any picture of a graft tangle T should replace the three *sources* of the removed tangle. This convention is especially useful when working over an unoriented skein module; otherwise, the simple fact that orientations should match leaves no choice.

For local types 2 and 6, one defines the *mid* point in the Reidemeister tangle as the source of the "middle" strand, which goes neither over the two other strands, nor under them. The convention is then to decorate one of the boundary points in the graft tangles with a letter "m", indicating that this point always replaces the mid point when performing a surgery – see Fig.6.

When working over an oriented skein module, only graft tangles with consistent orientations are allowed; note that this consistency condition depends on the local type.

2.1.4 Weights

There are two major ingredients in Fiedler's formulas. The first is the idea of performing surgeries that formally depend on some local and global parameters, i.e., the cochains $c_{T,l,g}$ defined so far. The second is to **weight these cochains with Gauss diagram formulas**. Very roughly, it amounts to considering the module freely generated by the elementary cochains $c_{T,l,g}$, not over $\mathbb{Z}$, but over a ring of arrow diagrams (as defined in Section 1).

As in Section 1, those arrow diagrams compute sums of products of writhes in the Gauss diagram that represents the long knot at the moment where each Reidemeister move is performed. The number of writhes involved in each product determines the *degree* of the weight. A weight of degree 1 (resp. 2) is called *linear* (resp. *quadratic*).

2.2 Results

Recall that when a 1-cochain in the space of knots is defined as an intersection form with Reidemeister moves, the condition for being a cocycle is to vanish on the meridians of the 2-codimensional strata defined by the higher order Reidemeister theorem (just like knot invariants, i.e., 0-cocycles, should vanish on the meridians of the 1-codimensional strata defined by the usual Reidemeister theorem). Two of these strata are by far the most complicated, and occupy a central place in [8]:

- The set of knots whose projection to the plane contains a quadruple point (which can be thought of as an arc sliding over a Reidemeister III move), denoted by $\divideontimes$. The equations associated with their meridians are called the *tetrahedron equations.*
- The set of knots whose projection contains a triple point with two tangent branches (an arc sliding over/under/through a Reidemeister II move), denoted by $\rtimes\!\!\!<$. The associated equations are called the *cube equations.*

The results are organized as follows: for each skein module (HOMFLYPT, pp.34-149 and Kauffman, pp.149-168), two 1-cocycle formulas are constructed: $R^{(1)}_{reg}$ and $\bar{R}^{(1)}$ in the HOMFLYPT case, $R^{(1)}_{F,reg}$ and $\bar{R}^{(1)}_F$ in the Kauffman case. Each of these are built step by step, by first solving the tetrahedron equation, then adjusting the solution so that it satisfies also the cube equations, then adjusting again so as to get all (when possible) remaining equations satisfied. Formulas whose name contains a subscript "reg" do *not* vanish on meridians involving Reidemeister I moves; however they vanish on all other meridians, which makes them *regular cocycles*, i.e., invariants of regular loops up to regular isotopy ([8] Theorem 3 p.94); they are both made of quantum cochains with linear weights. On the other hand, $\bar{R}^{(1)}$ and $\bar{R}^{(1)}_F$ satisfy all equations and define non-trivial cocycles in the space of long knots ([8] Theorem 4 p.127 and Theorem 5 p.165). They are made of a main part which is a quantum cochain with (at most) quadratic weights, and a corrective term which purely consists of a weight of degree 3 (and no surgery part).

2.2.1 From long knots to string links

There are many reasons for which one would like to generalize a construction related to long knots to string links in a 3-ball. For instance, producing families of invariants via knot cabling, or generalizing a formula computing the trivial knot invariant, which may prove to be non-trivial when applied to string links. When it comes to 1-cocycles, tangles in general are a little bit unfriendly: the moduli space of all tangles isotopic to a given one has very often a trivial first homology group. However, even a 1-cocycle formula that is null-cohomologous can lead to non-trivial invariants of tangles if evaluated on a *canonical arc which is not a loop.*

The way Fiedler extends his long knot cocycle formulas to arbitrary string links is extremely simple: just like knots, string links can be represented with Gauss diagrams, the only difference is that the base manifold is not a circle, but a collection of numbered arcs. Fix arbitrarily:

1. a way of gluing these arcs together so that they form a circle;

2. a point at infinity among the gluing points.

That is all. Now any sequence of Reidemeister moves applied to any string link diagram *looks like* a sequence applied to a long knot, and it makes sense to evaluate Fiedler's formulas there. Any quantum cocycle for long knots defines a family of cocycles in the moduli space of string links, indexed by the above choices of a circular permutation and a point at infinity.

2.2.2 The scan-arc and the scan-property

As mentioned earlier, there are no non-trivial loops in the moduli space of string links in general, except for particular cases such as long knots and their cables (for which there is a version of the Hatcher loop – [8], p.8).

Fiedler introduces a *scan-arc*, defined canonically for all string links, and denoted by scan(L). We represent it on Fig.9. It can be thought of as a generalization of the loop rot(K) defined for long knots: indeed, rot(K) consists basically of two consecutive "scans" (Fig.4). Notice that the little loop on the first frame is *not* a part of the string link, and its "creation" by Reidemeister I is *not* a part of the scan-arc either (although the sequence of Reidemeister II moves that lead to the second frame is included in the scan-arc): it follows that the scan-arc is regular (it does not involve any Reidemeister I move), hence it makes sense to evaluate $R^{(1)}_{reg}$ and $R^{(1)}_{F,reg}$ there.

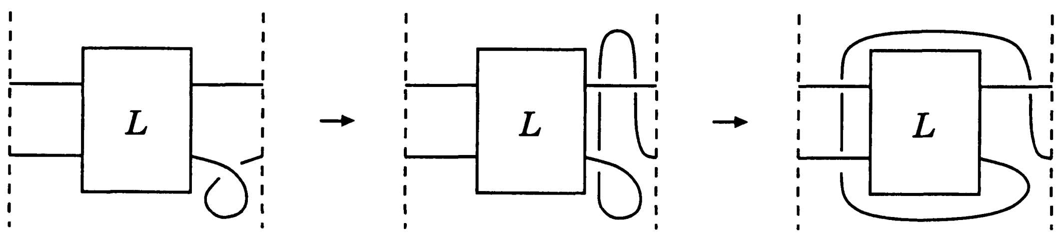


Figure 9: The scan-arc of a string link L

Now one would like a result of the type "Let c be a quantum cocycle formula. Then $c(\mathrm{scan}(L))$ is an isotopy invariant of the string link L.". It is not difficult to see that this holds true if c is a quantum formula with constant weights. However it fails with arbitrary weights.

By definition, a quantum cocycle formula c is said to have the *scan-property* if $c(\mathrm{scan}(L))$ defines an isotopy invariant of the string link L. All four formulas constructed in [8] have that property ([8] Theorems 3, 4, 5).

Among many examples, let us mention that if L is chosen to be the positive generator of the 2-strand braid group, and if the point at infinity is chosen correctly, then $R^{(1)}_{reg}(\mathrm{scan}(L))$ is equal to the HOMFLYPT polynomial of the right trefoil (times the class of the trivial braid in the HOMFLYPT skein module).

As for $\bar{R}^{(1)}$, the computations made lead to the following conjectures ([8], Conjectures 1 and 2).

Conjecture 2. *Let K be a long knot, let $v_2(K)$ be its Casson invariant and let P_K be* *its HOMFLYPT polynomial. Let δ denote the HOMFLYPT polynomial of the trivial 2-component link. Then*

$$\bar{R}^{(1)}(\mathrm{rot}(K)) = \delta v_2(K) P_K.$$

Conjecture 3. *Let K be a long knot with trivial framing and with non trivial Casson invariant. Then $\bar{R}^{(1)}(\mathrm{Hat}(K))$ is a non zero integer multiple of $\bar{R}^{(1)}(\mathrm{rot}(K))$ if and only if K is a torus knot.*

So far those two conjectures have been checked for knots up to 4 crossings.

2.2.3 Gradings

One specificity of the formulas $R^{(1)}_{reg}$ and $R^{(1)}_{F,reg}$ extended to string links is that they can be refined into a collection of cocycle formulas $R^{(1)}_{reg}(A)$ and $R^{(1)}_{F,reg}(A)$ indexed by a set of *gradings*, so that the original formulas are the sum of the refined formulas over all possible gradings.

A grading is defined for any Reidemeister II or III move. Just like $c_{T,l,g}$ is oblivious of Reidemeister moves that are not of type (l,g), $R^{(1)}_{reg}(A)$ and $R^{(1)}_{F,reg}(A)$ disregard the moves that do not have grading A. The full definition, however, is not very enlightening, so we do not mention it here; see [8], Definition 9 p.48.

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