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Iwasawa invariants of links

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1 Introduction

This is a survey article of a part of arithmetic topology, which is a theory on analogies between low-dimensional topology and number theory. This theory is based on regarding knots (links) in 3-manifolds as analogues of primes in number fields. In 1960's, Mazur [15] pointed out an analogy between Alexander-Fox theory and Iwasawa theory. From 1998, independently on the works of Kapranov, Reznikov *et al.*, the arithmetic topology has been developed by Morishita and his collaborators (cf. [17] [18] etc.). Morishita considered new analogies between link groups and Galois groups, which induced a new view point on analogies between Alexander-Fox theory and Iwasawa theory. In particular, Morishita [16] introduced an idea of *Iwasawa invariants* to knot theory, and Hillman, Matei and Morishita [6] defined the Iwasawa invariants of links in the 3-sphere S^3.

In this article, we survey the Iwasawa invariants of links and related analogies. Moreover, we discuss what is an analogue of Greenberg's conjecture, which is a problem (open in general) relating with Iwasawa invariants.

2 Motivations

First, we recall some basic analogies. Let M be an oriented connected closed 3-manifold, which is a finite cover of S^3 branched over some link. The analogue of M is a number field k, which is a finite dimensional algebraic extension of the rational number field $\mathbb{Q}$ ramified over some prime numbers. By regarding a closed path (i.e., a knot) in M as an analogue of a prime ideal of the ring $\mathcal{O}_k$ of algebraic integers in k, the first homology group $H_1(M,\mathbb{Z})$ is considered as a natural analogue of the ideal class group $Cl(k)$ of k. As an analogue of Hurewicz isomorphism $H_1(M,\mathbb{Z}) \simeq \pi_1(M)^{ab}$, we have an isomorphism $Cl(k) \simeq \mathrm{Gal}(k^{ur}/k)^{ab}$ by class field theory, where k^{ur} is the maximal unramified extension of k. It is well known that $Cl(k)$ is a finite abelian group, while $H_1(M,\mathbb{Z})$ is not necessarily finite.

The ideal class group $Cl(k)$ is one of the most interesting objects in number theory, since $Cl(k)$ describes how far from a principal ideal domain $\mathcal{O}_k$ is. In fact, $Cl(k)=\{1\}$ if and only if $\mathcal{O}_k$ is a principal ideal domain. For example, $\mathcal{O}_{\mathbb{Q}(\zeta_4)} = \mathbb{Z}[\sqrt{-1}]$ is a principal ideal domain, and hence $Cl(\mathbb{Q}(\zeta_4)) = \{1\}$, where $\mathbb{Q}(\zeta_n)$ denotes the nth cyclotomic fields.

The divisibility of the cardinality $\#Cl(k)$ by a specific prime number p is also interests in number theory. Before Wiles proved Fermat's last theorem, it has been known that the Fermat equation $x^p+y^p = z^p$ has no nontrivial integer solution if $\#Cl(\mathbb{Q}(\zeta_p)) \not\equiv 0 \pmod p$ and $p > 2$ (cf. e.g. [25]). It is known as a famous example that $\#Cl(\mathbb{Q}(\zeta_{37})) \equiv 0 \pmod{37}$. Iwasawa's class number formula, which is the origin of Iwasawa theory, describes the growth of the p-parts of $\#Cl(k_n)$ in a tower of cyclic extensions k_n of degree p^n over k, e.g., $k_n = \mathbb{Q}(\zeta_{p^{n+1}})$.

Our motivation is to consider the analogous subject, the p-adic growth of the order of $H_1(M_{p^n}, \mathbb{Z})$ in a tower of cyclic branched covers M_{p^n} of degree p^n over M. Therefore

- we fix a prime number p, and
- we assume that $H_1(M, \mathbb{Z})$ is finite, i.e., M is a rational homology 3-sphere

in the following. Since $Cl(k_n)$ is finite, we will assume that $H_1(M_{p^n}, \mathbb{Z})$ are also finite.

3 Iwasawa invariants

Let $L = K_1 \cup \cdots \cup K_r$ be an r-component link in a rational homology 3-sphere M. Put $G_L = \pi_1(X, *)$ the link group of L, i.e., the fundamental group of the exterior X of L with the base point $*$. For a surjective homomorphism $\sigma : G_L \to \mathbb{Z}$, we obtain an infinite cyclic cover X_σ of X corresponding to the kernel: $\operatorname{Ker}\sigma = \pi_1(X_\sigma)$. Let X_{σ,p^n} be the subcover of degree p^n over X, and M_{σ,p^n} the Fox completion. Thus we obtain a tower of cyclic branched covers M_{σ,p^n} of degree p^n over M, which are unbranched outside L. Iwasawa invariants of L are defined for each σ (and fixed p) such that $H_1(M_{\sigma,p^n}, \mathbb{Z})$ are finite for all $n \geq 0$.

Analogously, let S be a finite set of prime ideals of $\mathcal{O}_k$ such that $S_p \subset S$, where S_p denotes the set of all prime ideals $\wp$ of $\mathcal{O}_k$ such that $p \in \wp$. Put $G_S = \operatorname{Gal}(k_S/k)^{\text{pro-}p}$ the pro-p completion[1] of the Galois group of the maximal algebraic extension k_S of k unramified outside S. For a surjective homomorphism $G_S \to \mathbb{Z}_p$, we obtain an infinite cyclic pro-p-extension k_∞ of k corresponding to the kernel, which is called a $\mathbb{Z}_p$-extension of k, where $\mathbb{Z}_p$ denotes (the additive group of) the ring of p-adic integers. Note that $\mathbb{Z}_p \neq \mathbb{Z}/p\mathbb{Z}$. Then $\operatorname{Gal}(k_\infty/k) \simeq \mathbb{Z}_p$, and hence k_∞ can be regarded as a tower of cyclic subextensions k_n of degree p^n over k. Since any $\mathbb{Z}_p$-extensions are unramified outside S_p, we may assume that $S = S_p$. In [7], Iwasawa showed that for each k_∞ there is a triple $(\lambda_{k_\infty}, \mu_{k_\infty}, \nu_{k_\infty}) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \times \mathbb{Z}$ of integers such that

$$v_p(\#Cl(k_n)) = \lambda_{k_\infty} n + \mu_{k_\infty} p^n + \nu_{k_\infty}$$

for all sufficiently large $n \gg 0$, where v_p is the p-adic additive valuation normalized as

[1]The pro-p completion $G^{\text{pro-}p}$ of a group G is the projective limit of quotient p-groups of G. If G is a finite abelian group, $G^{\text{pro-}p}$ is isomorphic to the p-Sylow subgroup. $\mathbb{Z}^{\text{pro-}p} = \mathbb{Z}_p$ is the additive group of p-adic integers.

$v_p(p) = 1$. The original Iwasawa invariants are λ_{k_∞}, μ_{k_∞} and ν_{k_∞} above. If $p = 37$ and $k_n = \mathbb{Q}(\zeta_{p^{n+1}})$, it is known that $\lambda_{k_\infty} = 1$ and $\mu_{k_\infty} = 0$.

We obtain the following analogous formula, assuming the finiteness of $H_1(M_{\sigma,p^n}, \mathbb{Z})$.

Theorem 1 ([16] [6] [11]). *Assume that $H_1(M_{\sigma,p^n}, \mathbb{Z})$ are finite for all $n \geq 0$. Then there is a triple $(\lambda_{L,\sigma}, \mu_{L,\sigma}, \nu_{L,\sigma}) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \times \mathbb{Z}$ of integers such that*

$$v_p(\#H_1(M_{\sigma,p^n}, \mathbb{Z})) = \lambda_{L,\sigma} n + \mu_{L,\sigma} p^n + \nu_{L,\sigma}$$

for all sufficiently large $n \gg 0$.

We call $\lambda_{L,\sigma}$, $\mu_{L,\sigma}$ and $\nu_{L,\sigma}$ the Iwasawa invariants of L. Theorem 1 was firstly indicated by Morishita [16], and proved in [6] (resp. [11]) for the case where $M = S^3$ (resp. M is a rational homology 3-sphere) in the way of another proof of a part of Iwasawa's class number formula ([9], [25, Theorem 7.14]). Ueki [24] also gave another proof of Theorem 1 analogous to Iwasawa's original proof [7].

Iwasawa [8] pointed out that the invariant $\lambda_{k_\infty^{\mathrm{cyc}}}$ is an analogue of the genus of an algebraic curve, where k_∞^{cyc} denotes the cyclotomic $\mathbb{Z}_p$-extension, i.e., the unique $\mathbb{Z}_p$-extension contained in $\bigcup_{n=1}^{\infty} k(\zeta_{p^n})$. Based on this analogy, it is conjectured that $\mu_{k_\infty^{\mathrm{cyc}}} = 0$ in general, and Riemann-Hurwitz type formulas for $\lambda_{k_\infty^{\mathrm{cyc}}}$ were given by Kida [13] and Iwasawa [10]. Analogously, Ueki [24] gave Riemann-Hurwitz type formulas for $\lambda_{L,\sigma}$.

4 Calculations

Assume that $M = S^3$ for simplicity. Let $m_i \in G_L$ be the meridian of the component K_i of L. Then $G_L^{ab} = G_L/G_L' \simeq H_1(X, \mathbb{Z}) \simeq \mathbb{Z}^r$ is freely generated by $t_i = m_i G_L' \in G_L/G_L'$ $(1 \leq i \leq r)$. Put $z_i = \sigma(m_i) \in \mathbb{Z}$. Since σ is surjective, we have $\gcd(z_1, \cdots, z_r) = 1$. Since $z_i = 0$ if and only if K_i is unbranched in M_{σ,p^n} for all n, we may assume that $\prod_{i=1}^{r} z_i \neq 0$ by removing unbranched components.

Let

$$\Delta_L(t_1, \cdots, t_r) \in \Lambda = \mathbb{Z}[G_L/G_L'] = \mathbb{Z}[t_1^{\pm 1}, \cdots, t_r^{\pm 1}]$$

be the Alexander polynomial of L, and put

$$\Delta_{L,\sigma}(t) = (t-1)^{\min\{1, r-1\}} \Delta_L(t^{z_1}, \ldots, t^{z_r}) \in \mathbb{Z}[t^{\pm 1}] = \mathbb{Z}[\mathrm{Aut}(X_\sigma/X)]$$

the characteristic polynomial of the $\mathbb{Z}[t^{\pm 1}]$-module $H_1(X_\sigma, \mathbb{Z}) = (\mathrm{Ker}\,\sigma)^{ab}$. Now we embed $\mathbb{Z}[t^{\pm 1}]$ into the formal power series ring $\mathbb{Z}_p[[T]]$ via $t = 1 + T$. By the p-adic Weierstrass preparation theorem (cf. [25, Theorem 7.3]), $\Delta_{L,\sigma}(1+T)$ can be written in the form

$$\Delta_{L,\sigma}(1+T) = p^{\mu} P_{L,\sigma}(T) u(T)$$

with $0 \leq \mu \in \mathbb{Z}$, monic $P_{L,\sigma}(T) \in \mathbb{Z}_p[T]$ such that $P_{L,\sigma}(T) \equiv T^{\deg P_{L,\sigma}} \pmod p$ and $u(T) \in \mathbb{Z}_p[[T]]^\times$. Then μ and $P_{L,\sigma}(T)$ are uniquely determined for $\Delta_{L,\sigma}(t)$. Theorem 1 for $M = S^3$ is obtained by taking v_p of the following formula, and hence one can see that

$$\lambda_{L,\sigma} = \deg P_{L,\sigma}(T), \quad \mu_{L,\sigma} = \mu.$$

For the case $M \neq S^3$, we need [22, Theorem 3] instead of the following formula.

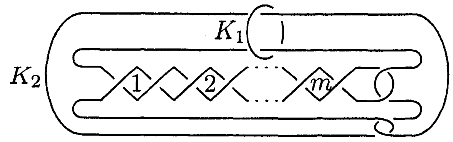
Theorem 2 **([14] [21]).** *Suppose that $M = S^3$, and put $v = \max_i v_p(z_i)$. Then we have*

$$|H_1(M_{\sigma,p^n},\mathbb{Z})| = |H_1(M_{\sigma,p^v},\mathbb{Z})| \cdot \Big| \prod_{\substack{\zeta^{p^n}=1 \\ \zeta^{p^v}\neq 1}} \Delta_{L,\sigma}(\zeta) \Big|$$

for all $n \geq v$, where $|H|$ denotes the order of a $\mathbb{Z}$-module H, i.e., $|H| = \#H$ if $\#H < \infty$, and $|H| = 0$ if $\#H = \infty$.

Moreover, one can check whether $H_1(M_{\sigma,p^n},\mathbb{Z})$ is finite or not by this formula. Therefore one can calculate Iwasawa invariants with the check of the assumption of Theorem 1 from the calculation of Alexander polynomials.

Example 1 **([12]).** Let $L = K_1 \cup K_2 \subset M = S^3$ be the following link.



Then $\Delta_L(t_1,t_2) = m(t_1-1)(t_2-1)^3$, and hence

$$\begin{aligned} \Delta_{L,\sigma} &= m(t-1)(t^{z_1}-1)(t^{z_2}-1)^3 \\ &= p^{v_p(m)}T\big((1+T)^{p^{v_p(z_1)}}-1\big)\big((1+T)^{p^{v_p(z_2)}}-1\big)^3 u(T). \end{aligned}$$

Since M_{σ,p^v} is a branched cover of S^3 along a knot, we have $\#H_1(M_{\sigma,p^v},\mathbb{Z}) < \infty$. Moreover, $\Delta_{L,\sigma}(t)$ has no common factors with the p^nth cyclotomic polynomials for all $n > v = v_p(z_1z_2)$. Therefore $\#H_1(M_{\sigma,p^n},\mathbb{Z}) < \infty$ for all $n \geq 0$, and

$$\lambda_{L,\sigma} = 1 + p^{v_p(z_1)} + 3p^{v_p(z_2)}, \quad \mu_{L,\sigma} = v_p(m).$$

The analogue of $\Delta_{L,\sigma}(t)$ is the Iwasawa polynomial $p^{\mu_{k_\infty}}P_{k_\infty}(T) \in \mathbb{Z}_p[T]$, which is the characteristic polynomial of the module Y_{k_∞} over $\mathbb{Z}_p[[T]] \simeq \mathbb{Z}_p[[\mathrm{Gal}(k_\infty/k)]]$ such that $P_{k_\infty}(T)$ is monic and $P_{k_\infty}(T) \equiv T^{\lambda_{k_\infty}} \pmod p$, where $Y_{k_\infty} \simeq \varprojlim Cl(k_n)^{\text{pro-}p}$ is the unramified quotient of $\mathrm{Ker}(G_S \to \mathbb{Z}_p)^{ab}$. Theorem 2 is based on the close relation between the structures of $H_1(M_{\sigma,p^n},\mathbb{Z})$ and the torsion part of $H_1(X_\sigma,\mathbb{Z})/(t^{p^n}-1)$. Analogously, there is a close relation between the structures of $Cl(k_n)^{\text{pro-}p}$ and $Y_{k_\infty}/((1+T)^{p^n}-1)$.

Iwasawa main conjecture (Mazur-Wiles' theorem) describes explicitly the close relation between Iwasawa polynomials $P_{k_\infty^{\rm cyc}}(T)$ and p-adic L-functions. An analogue of Iwasawa main conjecture has been given by Sugiyama [23].

If $\mathrm{Gal}(k/\mathbb{Q})$ is abelian, $\lambda_{k_\infty^{\rm cyc}}$ can be partially calculated via Iwasawa main conjecture. While there are some partial results ([1] [20]), it is still a difficult problem to determine the possible values of λ_{k_∞}, μ_{k_∞} and ν_{k_∞}. Motivated by this problem, the authors obtained the following theorem (cf. [12, Theorem 2.2 and Theorem 3.4]).

Theorem 3 ([12]). *Assume that $M = S^3$ and put*

$$\mathbf{P}_r = \left\{ (\lambda_{L,\sigma}, \mu_{L,\sigma}) \;\middle|\; L \text{ is } r\text{-component}, \prod_{i=1}^{r} z_i \neq 0,\ \#H_1(M_{\sigma,p^n}, \mathbb{Z}) < \infty \text{ for all } n \geq 0 \right\}.$$

Then we have

(1) $\mathbf{P}_1 = \{(0,0)\}$,

(2) $\mathbf{P}_r = (r - 1 + 2\mathbb{Z}_{\geq 0}) \times \mathbb{Z}_{\geq 0}$ *if* $p \neq 2$ *and* $r \geq 2$,

(3) $\mathbf{P}_2 = \mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 0}$ *if* $p = 2$.

The claim (1) is immediately obtained, since $\Delta_K(1) = \pm 1$ for a knot K. The $\subset$-parts of (2) and (3) are obtained by the Torres conditions. The $\supset$-parts need some results on the existence of a link with prescribed Alexander polynomials (cf. [12]).

5 More analogies

We also assume that $M = S^3$ in the following. Then $G_L^{ab} = G_L/G'_L \simeq \mathbb{Z}^r$, and $G_S^{ab}/\mathrm{Tor}\, G_S^{ab} \simeq \mathbb{Z}_p^{r_2+1} \simeq \mathrm{Gal}(\widetilde{k}/k) \simeq G_S/G'_S$ (assuming Leopoldt's conjecture, cf [25, Theorem 13.4]) with the corresponding subgroup G'_S, where $\widetilde{k}$ is the maximal free abelian pro-p-extension of k which is an analogue of the maximal free abelian cover $\pi : \widetilde{X} \to X$.

We suppose that σ satisfies $\prod_{i=1}^r z_i \neq 0$. As an analogous condition, we suppose that any $\wp \in S = S_p$ ramifies in k_∞/k. Then, by Theorem 3, we have $\lambda_{L,\sigma} \geq r - 1$. On the other hand, it is known that $\lambda_{k_\infty} \geq r_2$ if $\#S_p = \dim_{\mathbb{Q}} k$ (cf. [3]), where r_2 is the half of the number of embeddings $\iota : k \hookrightarrow \mathbb{C}$ such that $\iota(k) \not\subset \mathbb{R}$.

If we regard r_2 as an analogue of $r-1$, an analogue of a 2-component link is $S_p = \{\wp_1, \wp_2\}$ in the case where $\#S_p = \dim_{\mathbb{Q}} k = 2r_2 = 2$. For a 2-component link $L = K_1 \cup K_2$, one can easily see that $\#H_1(M_{\sigma,p^n}, \mathbb{Z}) < \infty$ for all $n \geq 0$ and $(\lambda_{L,\sigma}, \mu_{L,\sigma}) = (1,0)$ if and only if the linking number $\mathrm{lk}(K_1, K_2) \not\equiv 0 \pmod p$ ([12, Theoerm 3.2]). On the other hand, if we assume $\#Cl(k) \not\equiv 0 \pmod p$ (analogously to $\#H_1(S^3, \mathbb{Z}) = 1$) in the analogous case above, then it is known as Gold's theorem [2] that $(\lambda_{k_\infty}, \mu_{k_\infty}) = (1,0)$ if and only if $\wp_2^{\#Cl(k)} = \pi_2 \mathcal{O}_k$ for $\pi_2 \in \mathcal{O}_k$ which is not a pth power residue modulo $\wp_1^2$. This is one of the examples of analogies between linking numbers and power residue symbols.

From these points of view, S_p looks like an (r_2+1)-component link in the case where $\#S_p = \dim_{\mathbb{Q}} k$. However, while Example 1 shows the existence of infinitely many link $L = K_1 \cup K_2$ such that $\sup\{\lambda_{L,\sigma}\}_\sigma = \infty$ and $\mu_{L,\sigma} > 0$, Ozaki's theorem [19] states that $(\lambda_{k_\infty}, \mu_{k_\infty}) = (1,0)$ for almost all k_∞ if $\#S_p = \dim_{\mathbb{Q}} k = 2r_2 = 2$ and "Greenberg's conjecture" holds. Motivated by this difference, the authors [12] considered what is an analogue of Greenberg's conjecture. In the following, we shall recall and supplement the consideration.

Greenberg's original conjecture [3] states that $(\lambda_{k_\infty^{\rm cyc}}, \mu_{k_\infty^{\rm cyc}}) = (0,0)$ if $r_2 = 0$, i.e., $Y_{k_\infty^{\rm cyc}}$ is finite if k is a totally real number field. In the case where $M = S^3$, the analogue of this conjecture holds as Theorem 3 (1). If $r_2 = 0$, then $\widetilde{k} = k_\infty^{\rm cyc}$. Greenberg's generalized conjecture (cf. e.g. [4]) asserts that the unramified quotient $Y_{\widetilde{k}} = (G'_S)^{ab}/\sum_{\wp \in S} \widehat{\Lambda}\varphi(I_\wp \cap G'_S)$ of $(G'_S)^{ab}$ is *pseudonull*[2] (cf. [5]) as a module over $\widehat{\Lambda} = \mathbb{Z}_p[[G_S/G'_S]]$, where $\varphi : G'_S \to (G'_S)^{ab}$ is the natural mapping, and $I_\wp \subset G_S$ is an inertia group of a prime lying over $\wp \in S$ which is often regarded as an analogue of $\langle m_i \rangle \subset G_L$. Since $\langle m_i \rangle \cap G'_L = 1$, a strict analogue of Greenberg's conjecture is the following: *Is the link module $(G'_L)^{ab}$ pseudonull as a Λ-module ?* The answer is "no" in many cases, and this is a background of the difference between Example 1 and Ozaki's theorem. Since this question seems not so interesting, we modify an analogue of $Y_{\widetilde{k}}$ as follows.

Since $I_\wp \cap G'_S$ is equal to the inertia group $I_{\widetilde{\wp}} \subset G'_S$ of a prime lying over $\widetilde{\wp}$, where $\widetilde{\wp}$ is a prime of $\widetilde{k}$ lying over $\wp$, we regard the meridianal elements $[\widetilde{m_i}] \in H_1(\widetilde{X}, \pi^{-1}(*), \mathbb{Z})$ as analogues of $\varphi(I_{\widetilde{\wp}})$, where $\widetilde{m_i}$ is a lift of m_i with endpoints in $\pi^{-1}(*)$. Then we put $Y_L = (G'_L)^{ab}/\theta^{-1}(\sum_{i=1}^r \Lambda[\widetilde{m_i}])$ as an analogue of $Y_{\widetilde{k}}$, where $\theta : (G'_L)^{ab} \simeq H_1(\widetilde{X}, \mathbb{Z}) \hookrightarrow H_1(\widetilde{X}, \pi^{-1}(*), \mathbb{Z})$ is the natural embedding. Thus we obtain the following problem as a weak analogue of Greenberg's conjecture. Some examples has been given in [12].

Problem 1 ([12]). *Is Y_L pseudonull as a Λ-module ?*

A corrigendum to [12]. In [12, page 223, line 5], G_S should be replaced by G'_S. The author had confused $\langle I_\wp \cap G'_S \rangle_{\wp \in S}$ and $\langle I_\wp \rangle_{\wp \in S} \cap G'_S$.

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