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Continuous wavelet transforms on spaces of vector-valued functions

Kazuhide Oshiro
Graduate School of Mathematics, Nagoya University

Abstract

Let G be the semidirect product of a locally compact abelian group N with a closed subgroup H of $\mathrm{Aut}(N)$. We consider continuous wavelet transforms associated to unitary representations of G realized on spaces of vector-valued square integrable functions on N.

1 Introduction

Continuous wavelet transforms for the semidirect product group with a commutative normal subgroup have been studied by many authors. The simplest example is the one associated to a quasi-regular representation of the $ax+b$ group [2]. Furthermore wavelet transforms for a semidirect group with a unimodular, not necessarily commutative, normal subgroup are studied in [8].

Let G be the semidirect group $N\rtimes H$ of a locally compact abelian group N and a closed subgroup H of $\mathrm{Aut}(N)$. An element $g\in G$ is written as $g=(n,h)$ with $n\in N$ and $h\in H$. This group law is given by

$$(n,h)(n',h')=(n+hn',h'h)\quad(n,n'\in N,\ h,h'\in H).$$

Let $d\mu_H(h)$ denote a left Haar measure of H and dn a Lebesgue measure on N. We define the measure of G by

$$d\mu_G(g)=\delta(h)^{-1}dnd\mu_H(h),\quad g=(n,h)\in N\rtimes H,$$

where δ is the map from H to $\mathbb{R}_+$ such that $d(hn)=\delta(h)dn$. Then $d\mu_G$ is a left Haar measure of G. Let σ be an irreducible unitary representation of H

on a Hilbert space $\mathcal{H}_\sigma$. We define the unitary representation π of G on the space $L^2(N, \mathcal{H}_\sigma)$ of $\mathcal{H}_\sigma$-valued square integrable functions on N by

$$\pi(n,h)f(n_0) = \delta(h)^{-\frac{1}{2}}\sigma(h)f(h^{-1}(n_0 - n)) \quad (n, n_0 \in N, h \in H).$$

This representation is equivalent to the induced representation $\mathrm{Ind}_H^G \sigma$. In particular, when σ is trivial, π is called a *quasi-regular representation.* In this case, continuous wavelet transforms arising from π have been developed in various directions [7, 8, 11, 12, 13, 14]. In this paper, we consider a more general case. We introduce the wavelet transforms obtained from the unitary representation π with σ not necessarily finite dimensional.

2 Preliminaries

In this section, we recall basic notions about wavelet transform associated to a unitary representation of a locally compact group. Let G be a locally compact group and π an irreducible unitary representation of G defined on a complex separable Hilbert space $\mathcal{H}_\pi$. The representation π is said to be *square-integrable* if there exists a nonzero vector $\varphi \in \mathcal{H}_\pi$ such that the image of the map $\widetilde{W_\varphi} : \mathcal{H}_\pi \to C(G)$ given by

$$\widetilde{W_\varphi}\psi(g) = \langle \psi, \pi(g)\varphi \rangle \quad (\psi \in \mathcal{H}_\pi, \; g \in G)$$

is contained in $L^2(G)$, that is,

$$\int_G |\widetilde{W_\varphi}\psi(g)|^2 d\mu(g) < \infty$$

for all $\psi \in \mathcal{H}_\pi$. Then φ is called an *admissible vector.*

Theorem 1 ([1, Theorem 3.1]). *Suppose π is a square integrable representation of G defined on $\mathcal{H}_\pi$. There exists a unique positive self-adjoint operator C whose domain coincides with the set of admissible vectors such that*

$$\int_G \langle W_{\varphi_1}\psi_1(g), W_{\varphi_2}\psi_2(g) \rangle \, d\mu(g) = \langle \psi_1, \psi_2 \rangle \, \langle C\varphi_2, C\varphi_1 \rangle \quad (g \in G, \; \psi_1, \psi_2 \in \mathcal{H}_\pi).$$

for any admissible vectors φ_1 and φ_2.

For an admissibe vector φ, we define $C_\varphi = \langle C\varphi, C\varphi\rangle$. Applying $\varphi_1 = \varphi_2 = \psi_1 = \psi_2 = \varphi$ in Theorem 1, we have

$$C_\varphi = \frac{1}{\langle \varphi, \varphi\rangle} \int_G |\widetilde{W}_\varphi \varphi(g)|^2 d\mu(g) < \infty.$$

We define the map W_φ from $\mathcal{H}_\pi$ into $L^2(G)$ by

$$W_\varphi \psi = C_\varphi^{-\frac{1}{2}} \widetilde{W}_\varphi \psi \quad (\psi \in \mathcal{H}_\pi).$$

Then W_φ is isometry by Theorem 1, so that for any $\psi \in \mathcal{H}_\pi$ we have

$$\psi = \int_G W_\varphi \psi(g) \pi(g) \varphi d\mu(g)$$

in the weak sense. The map W_φ is called a *continuous wavelet transform.*

3 Construction of the wavelet transforms associated to π

From now on, let G be the semidirect product group as in Section 1. We denote by $\widehat{N}$ the unitary dual of N. Since N is commutative, any element of $\widehat{N}$ is one-dimensional. The dual action of G on $\widehat{N}$ is defined by

$$g \cdot \nu(n) = \nu(g^{-1} n g) \quad (g \in G, \nu \in \widehat{N}, n \in N).$$

For each $\nu \in \widehat{N}$, we denote by G_ν the stabilizer of ν, that is,

$$G_\nu = \{g \in G \ ; \ g \cdot \nu = \nu\},$$

which is a closed subgroup of G. We define $H_\nu = G_\nu \cap H$. Then $G_\nu = N \rtimes H_\nu$. We denote by $\mathcal{O}_\nu$ the G-orbit in $\widehat{N}$ through ν :

$$\mathcal{O}_\nu = \{g \cdot \nu, \ \ g \in G\}.$$

In this section, we construct the wavelet transforms associated to the unitary representation π after giving an irreducible decomposition of π.

For the study of irreducible subrepresentation of π, it is useful to introduce a unitary representation which is equivalent to π. We define the Fourier transform $\mathcal{F}$ on $L^2(N, \mathcal{H}_\sigma)$ by

$$\mathcal{F} f(\nu) = \widehat{f}(\nu) = \int_N \nu(n) f(n) dn \quad (\nu \in \widehat{N}).$$

Taking the conjugate of π by $\mathcal{F}$, we obtain the unitary representation $\widehat{\pi} = \mathcal{F} \circ \pi \circ \mathcal{F}^{-1}$ on $L^2(\widehat{N}, \mathcal{H}_\sigma)$. The representation $\widehat{\pi}$ is described as

$$\widehat{\pi}(n,h)\varphi(\nu) = \nu(n)\delta(h)^{\frac{1}{2}}\sigma(h)\varphi(h^{-1}\cdot\nu) \quad (\varphi \in L^2(\widehat{N}, \mathcal{H}_\sigma)). \tag{1}$$

Now let us assume the following [3, 8] :

(A1) The orbit space is *countably separated*, that is, there is a countable family $\{E_j\}$ of G-invariant Borel set in $\widehat{N}$ such that each orbit in $\widehat{N}$ is the intersection of all the $\{E_j\}$'s that contain it.

(A2) For each $\nu \in \widehat{N}$, the map $G/G_\nu \ni gG_\nu \mapsto g\cdot\nu \in \mathcal{O}_\nu$ is a homeomorphism.

(A3) Let μ be the Plancherel measure on $\widehat{N}$. There exists elements ν_k ($k \in K$) of $\widehat{N}$, indexed by some set K, such that $\mu(\mathcal{O}_{\nu_k}) > 0$ and $\mathcal{O}_{\nu_k} \cap \mathcal{O}_{\nu_{k'}} = \emptyset$ ($k \neq k'$) and $\mu(\widehat{N} \setminus \bigsqcup_{k\in K} \mathcal{O}_{\nu_k}) = 0$.

(A4) The stabilizer $H_{\nu_k} = \{h \in H \ ; \ h\cdot\nu_k = \nu_k\}$ at each $\nu_k \in \widehat{N}$ is compact.

(A5) For every $k \in K$, the restriction $\sigma|_{H_{\nu_k}}$ is multiplicity free. Namely, there exists a index set Λ_k such that $\sigma|_{H_{\nu_k}} = \bigoplus_{\alpha\in\Lambda_k} \rho_\alpha$ and $\rho_\alpha \not\simeq \rho_{\alpha'}$ ($\alpha \neq \alpha'$).

We say that G is *regular* if the two conditions (A1) and (A2) are satisfied. If $\nu \in \widehat{N}$ and ρ is an irreducible representation of H_ν, we define a unitary representation $\nu \otimes \rho$ of G_ν by

$$(\nu \otimes \rho)(n,h) = \nu(n)\rho(h) \quad (n \in N, h \in H_\nu).$$

Theorem 2 ([3, Theorem 6.42]). *Suppose G is regular. If $\nu \in \widehat{N}$ and ρ is an irreducible unitary representation of H_ν, then $\mathrm{Ind}_{G_\nu}^G \nu\otimes\rho$ is an irreducible representation of G. Every irreducible unitary representation of G is equivalent to one of this form. Moreover, $\mathrm{Ind}_{G_\nu}^G \nu\otimes\rho$ and $\mathrm{Ind}_{G_{\nu'}}^G \nu'\otimes\rho'$ are equivalent if and only if ν and ν' belong to the same orbit, say $\nu' = g\cdot\nu$, and $h \to \rho(h)$ and $h \to \rho'(g^{-1}hg)$ are equivalent representation of H_ν.*

The following theorem is useful in order to investigate whether $\mathrm{Ind}_{G_\nu}^G \nu\otimes\rho$ is square-integrable.

Theorem 3 ([10, Theorem 2]). *Let $\nu \in \widehat{N}$ and ρ be an irreducible unitary representation of H_ν. The representation $\mathrm{Ind}_{G_\nu}^G \nu \otimes \rho$ is square-integrable if and only if $\mu(\mathcal{O}_\nu) > 0$ and ρ is square-integrable.*

For $k \in K$, we regard $L^2(\mathcal{O}_{\nu_k}, \mathcal{H}_\sigma)$ as a subspace of $L^2(\widehat{N}, \mathcal{H}_\sigma)$ by zero extension. Thanks to (1), the space $L^2(\mathcal{O}_{\nu_k}, \mathcal{H}_\sigma)$ is G-invariant. We denote by $\widehat{\pi}_k$ the subrepresentation $\widehat{\pi}|_{L^2(\mathcal{O}_{\nu_k}, \mathcal{H}_\sigma)}$. By the assumption (A3), we have $\widehat{\pi} = \bigoplus_{k\in K} \widehat{\pi}_k$.

Proposition 1. *The unitary representation $\widehat{\pi}_k$ is equivalent to $\mathrm{Ind}_{G_{\nu_k}}^G \nu_k \otimes \sigma|_{H_{\nu_k}}$.*

Proof. We denote by Π_k the unitary representation $\mathrm{Ind}_{G_{\nu_k}}^G \nu_k \otimes \sigma|_{H_{\nu_k}}$. Let q be the canonical quotient map from G to G_{ν_k}. The unitary representation Π_k is the left-regular representation on the Hilbert space completion $\widetilde{\mathcal{L}}_{k,\sigma}$of the space $\mathcal{L}_{k,\sigma}$ defined by

$$\begin{aligned}\mathcal{L}_{k,\sigma} = \{F : G \to \mathcal{H}_\sigma; \quad & q(\mathrm{supp} F) \text{ is compact and} \\ & F((n,h)(n',h')) = \nu_k(n')^{-1}\sigma(h')^{-1}F(n,h) \text{ for} \\ & n, n' \in N, h \in H, h' \in H_{\nu_k}\}\end{aligned}$$

with the inner product

$$\langle F, F' \rangle = \int_{G/G_{\nu_k}} \langle F(g), F'(g) \rangle_\sigma \, d\mu_{G/G_{\nu_k}}(gG_{\nu_k}).$$

We define the map Φ from $\mathcal{L}_{k,\sigma}$ to $L^2(\mathcal{O}_{\nu_k}, \mathcal{H}_\sigma)$ by

$$\Phi(F)(\nu) = \delta(h)^{\frac{1}{2}}\sigma(h)F(0,h) \quad (\nu = h \cdot \nu_k).$$

The inverse map Φ^{-1} is given by

$$\Phi^{-1}\varphi(n,h) = \delta(h)^{-\frac{1}{2}} h \cdot \nu_k^{-1}(n)\sigma(h)^{-1}\varphi(h \cdot \nu_k).$$

The map Φ extends to a unitary operator from $\widetilde{\mathcal{L}}_{k,\sigma}$ onto $L^2(\mathcal{O}_{\nu_k}, \mathcal{H}_\sigma)$. Therefore, it suffices to show that $\widehat{\pi}_k(n,h) \circ \Phi = \Phi \circ \Pi_k(n,h)$ for all $(n,h) \in G$. For any $F \in \mathcal{L}_{k,\sigma}$, we have

$$\widehat{\pi}_k(n,h) \circ \Phi F(\nu) = \nu(n)\delta(h)^{\frac{1}{2}}\sigma(h)\Phi(F)(h^{-1} \cdot \nu).$$

On the other hand, we have

$$\begin{aligned}\Phi \circ \Pi_k(n,h)F(\nu) &= \delta(h')^{\frac{1}{2}}\sigma(h')\Pi_k(n,h)F(0,h')\\ &= \delta(h)^{\frac{1}{2}}\sigma(h')\varphi(-h^{-1}n, h^{-1}h')\\ &= \delta(h)^{\frac{1}{2}}h^{-1}h'\cdot\nu_k(h^{-1}n)\sigma(h)\Phi(F)(h^{-1}h'\cdot\nu_k)\\ &= \delta(h)^{\frac{1}{2}}\nu(n)\sigma(h)\Phi(F)(h^{-1}\cdot\nu),\end{aligned}$$

where $\nu = h'\cdot\nu_k$. Therefore we see that Φ intertwines $\widehat{\pi}_k$ and Π_k. □

Proposition 2 ([3, Proposition 6.9]). *Let G' be a closed subgroup of G. If $\{\tau_\beta\}$ is any family of unitary representations of G', then $\mathrm{Ind}_{G'}^G(\bigoplus\tau_\beta)$ is equivalent to $\bigoplus\mathrm{Ind}_{G'}^G\tau_\beta$.*

By Proposition 1 and Proposition 2, the unitary representation $\widehat{\pi}_k$ is equivalent to $\bigoplus_{\alpha\in\Lambda_K}\mathrm{Ind}_{G_{\nu_k}}^G\nu_k\otimes\rho_\alpha$. Combining Theorem 2 with the remarks following Theorem 3 and the assumption (A5), we see that $\widehat{\pi}$ is multiplicity free and $\widehat{\pi} = \bigoplus_{k\in K}\bigoplus_{\alpha\in\Lambda_K}\mathrm{Ind}_{G_{\nu_k}}^G\nu_k\otimes\rho_\alpha$. By Theorem 3, an irreducible unitary representation $\mathrm{Ind}_{G_{\nu_k}}^G\nu_k\otimes\rho_\alpha$ is square-integrable by the assumption (A4) because every irreducible unitary representation of a compact group is square-integrable. Therefore we obtain the following proposition :

Proposition 3. *Irreducible decomposition of the unitary representation $\widehat{\pi}$ into $\bigoplus_{k\in K}\bigoplus_{\alpha\in\Lambda_K}\mathrm{Ind}_{G_{\nu_k}}^G\nu_k\otimes\rho_\alpha$ is multiplicity free. Moreover, for each $k\in K$ and $\alpha\in\Lambda_K$, the induced representation $\mathrm{Ind}_{G_{\nu_k}}^G\nu_k\otimes\rho_\alpha$ is square-integrable.*

We construct the representation space of $\mathrm{Ind}_{G_{\nu_k}}^G\nu_k\otimes\rho_\alpha$. By the assumption (A5), $\sigma|_{H_{\nu_k}}$ is decomposed into $\bigoplus_{\alpha\in\Lambda_k}\rho_\alpha$ and each ρ_α is finite dimensional representation on the Hilbert space $\mathcal{H}_{\rho_\alpha}$. Therefore $\mathcal{H}_\sigma$ is a direct sum of irreducible H_{ν_k}-invariant subspaces, that is,

$$\mathcal{H}_\sigma = \bigoplus_{\alpha\in\Lambda_k}\mathcal{H}_{\rho_\alpha}. \tag{2}$$

We define the invariant subspace $\mathcal{L}_{k,\sigma,\alpha}$ of $\mathcal{L}_{k,\sigma}$ by

$$\mathcal{L}_{k,\sigma,\alpha} = \{\varphi\in\mathcal{L}_{k,\sigma}\ ;\ \varphi(n,h)\in\mathcal{H}_{\rho_\alpha},\ \text{a.a. } (n,h)\in G\}.$$

The Hilbert completion $\widetilde{\mathcal{L}}_{k,\sigma,\alpha}$ is the representation space of $\mathrm{Ind}_{G_{\nu_k}}^G\nu_k\otimes\rho_\alpha$. By (2), the space $\widetilde{\mathcal{L}}_{k,\sigma}$ is decomposed as $\bigoplus_{\alpha\in\Lambda_K}\widetilde{\mathcal{L}}_{k,\sigma,\alpha}$. Now we denote by $\mathcal{H}_{k,\sigma,\alpha}$ the subspace $\Phi(\widetilde{\mathcal{L}}_{k,\sigma,\alpha})$ of $L^2(\mathcal{O}_{\nu_k},\mathcal{H}_\sigma)$.

Lemma 1. *For any $\nu \in \mathcal{O}_{\nu_k}$, we define*

$$\mathcal{H}_{\alpha,\nu} = \sigma(h)\mathcal{H}_{\rho_\alpha},$$

where $\nu = h \cdot \nu_k$ $(h \in H)$. Then $\mathcal{H}_{\alpha,\nu}$ is well-defined. Moreover $\mathcal{H}_{k,\sigma,\alpha}$ is described as

$$\mathcal{H}_{k,\sigma,\alpha} = \{\varphi \in L^2(\mathcal{O}_{\nu_k}, \mathcal{H}_\sigma) \ ; \ \varphi(\nu) \in \mathcal{H}_{\alpha,\nu} \ a.a. \ \nu\}.$$

Proof. For any element $\varphi \in \mathcal{H}_{k,\sigma,\alpha}$ there exists $F \in \widetilde{\mathcal{L}}_{k,\sigma,\alpha}$ such that $\varphi = \Phi(F)$. Then

$$\varphi(\nu) = \Phi(F)(\nu) = \delta(h)^{\frac{1}{2}}\sigma(h)\varphi(0,h) \in \sigma(h)\mathcal{H}_{\rho_\alpha},$$

therefore we have

$$\mathcal{H}_{k,\sigma,\alpha} \subset \{F \in L^2(\mathcal{O}_{\nu_k}, \mathcal{H}_\sigma) \ ; \ \varphi(\nu) \in \mathcal{H}_{\alpha,\nu} \ a.a. \ \nu\}.$$

On the other hand, for any $\varphi \in L^2(\mathcal{O}_{\nu_k}, \mathcal{H}_\sigma)$ satisfing $\varphi(\nu) \in \mathcal{H}_{\alpha,\nu}$ *a.a.* ν, we have

$$\Phi^{-1}\varphi(n,h) = \delta(h)^{-\frac{1}{2}}h \cdot \nu_k(n)\sigma(h)\varphi(h \cdot \nu_k) \in \mathcal{H}_{\rho_\alpha}.$$

Therefore we see that $\Phi^{-1}\varphi \in \widetilde{\mathcal{L}}_{k,\sigma,\rho}$, so that $\varphi \in \mathcal{H}_{k,\sigma,\alpha}$. □

Proposition 4. *Irreducible decomposition of the space $L^2(\widehat{N}, \mathcal{H}_\sigma)$ into $\bigoplus_{k\in K}\bigoplus_{\alpha\in\Lambda_K}\mathcal{H}_{k,\sigma,\alpha}$ is multiplicity free.*

Let us construct the wavelet transforms associated to π. We choose an admissible vector $\varphi_{k,\alpha} \in \mathcal{H}_{k,\sigma,\alpha}$ such that $C_{\varphi_{k,\alpha}}$=1 for each k and α. We assume that

(A6) $\varphi = \sum_{k\in K}\sum_{\alpha\in\Lambda_K}\varphi_{k,\alpha}$ converge in $L^2(\widehat{N}, \mathcal{H}_\sigma)$.

Theorem 4. *Put $f = \mathcal{F}^{-1}\varphi \in L^2(N, \mathcal{H}_\sigma)$. We can define the map W_f from $L^2(N, \mathcal{H}_\sigma)$ to $L^2(G)$ by*

$$W_f\psi(g) = \langle\psi, \pi(g)f\rangle \quad (\psi \in L^2(\widehat{N}, \mathcal{H}_\sigma)).$$

Then W_f is isometry, and for any $\psi \in L^2(N, \mathcal{H}_\sigma)$ we have

$$\psi = \int_G W_f\psi(g)\pi(g)f d\mu_G(g)$$

in the weak sense.

Proof. For any $\psi = \mathcal{F}^{-1}\phi \in L^2(N, \mathcal{H}_\sigma)$ ($\phi \in L^2(\widehat{N}, \mathcal{H}_\sigma)$), we have

$$\int_G |W_f\psi(g)|^2 d\mu_G(g) = \int_G |\langle \psi, \pi(n,h)f\rangle|^2 d\mu_G(g) = \int_G |\langle \phi, \widehat{\pi}(n,h)\varphi\rangle|^2 d\mu_G(g).$$

By Proposition 4 and the orthogonality formula, the last term equals

$$\sum_{k\in K}\sum_{\alpha\in\Lambda_K}\int_G |\langle \phi_{k,\alpha}, \widehat{\pi}(n,h)\varphi_{k,\alpha}\rangle|^2 d\mu_G(g),$$

where $\phi = \sum_{k\in K}\sum_{\alpha\in\Lambda_K}\phi_{k,\alpha}$ ($\phi_{k,\alpha} \in \mathcal{H}_{k,\sigma,\alpha}$). Theorem 1 tell us that the expression above equals

$$\sum_{k\in K}\sum_{\alpha\in\Lambda_K} C_{\varphi_{k,\alpha}}\langle \phi_{k,\alpha}, \phi_{k,\alpha}\rangle = \langle \phi, \phi\rangle = \langle \psi, \psi\rangle$$

since $C_{\varphi_{k,\alpha}} = 1$. Therefore we have

$$\int_G |W_f\psi(g)|^2 d\mu_G(g) = \langle \psi, \psi\rangle$$

for any $\psi \in L^2(N, \mathcal{H}_\sigma)$. Hence, Theorem 4 is proved. □

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Graduate School of Mathematics
Nagoya university
Nagoya 464-8602
JAPAN