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IMPLICIT VISCOSITY ITERATIVE ALGORITHM FOR THE SPLIT EQUILIBRIUM PROBLEM AND THE FIXED POINT PROBLEM FOR ONE-PARAMETER NONEXPANSIVE SEMIGROUPS[†]

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ABSTRACT. In this paper, we introduce implicit iterative scheme for finding a common element of the split equilibrium problem and the fixed point problem for a set of one-parameter nonexpansive semigroup $\{T(s)|0 \leq s < \infty\}$ in real Hilbert spaces. We prove the sequence generated by the implicit viscosity iterative algorithm in Hilbert spaces under certain mild condition converge strongly to the common solution of the split equilibrium problem and the fixed point problem for a set of one-parameter nonexpansive semigroups, which is the unique solution of a variational inequality problem.
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1. INTRODUCTION

Throughout the paper, unless otherwise stated, let H_1 and H_2 be real Hilbert spaces with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. Let C and Q be nonempty closed convex subsets of H_1 and H_2, respectively. Recall, a mapping T with domain $D(T)$ and range $R(T)$ in H is called *nonexpansive* iff for all $x, y \in D(T)$, $\|Tx - Ty\| \leq \|x - y\|$. A family $\mathcal{S} = \{T(s)|0 \leq s < \infty\}$ of mappings of C into itself is called a *one-parameter nonexpansive semigroup* on C iff it satisfies the following conditions:

(a) $T(s+t) = T(s)T(t)$ for all $s, t \geq 0$ and $T(0) = I$;
(b) $\|T(s)x - T(s)y\| \leq \|x - y\|$ for all $x, y \in C$ and $s \geq 0$;
(c) the mapping $T(\cdot)x$ is continuous, for each $x \in C$.

The set of all the common fixed points of a family $\mathcal{S}$ is denoted by $Fix(\mathcal{S})$, i.e., $Fix(\mathcal{S}) := \{x \in C : T(s)x = x, 0 \leq s < \infty\} = \bigcap_{0\leq s<\infty} Fix(T(s))$, where $Fix(T(s))$ is the set of fixed points of $T(s)$. It is well known that $Fix(\mathcal{S})$ is closed and convex. It is clear that $T(s)T(t) = T(s+t) = T(t)T(s)$ for $s, t \geq 0$.

Recall that f is called to be *weakly contractive* [1] iff for all $x, y \in D(T)$, $\|f(x) - f(y)\| \leq \|x - y\| - \varphi(\|x - y\|)$, for some $\varphi : [0, +\infty) \to [0, +\infty)$ is a continuous and nondecreasing function such that φ is positive on $(0, +\infty)$ and $\varphi(0) = 0$. If $\varphi(t) = (1-k)t$ for a constant k with $0 < k < 1$ then f is called to be contraction. If $\varphi(t) \equiv 0$, then f is said to be nonexpansive.

Let C be a nonempty closed convex subset of H and $F : C \times C \to \mathbb{R}$ be a bifunction, where $\mathbb{R}$ is the set of real numbers. The equilibrium problem (for short, EP) to find $x \in C$ such that for all $y \in C$,

$$F(x, y) \geq 0. \tag{1.1}$$

The set of solutions of (1.1) is denoted by $EP(F)$. Given a mapping $T : C \to H$, let $F(x, y) = \langle Tx, y - x\rangle$ for all $x, y \in C$. Then $x \in EP(F)$ if and only if $x \in C$ is a solution of the variational inequality $\langle Tx, y - x\rangle \geq 0$ for all $y \in C$.

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To study the equilibrium problems, we assume that the bifunction $F : C \times C \to \mathbb{R}$ satisfies the following conditions:

(A1) $F(x, y) = 0$ for all $x \in C$;
(A2) F is monotone, i.e., $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;
(A3) for each $x, y, z \in C$, $\limsup_{t \to 0} F(tz + (1-t)x, y) \leq F(x, y)$;
(A4) for each $x \in C$ fixed, the function $y \mapsto F(x, y)$ is convex and lower semicontinuous.

Iterative methods for nonexpansive mappings have recently been applied to solve convex minimization problems; see, e.g., [2, 3] and the references therein. Let B be a strongly positive linear bounded operator (i.e., there is a constant $\bar{\gamma} > 0$ such that $\langle Bx, x \rangle \geq \bar{\gamma}\|x\|^2, \forall x \in H$), and T be a nonexpansive mapping on H. A typical problem is to minimize a quadratic function over the set of the fixed points of a nonexpansive mapping on a real Hilbert space H:

$$\min_{x \in F(T)} \frac{1}{2}\langle Bx, x \rangle - \langle x, b \rangle \tag{1.2}$$

where $F(T)$ is the fixed point set of the mapping T on H and b is a given point in H. Starting with an arbitrary initial $x_0 \in H$, define a sequence $\{x_n\}$ recursively by

$$x_{n+1} = (I - \alpha_n B)Tx_n + \alpha_n b, \ n \geq 0 \tag{1.3}$$

It is proved [3] (see also [4]) that the sequence $\{x_n\}$ generated by (1.3) converges strongly to the unique solution of the minimization problem (1.2) provided the sequence an satisfies certain conditions.

Recently, Moudafi [5] introduced the following *split equilibrium problem* (SEP): Let $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ be nonlinear bifunctions and $A : H_1 \to H_2$ be a bounded linear operator, then the SEP is to find $x^* \in C$ such that

$$F_1(x^*, x) \geq 0, \ \forall x \in C, \tag{1.4}$$

and such that

$$y^* = Ax^* \in Q \text{ solves } F_2(y^*, y) \geq 0, \ \forall y \in Q. \tag{1.5}$$

When looked separately, (1.4) is the classical EP, and we denoted its solution set by $EP(F_1)$. SEP (1.4)-(1.5) constitutes a pair of equilibrium problems which have to be solved so that the image $y^* = Ax^*$, under a given bounded linear operator A, of the solution x^* of EP (1.4) in H_1 is the solution of another EP (1.5) in another space H_2, and we denote the solution set of EP (1.5) by $EP(F_2)$.

The solution set of SEP (1.4)-(1.5) is denoted by $\Omega = \{p \in EP(F_1) : Ap \in EF(F_2)\}$. The SEP (1.4)-(1.5) includes the split variational inequality problem which is the generalization of the split zero problem and the split feasibility problem (see, for instance, [5, 6, 7]).

In 2013, Kazmi and Rizvi [8] introduced implicit iteration method for finding a common solution of split equilibrium problem and fixed point problem for a nonexpansive semigroup.

Motivated by works of Moudafi [5], Kazmi and Rizvi [8], we suggest and analyze an implicit iterative method for approximation of a common solution of the split equilibrium problem and the fixed point problem for one-parameter nonexpansive semigroup in a real Hilbert space.

2. Preliminaries

Definition 2.1. A mapping $U : H_1 \to H_1$ is said to be

(i) *monotone*, if $\langle Ux - Uy, x - y \rangle \geq 0, \forall x, y \in H_1$;
(ii) *α-inverse strongly monotone* (or, *α-ism*), if there exists a constant $\alpha > 0$ such that $\langle Ux - Uy, x - y \rangle \geq \alpha\|Ux - Uy\|^2, \forall x, y \in H_1$;
(iii) *firmly nonexpansive*, if is 1-*ism*.

Definition 2.2. A mapping $U : H_1 \to H_1$ is said to be *averaged* if and only if it can be written as the average of the identity mapping and a nonexpansive mapping, i.e., $U := (1 - \alpha)I + \alpha V$, where $\alpha \in (0, 1)$ and $V : H_1 \to H_1$ is nonexpansive and I is the identity operator on H_1.

Proposition 2.3. [5] *Let $U : H_1 \to H_1$ be a nonlinear mapping. Then,*

(i) *If $U = (1-\alpha)D + \alpha V$, where $D : H_1 \to H_1$ is averaged, $V : H_1 \to H_1$ is nonexpansive and $\alpha \in (0,1)$, then U is averaged;*
(ii) *The composite of finite many averaged mappings is averaged;*
(iii) *If U is τ-ism, then for $\gamma > 0, \gamma U$ is $\frac{\tau}{\gamma}$-ism;*
(iv) *U is averaged if and only if, its complement $I - U$ is τ-ism for some $\tau > \frac{1}{2}$.*

For every point $x \in H_1$, there exists a unique nearest point in C denoted by $P_C x$ such that

$$\|x - P_C x\| \le \|x - y\|, \ \forall y \in C. \tag{2.1}$$

P_C is called the *metric projection* of H_1 onto C. It is well known that P_C is a nonexpansive mapping and is characterized by the following property:

$$\langle x - P_C x, y - P_C x\rangle \le 0, \ \forall x \in H_1, y \in C. \tag{2.2}$$

Further, it is well known that every nonexpansive operator $T : H_1 \to H_1$ satisfies, for all $(x, y) \in H_1 \times H_1$,

$$\langle (x - T(x)) - (y - T(y)), T(y) - T(x)\rangle \le \frac{1}{2}\|(T(x) - x) - (T(y) - y)\|^2 \tag{2.3}$$

and therefore, we get, for all $(x, y) \in H_1 \times Fix(T)$,

$$\langle x - T(x), y - T(y)\rangle \le \frac{1}{2}\|T(x) - x\|^2. \tag{2.4}$$

A set valued mapping $M : H_1 \to 2^{H_1}$ is called *monotone* if for all $x, y \in H_1, u \in Mx$ and $v \in My$ imply $\langle x - y, u - v\rangle \ge 0$. A monotone mapping $M : H_1 \to 2^{H_1}$ is *maximal* if the graph $G(M)$ of M is not properly contained in the graph of any other monotone mappings. It is know that a monotone mapping M is maximal if and only if for $(x, u) \in H_1 \times H_1, \langle x - y, u - v\rangle \ge 0$, for every $(y, v) \in G(M)$ implies $u \in Bx$. Let $D : C \to H_1$ be an inverse strongly monotone mapping and let $N_C x$ be the normal cone to C at $x \in C$, i.e., $N_C x := \{z \in H_1 : \langle y - x, z\rangle \ge 0, \forall y \in C\}$. Define

$$Mv = \begin{cases} Dv + N_C x, & \text{if } x \in C, \\ \emptyset, & \text{if } x \notin C. \end{cases}$$

Then, M is maximal monotone and $0 \in Mx$ if and only if $v \in VI(C, M)$ (see [9] for more details).

Lemma 2.4. [8] *Let C be a nonempty closed convex subset of H_1 and let $F_1 : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1)-(A4). For $r > 0$ and for all $x \in H_1$, define a mapping $T_r^{F_1} : H_1 \to C$ as follows:*

$$T_r^{F_1} x = \{z \in C : F_1(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \ge 0, \forall y \in C\}.$$

Then the following hold:
(i) *$T_r^{F_1}(x) \ne \emptyset$ for each $x \in H_1$;*
(ii) *$T_r^{F_1}$ is single-valued;*
(iii) *$T_r^{F_1}$ is firmly nonexpansive, i.e., $\|T_r^{F_1}x - T_r^{F_1}y\|^2 \le \langle T_r^{F_1}x - T_r^{F_1}y, x - y\rangle, \forall x, y \in H_1$;*
(iv) *$Fix(T_r^{F_1}) = EP(F_1)$;*
(v) *$EP(F_1)$ is closed and convex.*

Further, assume that $F_2 : Q \times Q \to \mathbb{R}$ satisfying (A1)-(A4). For $s > 0$ and for all $w \in H_2$, define a mapping $T_s^{F_2} : H_2 \to Q$ as follows:

$$T_s^{F_2}(w) = \{d \in Q : F_2(d, e) + \frac{1}{s}\langle e - d, d - w\rangle \ge 0, \forall e \in Q\}.$$

Then, we easily observe that $T_s^{F_2}(w) \ne \emptyset$ for each $w \in Q$; $T_s^{F_2}$ is single-valued and firmly nonexpansive; $EP(F_2, Q)$ is closed and convex and $Fix(T_s^{F_2}) = EP(F_2, Q)$, where $EP(F_2, Q)$ is solution set of the following equilibrium problem: Find $y^* \in Q$ such that $F_2(y^*, y) \ge 0, \forall y \in Q$. We observe that $EP(F_2) \subset EP(F_2, Q)$. Further, it is easy to prove that Ω is closed and convex set.

Lemma 2.5. [10] *Assume A is a strongly positive linear bounded operator on a Hilbert space H with coefficient $\bar{\gamma} > 0$ and $0 < \rho \le \|A\|^{-1}$. Then $\|I - \rho A\| \le 1 - \rho\bar{\gamma}$.*

Lemma 2.6. [11] *Let C be a nonempty bounded closed convex subset of H and let $\mathcal{S} = \{T(s) : 0 \le s < \infty\}$ be a nonexpansive semigroup on C, then for any $h \ge 0$, $\lim_{t\to\infty} \sup_{x\in C} \|\frac{1}{t}\int_0^t T(s)xds - T(h)(\frac{1}{t}\int_0^t T(s)xds)\| = 0$.*

Lemma 2.7. [12] *Let C be a nonempty bounded closed convex subset of a Hilbert space H and $\mathcal{S} = \{T(t) : 0 \le t < \infty\}$ be a nonexpansive semigroup on C. If $\{x_n\}$ is a sequence in C satisfying the properties: (i) $x_n \rightharpoonup z$; (ii) $\limsup_{t\to\infty} \limsup_{n\to\infty} \|T(t)x_n - x_n\| = 0$, where $x_n \rightharpoonup z$ denote that $\{x_n\}$ converges weakly to z, then $z \in Fix(\mathcal{S})$.*

Lemma 2.8. [13] *Let $\{\lambda_n\}$ and $\{\beta_n\}$ be two nonnegative real number sequences and $\{\alpha_n\}$ a positive real number sequence satisfying the conditions $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $\lim_{n\to\infty} \frac{\beta_n}{\alpha_n} = 0$ or $\sum_{n=0}^{\infty} \beta_n < \infty$. Let the recursive inequality $\lambda_{n+1} \le \lambda_n - \alpha_n \psi(\lambda_n) + \beta_n$, $n = 0, 1, 2\cdots$, be given, where $\psi(\lambda)$ is a continuous and strict increasing function for all $\lambda \ge 0$ with $\psi(0) = 0$. Then $\{\lambda_n\}$ converges to zero, as $n \to \infty$.*

3. Implicit viscosity iterative algorithm

Theorem 3.1. *Let H_1 and H_2 be two real Hilbert spaces and let $C \subset H_1$ and $Q \subset H_2$ nonempty closed convex sets. Let $A : H_1 \to H_2$ be a bounded linear operator . Assume that $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ are the bifunctions satisfying (A1)-(A4) and F_2 is upper semicontinuous. Let f be a weakly contractive mapping with a function φ on H_1, B a strongly positive linear bounded self-adjoint operator with coefficient $\bar{\gamma} > 0$ on H_1, $\mathcal{S} = \{T(s) : s \ge 0\}$ a one parameter nonexpansive semigroup on C, respectively. Assume that $Fix(\mathcal{S}) \cap \Omega \neq \emptyset$, then for any $0 < \gamma \le \bar{\gamma}$ and let sequences $\{x_n\}, \{u_n\}$ and $\{z_n\}$ be generated by the following iterative algorithm:*

$$\begin{cases} u_n = J_{r_n}^{F_1}(x_n + \delta A^*(J_{r_n}^{F_2} - I)Ax_n), \\ z_n = (1-\beta_n)\frac{1}{t_n}\int_0^{t_n} T(s)u_n ds + \beta_n u_n, \\ x_n = (I - \alpha_n B)z_n + \alpha_n \gamma f(x_n), \forall n \ge 1, \end{cases} \tag{3.1}$$

where $r_n \subset (0, \infty)$ and $\delta \in (0, \frac{1}{L})$, L is the spectral radius of the operator A^*A and A^* is the adjoint of A and $\{\alpha_n\}, \{\beta_n\} \subset (0, 1)$, $\{t_n\} \subset (0, \infty)$ are real sequences satisfying the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$; (ii) $\lim_{n\to\infty} \beta_n = 0$; (iii) $\lim_{n\to\infty} t_n = \infty$; (iv) $\liminf_{n\to\infty} r_n > 0$.

Furthermore, the sequence $\{x_n\}$ converges strongly to $z^* \in Fix(\mathcal{S}) \cap \Omega$ which is uniquely solves the following variational inequality

$$\langle (\gamma f - B)z^*, p - z^* \rangle \le 0, \ \forall p \in Fix(\mathcal{S}) \cap \Omega. \tag{3.2}$$

Proof. **Step 1.** We will show that the sequence $\{x_n\}$ generated from (3.1) is well defined and $\{x_n\}$ is bounded.

Since $\alpha_n \to 0$ as $n \to \infty$, we may assume, with no loss of generality, that $\alpha_n < \|B\|^{-1}$ for all $n \ge 1$. Then, $\alpha_n < \frac{1}{\gamma}$ for all $n \ge 1$.

First, we show that the sequence $\{x_n\}$ generated from (3.1) is well defined. For each $n \ge 1$, define a mapping S_n^f in H_1 as follows

$$\begin{aligned} S_n^f x \ &:= \ (I - \alpha_n B)[(1-\beta_n)\frac{1}{t_n}\int_0^{t_n} T(s)(J_{r_n}^{F_1}(x + \delta A^*(J_{r_n}^{F_2} - I)Ax))ds \\ &\quad + \beta_n (J_{r_n}^{F_1}(x + \delta A^*(J_{r_n}^{F_2} - I)Ax))] + \alpha_n \gamma f(x). \end{aligned}$$

Indeed, since $J_{r_n}^{F_1}$ and $J_{r_n}^{F_2}$ both are firmly nonexpansive, they are averaged. For $\delta \in (0, \frac{1}{L})$, the mapping $(I + \delta A^*(J_{r_n}^{F_2} - I)A)$ is averaged, see [5]. It follow from Proposition 2.3 (ii) that the mapping $J_{r_n}^{F_1}(I + \delta A^*(J_{r_n}^{F_2} - I)A)$ is averaged and hence nonexpansive. For any $x, y \in H$, we compute

$$\begin{aligned} &\|S_n^f x - S_n^f y\| \le \|(I - \alpha_n B)\|[(1-\beta_n)\frac{1}{t_n}\int_0^{t_n} \|T(s)(J_{r_n}^{F_1}(x + \delta A^*(J_{r_n}^{F_2} - I)Ax)) \\ &- T(s)(J_{r_n}^{F_1}(y + \delta A^*(J_{r_n}^{F_2} - I)Ay))\|ds + \beta_n \|(J_{r_n}^{F_1}(x + \delta A^*(J_{r_n}^{F_2} - I)Ax)) \end{aligned}$$

$$-(J_{r_n}^{F_1}(y+\delta A^*(J_{r_n}^{F_2}-I)Ay))\|] + \alpha_n\gamma\|f(x)-f(y)\| \leq (1-\alpha_n\bar{\gamma})[(1-\beta_n)\|x-y\| + \beta_n\|x-y\|]$$
$$+\alpha_n\gamma\|f(x)-f(y)\| \leq [1-\alpha_n(\bar{\gamma}-\gamma)]\|x-y\| - \alpha_n\gamma\varphi(\|x-y\|) \leq \|x-y\| - \psi(\|x-y\|),$$

where $\psi(\|x-y\|) := \alpha_n\gamma\varphi(\|x-y\|)$. This shows that S_n^f is a weakly contractive mapping with a function ψ on H_1 for each $n \geq 1$. Therefore, by Theorem 5 of [14], S_n^f has a unique fixed point (say) $x_n \in H_1$. This means (3.1) has a unique solution for each $n \geq 1$, namely,

$$x_n = (I-\alpha_n B)[(1-\beta_n)\frac{1}{t_n}\int_0^{t_n} T(s)u_n ds + \beta_n u_n] + \alpha_n\gamma f(x_n).$$

Next, we show that $\{x_n\}$ is bounded. Indeed, for any $p \in Fix(\mathcal{S}) \cap \Omega$, we have $p = J_{r_n}^{F_1}p, Ap = J_{r_n}^{F_2}Ap$ and $p = T(s)p$. We estimate

$$\begin{aligned} \|u_n - p\|^2 &= \|J_{r_n}^{F_1}(x_n + \delta A^*(J_{r_n}^{F_2}-I)Ax_n) - J_{r_n}^{F_1}p\|^2 \leq \|x_n + \delta A^*(J_{r_n}^{F_2}-I)Ax_n - p\|^2 \\ &\leq \|x_n - p\|^2 + \delta^2\|A^*(J_{r_n}^{F_2}-I)Ax_n\|^2 + 2\delta\langle x_n - p, A^*(J_{r_n}^{F_2}-I)Ax_n\rangle. \end{aligned} \tag{3.3}$$

Thus, we have

$$\|u_n-p\|^2 \leq \|x_n-p\|^2+\delta^2\langle (J_{r_n}^{F_2}-I)Ax_n, AA^*(J_{r_n}^{F_2}-I)Ax_n\rangle+2\delta\langle x_n-p, A^*(J_{r_n}^{F_2}-I)Ax_n\rangle. \tag{3.4}$$

Now, we have

$$\delta^2\langle (J_{r_n}^{F_2}-I)Ax_n, AA^*(J_{r_n}^{F_2}-I)Ax_n\rangle \leq L\delta^2\langle (J_{r_n}^{F_2}-I)Ax_n, (J_{r_n}^{F_2}-I)Ax_n\rangle = L\delta^2\|(J_{r_n}^{F_2}-I)Ax_n\|^2. \tag{3.5}$$

Denoting $\Lambda := 2\delta\langle x_n - p, A^*(J_{r_n}^{F_2}-I)Ax_n\rangle$ and using (2.4), we have

$$\begin{aligned} \Lambda &= 2\delta\langle x_n - p, A^*(J_{r_n}^{F_2}-I)Ax_n\rangle = 2\delta\langle A(x_n-p), (J_{r_n}^{F_2}-I)Ax_n\rangle \\ &= 2\delta\langle A(x_n-p) + (J_{r_n}^{F_2}-I)Ax_n - (J_{r_n}^{F_2}-I)Ax_n, (J_{r_n}^{F_2}-I)Ax_n\rangle \\ &= 2\delta\Big\{\langle J_{r_n}^{F_2}Ax_n - Ap, (J_{r_n}^{F_2}-I)Ax_n\rangle - \|(J_{r_n}^{F_2}-I)Ax_n\|^2\Big\} \\ &\leq 2\delta\Big\{\frac{1}{2}\|(J_{r_n}^{F_2}-I)Ax_n\|^2 - \|(J_{r_n}^{F_2}-I)Ax_n\|^2\Big\} \leq -\delta\|(J_{r_n}^{F_2}-I)Ax_n\|^2. \end{aligned} \tag{3.6}$$

Using (3.4), (3.5) and (3.6), we obtain

$$\|u_n-p\|^2 \leq \|x_n-p\|^2 + \delta(L\delta-1)\|(J_{r_n}^{F_2}-I)Ax_n\|^2. \tag{3.7}$$

Since $\delta \in (0, \frac{1}{L})$, we obtain

$$\|u_n-p\|^2 \leq \|x_n-p\|^2. \tag{3.8}$$

Now, setting $g_n := \frac{1}{t_n}\int_0^{t_n} T(s)u_n ds$, we obtain

$$\|g_n-p\| = \left\|\frac{1}{t_n}\int_0^{t_n} T(s)u_n ds - p\right\| \leq \frac{1}{t_n}\int_0^{t_n}\|T(s)u_n - T(s)p\|ds \leq \|u_n-p\| = \|x_n-p\|. \tag{3.9}$$

By (3.8) and (3.9), we get

$$\|z_n-p\| = (1-\beta_n)\|g_n-p\|+\beta_n\|u_n-p\| \leq (1-\beta_n)\|u_n-p\|+\beta_n\|u_n-p\| = \|u_n-p\| \leq \|x_n-p\|. \tag{3.10}$$

Further, we estimate

$$\begin{aligned} \|x_n-p\|^2 &= \langle x_n-p, x_n-p\rangle \\ &= \langle (I-\alpha_n B)(z_n-p), x_n-p\rangle + \alpha_n\gamma\langle f(x_n)-f(p), x_n-p\rangle + \alpha_n\langle\gamma f(p) - Bp, x_n-p\rangle \\ &\leq [1-\alpha_n(\bar{\gamma}-\gamma)]\|x_n-p\|^2 + \alpha_n\langle\gamma f(p)-Bp, x_n-p\rangle - \alpha_n\gamma\varphi(\|x_n-p\|)\|x_n-p\| \\ &\leq \|x_n-p\|^2 + \alpha_n\langle\gamma f(p)-Bp, x_n-p\rangle - \alpha_n\gamma\varphi(\|x_n-p\|)\|x_n-p\|. \end{aligned} \tag{3.11}$$

Therefore, $\varphi(\|x_n-p\|) \leq \frac{1}{\gamma}\|\gamma f(p) - Bp\|$, which implies that $\{\varphi(\|x_n-p\|)\}$ is bounded. We obtain that $\{(\|x_n-p\|)\}$ is bounded by property of φ. So $\{x_n\}$ is bounded and so are $\{u_n\}, \{z_n\}, \{g_n\}, \{Bz_n\}$ and $\{f(x_n)\}$.

Step 2. We claim that there exists a subsequence $\{n_k\}$ of $\{n\}$ such that $x_{n_k} \rightharpoonup z^*$ and $z^* \in Fix(\mathcal{S})$. Indeed, for $p \in Fix(\mathcal{S}) \cap \Omega$ and from (3.9), then $\|g_n - p\| \leq \|u_n - p\| \leq \|x_n - p\|$.

Since $\{u_n\}, \{g_n\}, \{Bz_n\}, \{f(x_n)\}$ are bounded and the conditions $\lim_{n\to\infty} \alpha_n = 0 = \lim_{n\to\infty} \beta_n$, we see that

$$\|z_n - g_n\| = \|(1-\beta_n)g_n + \beta_n u_n - g_n\| = \beta_n\|u_n - g_n\| \to 0 \ (n \to \infty) \tag{3.12}$$

and

$$\|x_n - z_n\| = \|(I - \alpha_n B)z_n + \alpha_n \gamma f(x_n) - z_n\| = \alpha_n\|\gamma f(x_n) - Bz_n\| \to 0 \ (n \to \infty). \tag{3.13}$$

In view of (3.12) and (3.13), we obtain that

$$\|x_n - g_n\| \le \|x_n - z_n\| + \|z_n - g_n\| \to 0 \ (n \to \infty). \tag{3.14}$$

Let $K_1 = \{\omega \in C : \varphi(\|\omega - p\|) \le \frac{1}{\gamma}\|\gamma f(p) - Bp\|\}$, then K_1 is a nonempty bounded closed convex subset of C which is $T(s)$-invariant for each $0 \le s < \infty$ and contain $\{x_n\} \subset K_1$. So without loss of generality, we may assume that $\mathcal{S} := \{T(s) : 0 \le s < \infty\}$ is nonexpansive semigroup on K_1. By Lemma 2.6, we have

$$\limsup_{s\to\infty}\limsup_{n\to\infty} \|g_n - T(s)g_n\| = 0. \tag{3.15}$$

From (3.14) and (3.15), we obtain that $\|x_n - T(s)x_n\| \le \|x_n - g_n\| + \|g_n - T(s)g_n\| + \|T(s)g_n - T(s)x_n\| \le \|x_n - g_n\| + \|g_n - T(s)g_n\| + \|g_n - x_n\| \le 2\|x_n - g_n\| + \|g_n - T(s)g_n\|$, we arrive at

$$\limsup_{s\to\infty}\limsup_{n\to\infty} \|x_n - T(s)x_n\| = 0. \tag{3.16}$$

On the other hand, since $\{x_n\}$ is bounded, we know that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup z^*$. By Lemma 2.7 and (3.16), we arrive at $z^* \in Fix(\mathcal{S})$.

In (3.11), interchange z^* and p to obtain $\psi(\|x_{n_k} - z^*\|) \le \langle \gamma f(z^*) - Bz^*, x_{n_k} - z^*\rangle$, where $\psi(\|x_{n_k} - z^*\|) := \gamma\varphi(\|x_{n_k} - z^*\|)\|x_{n_k} - z^*\|$. From $x_{n_k} \rightharpoonup z^*$, we get that

$$\limsup_{k\to\infty} \psi(\|x_{n_k} - z^*\|) \le \limsup_{n\to\infty}\langle \gamma f(z^*) - Bz^*, x_{n_k} - z^*\rangle = 0.$$

Namely, $\psi(\|x_{n_k} - z^*\|) \to 0$ $(k \to \infty)$ which implies that $x_{n_k} \to z^*$ as $k \to \infty$ by the property of ψ and since $\|x_n - z_n\| \to 0$ thus $z_{n_k} \to z^*$.

Step 3. We will show that $\lim_{n\to\infty} \|u_n - x_n\| = 0$.

Further, we estimate by (3.1), (3.7) and (3.8), we have

$$\begin{aligned}
\|x_n - p\|^2 &= \|(I - \alpha_n B)(z_n - p) + \alpha_n(\gamma f(x_n) - Bp)\|^2 \\
&\le (1 - \alpha_n\bar{\gamma})^2\|z_n - p\|^2 + 2\alpha_n\langle \gamma f(x_n) - Bp + \gamma f(p) - \gamma f(p), x_n - p\rangle \\
&\le (1 + (\alpha_n\bar{\gamma})^2 - 2\alpha_n\bar{\gamma})\|u_n - p\|^2 + 2\alpha_n\gamma\varphi\|x_n - p\|^2 + 2\alpha_n\langle \gamma f(p) - Bp, x_n - p\rangle \\
&\le \|u_n - p\|^2 + (\alpha_n\bar{\gamma})^2\|u_n - p\|^2 + 2\alpha_n\gamma\varphi\|x_n - p\|^2 + 2\alpha_n\|\gamma f(p) - Bp\|\|x_n - p\| \\
&\le \|x_n - p\|^2 + \delta(L\delta - 1)\|(J_{r_n}^{F_2} - I)Ax_n\|^2 + (\alpha_n\bar{\gamma})^2\|x_n - p\|^2 + 2\alpha_n\gamma\varphi\|x_n - p\|^2 \\
&\quad + 2\alpha_n\|\gamma f(p) - Bp\|\|x_n - p\|.
\end{aligned} \tag{3.17}$$

Since $\{x_n\}$ is bounded, we may assume that $\rho := sup_{0<n<1}\|x_n - p\|$. Therefore, (3.17) reduces to $\delta(1 - L\delta)\|(J_{r_n}^{F_2} - I)Ax_n\|^2 \le \alpha_n^2\bar{\gamma}^2\rho^2 + 2\alpha_n\gamma\varphi\rho^2 + 2\alpha_n\|\gamma f(p) - Bp\|\rho = \alpha_n[\alpha_n\bar{\gamma}^2\rho^2 + 2\gamma\varphi\rho^2 + 2\|\gamma f(p) - Bp\|\rho]$. Further, since $\delta(1 - L\delta) > 0, \alpha_n \to 0$, preceding inequality implies that

$$\lim_{n\to\infty} \|(J_{r_n}^{F_2} - I)Ax_n\| = 0. \tag{3.18}$$

Next, we observe that

$$\begin{aligned}
&\|u_n - p\|^2 \\
= \ &\|J_{r_n}^{F_1}(x_n + \delta A^*(J_{r_n}^{F_2} - I)Ax_n) - T_{r_n}^{F_1}p\|^2 \le \langle u_n - p, x_n + \delta A^*(J_{r_n}^{F_2} - I)Ax_n - p\rangle \\
= \ &\frac{1}{2}\Big\{\|u_n - p\|^2 + \|x_n + \delta A^*(J_{r_n}^{F_2} - I)Ax_n - p\|^2 - \|(u_n - p) - [x_n + \delta A^*(J_{r_n}^{F_2} - I)Ax_n - p]\|^2\Big\} \\
= \ &\frac{1}{2}\Big\{\|u_n - p\|^2 + \|x_n - p\|^2 - \|u_n - x_n - \delta A^*(J_{r_n}^{F_2} - I)Ax_n\|^2\Big\} \\
\le \ &\frac{1}{2}\Big\{\|u_n - p\|^2 + \|x_n - p\|^2 - \|u_n - x_n\| - \delta^2\|A^*(J_{r_n}^{F_2} - I)Ax_n\|^2 + 2\delta\|A(u_n - x_n)\|\|(J_{r_n}^{F_2} - I)Ax_n\|\Big\}.
\end{aligned}$$

Hence, we have

$$\begin{aligned}\|u_n-p\|^2 &\le \|x_n-p\|^2-\|u_n-x_n\|^2-\delta^2\|A^*(J_{r_n}^{F_2}-I)Ax_n\|+2\delta\|A(u_n-x_n)\|\|(J_{r_n}^{F_2}-I)Ax_n\| \\ &\le \|x_n-p\|^2-\|u_n-x_n\|^2+2\delta\|A(u_n-x_n)\|\|(J_{r_n}^{F_2}-I)Ax_n\|. \qquad (3.19)\end{aligned}$$

Since $\{x_n\}$ and $\{u_n\}$ are bounded and A is a bounded linear operator then $\|A(u_n-x_n)\|$ is bounded and hence we may assume that $l := sup_{0<n<1}\|A(u_n-x_n)\|$. If follows from (3.17) and (3.19) that

$$\begin{aligned}\|x_n-p\|^2 &\le \|u_n-p\|^2+\alpha_n^2\bar{\gamma}^2\|x_n-p\|^2+2\alpha_n\gamma\varphi\|x_n-p\|^2+2\alpha_n\|\gamma f(p)-Bp\|\|x_n-p\| \\ &\le [\|x_n-p\|^2-\|u_n-x_n\|^2+2\delta\|A(u_n-x_n)\|\|(J_{r_n}^{F_2}-I)Ax_n\|]+\alpha_n^2\bar{\gamma}^2\|x_n-p\|^2 \\ &\quad +2\alpha_n\gamma\varphi\|x_n-p\|^2+2\alpha_n\|\gamma f(p)-Bp\|\|x_n-p\| \\ &= \|x_n-p\|^2-\|u_n-x_n\|^2+2\delta l\|(J_{r_n}^{F_2}-I)Ax_n\|+\alpha_n Q,\end{aligned}$$

where $Q := (2\gamma\varphi+\alpha_n\bar{\gamma}^2)\rho^2+2\|\gamma f(p)-Bp\|\rho$. Therefore, from (3.18) and $\alpha_n \to 0$, we obtain

$$\|u_n-x_n\|^2 \le 2\delta l\|(J_{r_n}^{F_2}-I)Ax_n\|+\alpha_n Q \to 0,\ (n\to\infty).$$

This implies that $\|u_n-x_n\| \to 0,\ (n\to\infty)$.

Step 4. We will show that $z^* \in \Omega$, where z^* is obtain in Step 2.

First, we show that $z^* \in EP(F_1)$. Since $u_n = J_{r_n}^{F_1}x_n$, we have $F_1(u_n,y)+\frac{1}{r_n}\langle y-u_n, u_n-x_n\rangle \ge 0$, $\forall y \in C$. It follows from monotonicity of F_1 that $\frac{1}{r_n}\langle y-u_n, u_n-x_n\rangle \ge F_1(y,u_n)$ and hence $\left\langle y-u_{n_k}, \frac{u_{n_k}-x_{n_k}}{r_{n_k}}\right\rangle \ge F_1(y,u_{n_k})$. Since $\|u_n-x_n\| \to 0$ and $x_{n_k} \to z^*$, we get $u_{n_k} \to z^*$ and $\frac{u_{n_k}-x_{n_k}}{r_{n_k}} \to 0$. It follows by (A4) that $0 \ge F_1(y,z^*), \forall z^* \in C$. For t with $0<\zeta\le 1$ and $y \in C$, let $y_\zeta = \zeta y+(1-\zeta)z^*$. Since $y \in C$, $z^* \in C$, we get $y_\zeta \in C$, and hence, $F_1(y_\zeta, z^*) \le 0$. So, from (A1) and (A4), we have $0 = F_1(y_\zeta, y_\zeta) \le \zeta F_1(y_\zeta, y)+(1-\zeta)F_1(y_\zeta, z^*) \le \zeta F_1(y_\zeta, y)$. Therefore $0 \le F_1(y_\zeta, y)$. From (A3), we have $0 \le F_1(z^*, y)$. This implies that $z^* \in EP(F_1)$.

Next, we show that $Az^* \in EP(F_2)$. Since $x_{n_k} \to z^*$ and A is bounded linear operator, $Ax_{n_k} \to Az^*$. Now, setting $v_{n_k} = Ax_{n_k}-J_{r_{n_k}}^{F_2}Ax_{n_k}$. It follows that from (3.18) that $\lim_{k\to\infty} v_{n_k} = 0$ and $Ax_{n_k}-v_{n_k} = J_{r_{n_k}}^{F_2}Ax_{n_k}$. Therefore from Lemma 2.4, we have $F_2(Ax_{n_k}-v_{n_k}, z)+\frac{1}{r_{n_k}}\langle z-(Ax_{n_k}-v_{n_k}), (Ax_{n_k}-v_{n_k})-Ax_{n_k}\rangle \ge 0,\ \forall z \in Q$. Since F_2 is upper semicontinuous in the first argument, taking lim sup to above inequality as $k\to\infty$ and using condition (iv), we obtain $F_2(Az^*, z) \ge 0,\ \forall z \in Q$, which means that $Az^* \in EP(F_2)$ and hence $z^* \in \Omega$.

Step 5. We claim that z^* is the unique solution of the variational inequality (3.2).

Firstly, we show the uniqueness of the solution to the variational inequality (3.2) in $Fix(\mathcal{S})\cap\Omega$. In fact, suppose that $a, b \in Fix(\mathcal{S})\cap\Omega$ satisfy (3.2), we see that

$$\langle (B-\gamma f)a, a-b\rangle \le 0, \qquad (3.20)$$

$$\langle (B-\gamma f)b, b-a\rangle \le 0. \qquad (3.21)$$

Adding these two inequalities (3.20) and (3.21) yields

$$0 \ge \langle B(a-b), a-b\rangle - \gamma\langle f(a)-f(b), a-b\rangle \ge (\bar{\gamma}-\gamma)\|a-b\|^2+\gamma\varphi(\|a-b\|)\|a-b\|,$$

thus $\varphi(\|a-b\|) \le \frac{\gamma-\bar{\gamma}}{\gamma}\|a-b\|$. From $\frac{\gamma-\bar{\gamma}}{\gamma} \le 0$, we get that $\varphi(\|a-b\|) \le 0$. By the property of φ, we must have $a=b$ and the uniqueness is proved.

Next, we show that z^* is a solution in $Fix(\mathcal{S})\cap\Omega$ to the variational inequality (3.2). Indeed, since $x_n = (I-\alpha_n B)(1-\beta_n)\frac{1}{t_n}\int_0^{t_n} T(s)u_n ds+(I-\alpha_n B)\beta_n u_n+\alpha_n\gamma f(x_n)$, we can rewrite that $Bx_n-\gamma f(x_n) = -\frac{1}{\alpha_n}(I-\alpha_n B)(1-\beta_n)(I-\frac{1}{t_n}\int_0^{t_n} T(s)ds)u_n+\frac{1}{\alpha_n}[(I-\alpha_n B)u_n-(I-\alpha_n B)x_n]$.

For any $p \in Fix(\mathcal{S})\cap\Omega$, it follows that

$$\begin{aligned}&\langle B(x_n)-\gamma f(x_n), u_n-p\rangle \\ =\ & -\frac{1-\beta_n}{\alpha_n}\langle (I-\frac{1}{t_n}\int_0^{t_n} T(s)ds)u_n-(I-\frac{1}{t_n}\int_0^{t_n} T(s)ds)p, u_n-p\rangle \qquad (3.22)\end{aligned}$$

$$+(1-\beta_n)\langle B(I-\frac{1}{t_n}\int_0^{t_n}T(s)ds)u_n,u_n-p\rangle+\frac{1}{\alpha_n}\langle u_n-x_n,u_n-p\rangle+\langle Bx_n-Bu_n,u_n-p\rangle.$$

Now, we consider the right side of (3.22), $\langle u_n-x_n,u_n-p\rangle \le r_nF_1(u_n,p)$. Note from $p\in Fix(\mathcal{S})\cap\Omega$, we see that $F_1(p,u_n)\ge 0$, then $F_1(u_n,p)\le -F_1(p,u_n)\le 0$, which implies that $\frac{1}{\alpha_n}\langle u_n-x_n,u_n-p\rangle\le 0$. On the other hand, we see that $I-\frac{1}{t_n}\int_0^{t_n}T(s)ds$ is monotone, that is, $\langle (I-\frac{1}{t_n}\int_0^{t_n}T(s)ds)u_n-(I-\frac{1}{t_n}\int_0^{t_n}T(s)ds)p,u_n-p\rangle\ge 0$. Thus, we obtain from (3.22) that

$$\langle B(x_n)-\gamma f(x_n),u_n-p\rangle\le(1-\beta_n)\langle B(I-\frac{1}{t_n}\int_0^{t_n}T(s)ds)u_n,u_n-p\rangle+\langle Bx_n-Bu_n,u_n-p\rangle. \quad (3.23)$$

Also, we notice from $\|x_n-u_n\|\to 0\ (n\to\infty)$ and $x_{n_k}\to z^*\in Fix(\mathcal{S})\cap\Omega$ that

$$\limsup_{k\to\infty}\langle B(I-\frac{1}{t_{n_k}}\int_0^{t_{n_k}}T(s)ds)u_{n_k},u_{n_k}-p\rangle=0, \quad (3.24)$$

and

$$\limsup_{k\to\infty}\langle B(x_{n_k}-u_{n_k},u_{n_k}-p\rangle=0. \quad (3.25)$$

Now replacing n in (3.23) with n_k and take lim sup, we have from (3.24) and (3.25) that

$$\langle (B-\gamma f)z^*,z^*-p\rangle\le 0, \quad (3.26)$$

for any $p\in Fix(\mathcal{S})\cap\Omega$. This is, $z^*\in Fix(\mathcal{S})\cap\Omega$ is unique solution of (3.2).

Step 6. We claim that

$$\limsup_{n\to\infty}\langle\frac{1}{t_n}\int_0^{t_n}T(s)u_nds-z^*,\gamma f(z^*)-Bz^*\rangle\le 0. \quad (3.27)$$

To show (3.27), we may choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle\frac{1}{t_n}\int_0^{t_n}T(s)u_nds-z^*,\gamma f(z^*)-Bz^*\rangle=\limsup_{i\to\infty}\langle\frac{1}{t_{n_i}}\int_0^{t_{n_i}}T(s)u_{n_i}ds-z^*,\gamma f(z^*)-Bz^*\rangle. \quad (3.28)$$

Since $\{x_{n_i}\}$ is bounded, we can choose a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ converges weakly to p.

We may assume without loss of generality, that $x_{n_i}\rightharpoonup p$, then $u_{n_i}\rightharpoonup p$, note from Step 2 and Step 3 that $p\in Fix(\mathcal{S})\cap\Omega$ and thus $\frac{1}{t_{n_i}}\int_0^{t_{n_i}}T(s)u_{n_i}ds\rightharpoonup p$. It follows from (3.28) that $\limsup_{n\to\infty}\langle\frac{1}{t_n}\int_0^{t_n}T(s)u_nds-z^*,\gamma f(z^*)-Bz^*\rangle=\langle p-z^*,\gamma f(z^*)-Bz^*\rangle\le 0$. So (3.27) holds, thanks to (3.2).

Step 7. We claim that $x_n\to z^*$ as $n\to\infty$.

First, from (3.14) and (3.27) we conclude that

$$\limsup_{n\to\infty}\langle\gamma f(z^*)-Bz^*,x_n-z^*\rangle\le 0. \quad (3.29)$$

Now we compute $\|x_n-z^*\|^2$ and the following estimates:

$$\begin{aligned}
&\|x_n-z^*\|^2\\
\le\ &(1-\alpha_n\bar{\gamma})^2\|z_n-z^*\|^2+2\alpha_n\langle\gamma f(x_n)-Bz^*,x_n-z^*\rangle\\
\le\ &(1-\alpha_n\bar{\gamma})^2\|z_n-z^*\|^2+2\alpha_n\gamma\|x_n-z^*\|^2+2\alpha_n\langle\gamma f(z^*)-Bz^*,x_n-z^*\rangle-2\alpha_n\gamma\varphi(\|x_n-z^*\|)\\
\le\ &(1-\alpha_n\bar{\gamma})^2\|x_n-z^*\|^2+2\alpha_n\gamma\|x_n-z^*\|^2+2\alpha_n\langle\gamma f(z^*)-Bz^*,x_n-z^*\rangle-2\alpha_n\gamma\varphi(\|x_n-z^*\|)\\
\le\ &(1+(\alpha_n\bar{\gamma})^2-2\alpha_n\bar{\gamma})\|x_n-z^*\|^2+2\alpha_n\gamma\|x_n-z^*\|^2+2\alpha_n\langle\gamma f(z^*)-Bz^*,x_n-z^*\rangle-2\alpha_n\gamma\varphi(\|x_n-z^*\|)\\
\le\ &(1+(\alpha_n\bar{\gamma})^2)\|x_n-z^*\|^2+2\alpha_n\langle\gamma f(z^*)-Bz^*,x_n-z^*\rangle-2\alpha_n\gamma\varphi(\|x_n-z^*\|).
\end{aligned}$$

It follows that $\varphi(\|x_n-z^*\|)\le\frac{\bar{\gamma}^2}{2\gamma}\alpha_n\|x_n-z^*\|^2+\frac{1}{\gamma}\langle\gamma f(z^*)-Bz^*,x_n-z^*\rangle$.

By virtue of the boundedness of $\{x_n\}$, (3.29) and the condition $\alpha_n\to 0\ (n\to\infty)$, we can conclude that $\lim_{n\to\infty}\varphi(\|x_n-z^*\|)=0$. By the property of φ, we obtain that $x_n\to z^*\in Fix(\mathcal{S})\cap\Omega$ as $n\to\infty$. This complete the proof of Theorem 3.1. □

From Theorem 3.1, setting one parameter nonexpansive semigroup for a single nonexpansive mapping T.

Corollary 3.2. *Let H_1 and H_2 be two real Hilbert spaces and let $C \subset H_1$ and $Q \subset H_2$ nonempty closed convex sets. Let $A : H_1 \to H_2$ be a bounded linear operator . Assume that $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ are the bifunctions satisfying (A1)-(A4) and F_2 is upper semicontinuous. Let f be a weakly contractive mapping with a function φ on H_1, B a strongly positive linear bounded self-adjoint operator with coefficient $\bar{\gamma} > 0$ on H_1, T a nonexpansive on C, respectively. Assume that $Fix(T) \cap \Omega \neq \emptyset$, then for any $0 < \gamma \leq \bar{\gamma}$ and let the iterative sequences $\{x_n\}, \{u_n\}$ and $\{z_n\}$ be generated by iterative algorithm:*

$$\begin{cases} u_n = J_{r_n}^{F_1}(x_n + \delta A^*(J_{r_n}^{F_2} - I)Ax_n), \\ z_n = (1 - \beta_n)Tu_n + \beta_n u_n, \\ x_n = (I - \alpha_n B)z_n + \alpha_n \gamma f(x_n), \forall n \geq 1, \end{cases} \tag{3.30}$$

where $r_n \subset (0, \infty)$ and $\delta \in (0, \frac{1}{L}), L$ is the spectral radius of the operator A^*A and A^* is the adjoint of A and $\{\alpha_n\}, \{\beta_n\} \subset (0, 1)$ be real sequences satisfying the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$; *(ii)* $\lim_{n\to\infty} \beta_n = 0$; *(iii)* $\liminf_{n\to\infty} r_n > 0$.

Then, the sequence $\{x_n\}$ converges strongly to $z^* \in Fix(T) \cap \Omega$ which is uniquely solves the following variational inequality (3.2).

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