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Quasi-symmetric numerical semigroups and double covers of curves [1]

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Abstract

We characterize the quasi-symmetric numerical semigroups H through the numerical semigroups $d_2(H)=\{h \mid 2h \in H\}$. In the case where $d_2(H)$ is generated by $d-1$ and d we investigate whether the quasi-symmetric numerical semigroup H is obtained from the Weierstrass semigroup of a ramification point of a double covering of a curve.

1 Notations and terminologies

Let $\mathbb{N}_0$ be the additive monoid of non-negative integers. A submonoid H of $\mathbb{N}_0$ is called a *numerical semigroup* if the complement $\mathbb{N}_0 \backslash H$ is finite. The cardinality of $\mathbb{N}_0 \backslash H$ is called the *genus* of H, denoted by $g(H)$. In this paper H always stands for a numerical semigroup of genus g. We set

$$c(H)=\min\{c \in \mathbb{N}_0 \mid c+\mathbb{N}_0 \subseteqq H\},$$

which is called the *conductor* of H. We have $g(H)+1 \leqq c(H) \leqq 2g(H)$. A numerical semigroup H is said to be *symmetric* and *quasi-symmetric* if $c(H)=2g(H)$ and $c(H)=2g(H)-1$, respectively. We set $d_2(H)=\{h \mid 2h \in H\}$, which is also a numerical semigroup. A *curve* means a complete non-singular irreducible algebraic curve over an algebraically closed field k of characteritic 0. For a pointed curve (C,P) we set

$$H(P)=\{n \in \mathbb{N}_0 \mid \exists f \in k(C) \text{ such that } (f)_\infty=nP\},$$

where $k(C)$ is the field of rational functions on C.

Remark 1.1 *Let $\pi: C \longrightarrow C'$ be a double covering of a curve with a ramification point $P \in C$. Then $d_2(H(P))=H(\pi(P))$.*

H is said to be *of double covering type*, which is abbreviated to *DC*, if there exists a double covering $\pi: C \longrightarrow C'$ with a ramification point P satisfying $H=H(P)$.

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2 Symmetric numerical semigroups and double covering type

Remark 2.1 ([2]) *Let H be symmetric. Then we have*

$$H = 2d_2(H) \cup \{2g(H) - 1 - 2t \mid t \in \mathbb{Z}\backslash d_2(H)\}.$$

Conversely, let H' be any numerical semigroup and g any integer with $g \geqq 3g(H')$. We set

$$S(H', g) = 2H' \cup \{2g - 1 - 2t \mid t \in \mathbb{Z}\backslash H'\}.$$

Then $S(H', g)$ is a symmetric numerical semigroup of genus g with $d_2(S(H', g)) = H'$.

H is said to be *Weierstrass* if there exists a pointed curve (C, P) with $H(P) = H$.

Theorem 2.2 *Assume that H is symmetric and $g \geqq \max\{3g(d_2(H)), 2c(d_2(H))\}$. If $d_2(H)$ is Weierstrass, then H is DC.*

For the proof see [3].

Example 2.1 Let $H = \langle 6, 8, n\rangle$ with odd $n \geqq 7$. Then $d_2(H) = \langle 3, 4\rangle$ and $g(d_2(H)) = 3$. Moreover, $g(H) = 6 + (n-1)/2 \geqq 9$ and H is symmetric. Indeed, $H = 2d_2(H) \cup \{2g(H) - 1 - 2t \mid t \in \mathbb{Z}\backslash d_2(H)\}$. Since $d_2(H) = \langle 3, 4\rangle$ is Weierstrass, H is DC for $n \geqq 13$.

3 Quasi-symmetric numerical semigroups

Theorem 3.1 *Assume that $g = g(H)$ is even. Then the following are equivalent:*
i) *H is quasi-symmetric.*
ii) *$d_2(H)$ is a symmetric numerical semigroup of genus $g(H)/2$.*

For the proof see [5].

Example 3.1 Let $H = \langle 5, 8, 11, 12, 19\rangle$. Then $g(H) = 8$ and $c(H) = 15 = 2g(H) - 1$, which implies that H is quasi-symmetric. In this case $d_2(H) = \langle 4, 5, 6\rangle$, which is symmetric and whose genus is $4 = g(H)/2$.

Proposition 3.2 *Let H' be a symmetric numerical semigroup. We set*

$$n = \min\{h' \in H' \mid h' \textit{ is odd}\}$$

and

$$s_i = \min\{h' \in H' \mid h' \equiv i \textit{ mod } n\}$$

for all $i = 1, \ldots, n-1$. We set

$$\{s_1, \ldots, s_{n-1}\} = \{s^{(1)} < \cdots < s^{(n-1)}\}.$$

Let
$$H = \langle n, 2s^{(1)}, \ldots, 2s^{(\frac{n-1}{2})}, 2s^{(\frac{n+1}{2})} - n, \ldots, 2s^{(n-1)} - n \rangle.$$
Then H is a quasi-symmetric numerical semigroup of genus $2g(H')$ with $d_2(H) = H'$.

For the proof see [5].

Theorem 3.3 *Assume that $g = g(H)$ is odd. Then the following are equivalent:*
i) *H is quasi-symmetric.*
ii) *$d_2(H)$ is a quasi-symmetric numerical semigroup of genus $(g(H) + 1)/2$.*

For the proof see [5].

Example 3.2 Let $H = \langle 3, 11, 19 \rangle$. Then $g(H) = 9$ and $c(H) = 17 = 2g(H) - 1$, which implies that H is quasi-symmetric. In this case $d_2(H) = \langle 3, 7, 11 \rangle$, which is quasi-symmetric and whose genus is $5 = (g(H) + 1)/2$.

Proposition 3.4 *Let H' be a quasi-symmetric numerical semigroup. We set*
$$n = \min\{h' \in H' \mid h' \text{ is odd}\}$$
and
$$s_i = \min\{h' \in H' \mid h' \equiv i \bmod n\}$$
for all $i = 1, \ldots, n-1$. We set
$$\{s_1, \ldots, s_{n-1}\} = \{s^{(1)} < \cdots < s^{(n-1)}\}.$$
Let
$$H = \langle n, 2s^{(1)}, \ldots, 2s^{(\frac{n-3}{2})}, 2s^{(\frac{n-1}{2})} - n, \ldots, 2s^{(n-1)} - n \rangle.$$
Then H is a quasi-symmetric numerical semigroup of genus $2g(H')-1$ with $d_2(H) = H'$.

For the proof see [5].

4 Quasi-symmetric numerical semigroups over $\langle d-1, d \rangle$ and double covering type

Remark 4.1 ([6]) *Let H be a Weierstrass numerical semigroup with $g(H) \geqq 6g(d_2(H))+4$. Then H is DC.*

Theorem 4.2 *Let H be a Weierstrass numerical semigroup.*
i) *If $g(H) = 6g(d_2(H)) + 3$, then H is DC.*
ii) *If $g(H) = 6g(d_2(H)) + 2$ and $g(d_2(H)) \geqq 4$, then H is DC.*
iii) *If $g(H) = 6g(d_2(H)) + 1$ and $g(d_2(H)) \geqq 6$, then H is DC*

For the proof see [4].

Remark 4.3 *If $g(H) \leqq 2g(d_2(H)) - 1$, then H is not DC by Riemann-Huruwitz' Formula .*

Problem 4. 1 Let H be a Weierstrass numerical semigroup. Assume that $2g(d_2(H)) \leqq g(H) \leqq 6g(d_2(H))$. Is every H DC?

We are in the following situation: Let d be an integer which is larger than 2. We set $H' = \langle d-1, d\rangle$, which is symmetric and Weierstrass. Indeed, let (C, P) be a pointed plane curve of degree d with a total flex P, i.e., $C.T_P = dP$ where T_P is the tangent line at P on C and $C.T_P$ denotes the intersection divisor of C with T_P. Then $H(P) = \langle d-1, d\rangle$, and vice versa. Assume that $d_2(H) = H'$ and $g(H) = 2g(H')$. Then by Theorem 3.1 H is quasi-symmetric.

Problem 4. 2 Let H be a numerical semigroup satisfying $d_2(H) = \langle d-1, d\rangle$ with $d \geqq 3$. Assume that $g(H) = 2g(d_2(H))$. Is H DC?

Remark 4.4 *Let $H' = \langle d-1, d\rangle$. If d is even (reps. odd), then* Proposition 3.2 *gives the quasi-symmetric numerical semigroup $H = \langle d-1, 2d, d^2 - d + 1\rangle$ (resp. $H = \langle d, 2(d-1), d^2 - d - 1\rangle$). In this case, $d_2(H) = H'$ and $g(H) = 2g(H')$.*

Theorem 4.5 *The numerical semigroups in* Remark 4.4 *are DC.*

For the proof see [5].

Example 4. 1 Let $d = 3$, i.e., $H' = \langle 2, 3\rangle$. Then $H = \langle 3, 4, 5\rangle$, which is a unique semigroup with $d_2(H) = H'$ and $g(H) = 2g(H')$. By Theorem 4.5, Problem 4.2 is solved affirmatively.

Example 4. 2 Let $d = 4$, i.e., $H' = \langle 3, 4\rangle$. A numerical semigroup H with $d_2(H) = H'$ and $g(H) = 2g(H')$ is either $\langle 3, 8, 13\rangle$ or $\langle 6, 7, 8, 9, 11\rangle$.

Remark 4.6 *Let $H' = \langle d-1, d\rangle$ with $d \geqq 4$. A numerical semigroup H with $d_2(H) = H'$ and $g(H) = 2g(H')$ is not uniquely determined.*

Remark 4.7 ([1]) *Every numerical semigroup H of genus 6 with $g(d_2(H)) = 3$ is DC.*

Hence, Problem 4.2 is solved affirmatively in the case $d = 4$.

Example 4. 3 Let $d = 5$, i.e., $H' = \langle 4, 5\rangle$. A numerical semigroup H with $d_2(H) = H'$ and $g(H) = 2g(H')$ is one of the following:
i) $H = \langle 5, 8, 19\rangle$ (Remark 4.4, $d = 5$),
ii) $H = \langle 8, 9, 10, 15, 21\rangle$,
iii) $H = \langle 8, 10, 13, 15, 17, 19\rangle$.

The semigroup $H = \langle 8, 9, 10, 15, 21\rangle$ in Example 4.3 ii) is generalized and the generalized semigroup is DC as follows:

Theorem 4.8 *For $d \geqq 5$ we set*

$$H = \langle 2(d-1), 2d-1, 2d, (d-1)^2 - 1, (d-1)^2 + d \rangle.$$

Then H is a quasi-symmetric numerical semigroup with $d_2(H) = \langle d-1, d \rangle$ and $g(H) = 2g(d_2(H))$, which is DC.

See [5] for the proof.

Proposition 4.9 *The semigroup $H = \langle 8, 10, 13, 15, 17, 19 \rangle$ in* Example 4.3 iii) *is DC.*

See [5] for the proof.

Theorem 4.10 Problem 4.2 *is solved affirmatively in the case $d = 5$.*

Proposition 4.11 *The semigroup $H = \langle 8, 10, 13, 15, 17, 19 \rangle$ in* Example 4.3 iii) *is generalized to*

$$H = \langle 2(d-1), 2d, (d-2)(d-1)+1, (d-2)(d-1)+3, \ldots, (d-2)(d-1)+2(d-2)+1 \rangle$$

for $d \geqq 4$, which is quasi-symmetric. Moreover, we have $d_2(H) = \langle d-1, d \rangle$ and $g(H) = 2g(d_2(H))$.

Problem 4. 3 Let $d \geqq 6$. Is

$$H = \langle 2(d-1), 2d, (d-2)(d-1)+1, (d-2)(d-1)+3, \ldots, (d-2)(d-1)+2(d-2)+1 \rangle$$

DC?

Example 4. 4 Let $d = 6$, i.e., $H' = \langle 5, 6 \rangle$. A numerical semigroup H with $d_2(H) = H'$ and $g(H) = 2g(H')$ is one of the following:
i) $H = \langle 5, 12, 31 \rangle$, which is DC by Theorem 4.5,
ii) $H = \langle 10, 11, 12, 25, 29 \rangle$, which is DC by Theorem 4.8,
iii) $H = \langle 10, 12, 21, 23, 25, 27, 29 \rangle$, which is given in Proposition 4.11 with $d = 6$,
iv) $H = \langle 10, 12, 15, 21, 29 \rangle$,
v) $H = \langle 10, 12, 15, 17, 29, 31, 33 \rangle$,
vi) $H = \langle 10, 12, 17, 23, 25, 31 \rangle$.

Problem 4. 4 Let $H' = \langle d-1, d \rangle$ with $d \geqq 6$. Let H be a numerical semigroup of genus $2g(H') = (d-1)(d-2)$ with $d_2(H) = H'$. Then is H DC?

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