Addendum to "Fibered knots with the same 0-surgery and the slice-ribbon conjecture"

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1 Introduction

This note is a summary of our works for the slice-ribbon conjecture and Akbulut-Kirby's conjecture on knot concordance.

The slice-ribbon conjecture is one of the most important problems in knot theory given by Fox ([7]). It asks whether any slice knot is a ribbon knot. There are many studies on this conjecture. On the other hand, until recently, few direct consequences of this conjecture were known. This situation has been changed by the recent Baker's work ([3]). In this note, we explain the consequence of the slice-ribbon conjecture given in [2]. This is the report of the second author's talk at the conference "Intelligence of Low-dimensional Topology" held in RIMS in May, 2015. Throughout this note, we work in smooth category.

2 Preliminaries

In this section, we will explain terminologies used in this note.

Let K_0 and K_1 be oriented knots in S^3. Then, K_1 is *concordant* to K_0 (denoted $K_1 \sim K_0$) if there exists a properly embedded oriented annulus $A \subset S^3 \times [0,1]$ such that $\partial(S^3 \times [0,1], A) = (S^3, K_1) \sqcup (-S^3, -K_0)$, where $-S^3$ and $-K_0$ are the reverses of S^3 and K_0, respectively. It is well known that the set of concordance classes forms an abelian group under the group operation induced by connected sum. The group is called *the knot concordance group* and denoted by $\mathrm{Conc}(S^3)$. An oriented knot is *slice* if its concordance class is the unit in $\mathrm{Conc}(S^3)$, that is, the knot is concordant to the unknot.

Let $p\colon S^3 \times [0,1] \to [0,1]$ be the natural projection. Let K_0 and K_1 be oriented knots in S^3. Then, K_1 is *ribbon concordant* to K_0 (denoted $K_1 \geq K_0$) if there exists a properly embedded oriented annulus $A \subset S^3 \times [0,1]$ such that $\partial(S^3 \times [0,1], A) = (S^3, K_1) \sqcup (-S^3, -K_0)$ and the restriction map $p|_A\colon A \to [0,1]$ is a Morse function without local maxima ([11]). We call this annulus A *a ribbon concordance from* K_1 *to* K_0. An oriented knot is *ribbon* if the knot is ribbon concordant to the unknot. Obviously, the relation $\geq$ is reflexive and transitive. Gordon ([11]) conjectured that it is antisymmetric, that is, $K_1 \geq K_0$ and $K_0 \geq K_1$ imply $K_1 = K_0$. In particular, he conjectured the relation

$\geq$ is a partial ordering on the set of oriented knots in S^3. By definitions, it is clear that ribbon knots are slice. The slice-ribbon conjecture ([7]) asks whether the converse is true. On ribbon concordance, Gordon proved the following:

Theorem 2.1 ([11, Lemmas 3.1 and 3.4]). *Let K_0 and K_1 be oriented knots, and $A \subset S^3 \times [0,1]$ be a ribbon concordance from K_1 to K_0. Put $X_i := S^3 \setminus K_i$ $(i = 0, 1)$ and $Y := S^3 \times [0,1] \setminus A$. Let $\widetilde{X}_i$ be the infinite cyclic cover of X_i, and $\widetilde{Y}$ be the infinite cyclic cover of Y. Then, we obtain the following:*

- *the homomorphism $\pi_1(X_1) \to \pi_1(Y)$ induced by the inclusion is surjective,*
- *the homomorphism $\pi_1(X_0) \to \pi_1(Y)$ induced by the inclusion is injective,*
- $\dim \mathrm{H}_1(\widetilde{X}_1; \mathbf{Q}) \geq \dim \mathrm{H}_1(\widetilde{Y}; \mathbf{Q}) \geq \dim \mathrm{H}_1(\widetilde{X}_0; \mathbf{Q})$,
- *if* $\dim \mathrm{H}_1(\widetilde{X}_1; \mathbf{Q}) = \dim \mathrm{H}_1(\widetilde{X}_0; \mathbf{Q})$ *and $\pi_1(\widetilde{X}_1)$ is residually nilpotent, then $K_1 = K_0$.*

Here, a group is residually nilpotent if the intersection of all the terms of its lower central series are trivial group. For example, if K_1 is fibered knot then $\pi_1(\widetilde{X}_1)$ is residually nilpotent because it is known that the commutator subgroup of the knot group of a fibered knot is free and free groups are residually nilpotent.

Miyazaki ([15]) introduced the notion of homotopically ribbon concordance. Let K_i be an oriented knot in an integral homology 3-sphere M_i $(i = 0, 1)$. Then, K_1 is *homotopically ribbon concordant* to K_0 (denoted $K_1 \geq' K_0$) if there exist a compact oriented 4-manifold V with $\mathrm{H}_*(V) \cong \mathrm{H}_*(S^3 \times [0,1])$ and a properly embedded oriented annulus $A \subset V$ such that they satisfy the following:

- $\partial(V, A) = (M_1, K_1) \sqcup (-M_0, -K_0)$,
- the homomorphism $\pi_1(M_1 \setminus K_1) \to \pi_1(V \setminus A)$ induced by the inclusion is surjective,
- the homomorphism $\pi_1(M_0 \setminus K_0) \to \pi_1(V \setminus A)$ induced by the inclusion is injective.

By Theorem 2.1, $K_1 \geq K_0$ implies $K_1 \geq' K_0$. Moreover, the third and the forth properties in Theorem 2.1 hold for homotopically ribbon concordance because to prove Theorem 2.1 we only use homotopical properties of ribbon concordance. An oriented knot in an integral homology 3-sphere is *homotopically ribbon* if it is homotopically ribbon concordant to the unknot in S^3. Originally, the definition of homotopically ribbon knots was given by Casson and Gordon ([4]). On homotopically ribbon concordance, Miyazaki proved the following:

Theorem 2.2 (a corollary of [15, Theorem 5.5]). *Let K_i be an oriented knot in S^3 $(i = 0, 1)$. Suppose that each K_i satisfies either* (1) *or* (2) *below:*

(1) K_i is minimal with respect to $\geq'$ among all fibered knots in integral homology 3-*spheres,*

(2) there is no $f(t) \in \mathbf{Z}[t] \setminus \{\pm t^k\}_{k \geq 0}$ such that $f(t)f(t^{-1}) | \Delta_{K_i}(t)$, where $\Delta_{K_i}(t)$ is the Alexander polynomial of K_i.

Then, if $K_1 \sharp \overline{K_0} \geq' 0$, we obtain $K_1 = K_0$, where 0 *is the unknot.*

Corollary 2.3. *Let K_0 and K_1 be oriented knots in S^3 with irreducible Alexander polynomials. Then, if $K_1 \sharp \overline{K_0} \geq 0$, we obtain $K_1 = K_0$.*

Remark 2.4 (cf. [3, the proof of Theorem 3]). *Baker observed that if $K_i \subset S^3$, we may replace the condition* (1) *in Theorem 2.2 with the following condition* $(1)'$.

(1)' K_i is minimal with respect to $\geq'$ among all fibered knots in S^3.

3 Baker's work on knot concordance

In this section, we mention Baker's work on knot concordance ([3]).

A fibered knot in a 3-manifold is *tight* if it is the binding of an open book decomposition of the 3-manifold which supports a tight contact structure. Hedden gave equivalent conditions to be tight as follows.

Theorem 3.1 ([12, Proposition 2.1]). *Let K be a fibered knot in S^3. Then the following are equivalent:*

- *K is tight.*
- *K is strongly quasipositive.*
- *$c(\xi_K) = 0$, where $c(\xi_K)$ is the Ozsváth-Szabó contact invariant associated to the contact structure ξ_K coming from the fibered knot K.*
- *K satisfies $g(K) = \tau(K)$, where $\tau(K)$ is Ozsváth-Szabó's knot concordance invariant τ of K.*

Remark 3.2. *It is known that all algebraic knots are fibered and their monodromies are products of positive Dehn twists. Hence, any algebraic knot is tight fibered. Other examples of tight fibered knots are introduced in [2, Lemma 3.2].*

On tight fibered knots, Baker proved the following:

Theorem 3.3 ([3, Lemma 2]). *Let K be a tight fibered knot in S^3. Then, K is minimal with respect to homotopically ribbon concordance $\geq'$ among fibered knots in S^3.*

Proof. For the sake of completeness, we give the proof. Let J be a fibered knot in S^3. Let $\widetilde{X}_1$ and $\widetilde{X}_0$ be the infinite cyclic covers of $S^3 \setminus K$ and $S^3 \setminus J$, respectively. Assume that $K \geq' J$. Then, by Theorem 2.1, we obtain

$$2g(K) = \dim \mathrm{H}_1(\widetilde{X}_1; \mathbf{Q}) \geq \dim \mathrm{H}_1(\widetilde{X}_0; \mathbf{Q}) = 2g(J). \tag{1}$$

By Theorem 3.1 and properties of Ozsváth-Szabó's τ-invariant, we have

$$g(J) \geq \tau(J) = \tau(K) = g(K). \tag{2}$$

Hence, by (1) and (2), we obtain

$$\dim \mathrm{H}_1(\widetilde{X}_1; \mathbf{Q}) = \dim \mathrm{H}_1(\widetilde{X}_0; \mathbf{Q}).$$

Here, by the remark after Theorem 2.1, $\pi_1(\widetilde{X}_1)$ is residually nilpotent. By Theorem 2.1, we have $K = J$. This implies K is minimal with respect to $\geq'$ among all fibered knots in S^3. □

As a corollary of Baker's theorem, we obtain the following consequence of the slice-ribbon conjecture.

Corollary 3.4 (cf. [3, Corollary 4], [2, Lemma 3.1]). *Suppose that the slice-ribbon conjecture is true. Let $K_1, \dots, K_n$ be prime, mutually distinct tight fibered knots. Then, $K_1, \dots, K_n$ are linearly independent in* $\mathrm{Conc}(S^3)$.

Proof. Suppose that for some $a_1, \dots, a_m \geq 0$ and $a_{m+1}, \dots, a_n \leq 0$, we have

$$a_1K_1\sharp\dots\sharp a_nK_n := a_1K_1\sharp\dots\sharp a_mK_m\sharp\overline{(-a_{m+1})K_1\sharp\dots\sharp(-a_n)K_n} \sim 0.$$

By the slice-ribbon conjecture, we obtain

$$a_1K_1\sharp\dots\sharp a_mK_m\sharp\overline{(-a_{m+1})K_1\sharp\dots\sharp(-a_n)K_n} \geq 0.$$

It is known that the connected sum of two strongly quasipositive fibered knots is also strongly quasipositive ([8, 18]). By Theorem 3.1, the knots

$$a_1K_1\sharp\dots\sharp a_mK_m \text{ and } (-a_{m+1})K_1\sharp\dots\sharp(-a_n)K_n$$

are tight fibered. By Theorem 2.2 and Remark 2.4, we obtain

$$a_1K_1\sharp\dots\sharp a_mK_m = (-a_{m+1})K_1\sharp\dots\sharp(-a_n)K_n.$$

By the prime decomposition theorem, $a_1 = \dots = a_n = 0$. This implies $K_1, \dots, K_n$ are linearly independent in $\mathrm{Conc}(S^3)$. □

Rudolph gave a question which asks whether the set of algebraic knots are linearly independent in $\mathrm{Conc}(S^3)$ ([17]). Motivated by Rudolph's question, Baker conjectured that if two tight fibered knots are concordant then they are the same. This conjecture is equivalent to the following:

Conjecture 3.5 (cf. [3, Conjecture 1]). *Prime tight fibered knots are linearly independent in* $\mathrm{Conc}(S^3)$.

By Corollary 3.4, if the slice-ribbon conjecture is true, Conjecture 3.5 is also true.

4 Akbulut-Kirby's conjecture

In this section, we show our main theorem given in [2]. In particular, we prove that if the slice-ribbon conjecture is true, modified Akbulut-Kirby's conjecture (Conjecture 4.1) is false.

Conjecture 4.1 ([13, Problem 1.19]). *If* 0*-surgeries on two unoriented knots give the same* 3*-manifold, then the knots with relevant orientations are concordant.*

Remark 4.2. *In the original statement of Akbulut-Kirby's conjecture is the following: If* 0*-framed surgeries on two knots give the same* 3*-manifold, then the knots are concordant. Livingston ([14]) showed that there exists an oriented knot K such that it is not concordant to its reverse. Hence, we need to modify the claim of the original Akbulut-Kirby's conjecture as Conjecture 4.1.*

We consider the following conjecture instead of Conjecture 4.1.

Conjecture 4.3. *If 0-surgeries on two unoriented knots K_0 and K_1 give the same 3-manifold, then $K_0 \sharp \overline{K_1}$ is ribbon by giving relevant orientations.*

Since $K_0 \sharp \overline{K_1}$ is slice if and only if K_0 and K_1 are concordant, Conjecture 4.3 implies Conjecture 4.1. Moreover, if the slice-ribbon conjecture is true, Conjecture 4.1 and Conjecture 4.3 are equivalent. By the following theorem, we see that Conjecture 4.3 is false.

Theorem 4.4 (cf. [2, Theorem 1.6]). *Conjecture 4.3 is false.*

Proof. Let K_0 and K_1 be knots satisfying the following conditions:

- $K_0 \neq K_1$,
- K_0 and K_1 have the same 0-surgery,
- K_0 and K_1 are fibered, and
- K_0 and K_1 have irreducible Alexander polynomials.

For example, the knots depicted in Figure 1 satisfy these conditions. By Corollary 2.3,

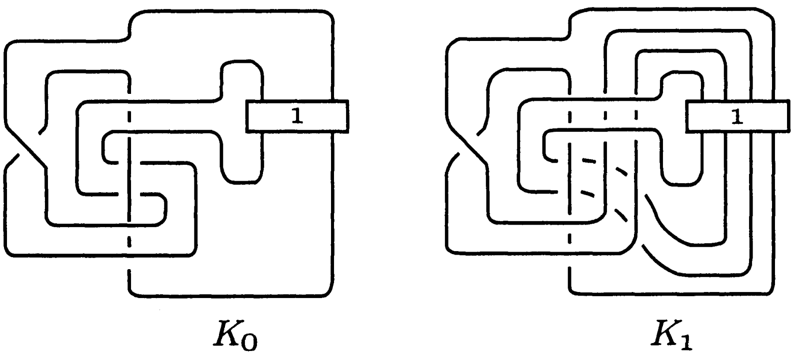


Figure 1: The definitions of K_0 and K_1. Each rectangle labeled 1 implies a full-twist.

$K_0 \sharp \overline{K_1}$ is not ribbon for any orientations of K_0 and K_1. Hence, the pair (K_0, K_1) is a counterexample of Conjecture 4.3. □

As a corollary of this result, we obtain the following:

Corollary 4.5 ([2, Theorem 1.6]). *If the slice-ribbon conjecture is true, Conjecture 4.1 is false.*

Proof. If the slice-ribbon conjecture is true, Conjecture 4.1 and Conjecture 4.3 are equivalent. By Theorem 4.4, Conjecture 4.1 is false. □

Remark 4.6. *Recently, Kouichi Yasui ([20]) proved that there are infinitely many counterexamples of Conjecture 4.1.*

5 Construction of counterexamples

In this section, we give a method to find pairs of knots satisfying the conditions in the proof of Theorem 4.4. First, we recall Osoinach's annular twisting techniques ([16]).

5.1 Annulus twists and annulus presentations

Let $A \subset S^3$ be an embedded annulus and $\partial A = c_1 \cup c_2$. Note that A may be knotted and twisted. In Figure 2, we draw an unknotted and twisted annulus. An *n-fold annulus twist along A* is to apply $(+1/n)$-surgery along c_1 and $(-1/n)$-surgery along c_2 with respect to the framing determined by the annulus A. For simplicity, we call a 1-fold annulus twist along A an *annulus twist along A*.

Remark 5.1. *An n-fold annulus twist does not change the ambient 3-manifold S^3 (see [16, Theorem 2.1]).*

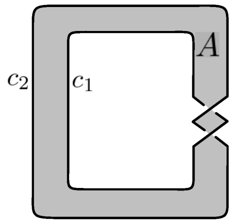


Figure 2: An unknotted annulus $A \subset S^3$ with a $+1$ full-twist.

Abe, Jong, Omae and Takeuchi ([1]) introduced the notion of an annulus presentation of a knot (in their paper it is called "band presentation"). Here, we extend the definition of annulus presentations of knots.

Let $A \subset S^3$ be an embedded annulus with $\partial A = c_1 \cup c_2$, which may be knotted and twisted. Take an embedding of a band $b\colon I \times I \to S^3$ such that

- $b(I \times I) \cap \partial A = b(\partial I \times I)$,
- $b(I \times I) \cap \operatorname{Int} A$ consists of ribbon singularities, and
- $A \cup b(I \times I)$ is an immersion of an orientable surface,

where $I = [0, 1]$. If a knot K is isotopic to the knot $(\partial A \setminus b(\partial I \times I)) \cup b(I \times \partial I)$, then we say that K admits an *annulus presentation* (A, b).

Example 5.2. *The knot* 6_3 *(with an arbitrary orientation) admits an annulus presentation (A, b), see Figure 3.*

Let K be a knot admitting an annulus presentation (A, b). Then, by $A^n(K)$, we denote the knot obtained from K by n-fold annulus twist along $\widetilde{A}$ with $\partial\widetilde{A} = \widetilde{c}_1 \cup \widetilde{c}_2$, where $\widetilde{A} \subset A$ is a shrunken annulus. Namely, $\overline{A \setminus \widetilde{A}}$ is a disjoint union of two annuli, each $\widetilde{c}_i$ is isotopic to c_i in $\overline{A \setminus \widetilde{A}}$ for $i = 1, 2$ and $A \setminus (\partial A \cup \widetilde{A})$ does not intersect $b(I \times I)$. For simplicity, we denote $A^1(K)$ by $A(K)$.

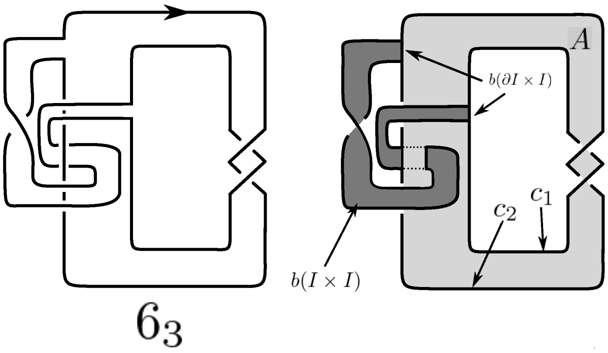


Figure 3: The definitions of the knot 6_3 (left) and its annulus presentation (right).

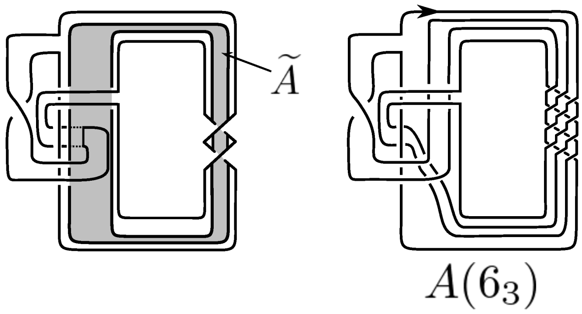


Figure 4: A shrunken annulus $\tilde{A}$ for the annulus presentation of 6_3 (left) and the knot $A(6_3)$ (right).

Example 5.3. *We consider the knot* 6_3 *with the annulus presentation* (A, b) *in Figure 3. Then* $A(6_3)$ *is the right picture in Figure 4.*

Then, Osoinach proved that the 0-surgery on $A^n(K)$ is diffeomorphic to that of K (though he did not use the notion of an annulus presentation).

Lemma 5.4 ([16]). *Let* K *be a knot admitting an annulus presentation* (A, b). *Then, the* 3*-manifold obtained by* 0*-surgery on* $A^n(6_3)$ *does not depend on* $n \in \mathbf{Z}$.

5.2 Construction

In this subsection, we construct counterexamples of Conjecture 4.3. Let K be a knot satisfying the following:

- K is fibered,
- K has an irreducible Alexander polynomial, and
- K admits an annulus presentation (A, b).

For example, 6_3 depicted in Figure 3 satisfies the conditions. Then, by Lemma 5.4, the 0-surgeries on $A^n(K)$ and K are the same. Hence, by Gabai's result ([9]), $A^n(K)$ is also fibered. It is known that the Alexander module of a knot is isomorphic to the first homology of the infinite cyclic cover of the 0-surgery on the knot as $\mathbf{Z}[t, t^{-1}]$-modules. Hence, the Alexander polynomial of $A^n(K)$ is equal to that of K, and it is irreducible. As a result,

- $A^n(K)$ and K have the same 0-surgery,
- $A^n(K)$ is fibered, and
- $A^n(K)$ has an irreducible Alexander polynomial.

By the proof of Theorem 4.4, if $A^n(K) \neq A^m(K)$, then $(A^n(K), A^m(K))$ is a counterexample of Conjecture 4.3.

Corollary 5.5. *Let K be as above. Then, if $A^n(K) \neq A^m(K)$, the pair $(A^n(K), A^m(K))$ is a counterexample of Conjecture 4.3.*

5.3 Infinitely many counterexamples

In this subsection, we construct infinitely many counterexamples of Conjecture 4.3. By Corollary 5.5 and Lemma 5.6 below, we obtain infinitely many counterexamples of Conjecture 4.3.

Lemma 5.6 (cf. [2, Remark 5.11]). *The knots $A^n(6_3)$ and $A^m(6_3)$ are ambient isotopic as unoriented knots if and only if $n = m$ or $n + m = -1$.*

Proof. Suppose that $n + m = -1$. Then, by Figure 5, we see that $A^n(6_3)$ and $A^m(6_3)$ are ambient isotopic as unoriented knots.

Conversely, suppose that $A^n(6_3)$ and $A^m(6_3)$ are ambient isotopic. Orient $A^n(6_3)$ arbitrarily and give $A^m(6_3)$ the corresponding orientation. Recall that $A^n(6_3)$ and $A^m(6_3)$ are fibered. Let $f_i\colon F \to F$ be the monodromy of $A^i(6_3)$ for $i = n, m$. Then, (F, f_i) gives an open book decomposition of S^3. Let ξ_i be a contact structure supported by the open book decomposition (F, f_i). By the assumption, we see that f_n and f_m are conjugate. In particular, ξ_n and ξ_m are isotopic. Let d_3 be the homotopy invariant of plane fields given by Gompf ([10]). Then, by Section 6, we obtain

$$d_3(\xi_n) = -n^2 - n + \frac{3}{2},$$

for $n \in \mathbf{Z}$. Note that the result of our computation is independent of the choice of the orientation of $A^n(6_3)$. Now, $d_3(\xi_n) = d_3(\xi_m)$ since ξ_n and ξ_m are isotopic. Hence, we obtain $n = m$ or $n + m = -1$. □

6 Computation of $d_3(\xi_n)$

In this section, we compute $d_3(\xi_n)$ for the contact structure ξ_n given in the proof of Lemma 5.6. In Section 6.1, we recall the definition of monodromies. In Section 6.2, we introduce the notion of annulus presentations compatible with fiber surfaces. In Section 6.3, we give the monodromy f_n of $A^n(6_3)$, and compute $d_3(\xi_n)$.

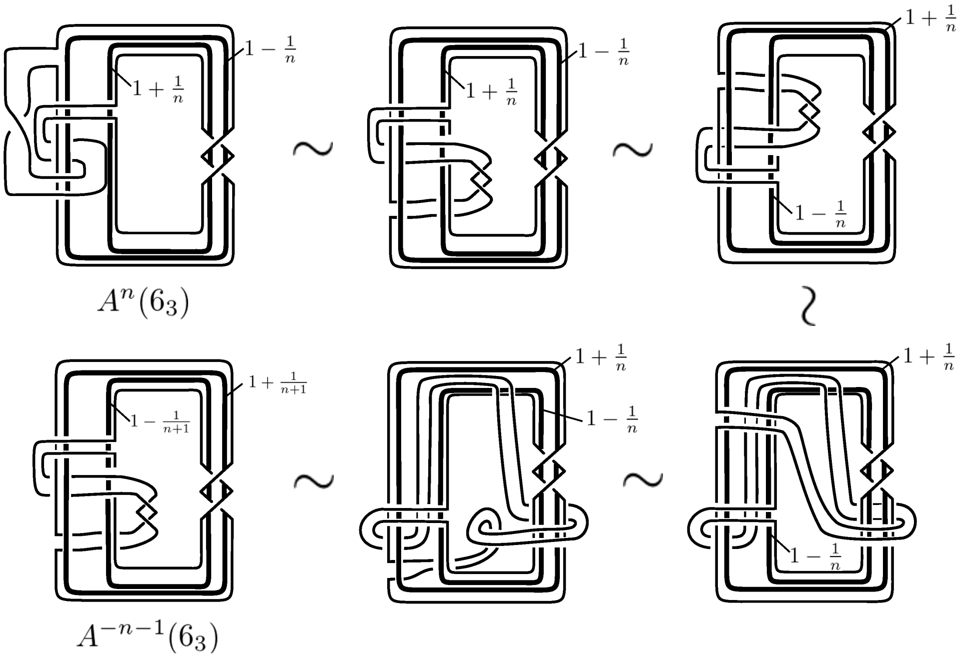


Figure 5: $A^n(6_3)$ is ambient isotopic to $A^{-n-1}(6_3)$.

6.1 Open book decompositions

Let F be an oriented surface with boundary and $f\colon F \to F$ a diffeomorphism on F fixing the boundary. Consider the pinched mapping torus

$$\widehat{M_f} = F \times [0,1]/_{\sim},$$

where the equivalent relation $\sim$ is defined as follows: $(x,1) \sim (f(x),0)$ for $x \in F$, and $(x,t) \sim (x,t')$ for $x \in \partial F$ and t, $t' \in [0,1]$. Here, we orient $[0,1]$ from 0 to 1 and we give an orientation of $\widehat{M_f}$ by the orientations of F and $[0,1]$. Let M be a closed oriented 3-manifold. If there exists an orientation-preserving diffeomorphism from $\widehat{M_f}$ to M, the pair (F,f) is called an *open book decomposition* of M. The map f is called the *monodromy* of (F,f). Note that we can regard F as a surface in M. The boundary of F in M, denoted by L, is called a *fibered link* in M, and F is called a *fiber surface* of L. The *monodromy of* L is defined by the monodromy f of the open book decomposition (F,f). Let M be a closed oriented 3-manifold, and (F,f) an open book decomposition of M. Let C be a simple closed curve on a fiber surface $F \subset M$. Then, *a twisting along* C *of order* n is defined as performing $(1/n)$-surgery along C with respect to the framing determined by F. Then we obtain the following.

Lemma 6.1 (Stallings). *The resulting manifold obtained from* M *by a twisting along* C *of order* n *is (orientation-preservingly) diffeomorphic to* $\widehat{M}_{t_C^{-n}\circ f}$*, where* t_C *is the right-handed Dehn twist along* C*.*

6.2 Compatible annulus presentations

Let $K \subset S^3$ be a fibered knot admitting an annulus presentation (A, b), and F a fiber surface of K. We say that (A, b) is *compatible with* F if there exist simple closed curves c_1' and c_2' on F such that

- $\partial\widetilde{A} = \widetilde{c}_1 \cup \widetilde{c}_2$ is isotopic to $c_1' \cup c_2'$ in $S^3 \setminus K$, where $\widetilde{A} \subset A$ is a shrunken annulus defined in Section 5.1, and
- each annular neighborhood of c_i' in F $(i = 1, 2)$ is isotopic to A in S^3.

Let $\widetilde{c}_1 \cup \widetilde{c}_2$ be the framed link with framing $(1/n, -1/n)$ with respect to the framing determined by the annulus A, and $c_1' \cup c_2'$ the framed link with framing $(1/n, -1/n)$ with respect to the framing determined by the fiber surface F. Then, by the first compatible condition, $\widetilde{c}_1 \cup \widetilde{c}_2$ is equal to $c_1' \cup c_2'$ as links in $S^3 \setminus K$. Moreover, by the second compatible condition, their framings coincide. As a result, $\widetilde{c}_1 \cup \widetilde{c}_2$ is equal to $c_1' \cup c_2'$ as framed links in $S^3 \setminus K$. Hence, if K is a fibered knot with (A, b) which is compatible with the fiber surface F, then $A^n(K)$ is the knot obtained from K by twisting along c_1' and c_2' of order $+n$ and $-n$, respectively. In particular, by Lemma 6.1, $A^n(K)$ is a fibered knot and the monodromy of $A^n(K)$ is $t_{c_1'}^{-n} \circ t_{c_2'}^{n} \circ f$, where f is the monodromy of K. As a summary, we obtain the following.

Lemma 6.2. *Let $K \subset S^3$ be a fibered knot admitting a compatible annulus presentation (A, b). Then $A^n(K)$ is also fibered for any $n \in \mathbf{Z}$. Moreover, the monodromy of $A^n(K)$ is $t_{c_1'}^{-n} \circ t_{c_2'}^{n} \circ f$, where f is the monodromy of K, and c_1' and c_2' are simple closed curves which give the compatibility of (A, b).*

Remark 6.3. *Let K be a fibered knot admitting an annulus presentation (A, b) (which may not be compatible with the fiber surface for K). Then, by Lemma 5.4 and Gabai's work ([9]), $A^n(K)$ is also fibered.*

Example 6.4. *We consider the knot 6_3 with the annulus presentation (A, b) in Figure 3. It is known that 6_3 is fibered. We choose a fiber surface as in the left picture in Figure 6, and denote it by F. In this case, the annulus presentation (A, b) is compatible with F. Indeed we define simple closed curves c_1' and c_2' on F by $\widetilde{c}_1$ and $\widetilde{c}_2$, where $\partial\widetilde{A} = \widetilde{c}_1 \cup \widetilde{c}_2$. Then $c_1' \cup c_2'$ clearly satisfies the compatible conditions.*

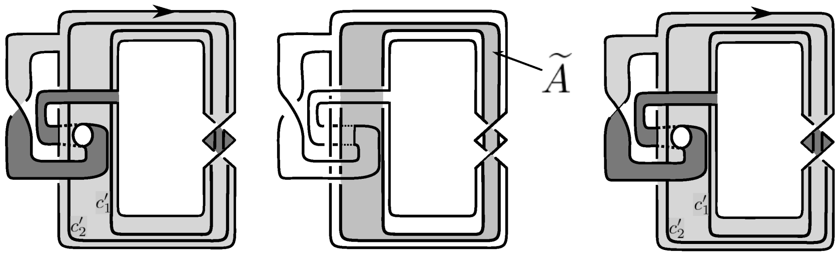


Figure 6: A fiber surface F of 6_3 (left) and a shrunken annulus $\widetilde{A}$ (center). The annulus presentation (A, b) of 6_3 is compatible with the fiber surface F (right).

6.3 The monodromy of $A^n(6_3)$

First, we describe the monodromy of 6_3. Orient 6_3 as in Figure 3. We draw a fiber surface of 6_3 as a plumbing of some Hopf bands (see Figure 7). By Figures 7 and 9, the monodromy of 6_3 is given by $t_d^{-1} \circ t_b \circ t_c^{-1} \circ t_a$ on $\Sigma_{2,1}$, where $\Sigma_{2,1}$ is the oriented surface depicted in Figure 9.

Now we describe the monodromy of $A^n(6_3)$. Suppose that $A^n(6_3)$ has the orientation derived from the orientation of 6_3. By Figures 8, 9, and Lemma 6.2, the monodromy f_n of $A^n(6_3)$ is given by $t_{c_1'}^{-n} \circ t_{c_2'}^{n} \circ t_d^{-1} \circ t_b \circ t_c^{-1} \circ t_a$ on $\Sigma_{2,1}$. If we give $A^n(6_3)$ the opposite orientation, the monodromy is given by $t_a \circ t_c^{-1} \circ t_b \circ t_d^{-1} \circ t_{c_2'}^{n} \circ t_{c_1'}^{-n}$ on $-\Sigma_{2,1}$, where $-\Sigma_{2,1}$ is the reverse of $\Sigma_{2,1}$. Then we obtain the following.

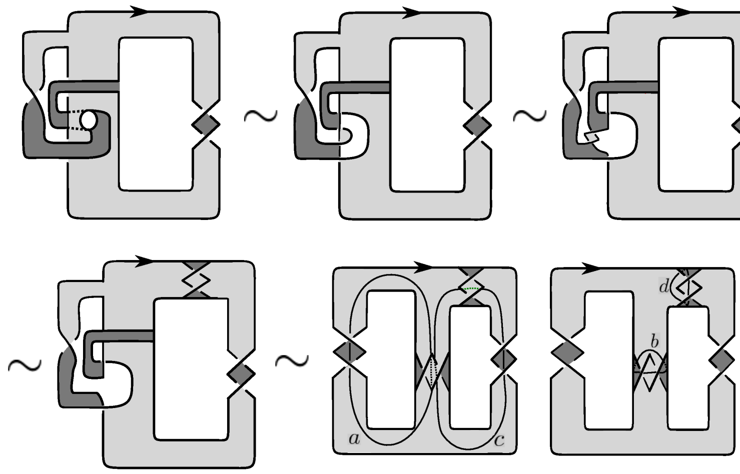


Figure 7: The bottom right pictures are fiber surfaces of 6_3 given by a plumbing of some Hopf bands. The loops a, b, c and d are core lines of these Hopf bands.

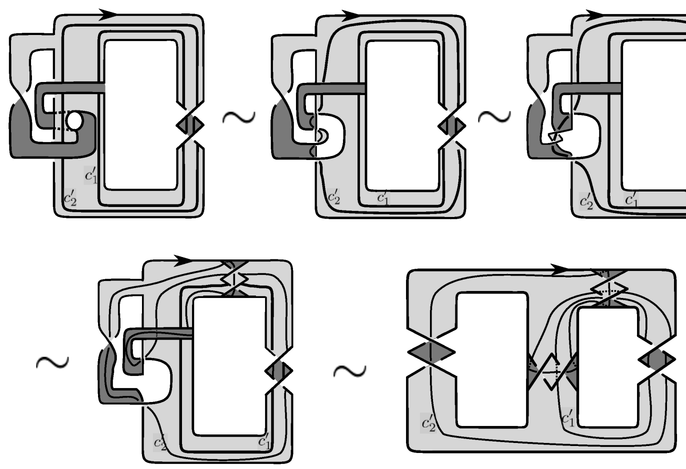


Figure 8: The simple closed curves c_1' and c_2' on the fiber surface of 6_3.

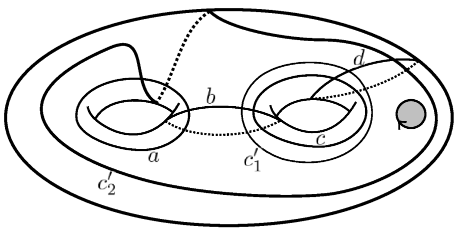


Figure 9: The monodromy f_n of $A^n(6_3)$ is $t_{c_2'}^n \circ t_{c_1'}^{-n} \circ t_d^{-1} \circ t_b \circ t_c^{-1} \circ t_a$ on $\Sigma_{2,1}$.

Lemma 6.5 (cf. [2, Remark 5.11]). *Let ξ_n be the contact structure on S^3 supported by the open book decomposition (F, f_n). Let d_3 is the invariant of plane fields given by Gompf [10]. Then, we obtain*

$$d_3(\xi_n) = -n^2 - n + \frac{3}{2}.$$

Moreover, even if we give $A^n(6_3)$ the opposite orientation, the value of d_3 does not change.

Proof. In order to compute $d_3(\xi_n)$, we use the formula for d_3 introduced in [5, 6]. By the above discussions, $f_n = t_{c_1'}^{-n} \circ t_{c_2'}^n \circ t_d^{-1} \circ t_b \circ t_c^{-1} \circ t_a$. This is conjugate to $f_n' = t_{c_1'}^{-n} \circ t_c^{-1} \circ t_{c_2'}^n \circ t_d^{-1} \circ t_b \circ t_a$.

First, we suppose that $n \geq 1$. Let X_n be the 4-manifold defined by the following: First, deform $\Sigma_{2,1}$ as in Figure 10 by using isotopies. Note that Figure 10 gives a handle decomposition of $\Sigma_{2,1}$. Second, from the handle decomposition, we draw the trivial D^2-bundle over $\Sigma_{2,1}$ as the union of a 0-handle and 4 1-handles as in Figure 11. Finally, attach 2-handles along the curves a, b, c, d, and n copies of c_1' and c_2' appearing in the factorization of f_n' as the top picture in Figure 12. Here, we denote the parallel copies of c_1' and c_2' by $c_1'^{(1)}, \dots, c_1'^{(n)}$ and $c_2'^{(1)}, \dots, c_2'^{(n)}$, respectively. Each framing is -1 if the factor is the right-handed Dehn twist, and $+1$ if the factor is the left-handed Dehn-twist, where we consider the framings with respect to the framing determined by $\Sigma_{2,1}$. Moreover, the under/over informations are given by the order: $c_1' > c > c_2' > d > b > a$.

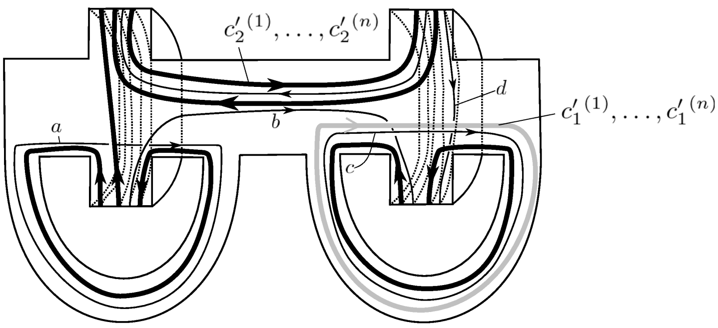


Figure 10: A handle decomposition of $\Sigma_{2,1}$.

By Kirby calculus, X_n is represented as the union of one 0-handle and $2n$ 2-handles as in Figure 12. For $i = 1, \dots, n$, put $e_i := c_1'^{(i-1)} - c_1'^{(i)}$ and for $j = 1, \dots, n-1$, put

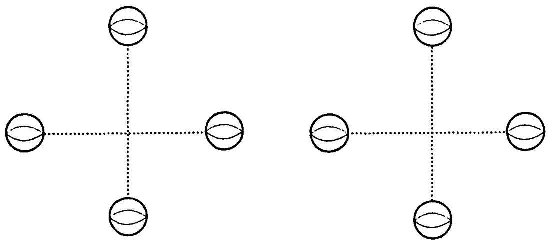


Figure 11: The trivial D^2-bundle over $\Sigma_{2,1}$.

$g_j := c_2'^{(j)} - c_2'^{(j+1)}$. Here, we put $c_1'^{(0)} = c$. Moreover, put $h := c_2'^{(1)} + a - b + d + c_1'^{(n)}$. Then, we can regard $\{e_1, \dots, e_n, h, g_1, \dots, g_{n-1}\}$ as a basis of $\mathrm{H}_2(X_n)$. The intersection form Q_{X_n} of X_n is represented by the following $2n \times 2n$-matrix with respect to the basis:

$$Q_{X_n} = \begin{pmatrix} 2 & -1 & & & & & & & \\ -1 & 2 & \ddots & & & & & & \\ & \ddots & \ddots & -1 & & & & & \\ & & -1 & 2 & -1 & & & & \\ & & & -1 & 0 & -1 & & & \\ & & & & -1 & -2 & 1 & & \\ & & & & & 1 & -2 & \ddots & \\ & & & & & & \ddots & \ddots & 1 \\ & & & & & & & 1 & -2 \end{pmatrix},$$

that is, the (i,j)-entry $(Q_{X_n})_{i,j}$ is given by

$$(Q_{X_n})_{i,j} = \begin{cases} 2 & \text{if } (i,j) = (k,k) \text{ for some } k \in \{1, \dots, n\}, \\ -1 & \text{if } (i,j) = (k,k+1), (k+1,k) \text{ for some } k \in \{1, \dots, n+1\}, \\ 0 & \text{if } (i,j) = (n+1, n+1), \\ -2 & \text{if } (i,j) = (k,k) \text{ for some } k \in \{n+2, \dots, 2n\}, \\ 1 & \text{if } (i,j) = (k,k+1), (k+1,k) \text{ for some } k \in \{n+2, \dots, 2n-1\}, \\ 0 & \text{otherwise.} \end{cases}$$

Note that

$$Q_{X_1} = \begin{pmatrix} 2 & -1 \\ -1 & 0 \end{pmatrix}.$$

Then, by the formula for d_3 given in [5, 6], for any $n \geq 1$, we obtain

$$d_3(\xi_n) = \frac{1}{4}(c^2(X_n) - 3\sigma(X_n) - 2\chi(X_n)) + q,$$

where $\sigma(X_n)$ is the signature of X_n, $\chi(X_n)$ is the Euler number of X_n and q is the number of the left-handed Dehn twists appearing in the factors of f_n'. Moreover, $c^2(X_n)$ is given

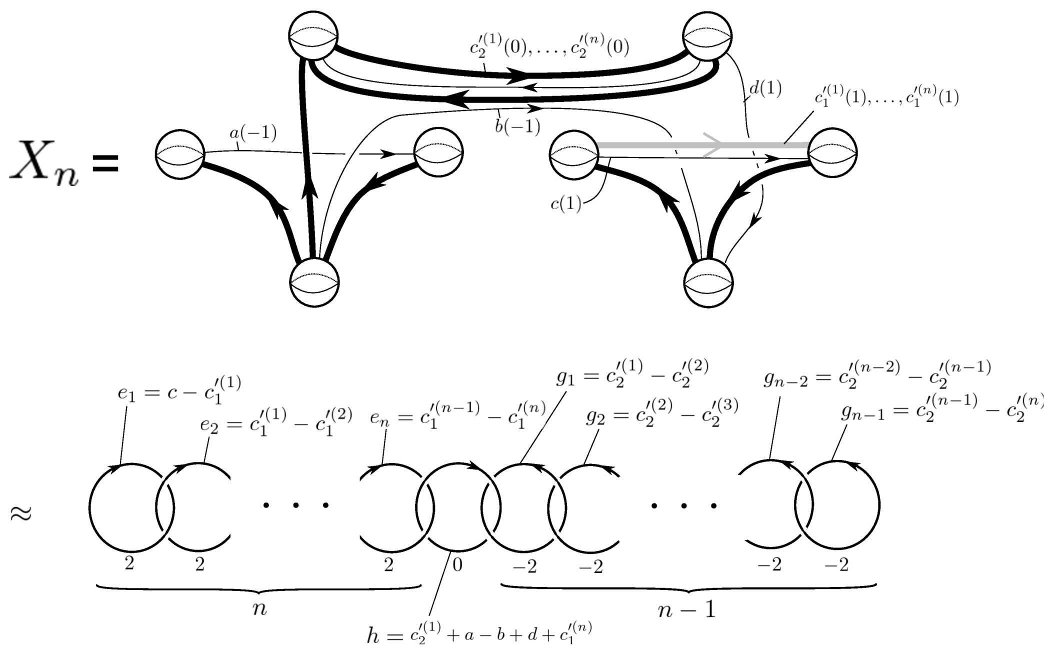


Figure 12: A Kirby diagram of X_n. The bold arc represents the parallel copies $c_2'^{(1)}, \dots, c_2'^{(n)}$ of c_2' and the gray arc represents the parallel copies $c_1'^{(1)}, \dots, c_1'^{(n)}$ of c_1'. In the top picture, the number in "()" on the right of each curve represents the framing. The bottom picture is obtained from the top by Kirby calculus as in the pictures depicted in the end of this manuscript. The bottom picture is a framed link with $2n$ components and the numbers $2, 2, \dots, 2, 0, -2, -2, \dots, -2, -2$ represent the framing.

by the following:

$$c^2(X_n) = (\mathrm{rot}(e_1), \dots, \mathrm{rot}(e_n), \mathrm{rot}(h), \mathrm{rot}(g_1), \dots, \mathrm{rot}(g_{n-1}))Q_{X_n}^{-1} \begin{pmatrix} \mathrm{rot}(e_1) \\ \vdots \\ \mathrm{rot}(e_n) \\ \mathrm{rot}(h) \\ \mathrm{rot}(g_1) \\ \vdots \\ \mathrm{rot}(g_{n-1}) \end{pmatrix},$$

where for a simple closed curve γ in $\Sigma_{2,1}$, we define $\mathrm{rot}(\gamma)$ as the winding number of γ. Here, we fix the trivialization of the tangent bundle of $\Sigma_{2,1}$ derived from Figure 11 (for detail, see [6, Section 3.1]). Moreover, for some simple closed curves $\gamma_1, \dots, \gamma_m$ and $\varepsilon_1, \dots, \varepsilon_m \in \mathbf{Z}$, we define $\mathrm{rot}(\varepsilon_1\gamma_1 + \cdots + \varepsilon_m\gamma_m) := \varepsilon_1 \mathrm{rot}(\gamma_1) + \cdots + \varepsilon_m \mathrm{rot}(\gamma_m)$. Obviously, $\chi(X_n) = 1 + 2n$ and $q = n + 2$. By Lemma 6.6 below, $c^2(X_n) = -4n(n+1)$, $\sigma(X_n) = 0$. Hence, we have

$$d_3(\xi_n) = \frac{1}{4}(-4n(n+1) - 2(1+2n)) + n + 2 = -n^2 - n + \frac{3}{2}.$$

In the case $n < 1$, we can compute $d_3(\xi_n)$ similarly. By the similar discussion, the second claim also holds. □

Lemma 6.6. *We obtain $c^2(X_n) = -4n(n+1)$ and $\sigma(X_n) = 0$.*

Proof. Note that, in our orientations, $\mathrm{rot}(c_2') = 1$, $\mathrm{rot}(b) = -1$, $\mathrm{rot}(a) = \mathrm{rot}(c) = \mathrm{rot}(d) = \mathrm{rot}(c_1') = 0$. Hence, $\mathrm{rot}(e_i) = \mathrm{rot}(g_j) = 0$ for any i, j, and $\mathrm{rot}(h) = 2$. Let f_{n+1} be the $2n$-dimensional vector whose entries are 0 except for the $n+1$-st entry where it is 2. By the definition, we have

$$\begin{aligned} c^2(X_n) &= {}^t f_{n+1} Q_{X_n}^{-1} f_{n+1} \\ &= 2 \times \frac{(\widetilde{Q}_{X_n})_{n+1,n+1}}{\det(Q_{X_n})} \times 2, \end{aligned}$$

where $(\widetilde{Q}_{X_n})_{n+1,n+1}$ is the $(n+1, n+1)$-cofactor of Q_{X_n}. By the cofactor expansion along the $(n+1)$-th row, we obtain

$$\begin{aligned} \det(Q_{X_n}) &= (-1)\det(A_{n-1})\det(-A_{n-1}) - \det(A_n)\det(-A_{n-2}) \\ &= (-1)^n n^2 - (-1)^n (n+1)(n-1) \\ &= (-1)^n, \end{aligned}$$

where A_n is the following $n \times n$-matrix, and its determinant $\det(A_n)$ is $n+1$:

$$A_n = \begin{pmatrix} 2 & -1 & & \\ -1 & 2 & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{pmatrix}.$$

Moreover, by the definition,

$$(\widetilde{Q}_{X_n})_{n+1,n+1} = \det(A_n)\det(-A_{n-1}) = (n+1)(-1)^{n-1} n.$$

Hence, $c^2(X_n) = 4 \times (n+1)(-1)^{n-1} n / (-1)^n = -4n(n+1)$.

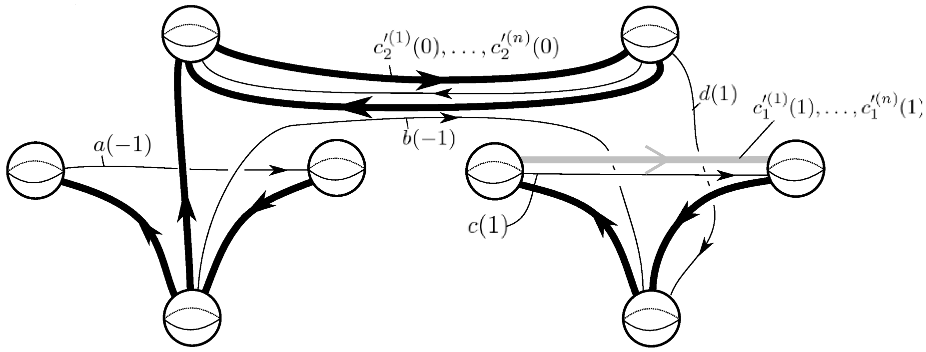
Next, we compute $\sigma(X_n)$. Let P_i be the $2n \times 2n$-matrix whose entries are 0 except for the $(i, i+1)$-entry where it is 1. Let E_{2n} be the $2n \times 2n$-unit matrix. Then, define the matrix $P_i(l)$ by $E_{2n} + lP_i$ for any $l \in \mathbf{R}$ and any $i = 1, \ldots, 2n-1$. For any $n > 1$, we define the matrix $P^{(2n)}$ by

$$P^{(2n)} := P_1(\frac{1}{2}) \cdots P_{n-1}(\frac{n-1}{n}) P_n(\frac{n}{n+1}) P_{n+1}(\frac{-(n+1)}{n}) P_{n+2}(\frac{n}{n-1}) \cdots P_{2n-1}(\frac{3}{2}).$$

If $n = 1$, we define $P^{(2)} := P_1(\frac{1}{2})$. Then,

$${}^t P^{(2n)} Q_{X_n} P^{(2n)} = \begin{pmatrix} 2 & & & & & & \\ & \ddots & & & & & \\ & & \frac{n+1}{n} & & & & \\ & & & \frac{-n}{n+1} & & & \\ & & & & \frac{-(n-1)}{n} & & \\ & & & & & \ddots & \\ & & & & & & \frac{-1}{2} \end{pmatrix}.$$

Hence, we obtain $\sigma(X_{f_n'}) = \sigma(Q_{X_n}) = \sigma({}^t P^{(2n)} Q_{X_n} P^{(2n)}) = 0$. □



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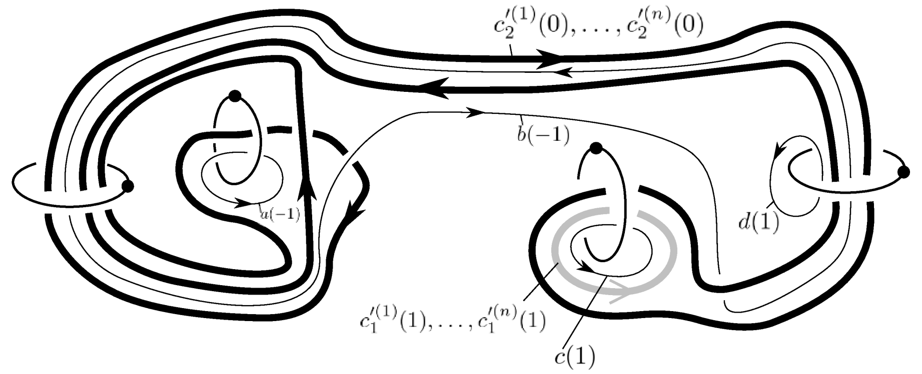

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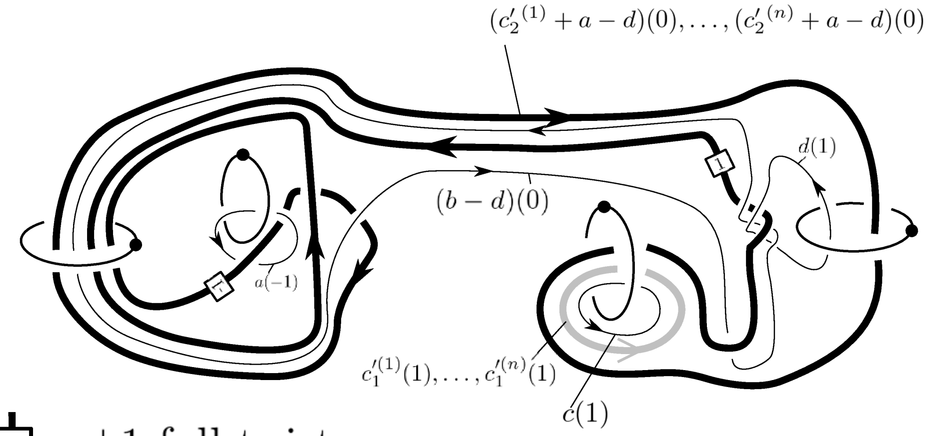

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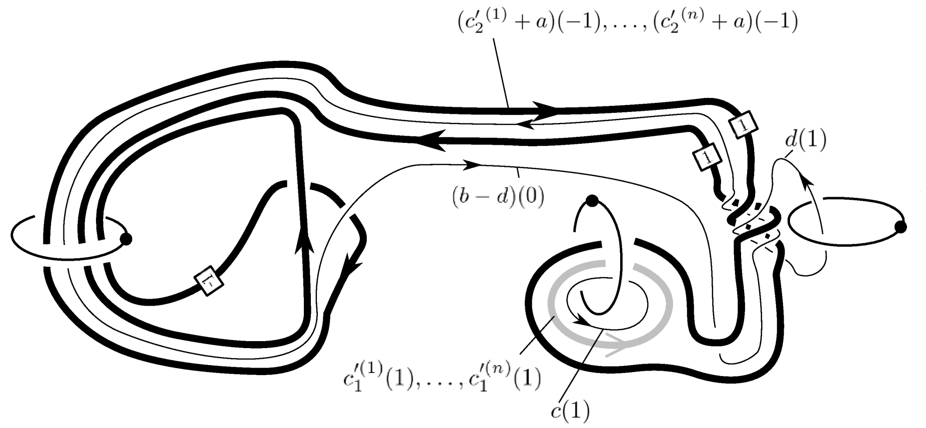
$(c_2'^{(1)}+a)(-1),\ldots,(c_2'^{(n)}+a)(-1)$
$d(1)$
$(b-d)(0)$
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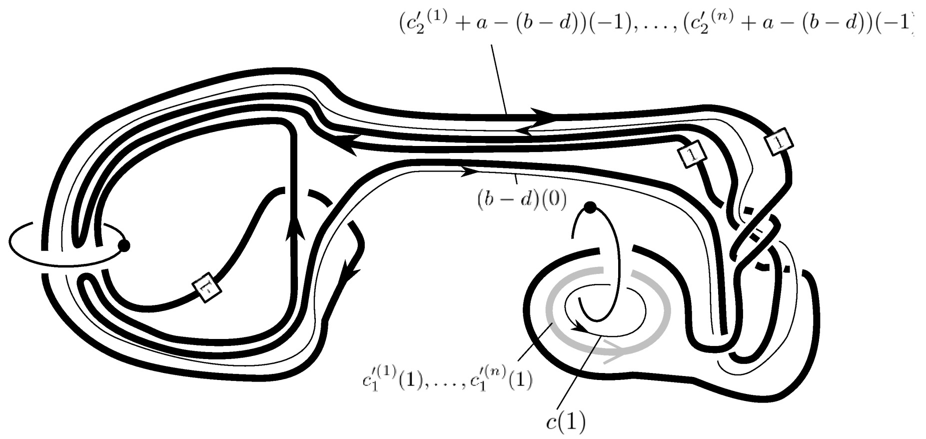
$(c_2'^{(1)}+a-(b-d))(-1),\ldots,(c_2'^{(n)}+a-(b-d))(-1$
$(b-d)(0)$
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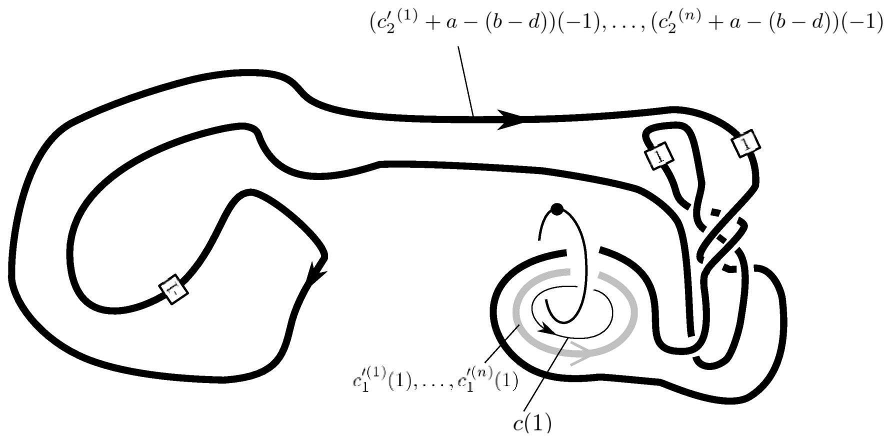
$(c_2'^{(1)}+a-(b-d))(-1),\ldots,(c_2'^{(n)}+a-(b-d))(-1)$
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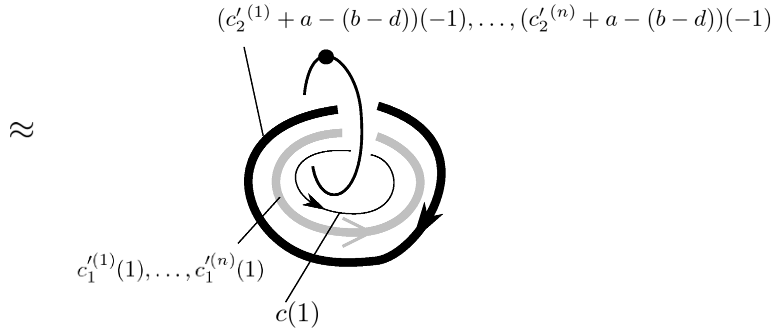
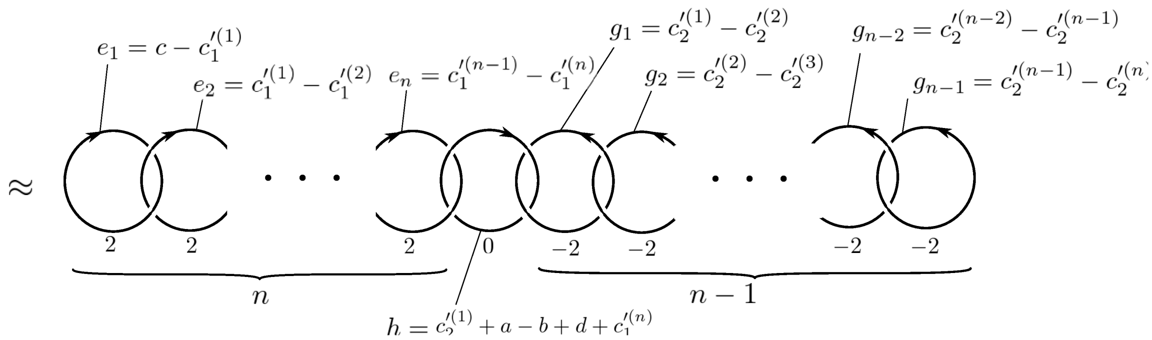
$c(1)$





Acknowledgements. The second author would like to thank Professor Tomotada Ohtsuki for inviting me the workshop "Intelligence of Low-dimensional Topology 2015". The authors deeply thank Takahiro Oba for telling them about the invariant d_3. They also thank Hidetoshi Masai for pointing out the gap of our main theorem in the first draft. The second author was supported by JSPS KAKENHI Grant number 15J01087.

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