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On certain line degenerated torus curves and their dual curves

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1 Introduction

Let $\mathcal{M}_d$ be the set of plane curves of degree d in $\mathbb{P}^2$ and let $\mathcal{M}_d(\Sigma) \subset \mathcal{M}_d$ be the set of plane curves which have fixed topological type of singularities Σ. For a given plane curve $C \in \mathcal{M}_d(\Sigma)$, we are interested in topological invariants of C. In particular, the fundamental group of the compliment $\pi_1(\mathbb{P}^2 \setminus C)$ and the Alexander polynomial $\Delta_C(t)$. It is known that they do not be determined by the configuration Σ and they are influenced by the location of singular points $\Sigma(C)$ and the form of its defining polynomial. For example, there exist two irreducible plane curve C_1, $C_2 \in \mathcal{M}_6(6A_2)$ such that there exists a smooth conic passing through the singular points of C_1 and there does not exist such a conic for C_2. Then $\pi_1(\mathbb{P}^2 \setminus C_1) = \mathbb{Z}_2 * \mathbb{Z}_3 \not\cong \mathbb{Z}_2 \oplus \mathbb{Z}_3 = \pi_1(\mathbb{P}^2 \setminus C_2)$ and $\Delta_{C_1}(t) = t^2 - t + 1 \neq 1 = \Delta_{C_2}(t)$. ([5]). We can observe that the defining polynomial of C_1 is written as $F_3^2 + F_2^3$ where $\deg F_j = j$. Such a pair (C_1, C_2) is called a *Zariski pair* which is studied by many authors.

There exists another interesting example. In [7], Duc Tai Pho constructs a new Zariski pair using *dual curves.* We recall this example. Let E_1 be the Fermat curve of degree 4 and E_2 be another smooth quartic which have 12 hyperflex points. Then their dual curves $\check{E}_1$ and $\check{E}_2$ in the space $\mathcal{M}_{12}(12E_6+16A_1)$ and the pair $(\check{E}_1, \check{E}_2)$ is an Alexander polynomial distinguished Zariski pair. We are interest in this phenomena.

In general, the configuration $\check{\Sigma}$ of singularities of the dual curve is not also determined by only the configuration Σ of singularities of the original curve. For example, we consider two plane curves $D_1 = \{G_1 = 0\}$ and $D_2 = \{G_2 = 0\}$ of degree 4 which are defined by

$$G_1(X,Y,Z) = (X-Y)^2(X+Y)^2 - Y^3Z, \quad G_2(X,Y,Z) = X^4 - Y^3Z.$$

Then D_1 and D_2 are contained in $\mathcal{M}_4(E_6)$. Note that D_1 is bi-tangent to the line at infinity $L_\infty := \{Z = 0\}$ and D_2 intersects L_∞ with multiplicity 4 where L_∞. Now we consider dual curves $\check{D}_1$ and $\check{D}_2$. They are in the different configurations spaces. That is $\check{D}_1 \in \mathcal{M}_4(2A_2 + A_1)$ and $\check{D}_2 \in \mathcal{M}_4(E_6)$ ([8, 6]). Their fundamental groups are $\pi_1(\mathbb{P}^2 \setminus C_1) \cong \pi_1(\mathbb{P}^2 \setminus C_2) \cong \mathbb{Z}/4\mathbb{Z}$. To compare their topologies, M. Oka introduces *the tangential fundamental group* and *the tangential Alexander polynomial* ([6]). We recall them briefly. Take a line $L \subset \mathbb{P}^2$ and we consider an affine space $\mathbb{C}^2_L := \mathbb{P}^2 \setminus L$. If $L = T_PC$ for some smooth point $P \in C$, then we call $\pi_1(\mathbb{C}^2_L \setminus C)$ the tangential fundamental group and $\Delta_C(t; L)$ the tangential Alexander polynomial ([6]). Moreover, M. Oka shows that

$$\pi_1(\mathbb{C}^2_{L_\infty} \setminus D_1) \not\cong \pi_1(\mathbb{C}^2_{L_\infty} \setminus D_2), \quad \Delta_{D_1}(t; L_\infty) \neq \Delta_{D_2}(t; L_\infty).$$

Moreover M. Oka studies *line degenerations* of irreducible plane curves and line degenerated torus curves is divided by two classes *visible* or *invisible*. ([6, 2]). The above example can be obtained by using visible line degenerated torus curves of degree 4.

In this note, we consider the configuration space $\mathcal{M}_{p+1}(B_{p+1,p})$ where p is a positive odd integer and $B_{p+1,p}$ is the Brieskorn singularity. Using line degenerated torus curves, we will construct certain family of line degenerated torus curves of degree $p+1$ which is in the space $\mathcal{M}_{p+1}(B_{p+1,p})$. We study their dual curves and the tangential fundamental groups.

2 Preliminaries

2.1 Line degenerated torus curves

Let $C = \{F = F_q^p + F_p^q = 0\} \in \mathcal{M}_{pq}$ be a projective (p,q) torus curve. Suppose that F has the following form:

$$F(X,Y,Z) = Z^j G(X,Y,Z) \tag{1.2}$$

where $G(X,Y,Z)$ is a reduced homogeneous polynomial of degree $pq-j$. We call a curve $D = \{G = 0\}$ *a line degenerated torus curve of type* (p,q) *of order* j and the line $L_\infty = \{Z = 0\}$ *the limit line of the degeneration.* Put $\mathcal{LT}_j(p,q;d)$ as the set of line degenerated torus curves of type (p,q) of order j and $\mathcal{LT}(p,q)$ is the union of $\mathcal{LT}_j(p,q;d)$ with respect to j.

We can divide the situation (1.2) into two cases which are called *visible degenerations* and *invisible degenerations.* Put the integer $r_k := \max\{r \in \mathbb{Z} \mid Z^r \text{ divides } F_k\}$ for $k = p,q$.

Visible case. Suppose that $r_p \cdot r_q \neq 0$ and $qr_p \neq pr_q$. Then F_q and F_p are written as $F_q(X,Y,Z) = F'_{q-r_q}(X,Y,Z)Z^{r_q}$ and $F_p(X,Y,Z) = F'_{p-r_p}(X,Y,Z)Z^{r_p}$. Putting $j := \min\{qr_p, pr_q\}$, we can factor F as $F(X,Y,Z) = Z^j G(X,Y,Z)$. Then G is written using F'_{p-r_p} and F'_{q-r_q} as

$$G(X,Y,Z) = \begin{cases} F'_{q-r_q}(X,Y,Z)^p + F'_{p-r_p}(X,Y,Z)^q Z^{qr_p - pr_q} & \text{if } j = pr_q, \\ F'_{q-r_q}(X,Y,Z)^p Z^{pr_q - qr_p} + F'_{p-r_p}(X,Y,Z)^q & \text{if } j = qr_p. \end{cases} \tag{1.3}$$

We call such a factorization *visible factorization* and D is called *a visible degeneration of* (p,q) *torus curve.* We denote the set of visible degenerations of order j by $\mathcal{LT}_j^V(p,q;pq-j)$ and the union $\cup_j \mathcal{LT}_j^V(p,q;pq-j)$ by $\mathcal{LT}^V(p,q)$.

Example 1. Let $D_1 = \{G_1 = 0\}$ and $D_2 = \{G_2 = 0\}$ be a plane curves of degree 4 which are defined in §1. Recall that the defining polynomials are $G_1(X,Y,Z) = (X-Y)^2(X+Y)^2 - Y^3 Z$ and $G_2(X,Y,Z) = X^4 - Y^3 Z$. We can check easily that D_1 and D_2 are in $\mathcal{LT}_2^V(3,2;4)$.

Invisible case. Either $r_p = 0$ or $r_q = 0$ but F can be written as (1.2). Then D is called *an invisible degeneration of* (p,q) *torus curve.*

In this case, write $F_p^q + F_q^p = \sum_{i=0}^{pq} C_i(X,Y)Z^i$. Then $C_i(X,Y) = 0$ for i is less than $j-1$ and therefore Z^j divides F. We denote the set of invisible degenerations of order j by $\mathcal{LT}_j^I(p,q;pq-j)$ and the union $\cup_j \mathcal{LT}_j^I(p,q;pq-j)$ by $\mathcal{LT}^I(p,q)$.

2.2 The divisibility of Alexander polynomials

Let U be an open neighborhood of 0 in $\mathbb{C}$ and let $\{C_s \mid s \in U\}$ be an analytic family of irreducible curves of degree d which degenerates into $C_0 := D + j\, L_\infty$ $(1 \leq j < d)$ where D is an irreducible curve of degree $d-j$ and L_∞ is a line. We assume that there is a point $B \in L_\infty \setminus L_\infty \cap D$ such that $B \in C_s$ and the multiplicity of C_s at P is j for any non-zero $s \in U$. We call such a degeneration *a line degeneration of order* j and we call L_∞ *the limit line* of the degeneration and B is called *the base point* of the degeneration. In [6], M. Oka showed that there exists a canonical surjection:

$$\varphi : \pi_1(\mathbb{C}^2 \setminus D) \to \pi_1(\mathbb{C}^2 \setminus C_s)$$

where s is a sufficiently small positive real number, $\mathbb{C}^2 = \mathbb{P}^2 \setminus L_\infty$ and as a corollary he showed the divisibility among the Alexander polynomials of a line degeneration family:

$$\Delta_{C_s}(t) \mid \Delta_{D_0}(t).$$

He also showed that a visible type of torus curve of type (p,q) can be expressed as a line degeneration of irreducible torus curves of degree pq. Hence the Alexander polynomial of visible degenerations are not trivial.

2.3 Dual curves and its singularities

Let Σ be a finite set of topological class of singularities and let $\mathcal{M}_d(\Sigma)$ be the configuration space of plane curves of degree d with a fixed singularity configuration Σ as in §1. We say that $\mathcal{I} = (n_1, \dots, n_k)$ is a partition of d if $\sum_{i=1}^k n_i = d$. Let $\mathcal{P}(d)$ be the set of partitions of the integer d. We take $C \in \mathcal{M}_d(\Sigma)$ and we denote the set of singular points $\operatorname{Sing} C$. Recall that *the class formula and the flex formula.* Let $\check{d}$ be the degree of the dual curve $\check{C}$ and $\mathcal{F}(C)$ be the number of the flex points. Then $\check{d}$ and $\mathcal{F}(C)$ are given by the following:

$$\check{d} = d(d-1) - \sum_{P \in \operatorname{Sing} C} (\mu(C,P) + m(C,P) - 1)$$

$$\mathcal{F}(C) = 3d(d-2) - \sum_{P \in \operatorname{Sing} C} I(C, \mathcal{H}(C); P)$$

where $\mu(C,P)$ is the Milnor number of C at P, $m(C,P)$ is the multiplicity of C at P and $\mathcal{H}(C) := \{H = 0\}$ is the Hessian curve of C where the defining polynomial H of the Hessian curve is defined by the determinant of the matrix

$$\begin{pmatrix} F_{X,X} & F_{X,Y} & F_{X,Z} \\ F_{Y,X} & F_{Y,Y} & F_{Y,Z} \\ F_{Z,X} & F_{Z,Y} & F_{Z,Z} \end{pmatrix}$$

where F is the defining polynomial of C and $F_{I,J}$ is the partial differential of variables I and J where $I,\ J \in \{X, Y, Z\}$.

Take a smooth point $P \in C$ and we consider the tangent line L_P of C at P and put the intersection points $L_P \cap C := \{R_1, \dots, R_k\}$. We consider the map $\psi : C \setminus \operatorname{Sing} C \to \mathcal{P}(d)$ which is defined by

$$\psi(P) = (I(C, L_P; R_1), \dots, I(C, L_P R_k)) \in \mathcal{P}(d)$$

where $I(C, L_P; R_j)$ is the intersection multiplicity of C and L_P at R_j. If necessary, we assume $I(C, L_P; R_i) \geq I(C, L_P; R_j)$ if $i < j$. A smooth point $P \in C$ is called *tangentially generic* if $\psi(P) = (2, 1, \dots, 1)$ where G is the Gauss map. Let $\Sigma^{ntg}(C) = \{P_{k+1}, \dots, P_{k+t}\}$ be the set of smooth points which are not tangentially generic and put $\tilde{\Sigma}(C) := \Sigma(C) \cup \Sigma^{ntg}(C)$. It is known that the singularities of dual curves are come from points in $\tilde{\Sigma}(C)$.

Recall basic properties of singularities of dual curves ([4, 3]). We take P in $\tilde{\Sigma}(C)$. First we assume that L_P is not a multi-tangent line of C and $L_P \cap \operatorname{Sing} C = \emptyset$. If (C, P) is topological equivalent to $B_{n,m}$ $(n > m)$ and the Puiseux order of C at P is n/m, then $(\check{C}, \check{P})$ is topological equivalent to $B_{n,n-m}$. Let P be a flex point of flex order $k-2$. Then $(\check{C}, \check{P})$ is topological equivalent to $B_{k,k-1}$ for $k \geq 3$.

Next we assume that $P \in \Sigma^{ntg}(C)$. Put $L_P \cap C := \{R_1, \dots, R_k\}$ and $\psi(P) := (n_1, \dots, n_k) \in \mathcal{P}(d)$. The following Lemma is important for our results.

Lemma 1. *Suppose that $R_1, \dots, R_k$ are smooth points of C and $n_1, \dots, n_k > 2$. Then the dual singularity $(\check{C}, L_P)$ satisfies the following conditions.*

(1) *The singularity $(\check{C}, L_P)$ has k-irreducible components $\check{C}_1, \dots, \check{C}_k$ such that $\check{C}_i$ and $\check{C}_j$ intersect with intersection multiplicity $(n_i - 1)(n_j - 1)$.*

(2) *The singularity $(\check{C}, L_P)$ is a degenerate singularity and its Milnor number of $(\check{C}, L^*)$ is $(d-k)^2 - d + 1$.*

Proof. We may assume that the multi-tangent line L_P is the line at infinity $L_\infty = \{Z = 0\}$. Put $R_i = (\alpha_i, 1, 0)$ for $i = 1, \dots, k$. Let $O^* = (0, 0, 1) \in \check{\mathbb{P}}^2$ be the Gauss image of L_∞.

As the multiplicities $n_1, \dots, n_k > 2$, $(\check{C}, O^*)$ has n-irreducible components which are defined by the union of the Gauss image of (C, R_i) for $i = 1, \dots, k$. Note that the tangent directions at O^* of irreducible components are mutually distinct and $(\check{C}, R_i)$ is topologically equivalent to B_{n_i-1,n_i} as R_i is a flex point with the flex order $n_i - 2$.

Let $(u, v) = (U/W, V/W)$ be the affine coordinate system in $\mathbb{C}^2_W = \check{\mathbb{P}}^2 \setminus \{W = 0\}$. Let $\check{f}(u, v)$ be the defining polynomial of the dual curve $\check{C}$ in this affine space. By the above considerations, we have

$$\check{f}(u, v) = \prod_{i=1}^{k} \check{f}_i(u, v), \quad \check{f}_i(u, v) = (v - \beta_i\, u)^{n_i - 1} + \gamma_i\, u^{n_i} + (\text{higher terms})$$

where $\beta_1, \dots, \beta_d$ are mutually distinct complex numbers. The Newton principal part of $\check{f}(u, v)$ is given by

$$\mathcal{N}(\check{f}; u, v) = \prod_{i=1}^{d} (v - \beta_i\, u)^{n_i - 1}.$$

Then the Newton boundary is consists of the degenerate face Δ with the weight vector $T := {}^t(1, 1)$. After taking toric modifications, we can count its Milnor number using A'Compo Theorem [1]. $\square$

3 Statement of Theorems

In this section, we assume that p and q are positive integers such that $q \mid p-1$ and we put $p_1 := p - \frac{p-1}{q} \in \mathbb{Z}$. Let $\mathcal{P}(p_1)$ be the set of partitions of p_1. For any partition $\mathcal{I} = (\iota_1, \dots, \iota_k) \in \mathcal{P}(p_1)$, we say that k is *the length* of $\mathcal{I}$ and it is denoted by $|\mathcal{I}|$. Put a subset $U(|\mathcal{I}|) \subset (\mathbb{C}^*)^{|\mathcal{I}|}$:

$$U(|\mathcal{I}|) := \{\alpha = (\alpha_1, \dots, \alpha_{|\mathcal{I}|}) \in (\mathbb{C}^*)^{|\mathcal{I}|} \mid \alpha_i \neq \alpha_j \, (i \neq j)\}.$$

For a fixed partition $\mathcal{I} = (\iota_1, \dots, \iota_k)$ in $\mathcal{P}(p_1)$ of the length k, we associate $\alpha = (\alpha_1, \dots, \alpha_k) \in U(|\mathcal{I}|)$ with a (p, q) torus curve $C(\alpha) = \{F_\alpha = 0\}$ where

$$F_\alpha(X,Y,Z) = F_{p,\alpha}(X,Y,Z)^q - F_q(X,Y,Z)^p,$$

$$F_{p,\alpha}(X,Y,Z) = \prod_{i=1}^{k} (X - \alpha_i Y)^{\iota_i} Z^{\frac{p-1}{q}}, \quad F_q(X,Y,Z) = Y^{q-1}Z.$$

Then we can factorize as the following:

$$\begin{aligned} F_\alpha(X,Y,Z) &= F_{p,\alpha}(X,Y,Z)^q - F_q(X,Y,Z)^p \\ &= Z^{p-1}\left(\prod_{i=1}^{k}(X-\alpha_i Y)^{q\iota_i} - Y^{p(q-1)}Z\right). \end{aligned}$$

We consider a visible degeneration $D(\alpha) = \{G_\alpha = 0\}$ of degree $p(q-1)+1$:

$$D(\alpha): \quad G_\alpha(X,Y,Z) = \prod_{i=1}^{k}(X-\alpha_i Y)^{q\iota_i} - Y^{p(q-1)}Z = 0.$$

By the definition, $D(\alpha)$ intersects to the limit line $L_\infty = \{Z = 0\}$ at d-points and $D(\alpha)$ is smooth on L_∞. That is L_∞ is a multi-tangent line of $D(\alpha)$.

For a generic $\alpha \in U(|\mathcal{I}|)$, $D(\alpha)$ has a unique singularity at O which is topological equivalent to the Brieskorn type $\mathcal{B} := B_{p(q-1)+1, p(q-1)}$. Thus $D(\alpha)$ is contained in the space $\mathcal{M} := \mathcal{M}_{p+1}(\mathcal{B})$. For an arbitrary partition $\mathcal{I}$, we define a subspace $\mathcal{M}(\mathcal{I}) \subset \mathcal{M}$ as

$$\mathcal{M}(\mathcal{I}) := \{D(\alpha) \in \mathcal{M} \mid \alpha \in U(|\mathcal{I}|)\}.$$

Example 2. Let D_1, $D_2 \in \mathcal{LT}_2^V(3,2,4)$ be plane curves in Example 1. By the definition, we can show that $D_1 \in \mathcal{M}(\mathcal{I}_g)$ and $D_2 \in \mathcal{M}(\mathcal{I}_m)$ where $\mathcal{I}_g = (1,1)$, $\mathcal{I}_m = (2) \in \mathcal{P}(2)$.

Now we consider degenerations of our curves. Let $\mathcal{I}_g$ be the generic partition p_1. The following Lemma is obviously holds by the definition.

Lemma 2. *For any partition $\mathcal{J} \in \mathcal{P}(p_1)$ of its length $1 \leq i \leq d$, there exists a family of line degenerated torus curves $\{D_t\}$ such that $D_0 \in \mathcal{M}(\mathcal{J})$ and $D_t \in \mathcal{M}(\mathcal{I}_g)$ for $t \neq 0$.*

We will study geometries of a visible torus curve $D(\alpha)$ in $\mathcal{M}(\mathcal{I})$:

1. The configurations of singularities of the dual curve $\check{D}(\alpha)$.
2. The tangential fundamental group $\pi_1(\mathbb{C}^2_{L_\infty} \setminus D(\alpha))$.
3. The tangential Alexander polynomial $\Delta_{D(\alpha)}(t, L_\infty)$.

Our main results are given as the following.

Theorem 1. *Let $\mathcal{I} = (\iota_1, \ldots, \iota_k) \in \mathcal{P}(p_1)$ be a partition of the length k and we take a generic $\alpha \in U(|\mathcal{I}|)$. If $D(\alpha) \in \mathcal{M}(\mathcal{I})$, then the dual singularities of the dual curve $\check{D}(\alpha)$ is generically given as the following:*

$$\Sigma(\check{D}(\alpha)) = [(2k-2)A_2, nA_1, (\check{D}(\alpha), O^*)]$$

where $n = \dfrac{1}{2}(k-1)(2p(q-1)-k-4)$. The dual singularity $(\check{D}(\alpha), O^*)$ satisfies the following:

(1) *It is a degenerate singularity.*

(2) *It has d-components and each components intersect with intersection multiplicity $(q\iota_i - 1)(q\iota_j - 1)$ and its Milnor number $(p(q-1)+1-k)^2 - p(q-1)$.*

Theorem 2. *Put $q = 2$. Let $\mathcal{I}_g = (1, \ldots, 1)$ be the generic partition and $\mathcal{I}_m = (\frac{1}{2}(p+1))$ be the maximal partition of $\frac{1}{2}(p+1)$. Then the tangential fundamental groups are not isomorphic and the tangential Alexander polynomials are different: $\pi_1(\mathbb{C}^2_{L_\infty} \setminus D_g) \not\cong \pi_1(\mathbb{C}^2_{L_\infty} \setminus D_m)$ and $\Delta_{D_g}(t, L_\infty) \neq \Delta_{D_m}(t, L_\infty)$ where $D_g \in \mathcal{M}(\mathcal{I}_g)$ and $D_m \in \mathcal{M}(\mathcal{I}_m)$.*

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