数理解析研究所講究録
第 1945 巻 2015 年 62-68

τ-tilting modules over Nakayama algebras

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1 Introduction

The notion of tilting modules plays a central role in representation theory of algebras. As an important result, there is a bijection between basic tilting modules and functorially finite faithful torsion classes for a given algebra. The bijection produces fruitful results for tilting modules. To give a class of modules corresponding bijectively functorially finite torsion classes, the author in [AIR] introduced the notion of (support) τ-tilting modules. Indeed, they showed that there is a bijection between basic support τ-tilting modules and functorially finite torsion classes. By the bijection, support τ-tilting modules hold various properties of tilting modules. Moreover, they correspond bijectively with many important objects in representation theory, *e.g.*, basic two-term silting complexes and basic cluster-tilting objects. Therefore it is important to give a classification of (support) τ-tilting modules.

In this report, we give a classification of τ-tilting modules over Nakayama algebras. The following theorem is our main result.

Theorem 1.1. *Let Λ be a Nakayama algebra with n simple modules. Assume that the length of every indecomposable projective Λ-module is at least n. Then there is a bijection between*

(1) *the set $\tau\text{-tilt}\Lambda$ of isomorphism classes of basic τ-tilting Λ-modules,*

(2) *the set $\mathcal{T}(n)$ of triangulations of an n-regular polygon with a puncture.*

Throughout this report, we use the following notation. By an algebra we mean basic, ring-indecomposable and finite dimensional algebra over an algebraically closed field K, and by a module we mean a finitely generated right module. For an algebra Λ, we denote by $\mathsf{mod}\Lambda$ the category of finitely generated right Λ-modules. We denote by $[i,j]$ the interval $\{i, i+1, \dots, j-1, j\}$ of integers $i \leq j$. Let $\{e_1, e_2, \dots, e_n\}$ be a complete set of primitive orthogonal idempotents of an algebra Λ. For each $i \in [1,n]$, we put $P_i = e_i\Lambda$ and $S_i = P_i/\mathrm{rad}P_i$.

2 Preliminaries

Let Λ be a finite dimensional K-algebra. In this section, we collect some results which are necessary in this report. We start with basic fact for representation theory of finite dimensional algebras.

(1) The category $\mathsf{mod}\Lambda$ is *Krull-Schmidt*, that is, any module M in $\mathsf{mod}\Lambda$ is isomorphic to a finite direct sum $M_1 \oplus M_2 \oplus \cdots \oplus M_m$, where $M_1, M_2, \ldots, M_m$ are indecomposable Λ-modules. Moreover, the direct sum is uniquely determined up to isomorphism and permutation. Then we let $|M| := m$. We say that M is *basic* if $M_1, \ldots, M_m$ are pairwise nonisomorphic.

(2) We say that Λ is *basic* if it is basic as a Λ-module. A basic algebra Λ is isomorphic to the bounded quiver algebra KQ/I, where Q is a finite quiver and I is an admissible ideal of the path algebra KQ. Then, for the vertex set $\{1, 2, \ldots, n\}$ in Q, we have a decomposition $\Lambda = P_1 \oplus P_2 \oplus \cdots \oplus P_n$ as a Λ-module, where P_i is an indecomposable projective Λ-module. Moreover, each indecomposable projective (respectively, simple) Λ-module is isomorphic to P_i (respectively, $S_i := P_i/\mathrm{rad}P_i$) for some $i \in \{1, 2, \ldots, n\}$.

(3) For a Λ-module M with a minimal projective presentation $P^{-1} \xrightarrow{p} P^0 \to M \to 0$, we define τM in $\mathsf{mod}\Lambda$ by an exact sequence

$$0 \to \tau M \to \nu P^{-1} \xrightarrow{\nu p} \nu P^0,$$

where $\nu := \mathrm{Hom}_K(\mathrm{Hom}_\Lambda(-, \Lambda), K)$. We call τ the AR translation of Λ. We have $\tau P = 0$ if P is a projective Λ-module.

2.1 τ-tilting modules

In this subsection, we recall the definition of τ-tilting modules.

Definition 2.1. (1) We call M in $\mathsf{mod}\Lambda$ *τ-rigid* if $\mathrm{Hom}_\Lambda(M, \tau M) = 0$.

(2) We call M in $\mathsf{mod}\Lambda$ *τ-tilting* if it is τ-rigid and $|M| = |\Lambda|$.

(3) We call M in $\mathsf{mod}\Lambda$ *support τ-tilting* if there exists an idempotent $e \in \Lambda$ such that M is a τ-tilting $(\Lambda/\Lambda e\Lambda)$-module.

We denote by τ-rigidΛ the set of isomorphism classes of indecomposable τ-rigid Λ-modules and by τ-tiltΛ the set of isomorphism classes of basic τ-tilting Λ-modules.

Example 2.2. Every projective Λ-module is a τ-rigid Λ-module by the definition of τ. In particular, Λ is a τ-tilting Λ-module.

By the following remark, τ-tilting modules are a generalization of tilting modules. A Λ-module T is said to be *tilting* if it satisfies three conditions: (a) the projective dimension is at most one, (b) rigid (*i.e.*, $\mathrm{Ext}^1_\Lambda(T, T) = 0$), and (c) $|T| = |\Lambda|$.

Remark 2.3. Every τ-rigid Λ-module is rigid and the converse holds if the projective dimension of a rigid Λ-module is at most one. Thus, if a Λ-module satisfies the condition (a) and (b) above, then it is τ-rigid. Hence tilting Λ-modules are τ-tilting Λ-modules. Moreover, if Λ is hereditary (*i.e.*, the global dimension of Λ is at most one), then τ-tilting Λ-modules are exactly tilting.

We obtain a close connection between support τ-tilting modules and other important objects in representation theory.

Theorem 2.4. ([AIR, Theorem 0.5]) *Let Λ be a finite dimensional K-algebra. Then there are bijections between*

(1) *the set of isomorphism classes of basic support τ-tilting Λ-modules,*

(2) *the set of functorially finite torsion classes in* $\mathsf{mod}\Lambda$,

(3) *the set of isomorphism classes of basic two-term silting complexes for Λ,*

(4) *the set of isomorphism classes of basic cluster-tilting objects in a 2-Calabi-Yau triangulated category $\mathcal{C}$ if Λ is an associated 2-Calabi-Yau tilted algebras.*

Hence it is an important to give a classification of (support) τ-tilting modules.

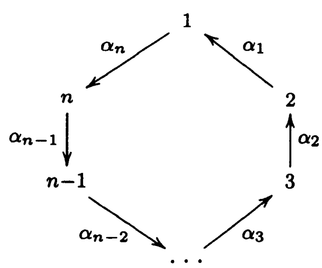
2.2 Nakayama algebras

In this subsection, we recall properties of Nakayama algebras. A module M is said to be *uniserial* if it has a unique composition series. A finite dimensional algebra is said to be *Nakayama* if every indecomposable projective module and every indecomposable injective module are uniserial. Nakayama algebras is given by the following quivers.

Proposition 2.5. ([ASS, V.3.2]) *A basic ring-indecomposable algebra is Nakayama if and only if its quiver is either $\vec{A}_n$ or $\vec{\Delta}_n$.*

$$\vec{A}_n : \quad n \xrightarrow{\alpha_{n-1}} n-1 \xrightarrow{\alpha_{n-2}} \cdots \xrightarrow{\alpha_2} 2 \xrightarrow{\alpha_1} 1$$

$\vec{\Delta}_n$:

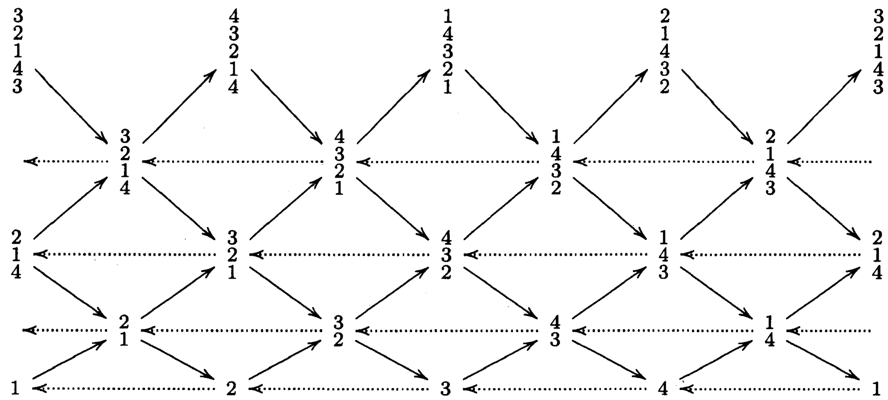


In the following, we assume that Λ is a basic ring-indecomposable Nakayama algebra with $n := |\Lambda|$. We give a concrete description of indecomposable Λ-modules. We denote by $\ell(M)$ the length of a Λ-module M.

Proposition 2.6. ([ASS, V.3.5, V.4.1 and V.4.2]) *For any indecomposable Λ-module M, there exist $i \in [1, n]$ and $l \in [1, \ell(P_i)]$ such that $M \simeq P_i/\mathrm{rad}^l P_i$ and $l = \ell(M)$. Moreover, if M is not projective, then we have $\tau M \simeq \mathrm{rad}P_i/\mathrm{rad}^{l+1}P_i$ and $\ell(\tau M) = \ell(M)$.*

By Proposition 2.6, each indecomposable Λ-module M is uniquely determined, up to isomorphism, by its simple top S_j and the length $l := \ell(M)$. In this case, M has a unique composition series with the associated composition factors $S_j, S_{j-1}, \ldots, S_{j-l+1}$.

We give an example of Nakayama algebras. We let $\Lambda_n^r := K\vec{\Delta}_n/J^r$, where J is the arrow ideal of $K\vec{\Delta}_n$. The Auslander-Reiten quiver of Λ_n^r can be drawn easily [ASS, V.4.1]. For example, the Auslander-Reiten quiver of Λ_4^5 is given by the following quiver, where the broken arrows mean the action of the AR translation τ:



2.3 Triangulations

In this subsection, we recall the definition and properties of triangulations. Let $\mathcal{G}_n$ be an n-regular polygon with a puncture. We label the points of $\mathcal{G}_n$ counterclockwise around the boundary by $1, 2, \ldots, n$.

Definition 2.7. Let $i, j \in [1, n]$.

(1) An *inner arc* $\langle i, j\rangle$ in $\mathcal{G}_n$ is a path from the point i to the point j homotopic to the boundary path $i, i+1, \ldots, i+l = j$ (mod n), where l is the smallest positive integer satisfying $i + l = j$ (mod n) and $l \geq 2$. Then we call i an *initial* point, j a *terminal* point, and $\ell(\langle i, j\rangle) := l$ the *length* of the inner arc. By definition, $2 \leq \ell(\langle i, j\rangle) \leq n$ holds for any inner arc in $\mathcal{G}_n$.

(2) A *projective arc* $\langle \bullet, j\rangle$ in $\mathcal{G}_n$ is a path from the puncture to the point j. Then we call j a *terminal* point.

(3) An *admissible arc* is an inner arc or a projective arc. Namely,

$$\mathrm{Arc}(n) := \{\text{admissible arcs in } \mathcal{G}_n\} = \{\langle i, j\rangle \mid i, j \in [1, n]\} \coprod \{\langle \bullet, j\rangle \mid j \in [1, n]\}.$$

Note that, if $i \neq j$, $\langle i, j\rangle$ and $\langle j, i\rangle$ are different arcs as the picture in Figure 1 shows.

Definition 2.8. (1) Two admissible arcs in $\mathcal{G}_n$ are called *compatible* if they do not intersect in $\mathcal{G}_n$ (except their initial and terminal points).

(2) A *triangulation* of $\mathcal{G}_n$ is a maximal set of distinct pairwise compatible admissible arcs. We denote by $\mathcal{T}(n)$ the set of triangulations of $\mathcal{G}_n$.

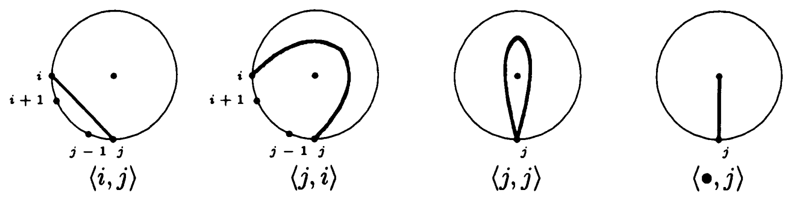


Figure 1: Admissible arcs in a polygon with a puncture

(3) For integers $l_1, l_2, \dots, l_n \geq 1$, we denote by $\mathcal{T}(n; l_1, l_2, \dots, l_n)$ the subset of $\mathcal{T}(n)$ consisting of triangulations such that the length of every inner arc with the terminal point j is at most l_j for any $j \in [1, n]$.

For example, the set of all projective arcs gives a triangulation of $\mathcal{G}_n$.

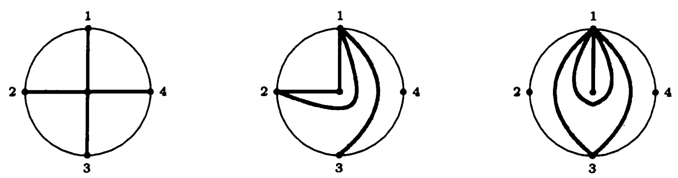


Figure 2: Triangulations of $\mathcal{G}_4$

By easy observation, triangulations have the following properties.

Proposition 2.9. *Each triangulation of $\mathcal{G}_n$ consists of exactly n admissible arcs and contains at least one projective arc.*

3 Main result

In this section, we give a proof of Theorem 1.1. First, we give a criterion for indecomposable modules to be τ-rigid.

Proposition 3.1. *Let M be an indecomposable nonprojective Λ-module. Then M is τ-rigid if and only if $\ell(M) < n$ holds.*

Proof. By Proposition 2.6, we may assume that M (respectively, τM) has a (unique) composition series with associated composition factors $\mathrm{top}M = S_j, S_{j-1}, \dots, S_{j-l+1} = \mathrm{soc}M$ (respectively, $\mathrm{top}\tau M = S_{j-1}, S_{j-2}, \dots, S_{j-l} = \mathrm{soc}\tau M$), where $l := \ell(M) = \ell(\tau M)$. If $l < n$ holds, then M (respectively, τM) does not have S_{j-l} (respectively, S_j) as a composition factor. Thus we have $\mathrm{Hom}_\Lambda(M, \tau M) = 0$, and hence M is τ-rigid. On the other hand, if $l \geq n$ holds, then M (respectively, τM) has S_{j-l} (respectively, S_j) as a composition factor. Thus there exists a non-zero morphism $M \to \tau M$ in $\mathsf{mod}\Lambda$, and hence M is not τ-rigid. □

Secondly, we give a correspondence between indecomposable τ-rigid modules and admissible arcs. By Proposition 3.1, every indecomposable nonprojective τ-rigid Λ-module M is uniquely determined by its simple top S_j and its simple socle S_k. Such an indecomposable τ-rigid module is denoted by $(k-2, j)$. Moreover, let $(\bullet, j) := P_j$.

Proposition 3.2. *Let Λ be a Nakayama algebra and $\ell_j := \ell(P_j)$. The following hold.*

(1) There is a bijection

$$f : \tau\text{-rigid}\Lambda \longrightarrow \{\langle \bullet, i\rangle \mid i \in [1,n]\} \coprod \{\langle i,j\rangle \mid i,j \in [1,n],\ \ell(\langle i,j\rangle) \leq \ell_j\}$$

given by $(i,j) \mapsto \langle i,j\rangle$ for $i \in [1,n] \coprod \{\bullet\}$ and $j \in [1,n]$.

(2) For any $i,k \in [1,n] \coprod \{\bullet\}$ and $j,l \in [1,n]$, $(i,j) \oplus (k,l)$ is τ-rigid if and only if $\langle i,j\rangle$ and $\langle k,l\rangle$ are compatible .

Proof. (1) By Proposition 3.1, every indecomposable Λ-module M is either a projective Λ-module or a Λ-module with $\ell(M) < \min\{\ell(P), n\}$, where P is a projective cover of M. Thus there are one-to-one correspondences

$$\{P_j \mid j \in [1,n]\} \longleftrightarrow \{\langle \bullet, j\rangle \mid j \in [1,n]\}$$

$$\{(i,j) \mid i,j \in [1,n],\ \ell((i,j)) < \min\{\ell_j, n\}\} \longleftrightarrow \{\langle i,j\rangle \mid i,j \in [1,n],\ \ell(\langle i,j\rangle) \leq \ell_j\}.$$

(2) Assume that $(i,j) \oplus (k,l)$ is not τ-rigid. We may assume that $\mathrm{Hom}_\Lambda((i,j), \tau(k,l)) \neq 0$ and $k \neq \{\bullet\}$. Then (i,j) (respectively, $\tau(k,l)$) has S_{k-1} (respectively, S_j) as a composition factor. Thus $\langle i,j\rangle$ and $\langle k,l\rangle$ are compatible. Conversely, we can easily check, if $\langle i,j\rangle$ and $\langle k,l\rangle$ are compatible, then $(i,j) \oplus (k,l)$ is τ-rigid. □

As a conclusion, we obtain the following theorem. This is a generalization of Theorem 1.1.

Theorem 3.3. ([Ad]) *Let Λ be a Nakayama algebra with n simple modules and $\ell_j := \ell(P_j)$ for any $j \in [1,n]$. Then there is a bijection*

$$\tau\text{-tilt}\Lambda \longrightarrow \mathcal{T}(n; \ell_1, \ell_2, \cdots, \ell_n)$$

given by $M = M_1 \oplus M_2 \oplus \cdots \oplus M_n \mapsto \{f(M_1), f(M_2), \cdots, f(M_n)\}$.

Proof. It follows from Proposition 3.2. □

As an application of Theorem 3.3, we give a proof of the following well-known result.

Corollary 3.4. *Let $\Lambda := K\vec{A}_n$ be a path algebra. Then there is a bijection between*

(1) *the set* tiltΛ *of isomorphism classes of basic tilting Λ-modules,*

(2) *the set of triangulations of an $(n+2)$-regular polygon (with no puncture).*

Proof. By Theorem 3.3 and $\ell(P_i) = i$ for any $i \in [1,n]$, we have a bijection

$$\tau\text{-tilt}\Lambda \longrightarrow \mathcal{T}(n; 1, 2, \ldots, n).$$

Since Λ is hereditary, we have τ-tiltΛ = tiltΛ by Remark 2.3. On the other hand, we show that

$$\mathcal{T}(n; 1, 2, \ldots, n) = \{X \in \mathcal{T}(n) \mid \langle \bullet, n\rangle \in X\}.$$

Indeed, assume that $X \in \mathcal{T}(n)$ with $\langle \bullet, n\rangle \in X$. Then we have $\ell(\langle i,j\rangle) \leq j$ for each inner arc $\langle i,j\rangle \in X$. Thus, we have $X \in \mathcal{T}(n; 1, 2, \ldots, n)$. Conversely, assume that $X \in \mathcal{T}(n; 1, 2, \ldots, n)$. Clearly, the projective arc $\langle \bullet, n\rangle$ is compatible with all admissible arc in X. Thus, we have $\langle \bullet, n\rangle \in X$. Note that $\mathcal{T}(n; 1, 2, \ldots, n)$ can identify the set of triangulations of an $(n+2)$-regular polygon (with no puncture). □

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