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Note on the space of polynomials with roots of bounded multiplicity

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Abstract

We study the homotopy type of the space $\mathrm{SP}_n^d(X)$ consisting of all d particles in X with multiplicity less than n. When $X = \mathbb{C}$, this space may be identified with the space SP_n^d of all monic complex coefficient polynomials $f(z) \in \mathbb{C}[z]$ of degree d without roots of multiplicity $\geq n$. In this paper we announce the main result given in [8] concerning to the homotopy stability dimension of this space which improves that obtained in the previous paper [3].

1 Introduction.

Basic definitions and notations. For spaces X and Y, let $\mathrm{Map}^*(X, Y)$ denote the space consisting of all continuous base-point preserving maps from X to Y with the compact-open topology. When X and Y are complex manifolds, we denote by $\mathrm{Hol}^*(X, Y)$ the subspace of $\mathrm{Map}^*(X, Y)$ consisting of all base-point preserving holomorphic maps.

For each integer $d \geq 1$, let $\mathrm{Map}_d^*(S^2, \mathbb{C}\mathrm{P}^{n-1}) = \Omega_d^2\mathbb{C}\mathrm{P}^{n-1}$ denote the space of all based continuous maps $f : (S^2, \infty) \to (\mathbb{C}\mathrm{P}^{n-1}, [1 : 1 : \cdots : 1])$ such that $[f] = d \in \mathbb{Z} = \pi_2(\mathbb{C}\mathrm{P}^{n-1})$, where we identify $S^2 = \mathbb{C} \cup \{\infty\}$ and choose $\infty \in S^2$ and $[1 : 1 : \cdots : 1] \in \mathbb{C}\mathrm{P}^{n-1}$ as the base points of S^2 and $\mathbb{C}\mathrm{P}^{n-1}$, respectively. Let $\mathrm{Hol}_d^*(S^2, \mathbb{C}\mathrm{P}^{n-1})$ denote the subspace of $\Omega_d^*\mathbb{C}\mathrm{P}^{n-1}$ consisting of all based holomorphic maps.

Let S_d denote the symmetric group of d letters. Then the group S_d acts on the space $X^d = X \times \cdots \times X$ (d-times) by the coordinate permutation and let $\mathrm{SP}^d(X)$ denote the d-th *symmetric product* of X given by the orbit space $\mathrm{SP}^d(X) = X^d/S_d$.

Let $F(X, d) \subset X^d$ denote the subspace consisting of all $(x_1, \cdots, x_n) \in X^d$ such that $x_i \neq x_j$ if $i \neq j$. Since $F(X, d)$ is S_d-invariant, we define the orbit space $C_d(X)$ by $C_d(X) = F(X, d)/S_d$. The space $C_n(X)$ is usually called *the configuration space of unordered n-distinct points in X*. Note that there is an inclusion $C_d(X) \subset \mathrm{SP}^d(X)$.

Let $\mathrm{P}^d(\mathbb{C})$ denote the space consisting of all monic polynomials

$$f(z) = z^d + a_1 z^{d-1} + \cdots + a_d \in \mathbb{C}[z]$$

of the degree d. Similarly, let SP^d_n denote the susbspace of $\mathrm{P}^d(\mathbb{C})$ consisting of all monic polynomials $f(z) \in \mathrm{P}^d(\mathbb{C})$ without root of multiplicity $\geq n$.

Definition 1.1. Note that each element $\alpha \in \mathrm{SP}^d(X)$ can be represented as the formal sum $\alpha = \sum_{k=1}^r n_k x_k$, where $\{x_k\}_{k=1}^r$ are mutually distinct points in X and each n_k is a positive integer such that $\sum_{k=1}^r n_k = d$.

Then by using the notation, we define the subspace $\mathrm{SP}^d_n(X) \subset \mathrm{SP}^d(X)$ by

$$\mathrm{SP}^d_n(X) = \Big\{ \sum_{k=1}^r n_k x_k \in \mathrm{SP}^d(X) : n_k < n \text{ for any } 1 \leq k \leq r \Big\}.$$

Note that there is an increasing filtration

$$\emptyset = \mathrm{SP}^d_1(X) \subset C_d(X) = \mathrm{SP}^d_2(X) \subset \mathrm{SP}^d_3(X) \subset \cdots \subset \mathrm{SP}^d_d(X) \subset \mathrm{SP}^d_{d+1}(X) = \mathrm{SP}^d(X).$$

Remark 1.2. (i) If $X = \mathbb{C}$ we can easily see that there is a natural homeomorphism $\mathrm{P}^d(\mathbb{C}) \cong \mathrm{SP}^d(\mathbb{C})$ by identifying $\mathrm{P}^d(\mathbb{C}) \ni \prod_{k=1}^r (z - \alpha_k)^{n_k} \mapsto \sum_{k=1}^r n_k \alpha_k \in \mathrm{SP}^d(\mathbb{C})$, where $(\alpha_1, \cdots, \alpha_r) \in F(\mathbb{C}, r)$ and $\sum_{k=1}^r n_k = d$. It is also easy to see that the there is a natural homemorphism $\mathrm{SP}^d_n \cong \mathrm{SP}^d_n(\mathbb{C})$ by using this identification.

(ii) It is easy to see that the space $\mathrm{Hol}^*_d(S^2, \mathbb{C}\mathrm{P}^{n-1})$ can be identified with the space consisting of all n-tuples $(f_1(z), \cdots, f_n(z)) \in \mathrm{P}^d(\mathbb{C})^n$ of monic polynomials of the same degree d such that polynomials $f_1(z), \cdots, f_n(z)$ have no common root. $\Box$

Definition 1.3. Define *the jet map* $j^d_n : \mathrm{SP}^d_n \to \Omega^2_d \mathbb{C}\mathrm{P}^{n-1} \simeq \Omega^2 S^{2n-1}$ by

$$j^d_n(f)(x) = \begin{cases} [f(x) : f(x) + f'(x) : f(x) + f''(x) : \cdots : f(x) + f^{(n-1)}(x)] & \text{if } x \in \mathbb{C} \\ [1 : 1 : \cdots : 1] & \text{if } x = \infty \end{cases}$$

for $(f, x) \in \mathrm{SP}^d_n \times S^2$, where we identify $S^2 = \mathbb{C} \cup \infty$.

Remark 1.4. A map $f : X \to Y$ is called *a homotopy equivalence* (resp. *a homology equivalence*) *up to dimension* D if the induced homomorphism $f_* : \pi_k(X) \to \pi_k(Y)$ (resp. $f_* : H_k(X, \mathbb{Z}) \to H_k(Y, \mathbb{Z})$) is an isomorphism for any $k < D$ and an epimorphism if $k = D$. Similarly, it is called *a homotopy equivalence* (resp. *a homology equivalence*) *through dimension* D if $f_* : \pi_k(X) \to \pi_k(Y)$ (resp. $f_* : H_k(X, \mathbb{Z}) \to H_k(Y, \mathbb{Z})$) is an isomorphism for any $k \leq D$. $\Box$

2 The main result.

The previous results. Let M_g denote closed Riemann surface of genus g, and let $* \in M_g$ be its base-point. Note that $M_g = S^2$ if $g = 0$. Then, recall the following two results given in [12] and [3].

Theorem 2.1 ([12]; the case $g \geq 1$). *If $g \geq 1$, there is a map* $\mathrm{SP}^d_n(M_g \setminus \{*\}) \to \mathrm{Map}^*_0(M_g, \mathbb{C}\mathrm{P}^{n-1})$ *which is a homology equivalence up to dimension $D(d,n)$, where $\lfloor x \rfloor$ is the integer part of a real number x and $D(d,n)$ denotes the positive integer given by*

$$D(d,n) = \begin{cases} \lfloor \frac{d}{2} \rfloor & \text{if } n = 2 \\ \lfloor \frac{d}{n} \rfloor - n + 3 & \text{if } n \geq 3 \end{cases} \quad \Box$$

Remark 2.2. Recently the much better stability dimension for the case $g \geq 1$ was obtained by A. Kupers and J. Miller in [?] (cf. [5], [6], [10]).

Theorem 2.3 ([3]; the case $g = 0$). *If $g = 0$, the jet map*

$$j^d_n : \mathrm{SP}^d_n \to \Omega^2_d \mathbb{C}\mathrm{P}^{n-1} \simeq \Omega^2 S^{2n-1}$$

is a homotopy equivalence up to dimension $(2n-3)\lfloor \frac{d}{n} \rfloor$ if $n \geq 3$ and it is a homology equivalence up to dimension $\lfloor \frac{d}{2} \rfloor$ if $n = 2$. $\Box$

Theorem 2.4 ([4], [11]). *There is a homotopy equivalence*

$$\mathrm{SP}^d_n \simeq \mathrm{Hol}^*_{\lfloor \frac{d}{n} \rfloor}(S^2, \mathbb{C}\mathrm{P}^{n-1}) \quad \text{if } n \geq 3$$

and there is a stable homotopy equivalence $\mathrm{SP}^d_2 \simeq_s \mathrm{Hol}_{\lfloor \frac{d}{2} \rfloor}(S^2, \mathbb{C}\mathrm{P}^1)$ *if $n = 2$.* $\Box$

The new result. We can improve the stability dimension of the above result for $n \geq 3$ as follows:

Theorem 2.5 ([8]). *If $n \geq 3$ and $g = 0$, the jet map $j^d_n : \mathrm{SP}^d_n \to \Omega^2_d \mathbb{C}\mathrm{P}^{n-1} \simeq \Omega^2 S^{2n-1}$ is a homotopy equivalence through dimension $D(d,n) = (2n-3)(\lfloor \frac{d}{n} \rfloor + 1) - 1$.* $\Box$

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