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NON-REGULAR SEMIGROUPS WHICH ARE AMALGAMATION BASES *

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In this paper, we study non-regular semigroups which are amalgamation basess for finite semigroups or for all semigroups.

1 Semigroup amalgamation bases

Definition. Let $\mathcal{A}$ be the class of finite semigroups or the class of all semigroups. Let S, T, U be semigroups in $\mathcal{A}$ such that U is a subsemigroup of S and T in common. Then a triple $[S, T; U]$ is called an amalgam of semigroups S, T with U as a core in $\mathcal{A}$. An amalgam $[S, T; U]$ of $\mathcal{A}$ is called to be *weakly embedable* in $\mathcal{A}$ if there exist a semigroup K belonging to $\mathcal{A}$ and monomorphisms $\xi_1 : S \to K$, $\xi_1 : T \to K$ such that the restrictions to U of ξ_1 and ξ_2 are equal to each other (that is, $\xi_1(S) \cap \xi_2(T) \supseteq \xi_1(U)$). An amalgam $[S, T; U]$ of $\mathcal{A}$ is called to be *strongly embeddable* in $\mathcal{A}$ if $\xi_1(S) \cap \xi_2(T) = \xi_1(U)$. A semigroup U in $\mathcal{A}$ is *amalgamation base* [resp. *weak* amalgamation base] if any amalgam with a core U in $\mathcal{A}$ is strongly embeddable [resp. weakly embeddable] in $\mathcal{A}$.

We have the following results which will be used later.

Result 1[[4], Theorem 12]. *Any finite semigroup U is an amalgamation base for finite semigroups if and only if U is a weak amalgamation base for finite semigroups [for all semigroup] .*

Result 2[[6], Theorem 1]. *If a finite semigroup U is an amalgamation base for finite semigroups, then all $\mathcal{J}$-classes of U form a chain.*

*This is an absrtact and the paper will appear elsewhere.

2 Non-regular amalgamation bases for all semigroups

In the several papers [1], [4], [12] and etc., the study of regular amalgamation bases for all semigroupshas been made.

From now on we discuss non-regular amalgamation bases for all semigroups.

The Kimura's counter-example was stated in the second volume of the book [2].

Example. Let $U = \{u, v, w, 0\}$ be a null semigroup in which all products are equal to 0. Let $S_1 = U \cup \{a\}$ where $a \notin U$, $au = ua = v$, and all the other products in S_1 are set equal to z. Let $S_2 = U \cup \{b\}$ where $b \notin U$, $bv = vb = w$, and all other products in S_2 are set equal to 0. Then the amalgam $[S_1, S_2; U]$ is not embeddable in the class of all semigroups.

Actually, $w = bv = b(ua) = (bu)a = 0a = 0$ in any oversemigroup, a contradiction.

Thus we have

Theorem 1. *Any null semigroups with at least 3 elements are not amalgamation bases for all semigroups*

On the other hand, it was known that

Resut 3[[4], Corollary 26]. *2-element semigroups are amalgamation bases for all semigroups.*

Result 4[Theorem 4.1, [4]]. *Let S be a semigroup with 0 consisting of a group G of units and a nilpotent ideal N. Then S is an amalgamation base for all semigroups if and only if there exists $a \in N$ such that $N = Ga \cup Ga^2 \cup \cdots \cup Ga^n \cup \{0\}$ and $Ga = aG$*

Definition. Let S be a comutative semigroup with only finitely many $\mathcal{J}$-classes, where $\mathcal{J}$ denotes the the Green's J-relation on S.

Define a quasi-order $\geq_{\mathcal{J}}$ on S by $s \geq_{\mathcal{J}} t$ if and only if $Ss \supseteq St$.

$s >_{\mathcal{J}} t$ if $S^1 s \supset S^1 t$.

Let a, b be elements of S. Then we say that the pair a and b are *$\mathcal{J}$-comparable* if $a \geq_{\mathcal{J}} b$ or $b \geq_{\mathcal{J}} a$. Otherwise, a, b are *$\mathcal{J}$-incomparable.* Also, $\mathcal{J}$-incomparable elements a, b of S are called *E-distinct* if there exists an idempotent $e \in S$ such that (i) either $a\mathcal{J}ea$, $b >_{\mathcal{J}} eb$ or (ii) $a >_{\mathcal{J}} ea$, $b\mathcal{J}eb$.

A subset A of S is called *E-distinct* if any pair of $\mathcal{J}$-incomparable elements of A are E-distinct.

Result 5[The main theorem, [10]]. *Let S be a commutative semigroup with only finitely many $\mathcal{J}$-classes. Then S is E-distinct if and only if S is an amalgamation base for all semigroups.*

Result 6[Corollary, [10]]. *Let S be a commutative semigroup with only finitely many $\mathcal{J}$-classes. Then S is an amalgamation base for all semigroups if and only if all factor semigroups of S are an amalgamation base for all semigroups.*

3 Non-regular amalgamation bases for finite semigroups

Result 7[Theorem 20, [4]]. *Any finite cyclic semigroup is an amalgamation base for*

semigroups and for finite semigroups

Theorem 2 *Let S be a semigroup with 0 consisting of a group G of units and a nilpotent ideal N. Then S is an amalgamation base for finite semigroups if and only if there exists $a \in N$ such that $N = Ga \cup Ga^2 \cup \cdots \cup Ga^n \cup \{0\}$ and $Ga = aG$.*

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