数理解析研究所講究録
第 1968 巻 2015 年 113-116

On orientations of fixed point sets of spin structure preserving involutions on manifolds.

Seiji Nagami
Academic Support Center
Setsunan University

1 Introduction

Let X be an oriented connected closed smooth manifold of dimension n with $n \geq 4$, and F an embedded closed submanifold of codimension 2 with $[F]_2 = 0 \in H_{n-2}(X; \mathbf{Z}_2)$, where $[F]_2$ denote the homology class represented by F in X with coefficients in $\mathbf{Z}_2$. Then we have a double branched covering map $\widetilde{X} \to X$ branched along F. In [2] and [3], we have obtained the following result;

Theorem 1.1 *Suppose that $H_1(X; \mathbf{Z}_2) = 0$. Then $\widetilde{X}$ admits a spin structure if and only if F admits an orientation such that $[F]^\sharp \in H^2(X; \mathbf{Z})$ is twice a cohomology class of which reduction modulo 2 coincides with the second Stifel-Whitney class. Here, $[F]^\sharp$ denote the Poincaré-dual of $[F]$.*

The assumption that $H_1(X; \mathbf{Z}_2) = 0$ is essential. As a generalizaition of the above theorem, we first obtain;

Theorem 1.2 *Let H be a connected closed surface smoothly embedded in X. Suppose that $H_1(X, \mathbf{Z}_2) = 0$, that $n = 4$, and that $[H]_m = 0 \in H_2(X; \mathbf{Z}_m)$, where $[H]_m$ denote the homology class represented by the oriented H. Then $\widetilde{X}$ is spin if and only if $[F]^\sharp \in H^2(X; \mathbf{Z})$ is m times a cohomology class of which reduction modulo 2 coincides with the second Stifel-Whitney class.*

Although Theorem 1.2 is the case for $n = 4$, it should hold for all positive integer n.
Next we have obtained the following ([3]) as an another generalization of Theorem 1.1;

Theorem 1.3 *$\widetilde{X}$ admits a spin structure that is preserved by the covering transformation map $T : \widetilde{X} \to \widetilde{X}$ if and only if $[F]^\sharp \in H^2(X; \mathbf{Z})$ is twice a cohomology class of which reduction modulo 2 coincides with the second Stifel-Whitney class.*

Suppose that F admits an orientation such that $[F]^\sharp = 2w \in H^2(X; \mathbf{Z})$ with $(w)_2 = w_2(X)$, where $(w)_2 \in H^2(X; \mathbf{Z}_2)$ denotes the reduction modulo 2. Then we have

$$H_1(X; \mathbf{Z}) \cong \oplus_{i=1}^n \mathbf{Z}_2 \oplus_{i=1}^{N_0} \mathbf{Z}\langle p_i \rangle \oplus_{i=1}^{N_1} \mathbf{Z}_{2^{r_i}} \langle q_i \rangle \oplus_{i=1}^{N_2} \mathbf{Z}_{k_i},$$

where $r_i \geq 2$ and k_i odd. Therefore we obtain

$$H_1(X; \mathbf{Z}_2) \cong \oplus_{i=1}^n \mathbf{Z}_2 \oplus_{i=1}^{N_0} \mathbf{Z}_2\langle p_i \rangle \oplus_{i=1}^{N_1} \mathbf{Z}_2\langle q_i \rangle$$

Then the 1-st homology group $H_1(X-F;\mathbf{Z}_2)$ is isomorphic to $\mathbf{Z}_2\langle\mu_1,\dots\mu_s\rangle\oplus H_1(X;\mathbf{Z}_2)$, where μ_i is homology class represented by a meridian circle to F.

We choose a homomorphism $v: H_1(X-F;\mathbf{Z}_2)\to\mathbf{Z}_2$ so that $v(\mu_i)=1\in\mathbf{Z}_2$ holds for all $1\le i\le s$. Let $\Omega\subset X$ be an oriented closed $n-2$-submanifold of X such that $[\Omega]^\sharp=\omega\in H^2(X;\mathbf{Z})$. Let $L_i\subset X$ be an oriented loop such that $[L]_i=l_i\in H_1(X)$. Then fix an embedding $f_i: S^1\times D^{n-1}\to X$ so that

$$\begin{cases} f_i(S^1\times 0)=L_i \\ f_i(S^1\times D^{n-1})\cap(F\cup\Omega)=\emptyset. \end{cases}$$

Since $2l_i=0$, we can choose an emmbedded surface G such that $\partial G=f_i(S^1\times\{a,b\})$, where $a,b\in\partial D^{n-1}$. Then by setting $X'=\overline{X-f_i(S^1\times D^{n-1})}$, we have that $(G_i,\partial G_i)\subset(X',\partial X')$. Then define $v\in H^1(X-F;\mathbf{Z}_2)$ as

$$v(l_i)=[\Omega]\cdot[G_i,\partial G_i]-\frac{1}{2}([F]\cdot[G_i,\partial G_i]).$$

Here, $\cdot: H_{n-2}(X';\mathbf{Z})\times H_2(X',\partial X');\mathbf{Z}\to Z$ denote the intersection pairing. Then the coveing trnsformation map $T:\widetilde{X}\to\widetilde{X}$ of the double branched covering $\widetilde{X}\to X$ determinded by v is a spin structure preserving.

Remark 1.1 *Ono gives semi-orientation of fixed point set for each spin structure on $\widetilde{X}$ that is preserved by $T:\widetilde{X}\to\widetilde{X}$ as follows([6]): let $SO(\widetilde{X})\to\widetilde{X}$ denote the orthonormal frame bundle of $\widetilde{X}$ together with a spin structure $Spin(\widetilde{X})\to SO(\widetilde{X})$ that is presered by T. Then the differential $dT: SO(\widetilde{X})\to SO(\widetilde{X})$ has a lift $\widetilde{dT}: Spin(\widetilde{X})\to Spin(\widetilde{X})$. Since the restriction $\widetilde{dT}|_{\widetilde{F}}$ is a bundle automorphism, it is a section of the adjoint bundle $Ad(Spin(\widetilde{X}))\to\widetilde{X}$. Because $Ad(Spin(\widetilde{X}))$ is a subbundle of the Clliford algebra bundle $Cl(\widetilde{X})\to\widetilde{X}$, and $Cl(\widetilde{X})\to\widetilde{X}$ is isomorphic to the exterior bundle $\wedge^*T\widetilde{X}$, $\widetilde{dT}$ is a section of $\wedge^*T\widetilde{X}$. Moreover we can see that $\wedge^*T\widetilde{X}$ is a section of a exterior bundle of a normal bundle ν of $\widetilde{F}$ in $\widetilde{X}$. Thus $\widetilde{dT}$ determines an orientation of ν, and given $\widetilde{X}$ determines an orientation of $\widetilde{F}$. In [4], we have shown that the homology class $[F]^\sharp\in H^2(X;\mathbf{Z})$ represented by this orientaion on $F\approx\widetilde{F}$ is twice a characteristic cohomology class.*

Example 1.1 *Let $A\approx S^1\times(0,1)$ be an anulus embedded in $\mathbf{R}^2$ and $t: A\to A$ the involution given by $t(x,y)=(-x,y)$. Let the tangent bundle TA be trivialized so that its framing be bounding one. Then the differential $dt: TA|_{S^1\times 0}\approx S^1\times\mathbf{R}^2\to TA|_{S^1\times 0}$ of t has the following form;*

$$dt:\left(e^{\theta i},\begin{pmatrix}a\\b\end{pmatrix}\right)\to\left(e^{-\theta i},-R(-2\theta)\begin{pmatrix}a\\b\end{pmatrix}\right),$$

where $R(\theta)$ denote the matrix $\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$. Then the lift $\widetilde{dt}: Spin(A)\approx A\times Spin(2)\to Spin(A)$ of $dt: SO(A)\to SO(A)$ to the spin structure $Spin(A)\to SO(A)$ with respect to the given framing has the form;

$$\left(e^{\theta i},\xi\right)\to\left(e^{-\theta i},(\cos\theta-\sin\theta\frac{\partial}{\partial x}\frac{\partial}{\partial y})\xi\right).$$

Therefore at the north pole $N=((\begin{smallmatrix}1\\0\end{smallmatrix}),0)\in A$ (resp. south pole $S=((\begin{smallmatrix}-1\\0\end{smallmatrix}),0)\in A$), the given spin structure determines the orientation $\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y}$ (resp. $-\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y}$).

The above argument shows that the hyperelliptic involution $t : S^2 \to S^2$ together with the unique spin structure S^2 gives the orientation of the branched locus $N \cup S$ of the quotient space $S^2 \approx S^2/\langle t\rangle$ so that $[N \cup S]^\sharp = 0 \in \mathbf{Z} \cong H^2(S^2;\mathbf{Z})$, which is twice a characteristic cohomology class.

Next we consider the case for anulus. Set $T = S^1 \times S^1, l = S^1 \times *$ and $m = * \times S^1$. Then the cohomology classes represented by l and m generate the cohomology group $H^1(T;\mathbf{Z}_2) \cong (\mathbf{Z}_2)^2$. If we give a spin structure on T that restricts to Lie group spin structres on l and m, then the induced orientation of the branched locus F in the quotient space $S^2 \approx T/\langle t\rangle$ by the hyperelliptic involution $t : T \to T$ satisfies $[F]^\sharp = \pm 4 \in \mathbf{Z} \cong H^2(S^2;\mathbf{Z})$. If we consider a spin structure that restricts to Lie group spin structres on l and to bounding spin structure on m, then the induced orientation satisfies $[F]^\sharp = \pm 0 \in \mathbf{Z}$. These are orienations which are twice a characteristic cohomology class.

By considering the Gysin exact sequence for the double cover $\widetilde{X} - \widetilde{F} \to X - F$, we obtain the following([5]);

Theorem 1.4 *$\widetilde{X}$ is spin if and only there exists a class $w \in H^1(X - F;\mathbf{Z}_2)$ such that $v \cup w = w_2(X - F)$ and that $\langle v, \mu\rangle = 1 \in \mathbf{Z}_2$, where $v \in H^1(X - F;\mathbf{Z}_2)$ determines the double cover $\widetilde{X} - \widetilde{F} \to X - F$, and $\mu \in H_1(X - F;\mathbf{Z}_2)$ is a homology class represented by a meridian to F.*

Theorem 1.4 implies another way to state Theorem 1.3([5]);

Theorem 1.5 *$\widetilde{X}$ admits a spin structure that is preserved by the covering transformation map $T : \widetilde{X} \to \widetilde{X}$ if and only if $v \cup v = w_2(X - F)$.*

As a corollary, we have([5]);

Corollary 1.1 *Let $\widetilde{Y} \to Y$ be an unbranched double cover determinded by $\rho \in H^1(Y; Z_2)$. Then $\widetilde{Y}$ admits a spin structure that is preserved by the covering transformation map $t : \widetilde{Y} \to \widetilde{Y}$ of odd type if and only if $\rho \cup \rho = w_2(Y)$.*

Example 1.2

Let $\tau : S^n \to S^n$ denote the antipodal map with odd n and $q : S^n \to RP^n$ its quotient map. Then τ is a spin structure preserving map and q is the double caverng map that corresponds to $\rho = 1 \in \mathbf{Z}_2 \cong H^1(RP^n;\mathbf{Z}_2)$. Recall that τ is odd type with respect to the unique spin structure on S^n if and only if $n \equiv 3 \mod 4$ ([1]). Note that $n \equiv 3 \mod 4$ if and only if $\rho \cup \rho = w_2(RP^n)$ because $w_2(RP^n) = \frac{n(n+1)}{2} \in \mathbf{Z}_2 \cong H^2(RP^n;\mathbf{Z}_2)$.

References

[1] M. Atiyah and R. Bott:*A Lefschetz fixed point formula for elliptic complexes: II. Applications*, Ann. Maht., **88**(1968), 451–491.

[2] S.Nagami: *Existence of spin structures on double branched covering spaces over four-manifolds*, Osaka J. Math., **37**(2000), 425–440.

[3] S.Nagami: *A note on orientations of fixed point sets of spin structure preserving involutions*, Kobe J. Math., **20**(2003), 39–51.

[4] S.Nagami: *Existence of spin structures on cyclic branched covering spaces over four-manifolds*, Perspectives of Complex Analysis, Differential Geometry and Mathematical Physics: Proceedings of the 5th International Workshop on Complex Structures and Vector Fields: St. Konstantin, Bulgaria, 3-9 September 2000. World Scientific, (2001), 86–92.

[5] : S. Nagami: *On spin structures of double branched coveint spaces*, JP J. Geom. Top., **14**(2013), 119–147.

[6] K. Ono: *On a Theorem of Edmonds*, Progress in differential geometry, 243–245, Adv.Stud. Pure Math., **22**, Math Soc. Japan, Tokyo, 1993.