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Asymptotic Expansions for the Ground State Energy of a Model with a Massless Quantum Field

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Abstract

A new asymptotic perturbation theory for linear operators (A. Arai, *Ann. Henri Poincaré*, Online First, 2013, DOI 10.1007/s00023-013-0271-7) and its application to asymptotic expansions, in the coupling constant, of the ground state energy of a quantum system interacting with a massless quantum field are reviewed.

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1 Introduction

In a recent paper [3], the author presented a new asymptotic perturbation theory for linear operators and, as an application of it, derived asymptotic expansions, in the coupling constant, of the ground state energy of the generalized spin-boson model [4]. The purpose of the present article is to review some basic results in [3]. In this introduction we briefly describe some backgrounds and motivations behind the work [3].

As is well known, the Hamiltonian of a quantum system may have a parameter $\lambda \in \mathbb{R}$, called the coupling constant, which denotes the strength among microscopic objects constituting the quantum system (the case $\lambda = 0$ corresponds to the non-coupling case). Let us consider such a quantum system and $H(\lambda)$ be its Hamiltonian. Assume that $H(\lambda)$ is bounded below. Then one of the interesting quantities of the quantum system is the lowest energy $E_{\min}(\lambda)$ defined by

$$E_{\min}(\lambda) := \inf \sigma(H(\lambda)), \tag{1.1}$$

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where, for a linear operator A on a Hilbert space, $\sigma(A)$ denotes the spectrum of it. Basic problems on the lowest energy are as follows:

(P.1) Is $E_{\rm min}(\lambda)$ an eigenvalue of $H(\lambda)$? In that case, $H(\lambda)$ is said to have a ground state and $E_{\rm min}(\lambda)$ is called the ground state energy of $H(\lambda)$.[1] The non-zero vector in $\ker(H(\lambda) - E_{\rm min}(\lambda)$ is called a ground state of $H(\lambda)$.

(P.2) Properties of $E_{\rm min}(\lambda)$ as a function of λ. For example:
(i) Is it analytic in λ in a neighborhood of the origin ?
(ii) Does it have asymptotic expansions in λ as $\lambda \to 0$?

(P.3) To identify the spectra of $H(\lambda)$

Problems (P.1) and (P.2) have been part of the subjects of perturbation theories for linear operators (e.g., [15, 18]).[2] Problems (P1.)–(P.3) are non-trivial and difficult in general. In particular, in the case where the lowest energy $E_{\rm min}(0)$ of the unperturbed Hamiltonian $H_0 := H(0)$ is a non-isolated eigenvalue. This situation typically appears in models of *massless* quantum fields where $\sigma(H_0) = [E_{\rm min}(0), \infty)$.

In the case where $E_{\rm min}(0)$ is a non-isolated eigenvalue of H_0, one can not use the standard perturbation theories where the discreteness of the eigenvalue of H_0 to be considered is assumed [15, 18]. The perturbation problem in that case is a special case of the so-called embedded eigenvalue problems to which the standard perturbation theories can not be applied.

In the case where $H(\lambda)$ is a finite dimensional many-body Schrödinger operator, dilation analytic methods have been developed to solve the embedded eigenvalue problems (e.g., [18, §XII.6]). Okamoto and Yajima [16] extended the dilation analytic methods to the case of a *massive* quantum field Hamiltonian. But, the method has not been valid in the case of massless quantum fields.

In the second half of 1990's, however, some breakthroughs were made in treating embedded eigenvalue problems concerning Hamiltonians with a massless quantum field [4, 7, 8]. As for asymptotic expansions of embedded eigenvalues, Bach, Fröhlich and Sigal [7, 8] developed renormalization group methods and applied it to a model in non-relativistic quantum electrodynamics (QED) to prove the existence of a ground state and resonant states with second order asymptotic expansions in the coupling constant. Hainzl and Seiringer [13] derived the second order asymptotic expansion, in the coupling constant, of the ground state energy of a model in non-relativistic QED. Bach, Fröhlich and Pizzo [5, 6] discussed an "asymptotic-like" expansion up to any order in a model of non-relativistic

[1]In the case where one does not require the strict distinction for concepts, $E_{\rm min}$ also is called the ground state energy even if it is not an eigenvalue of $H(\lambda)$

[2](P.2) also applies to every eigenvalue of $H(\lambda)$.

QED. Recently Faupin, Møller and Skibsted [11] presented a general perturbation theory, up to the second order in the coupling constant, for embedded eigenvalues.

Some authors have obtained a stronger result that $E_{\min}(\lambda)$ is analytic in λ: Griesemer and Hasler [12](a model in non-relativistic QED); Abdesselam [1](the massless spin-boson model); Hasler and Herbst [14](the spin-boson model); Abdesselam and Hasler [2](the massless Nelson model).

The methods used in these studies, however, seem to be model-dependent. One of the motivations for the present work comes from seeking general structures (if any) of asymptotic perturbation theories for $E_{\min}(\lambda)$, keeping in mind the case where $E_{\min}(0)$ is a non-isolated eigenvalue of H_0. To be concrete, a basic question is: To what extent is it possible to develop a general asymptotic or analytic perturbation theory which can be applied to massless quantum field models including those mentioned above ? Of course, to develop such an asymptotic perturbation theory, a new idea is necessary. We find it in the so-called Brillouin–Wigner perturbation theory [9, 20, 21], which seems to be not so noted in the literature. An advantage of this perturbation theory lies in that the unperturbed eigenvalue under consideration is *not necessarily isolated*, although the multiplicity of it should be finite. On the other hand, in the standard perturbation theory (analytic or asymptotic) developed by T. Kato, Rellich and other people, which comes from heuristic perturbation theories by Rayleigh [17] and Schrödinger [19], the unperturbed eigenvalue under consideration must be isolated with a finite multiplicity. Then a natural question is: What is the mathematically rigorous form (X in the Table 1) of the Brillouin–Wigner perturbation theory ? The paper [3] gives a first step towards a complete answer to this question.

Heuristic perturbation theories	*Unperturbed eigenvalue (finite multiplicity)*	*Forms of rigorous theories*
Rayleigh, Schrödinger	isolated	T. Kato, Rellich, $\cdots$
Brillouin, Wigner	not necessarily isolated	X

Table 1: Comparison of two perturbation theories

2 Simultaneous Equations for an Eigenvalue and an Eigenvector

Let $\mathcal{H}$ be a complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$ (anti-linear in the first variable and linear in the second) and norm $\| \cdot \|$. As an unperturbed operator we take a symmetric (not necessarily self-adjoint) operator H_0 on $\mathcal{H}$ which obeys the following condition:

(H.1) H_0 has a simple eigenvalue $E_0 \in \mathbb{R}$.

We remark that E_0 is not necessarily an isolated eigenvalue. It may be allowed to be an embedded eigenvalue. This is a new point.

We fix a normalized eigenvector Ψ_0 of H_0 with eigenvalue E_0:

$$H_0\Psi_0 = E_0\Psi_0, \quad \|\Psi_0\| = 1.$$

We denote by P_0 the orthogonal projection onto the eigenspace

$$\mathcal{H}_0 := \{\alpha\Psi_0 | \alpha \in \mathbb{C}\}.$$

Then

$$Q_0 := I - P_0,$$

is the orthogonal projection onto the $\mathcal{H}_0^\perp$, the orthogonal complement of $\mathcal{H}_0$. Since H_0 is symmetric, it is reduced by $\mathcal{H}_0$ and $\mathcal{H}_0^\perp$. We denote by H_0' the reduced part of H_0 to $\mathcal{H}_0^\perp$.

A perturbation of H_0 is given by a linear operator H_{I} on $\mathcal{H}$ (H_{I} is not necessarily symmetric). Hence the perturbed operator (the total Hamiltonian) is defined by

$$H(\lambda) := H_0 + \lambda H_{\mathrm{I}} \quad (\lambda \in \mathbb{R})$$

For a linear operator A on $\mathcal{H}$, we denote by $D(A)$ and $\sigma_{\mathrm{p}}(A)$ the domain and the point spectrum (the set of eigenvalues) of A respectively.

Definition 2.1 (1) A vector $\Psi \in \mathcal{H}$ overlaps with a vector $\Phi \in \mathcal{H}$ if $\langle \Psi, \Phi \rangle \neq 0$.
(2) A vector $\Psi \in \mathcal{H}$ overlaps with a subset $\mathcal{D} \subset \mathcal{H}$ if there exists a vector $\Phi \in \mathcal{D}$ which overlaps with Ψ .

The next proposition describes basic structures for a new perturbation theory:

Proposition 2.2 *Assume (H.1). Let $\lambda \in \mathbb{R} \setminus \{0\}$ be fixed and E be a complex number with $E \notin \sigma_{\mathrm{p}}(H_0')$. Then:*

(i) *If $E \in \sigma_{\mathrm{p}}(H(\lambda))$ and Ψ_0 overlaps with $\ker(H(\lambda) - E)$, then there exists a vector $\Psi \in \ker(H(\lambda) - E)$ such that $Q_0 H_{\mathrm{I}} \Psi \in D((E - H_0')^{-1})$ and*

$$E = E_0 + \lambda \langle \Psi_0, H_{\mathrm{I}}\Psi \rangle , \tag{2.1}$$

$$\Psi = \Psi_0 + \lambda(E - H_0')^{-1} Q_0 H_{\mathrm{I}} \Psi. \tag{2.2}$$

(ii) (Converse of (i)) *If E and $\Psi \in D(H(\lambda)) \cap D(((E - H_0')^{-1} Q_0 H_{\mathrm{I}})$ satisfy (2.1) and (2.2), then $E \in \sigma_{\mathrm{p}}(H(\lambda))$ and $\Psi \in \ker(H(\lambda) - E) \setminus \{0\}$, overlapping with Ψ_0.*

Proof. See [3, Proposition 2.1]. ∎

Note that (2.1) and (2.2) can be viewed as a simultaneous equation for the pair (E, Ψ).

Under some additional conditions, (2.1) and (2.2) can be iterated to give an expression which suggests a form of asymptotic expansions of E and Ψ:

Corollary 2.3 *Assume (H.1). Let $E \notin \sigma_p(H_0')$ and suppose that $E \in \sigma_p(H(\lambda))$ and Ψ_0 overlaps with* $\ker(H(\lambda) - E)$. *Let Ψ be as in Proposition 2.2-(i). Suppose that, for some $n \geq 1$,*

$$\Psi_0 \in D([(E - H_0')^{-1} Q_0 H_I]^n).$$

Then $\Psi \in D([(E - H_0')^{-1} Q_0 H_I]^{n+1})$ and

$$\Psi = \Psi_0 + \sum_{k=1}^{n} \lambda^k [(E - H_0')^{-1} Q_0 H_I]^k \Psi_0 + \lambda^{n+1} [(E - H_0')^{-1} Q_0 H_I]^{n+1} \Psi.$$

$$\begin{aligned} E &= E_0 + \lambda \langle \Psi_0, H_I \Psi_0 \rangle + \sum_{k=1}^{n} \lambda^{k+1} \langle \Psi_0, H_I [(E - H_0')^{-1} Q_0 H_I]^k \Psi_0 \rangle \\ &\quad + \lambda^{n+2} \left\langle \Psi_0, H_I [(E - H_0')^{-1} Q_0 H_I]^{n+1} \Psi \right\rangle. \end{aligned}$$

Proof. An easy exercise. ∎

In applications to quantum field models, the following situation may occur:

(H.2) (i) H_I is symmetric and $\Psi_0 \in D(H(\lambda)) = D(H_0) \cap D(H_I)$.

(ii) There exists a constant $r > 0$ such that, for all $\lambda \in \mathbb{I}_r^\times := (-r, 0) \cup (0, r)$, $H(\lambda)$ has an eigenvalue $E(\lambda)$ with the following properties:

(a) $E(\lambda) \notin \sigma_p(H_0')$.

(b) Ψ_0 overlaps with $\ker(H(\lambda) - E(\lambda))$.

The next proposition immediately follows from Proposition 2.2:

Proposition 2.4 *Assume (H.1) and (H.2). Then, for each $\lambda \in \mathbb{I}_r^\times$, there exists a vector $\Psi(\lambda) \in \ker(H(\lambda) - E)$ such that $Q_0 H_I \Psi \in D((E(\lambda) - H_0')^{-1})$ and*

$$\begin{aligned} E(\lambda) &= E_0 + \lambda \langle \Psi_0, H_I \Psi(\lambda) \rangle, \\ \Psi(\lambda) &= \Psi_0 + \lambda (E(\lambda) - H_0')^{-1} Q_0 H_I \Psi(\lambda). \end{aligned}$$

3 Upper Bound for the Lowest Energy

In the case where $H_{\rm I}$ is symmetric, $H(\lambda)$ is Hermitian[3]. Hence one can define

$$\mathcal{E}_0(\lambda) := \inf_{\Psi \in D(H(\lambda)), \|\Psi\|=1} \langle \Psi, H(\lambda)\Psi \rangle \,,$$

the infimum of the numerical range of $H(\lambda)$.

We remark that, if $H(\lambda)$ is self-adjoint, then $\mathcal{E}_0(\lambda) = E_{\rm min}(\lambda)$ (see (1.1)).

A stronger condition for H_0 and E_0 is stated as follows:

(H.3) H_0 is self-adjoint and $E_0 = \inf \sigma(H_0)$.

Theorem 3.1 (An upper bound for $\mathcal{E}_0(\lambda)$) *Assume (H.1) and (H.3). Suppose that $H_{\rm I}$ is symmetric and*

$$\Psi_0 \in D(H_{\rm I}(H_0' - E_0)^{-1} Q_0 H_{\rm I}).$$

Let

$$\begin{aligned} N_0 &:= \|(H_0' - E_0)^{-1} Q_0 H_{\rm I} \Psi_0\|^2, \\ a &:= \left\langle Q_0 H_{\rm I}\Psi_0, (H_0' - E_0)^{-1} Q_0 H_{\rm I}\Psi_0 \right\rangle, \\ b &:= \left\langle (H_0' - E_0)^{-1} Q_0 H_{\rm I}\Psi_0, H_{\rm I}(H_0' - E_0)^{-1} Q_0 H_{\rm I}\Psi_0 \right\rangle. \end{aligned}$$

Then, for all $\lambda \in \mathbb{R}$,

$$\mathcal{E}_0(\lambda) \le E_0 + \frac{1}{1 + N_0\lambda^2}\left(\langle \Psi_0, H_{\rm I}\Psi_0 \rangle \lambda - a\lambda^2 + b\lambda^3 \right).$$

Proof. Take as a trial vector $\Psi_1 := \Psi_0 - \lambda(H_0' - E_0)^{-1} Q_0 H_{\rm I}\Psi_0$ which may be an "approximate ground state" of $H(\lambda)$. Then $\mathcal{E}_0(\lambda) \le \langle \Psi_1, H(\lambda)\Psi_1\rangle / \|\Psi_1\|^2$. The calculation of the right hand side yields the desired result. ∎

Remark 3.2 One may improve the upper bound by taking as a trial vector $\Psi_N := \Psi_0 + \sum_{n=1}^N \lambda^n ((E_0 - H_0')^{-1} Q_0 H_{\rm I})^n \Psi_0$.

Corollary 3.3 *Under the same assumption as in Theorem 3.1, consider the case where*

$$|\langle \Psi_0, H_{\rm I}\Psi_0\rangle| < |\lambda|(a - b\lambda).$$

Then

$$\mathcal{E}_0(\lambda) < E_0.$$

In particular, $\mathcal{E}_0(\lambda) \in \rho(H_0)$ (the resolvent set of H_0).

[3] Here we mean by "a linear operator A on $\mathcal{H}$ (not necessarily densely defined) is Hermitian" that $\langle \psi, A\phi \rangle = \langle A\psi, \phi\rangle$ for all $\psi, \phi \in D(A)$ (hence $\langle \psi, A\psi\rangle$ is a real number for all $\psi \in D(A)$).

4 Asymptotic Expansion to the Second Order in λ

For the reader's convenience, we first state a result on the asymptotic expansion to the second order in λ. For this purpose, we need additional conditions:

(H.4) (i) $\lim_{\lambda\to 0}\|\Psi(\lambda)\| = 1$. (ii) $E(\lambda) < E_0, \ \forall\lambda \in \mathbb{I}_r^{\times}$.

In what follows we assume (H.1)–(H.4). We introduce operator-valued functions of λ:

$$K(\lambda) := (E(\lambda) - H_0)^{-1}Q_0H_{\mathrm{I}},$$
$$G(\lambda) := H_{\mathrm{I}}(E(\lambda) - H_0)^{-1}Q_0.$$

Theorem 4.1 *Assume (H.1)–(H.4). Suppose that*

$$\Psi_0 \in D(G(\lambda)H_{\mathrm{I}}) \cap D((H_0' - E_0)^{-1/2}Q_0H_{\mathrm{I}})$$

for all $\lambda \in \mathbb{I}_r^{\times}$ *with* $\sup_{\lambda\in\mathbb{I}_r^{\times}}\|G(\lambda)H_{\mathrm{I}}\Psi_0\| < \infty$. *Then*

$$E(\lambda) = E_0 + \lambda\langle\Psi_0, H_{\mathrm{I}}\Psi_0\rangle - \lambda^2\|(H_0' - E_0)^{-1/2}Q_0H_{\mathrm{I}}\Psi_0\|^2 + o(\lambda^2) \quad (\lambda \to 0).$$

Proof. See [3, Theorem 3.5]. ∎

5 Asymptotic Expansion up to Any Finite Order in λ

Let

$$K_0 := (E_0 - H_0')^{-1}Q_0H_{\mathrm{I}}.$$

For each $\ell \in \mathbb{N}$, we define an operator-valued function K_ℓ on $\mathbb{R}^\ell$ by

$$K_\ell(x_1, \ldots, x_\ell) := \sum_{r=1}^{\ell}(-1)^r \sum_{\substack{j_1+\cdots+j_r=\ell \\ j_1,\ldots,j_r\geq 1}} x_{j_1}\cdots x_{j_r}(E_0 - H_0')^{-(r+1)}Q_0H_{\mathrm{I}}, \quad (x_1, \ldots, x_\ell) \in \mathbb{R}^\ell.$$

For a natural number $N \geq 2$, we define a sequence $\{a_n\}_{n=1}^N$ as follows:

$$a_1 := \langle\Psi_0, H_{\mathrm{I}}\Psi_0\rangle,$$
$$a_n = \sum_{\substack{q+\ell=n \\ q,\ell\geq 1}} \sum_{\substack{\ell_1+\cdots+\ell_q=\ell-1 \\ \ell_1,\ldots,\ell_q\geq 0}} \langle H_{\mathrm{I}}\Psi_0, K_{l_1}(a_1, \ldots, a_{\ell_1})\cdots K_{l_q}(a_1, \ldots, a_{\ell_q})\Psi_0\rangle, \quad n = 2, \ldots, N,$$

provided that

$$\Psi_0 \in \cap_{n=2}^N \cap_{\substack{q+\ell=n \\ q,\ell\geq 1}} \cap_{\substack{\ell_1+\cdots+\ell_q=\ell-1 \\ \ell_1,\ldots,\ell_q\geq 0}} \cap_{r_1=0}^{\ell_1} \cdots \cap_{r_q=0}^{\ell_q} D\left(\prod_{j=1}^q (E_0 - H_0')^{-(r_j+1)} Q_0 H_\mathrm{I}\right). \quad (5.1)$$

We have

$$a_2 = -\left\langle H_\mathrm{I}\Psi_0, (H_0' - E_0)^{-1} Q_0 H_\mathrm{I}\Psi_0\right\rangle \leq 0,$$

$$a_3 = \left\langle (H_0' - E_0)^{-1} H_\mathrm{I}\Psi_0, H_\mathrm{I}(H_0' - E_0)^{-1} Q_0 H_\mathrm{I}\Psi_0\right\rangle \\ - \langle \Psi_0, H_\mathrm{I}\Psi_0\rangle \|(H_0' - E_0)^{-1} Q_0 H_\mathrm{I}\Psi_0\|^2.$$

One of the main results in [3] is as follows:

Theorem 5.1 *Let $N \geq 2$ be a natural number. Assume (H.1)–(H.4). Suppose that (5.1) holds and $\Psi_0 \in \cap_{n=1}^{N-1} D(G(\lambda)^n H_\mathrm{I})$ with $\sup_{r\in \mathbb{I}_r^\times} \|G(\lambda)^n H_\mathrm{I}\Psi_0\| < \infty$, $n = 1, \ldots, N-1$. Then*

$$E(\lambda) = E_0 + \sum_{n=1}^N a_n \lambda^n + o(\lambda^N) \quad (\lambda \to 0).$$

Proof. See [3, Theorem 4.1]. ∎

6 The Generalized Spin-Boson Model

6.1 Definitions

The generalized spin-boson (GSB) model [4] describes a model of a general quantum system interacting with a Bose field. Let $\mathcal{K}$ be the Hilbert space of a general quantum system S and

$$\mathcal{F} := \oplus_{n=0}^\infty \otimes_\mathrm{s}^n L^2(\mathbb{R}^\nu) = \left\{\psi = \{\psi^{(n)}\}_{n=0}^\infty | \psi^{(n)} \in \otimes_\mathrm{s}^n L^2(\mathbb{R}^\nu), n \geq 0, \sum_{n=0}^\infty \|\psi^{(n)}\|^2 < \infty\right\}$$

be the boson Fock space over $L^2(\mathbb{R}^\nu)$ ($\nu \in \mathbb{N}$), where $\otimes_\mathrm{s}^n$ denotes n-fold symmetric tensor product with $\otimes_\mathrm{s}^0 L^2(\mathbb{R}^\nu) := \mathbb{C}$. Then Hilbert space of the composite system of S and the Bose field is given by

$$\mathcal{H} = \mathcal{K} \otimes \mathcal{F}.$$

We take a bounded below self-adjoint operator A on $\mathcal{K}$ as the Hamiltonian of the system S.

We denote by $\omega : \mathbb{R}^\nu \to [0,\infty)$ the one-boson energy function, which is assumed to satisfy $0 < \omega(k) < \infty$ a.e. (almost everywhere) $k \in \mathbb{R}^\nu$. For each $n \geq 1$, we define the function $\omega^{(n)}$ on $(\mathbb{R}^\nu)^n$ by

$$\omega^{(n)}(k_1,\dots,k_n) := \sum_{j=1}^{n} \omega(k_j), \quad \text{a.e.}(k_1,\dots,k_n) \in (\mathbb{R}^\nu)^n.$$

We denote the multiplication operator by the function $\omega^{(n)}$ by the same symbol. We set $\omega^{(0)} := 0$. Then the operator

$$d\Gamma(\omega) := \oplus_{n=0}^{\infty}\omega^{(n)}$$

on $\mathcal{F}$, the second quantization of ω, describes the free Hamiltonian of the Bose field.

The annihilation operator $a(f)$ ($f \in L^2(\mathbb{R}^\nu)$) is the densely defined closed operator on $\mathcal{F}$ such that its adjoint $a(f)^*$ is of the form

$$(a(f)^*\psi)^{(0)} = 0, \quad (a(f)^*\psi)^{(n)} = \sqrt{n}S_n(f \otimes \psi^{(n-1)}),\ n \geq 1,\ \psi \in D(a(f)^*),$$

where S_n is the symmetrization operator on $\otimes^n L^2(\mathbb{R}^\nu)$. The Segal field operator $\phi(f)$ is defined by

$$\phi(f) := \frac{1}{\sqrt{2}}(a(f)^* + a(f)).$$

The total Hamiltonian of the GSB model is of the form

$$H_{\mathrm{GSB}}(\lambda) = A \otimes I + I \otimes d\Gamma(\omega) + \lambda \sum_{j=1}^{J} B_j \otimes \phi(g_j) \quad (\lambda \in \mathbb{R}),$$

where $J \in \mathbb{N}$ and, for $j = 1,\dots,J$, B_j is a symmetric operator on $\mathcal{K}$ and $g_j \in L^2(\mathbb{R}^\nu)$. The unperturbed Hamiltonian is

$$H_0 := H_{\mathrm{GSB}}(0) = A \otimes I + I \otimes d\Gamma(\omega).$$

One says that, if $\omega_0 := \mathrm{ess.inf}_{k\in\mathbb{R}^\nu}\omega(k)$ (the essential infimum of ω) is strictly positive (resp. equal to zero), then the boson is massive (resp. massless).

If the boson is massless and $\omega(\mathbb{R}^\nu) = [0,\infty)$, then

$$\sigma(H_0) = [E_0,\infty) \quad (E_0 = \inf\sigma(H_0) = \inf\sigma(A))$$

Hence, in this case, all the eigenvalues of H_0 (if exist) are embedded eigenvalues. In particular, E_0 can not be an isolated eigenvalue of H_0. Thus the standard perturbation theory can not be applied to E_0 in the present case.

6.2 Some properties of the GSB model

Let

$$\Lambda := \{\lambda \in \mathbb{R} | H_{\mathrm{GSB}}(\lambda) \text{ is self-adjoint and bounded below}\}$$

and, for each $\lambda \in \Lambda$,

$$E(\lambda) := \inf \mathrm{spec}(H_{\mathrm{GSB}}(\lambda)),$$

the lowest energy of the GSB model. The next theorem tells us that the lowest energy $E(\lambda)$ is an even function of λ.

Theorem 6.1 *The set Λ is reflection symmetric with respect to the origin of $\mathbb{R}$ (i.e., $\lambda \in \Lambda \Longleftrightarrow -\lambda \in \Lambda$) and $E(\cdot)$ is an even function on Λ: $E(\lambda) = E(-\lambda), \quad \lambda \in \Lambda$.*

Proof. See [3, Theorem 5.1]. ∎

In what follows, we assume the following conditions:

(A.1) The operator A has compact resolvent. We set $\widetilde{A} := A - E_0 \geq 0$.

(A.2) Each B_j $(j = 1, \ldots, J)$ is $\widetilde{A}^{1/2}$-bounded.

(A.3) $g_j, g_j/\omega \in L^2(\mathbb{R}^\nu)$, $j = 1, \ldots, J$.

(A.4) The function ω is continuous on $\mathbb{R}^\nu$ with $\lim_{|k|\to\infty} \omega(k) = \infty$ and there exist constants $\gamma > 0$ and $C > 0$ such that

$$|\omega(k) - \omega(k')| \leq C|k - k'|^\gamma (1 + \omega(k) + \omega(k')), \quad k, k' \in \mathbb{R}^\nu.$$

Assumption (A.1) implies that A has a normalized ground state. We denote it by ψ_0:

$$A\psi_0 = E_0\psi_0, \quad \|\psi_0\| = 1.$$

The vector $\Omega_0 := \{1, 0, 0, \ldots\} \in \mathcal{F}$ is called the Fock vacuum. We denote by P_{Ω_0} the orthogonal projection onto $\{\alpha\Omega_0 | \alpha \in \mathbb{C}\}$. The orthogonal projection onto $\ker \widetilde{A} = \ker(A - E_0)$ is denoted by p_{ψ_0}.

Theorem 6.2 [4] *Assume (A.1)–(A.4). Then there exists a constant $r > 0$ independent of λ such that the following hold:*

(i) $(-r, r) \subset \Lambda$.

(ii) *For all $\lambda \in (-r, r)$, $H_{\mathrm{GSB}}(\lambda)$ has a ground state $\Psi_0(\lambda)$ and there exists a constant $M > 0$ independent of $\lambda \in (-r, r)$ such that, for all $|\lambda| < r$, $\|\Psi_0(\lambda)\| \leq 1$ and*

$$\langle \Psi_0(\lambda), p_{\psi_0} \otimes P_{\Omega_0} \Psi_0(\lambda) \rangle \geq 1 - \lambda^2 M^2 > 0$$

6.3 Second order asymptotic expansion of $E(\lambda)$ in λ

We need additional assumptions:

(A.5) The eigenvalue E_0 of A is simple and there exists a $j_0 \in \{1, \ldots, J\}$ such that $B_{j_0}\psi_0 \neq 0$.

(A.6) The set $\{g_1, \ldots, g_J\} \subset L^2(\mathbb{R}^\nu)$ is linearly independent.

Theorem 6.3 (Second order asymptotics) *Assume (A.1)–(A.6) and let*

$$a_{\text{GSB}} := \frac{1}{2}\sum_{j,\ell=1}^{J} \int_{\omega(k)>0} \left\langle B_j\psi_0, (\widetilde{A} + \omega(k))^{-1} B_\ell \psi_0 \right\rangle g_j(k)^* g_\ell(k) dk.$$

Then $a_{\text{GSB}} > 0$ *and*

$$E(\lambda) = E_0 - a_{\text{GSB}}\lambda^2 + o(\lambda^2) \quad (\lambda \to 0).$$

Proof. See [3, Theorem 5.13]. ∎

Remark 6.4 A similar asymptotic expansion is obtained for a massless Dereziński-Gérard model [10] by Faupin-Mϕller-Skibsted [11] and for the Pauli-Fierz model in nonrelativistic QED by Hainzl-Seiringer [13]. But the methods are quite different from our method.

6.4 Higher order asymptotics

In this section we use the following notation:

$$H_{\text{I}} := \sum_{j=1}^{J} B_j \otimes \phi(g_j), \quad Q_0 := I - p_{\psi_0} \otimes P_{\Omega_0},$$

$$H_0' := Q_0 H_0 Q_0 \ \ \text{(the reduced part of } H_0 \text{ to } [\ker(H_0 - E_0)]^\perp),$$

$$K_\ell(x_1, \ldots, x_\ell) := \sum_{r=1}^{\ell} (-1)^r \sum_{\substack{j_1+\cdots+j_r=\ell \\ j_1,\ldots,j_r \geq 1}} x_{j_1} \cdots x_{j_r} (E_0 - H_0')^{-(r+1)} Q_0 H_{\text{I}},$$

$$(x_1, \ldots, x_\ell) \in \mathbb{R}^\ell.$$

Theorem 6.5 (Asymptotic expansion up to any finite order) *Assume (A.1)–(A.6) and*

$$g_j, \ \frac{g_j}{\omega^{N-1}} \in L^2(\mathbb{R}^\nu), \quad j = 1, \ldots, J$$

with $N \geq 4$ *even. Let* $b_1 = 0$ *and*

$$b_n = \sum_{\substack{q+\ell=n \\ q,\ell \geq 1}} \sum_{\substack{\ell_1+\cdots+\ell_q=\ell-1 \\ \ell_1,\ldots,\ell_q \geq 0}} \langle H_{\text{I}}\psi_0 \otimes \Omega_0, K_{l_1}(b_1, \ldots, b_{\ell_1}) \cdots K_{l_q}(b_1, \ldots, b_{\ell_q})\psi_0 \otimes \Omega_0 \rangle,$$

$$n = 2, \ldots, N.$$

Then

$$b_{2n-1} = 0, \quad n = 1, \ldots, \frac{N}{2}$$

and

$$E(\lambda) = E_0 + \sum_{n=1}^{N/2} b_{2n}\lambda^{2n} + o(\lambda^N) \quad (\lambda \to 0).$$

Proof. See [3, Theorem 5.17]. ∎

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